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# FOLD-FORMS FOR FOUR-FOLDS

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This paper explains an application of Gromov's h-principle to prove the existence, on any orientable four-manifold, of a folded symplectic form. That is a closed two-form which is symplectic except on a separating hypersurface where the form singularities are like the pullback of a symplectic form by a folding map. We use the h-principle for folding maps (a theorem of Eliashberg) and the h-principle for symplectic forms on open manifolds (a theorem of Gromov) to show that, for orientable even-dimensional manifolds, the existence of a stable almost complex structure is necessary and sufficient to warrant the existence of a folded symplectic form.

## 1. Introduction

One says that a differential problem satisfies the h-principle if any formal solution (i.e., a solution for the associated algebraic problem) is homotopic to a genuine (i.e., differential) solution. Therefore, when the h-principle holds, one may concentrate on a purely topological question in order to prove the existence of a differential solution.

Differential problems are equations, inequalities or, more generally, relations [**13**] involving derivatives of maps. The following are examples of problems known to satisfy the h-principle: existence of immersions in strictly positive codimension (theorems of Whitney [**30**], Nash [**24**], Kuiper [**16**], Smale [**26**], Hirsch [**14**] and Poénaru [**25**]), existence of symplectic forms on open manifolds (theorem of Gromov [**12**], who built the general machinery of the h-principle as an obstruction theory for the sheaves of germs of maps) and existence of maps whose only singularities are folds (theorem of Eliashberg [**6**, **7**]).

This paper explains an application of the h-principle to prove the existence, on any compact orientable four-manifold, of a folded symplectic form, that is, a closed two-form with only fold singularities as defined below. According to the h-principle philosophy, this proof is divided in two steps:

(1) show that the h-principle holds for this problem, and
(2) show that a formal solution exists.

For the first step, the basic ingredients are the h-principle for maps whose only singularities are folds [**6**, **7**] and the h-principle for symplectic forms on open manifolds [**12**]. This combination is a shortcut based on an idea contained in a book by Eliashberg and Mishachev [**9**]. We thus avoid dealing with the h-principle in its generality.

Here is the flavor of Eliashberg's result. Let $Z$ be a hypersurface in a manifold $M$, that is, a codimension 1 embedded submanifold (this is the meaning of *hypersurface* throughout this paper). A map $f : M \to N$ between manifolds of the same dimension is called a *$Z$-immersion* (or said to *fold along the submanifold* $Z$) if it is regular (i.e., its derivative is invertible) on $M \setminus Z$, and if near any $p \in Z$ and near its image $f(p)$ there are coordinates centered at those points where $f$ becomes

$$(x_1, x_2, \dots, x_n) \longmapsto (x_1^2, x_2, \dots, x_n).$$

A homomorphism $F : TM \to TN$ between tangent bundles is called a *$Z$-monomorphism*, if it is injective on $T(M \setminus Z)$ and on $TZ$, and if there exists a fiber involution $\tau : \mathcal{T} \to \mathcal{T}$ on a tubular neighborhood $\mathcal{T}$ of $Z$ whose set of fixed points is $Z$ and such that $F \circ d\tau = F$. The differential $df : TM \to TN$ of a $Z$-immersion is a $Z$-monomorphism. Eliashberg [**6**] proved that, if every connected component of $M \setminus Z$ is open, then any $Z$-monomorphism $TM \to TN$ is homotopic (within $Z$-monomorphisms $TM \to TN$) to the differential of a $Z$-immersion. In the language of [**13**], the theorem says that, when $M \setminus Z$ is open, $Z$-immersions satisfy the (everywhere $C^0$-dense) h-principle; a $Z$-monomorphism is then called a *formal solution*. For the present application, we require a more general statement [**7**] dealing with foliated target manifolds.

A *folded symplectic form* on a $2n$-dimensional manifold $M$ is a closed two-form $\omega$ which is nondegenerate except on a hypersurface $Z$ called the *folding hypersurface* where, centered at every point $p \in Z$, there are coordinates for $M$ adapted to $Z$ where the form $\omega$ becomes

$$x_1 dx_1 \wedge dx_2 + dx_3 \wedge dx_4 + \cdots + dx_{2n-1} \wedge dx_{2n}.$$

The pullback of a symplectic form by a $Z$-immersion is a folded symplectic form with folding hypersurface $Z$.

A formal solution for the problem of existence of a folded symplectic form turns out to be a stable almost complex structure. Let $M$ be a $2n$-dimensional manifold with a structure of complex vector bundle on $TM \oplus \mathbb{R}^2$, where $\mathbb{R}^2$ denotes the trivial rank 2 real vector bundle over $M$. We will show that $M$ admits folded symplectic forms.

Here is how Gromov's theorem comes in. We embed $M$ as level zero in $M \times \mathbb{R}$. The given stable almost complex structure on $M$ yields a complex hyperplane field on $M \times \mathbb{R}$ and hence an almost complex structure on $M \times \mathbb{R}^2$. Since this manifold is open, Gromov's application of the h-principle [**12**] guarantees the existence of a symplectic form on $M \times \mathbb{R}^2$ inducing almost complex structures in the same homotopy class as the given one. Since $M \times \mathbb{R}$ sits here as a codimension one submanifold, the restriction $\omega_0$ of the symplectic form to this submanifold has maximal rank, i.e., has exactly a one-dimensional kernel at every point. Let $\mathcal{L}$ be the one-dimensional foliation determined by the kernel $L$ of $\omega_0$. The projection of $\omega_0$ to $T(M \times \mathbb{R})/L$ is well-defined and nondegenerate. Suppose that we could immerse $M$ in $M \times \mathbb{R}$ in a *good* way, meaning that locally the composition of that immersion with the projection to the local leaf space of $\mathcal{L}$ is a $Z$-immersion, for some hypersurface $Z$ in $M$. Since this leaf space is symplectic, by pullback we would obtain a folded symplectic form on $M$. Hence, we concentrate on deforming the initial embedding at level zero into a good immersion in order to prove:

**Theorem A.** *Let $M$ be a $2n$-dimensional manifold with a stable almost complex structure $J$. Then $M$ admits a folded symplectic form consistent with $J$ in any degree 2 cohomology class.*

The notion of consistency is explained in Section 2. The existence of a stable almost complex structure is a necessary condition for the existence of a folded symplectic form on an orientable manifold (see Section 2). Theorem A is then saying that it is also sufficient. This contrasts with the case of a (honest) symplectic form, for whose existence an almost complex structure is necessary, but only sufficient if the manifold is open [**12**]. The sphere $S^6$ is a trivial example (thanks to Stokes' theorem) and $\mathbb{CP}^2\#\mathbb{CP}^2\#\mathbb{CP}^2$ is an important example (thanks to Seiberg–Witten invariants [**28**]) of almost complex manifolds without any symplectic form.

To produce a formal solution for four-manifolds is easily accomplished. Hirzebruch and Hopf [**15**] showed that the integral Stiefel–Whitney class $W_3$ vanishes for any compact orientable four-manifold, or, in other words, such manifolds always have stable almost complex structures. (This is the same reason why such manifolds are spin-c [**17**, Theorem D.2].) Since we are in the stable range, it is enough to add a trivial $\mathbb{R}^2$ bundle to $TM$ for this to admit a structure of complex vector bundle. All this is also true when $M$ is not compact [**11**, Section 5.7]. We thus obtain the following relevant special case of Theorem A:

**Theorem B.** *Let $M$ be an orientable four-manifold. Then $M$ admits a folded symplectic form consistent with any given stable almost complex structure and in any degree 2 cohomology class.*

In higher dimensions, there are plenty of orientable manifolds that have no stable almost complex structures ($S^1 \times \mathrm{SU}(3)/\mathrm{SO}(3)$, for instance [**17**]), and hence cannot have folded symplectic forms. The condition $W_3(M) = 0$ is necessary and sufficient in dimensions 6 (since the next obstruction $W_7$ vanishes for dimensional reasons) and 8 (where Massey [**20**] proved that $W_7$ always vanishes). According to [**4**, **29**], until 1998 it was still not known general necessary and sufficient conditions (in terms of invariants such as characteristic classes and the cohomology ring) for the existence of a stable almost complex structure on manifolds of dimension $\geq$10.

As for the contents of this paper: Section 2 reviews folded symplectic manifolds and some *folded* tangent bundles associated to them; Section 3 describes the application of Gromov's theorem to guarantee a symplectic form starting from a structure of complex vector bundle; Section 4 proves the existence of an isomorphism between a *folded* tangent bundle and a suitable complex vector bundle; Section 5 describes the application of Eliashberg's theorem to produce folded symplectic forms; and Section 6 contains the conclusion of the proof of Theorems A and B.

## 2. Folded symplectic manifolds

Let $M$ be an oriented manifold of dimension $2n$, and let $\omega$ be a closed two-form on $M$. The highest wedge power $\omega^n$ is a section of the (trivial) orientation bundle $\wedge^{2n}T^*M$.

**Definition.** A *folded symplectic form* is a closed two-form $\omega$ such that $\omega^n$ intersects the 0-section of $\wedge^{2n}T^*M$ transversally, and such that $\imath^*\omega$ has maximal rank everywhere, where $\imath : Z \hookrightarrow M$ is the inclusion of the zero-locus, $Z$, of $\omega^n$.

By transversality, $Z$ is a codimension-1 submanifold of $M$, called the *folding hypersurface*. A *folded symplectic manifold* is a pair $(M, \omega)$ where $\omega$ is a folded symplectic form on $M$. The folding hypersurface $Z$ of a folded symplectic manifold $(M, \omega)$ separates $M$ into the regions $M^+$ and $M^-$, where the form matches or is opposite to the given orientation, respectively. Hence, $Z$ has a co-orientation depending on $\omega$ and on the choice of orientation on $M$. (The notion of folded symplectic form extends to arbitrary even-dimensional manifolds, not necessarily orientable, but we will not deal with those in this paper.)

The Darboux theorem for folded symplectic forms states that, if $(M,\omega)$ is a folded symplectic manifold and $p$ is any point on the folding hypersurface $Z$, then there is a coordinate chart $(\mathcal{U}, x_1, \ldots, x_{2n})$ centered at $p$ such that on $\mathcal{U}$

$$\omega = x_1 dx_1 \wedge dx_2 + dx_3 \wedge dx_4 + \ldots + dx_{2n-1} \wedge dx_{2n} \text{ and } Z \cap \mathcal{U} = \{x_1 = 0\}.$$

This follows, for instance, from a folded analog of Moser's trick [**3**].

Doubles of symplectic manifolds with $\omega$-convex [**8**] (or $\omega$-concave) boundary are easy examples of manifolds with folded symplectic forms. Simplest instances are the spheres $S^{2n}$, where a folded symplectic form is obtained by pulling back the standard symplectic form on $\mathbb{R}^{2n}$ via the folding map $S^{2n} \to D^{2n}$.

Starting in dimension 4, folded symplectic forms are not generic in the set of closed two-forms. Let $M$ be a (compact) oriented four-manifold, and let $\omega$ be a closed two-form on $M$. If $\gamma$ is a given volume form on $M$, then $\omega \wedge \omega = f\gamma$ for some $f \in C^\infty(M)$. A generic $\omega$ [**18**] is never 0, has rank 2 on a (compact) codimension-1 submanifold, $Z$, and is nondegenerate elsewhere. The hypersurface $Z$ is the 0-locus of $f$. Its complement $M \setminus Z$ is the disjoint union of the sets $M^+ = \{f > 0\}$ where $\omega$ matches the given orientation and $M^- = \{f < 0\}$ where $\omega$ induces the opposite orientation. For $\omega$ to be folded symplectic, we would need that $TZ$ and the rank 2 bundle over $Z$ given by ker $\omega$ intersect transversally as subbundles of $TM|_Z$. Yet generically $\omega$ is not folded symplectic, since its restriction to $Z$ vanishes along some codimension-2 submanifold $C$ (a union of circles), where ker $\omega$ is contained in $TZ$ [**18**]. Although a generic two-form on a three-manifold vanishes only at isolated points, here the three-manifold already depends on the two-form. Moreover, generically there are isolated *parabolic* points on those lines (circles), where the tangent space to those lines is contained in ker $\omega$. There is at least one continuous family of inequivalent neighborhoods of parabolic points [**1**, **10**].

Now let $M$ be an $m$-dimensional manifold with a separating hypersurface $Z$. For instance, $M$ could be an oriented manifold equipped with a folded symplectic form, and $Z$ its folding hypersurface.

The complement $M \setminus Z$ is the disjoint union of open sets $M^+$ and $M^-$. Over $Z$, the tangent bundle has a trivial line subbundle $V$, spanned by a vector field transverse to $Z$ pointing from $M^-$ to $M^+$. The quotient $TM/V$ is isomorphic to $TZ$, so that $TM|_Z \simeq TZ \oplus V$.

**Definition.** The *$Z$-tangent bundle* of $M$ is the rank $m$ real vector bundle ${}^Z TM$ over $M$ obtained by gluing $TM|_{M\setminus M^-}$ to $TM|_{M\setminus M^+}$ by the constant diagonal map $\text{Id} \oplus (-1) : Z \to \text{GL}(TZ \oplus V)$.

There are analytic and algebraic approaches to ${}^{Z}TM$, which enhance its geometry [**3**]. From its definition it follows that:

**Lemma 2.1.** *Let $M$ be an $m$-dimensional manifold with a separating hypersurface $Z$. Then there is an isomorphism of real vector bundles*

$$TM \oplus \mathbb{R} \simeq {}^{Z}TM \oplus \mathbb{R}.$$

A *complex structure* on a vector bundle $E$ over a manifold $M$ is a bundle homomorphism $J : E \to E$ such that $J^2 = -\mathrm{Id}$. If $E$ is an orientable rank $2m$ vector bundle, the existence of a complex structure on $E$ is equivalent to the existence of a section of the associated $(\mathrm{SO}(2m)/U(m))$-bundle. A *stable complex structure* on a vector bundle $E$ over $M$ is an equivalence class of complex structures on the vector bundles $E \oplus \mathbb{R}^k$ ($k \in \mathbb{Z}_0^+$), two complex structures, $J_1$ on $E\oplus\mathbb{R}^{k_1}$ and $J_2$ on $E\oplus\mathbb{R}^{k_2}$, being *equivalent* when there exist $m_1, m_2 \in \mathbb{Z}_0^+$ such that $((E\oplus\mathbb{R}^{k_1})\oplus\mathbb{C}^{m_1}, J_1\oplus \mathrm{i})$ and $((E\oplus\mathbb{R}^{k_2})\oplus\mathbb{C}^{m_2}, J_2\oplus \mathrm{i})$ are isomorphic complex vector bundles. A *stable almost complex structure* on $M$ is a stable complex structure on $TM$.

The $Z$-tangent bundle for the folding hypersurface $Z$ of a folded symplectic form $\omega$ has a canonical complex structure $J_0$ [**3**] *consistent* with $\omega$. We say that a folded symplectic form $\omega$ is *consistent* with a stable almost complex structure on $M$ if $({}^{Z}TM \oplus \mathbb{C}, J_0 \oplus i)$ belongs to the given equivalence class of complex structures on $TM \oplus \mathbb{R}^{2k}$, $k \in \mathbb{Z}_0^+$.

## 3. First instance of the h-principle

Let $M$ be a $2n$-dimensional manifold with a stable almost complex structure. The homotopy groups $\Pi_q(\mathrm{SO}(2m)/U(m))$ are isomorphic for fixed $q$ and variable $m$ such that $q < 2m - 1$ (this is the so-called *stable range* [**19**]). Hence, if there exists a complex structure on $TM \oplus \mathbb{R}^{2k}$, then there exists a complex structure on $TM \oplus \mathbb{R}^2$.

Let $J$ be a complex structure on $TM \oplus \mathbb{R}^2$. Let

$$\begin{array}{rccc} i: & M & \hookrightarrow & M \times \mathbb{R} \\ & p & \mapsto & (p,0) \end{array} \quad \text{and} \quad \begin{array}{rccc} \pi: & M \times \mathbb{R} & \twoheadrightarrow & M \\ & (p,t) & \mapsto & p \end{array}$$

be the embedding at level zero, and the projection to the first factor. By pullback, $i$ induces an isomorphism in cohomology.

Via the identification $T(M \times \mathbb{R}) \simeq \pi^*(TM) \oplus \mathbb{R}$, the structure $J$ induces a structure of complex vector bundle, still called $J$, on $T(M \times \mathbb{R}) \oplus \mathbb{R} \simeq \pi^*(TM) \oplus \mathbb{C}$. Then the complex subbundle

$$H_0 = T(M \times \mathbb{R}) \cap J(T(M \times \mathbb{R})) \subset T(M \times \mathbb{R}) \oplus \mathbb{R}$$

is a complex hyperplane field over $M \times \mathbb{R}$. Let $\omega_1$ be a two-form of maximal rank in $M \times \mathbb{R}$ *compatible with* $J$, that is,

$$\omega_1(u,v) = g(Ju,v) \ , \ \forall u,v \in H_0 \ , \ \text{and } \omega_1(u,\cdot) = 0 \ , \ \forall u \in H_0^{\perp},$$

for some riemannian metric $g$ on $TM \times \mathbb{R}$, where $H_0^{\perp}$ denotes the orthocomplement of $H_0$ with respect to $g$. A *regular homotopy* of 2 two-forms of maximal rank is a homotopy within two-forms of maximal rank.

**Lemma 3.2.** *Let $M$ be a manifold with a structure $J$ of complex vector bundle on $TM \oplus \mathbb{R}^2$. Then there exists in $M \times \mathbb{R}$ a closed two-form of maximal rank in any degree 2 cohomology class, which is regularly homotopic to any two-form of maximal rank compatible with $J$.*

This is an immediate consequence of the following proposition which was originally proved by McDuff [**21**]. The proof below is taken from Eliashberg-Mishachev [**9**]. We reproduce it since this result is not as widely known as the other applications of the h-principle and since the idea in this proof is crucial for the present paper's strategy. The key to this proof is Gromov's theorem [**12**] saying that, for every degree 2 cohomology class on any open manifold, any nondegenerate two-form is regularly homotopic to a symplectic form in that class; moreover, if two symplectic forms are regularly homotopic, then they are homotopic within symplectic forms. Recall that a manifold is *open* if there are no closed manifolds (i.e., compact and without boundary) among its connected components.

**Proposition ([21]).** *For any two-form of maximal rank on an odd-dimensional manifold and any degree 2 cohomology class, there exists a closed two-form of maximal rank in that class which is regularly homotopic to the given form.*

*Proof.* Let $\omega_1$ be a two-form of maximal rank on a $(2n+1)$-dimensional manifold $N$ and let $\alpha$ be a degree 2 cohomology class in $N$. By homotopy, the projection to the first factor $\pi : N \times \mathbb{R} \to N$ induces an isomorphism in cohomology.

If $N$ is orientable, then $\omega_1$ extends in a homotopically unique way compatible with orientations to a nondegenerate two-form, $\omega_2$, in $N \times \mathbb{R}$. Gromov's result [**12**] cited above guarantees the existence, in the class $\pi^*\alpha$, of a homotopically unique symplectic form $\omega_3$ in $N \times \mathbb{R}$ regularly homotopic to $\omega_2$. The restriction of $\omega_3$ to the zero level $M$ is a closed two-form of maximal rank.

If $N$ is not orientable, we replace $N \times \mathbb{R}$ in the previous argument by the total space of the real line bundle given by the kernel of $\omega_1$. □

## 4. Vector bundle isomorphism

Let $\widetilde{\omega}$ be a closed two-form of maximal rank in $M \times \mathbb{R}$, and let $L$ be the line field on $M \times \mathbb{R}$ given by the kernel of $\widetilde{\omega}$ at each point. By orientability of $M$, the line bundle $L$ is trivializable. Let $\mathcal{L}$ be the one-dimensional foliation corresponding to $L$. Choose a complementary hyperplane field $H$ so that $T(M \times \mathbb{R}) \simeq H \oplus L$.

Let $Z_0$ be a separating hypersurface in $M$ with a coorientation. Since by Lemma 2.1 we have that

$$^{Z_0}TM \oplus \mathbb{R} \simeq TM \oplus \mathbb{R} \simeq i^*(H \oplus L),$$

the restriction $i^*H$ is stably isomorphic to $^{Z_0}TM$. The Stiefel–Whitney classes are stable invariants, and the mod 2 reduction of the Euler class of an orientable rank $m$ real vector bundle $E$ coincides with the $m$th Stiefel–Whitney class of $E$ (see, for instance, [**23**]). Therefore, the Euler numbers (i.e., the evaluations of the Euler classes over the fundamental homology class) of $i^*H$ and of $^{Z_0}TM$ differ by an even integer, let us say

$$\chi(i^*H) = \chi(^{Z_0}TM) + 2k.$$

If two stably isomorphic orientable rank $2n$ real vector bundles over an $2n$-dimensional connected manifold have the same Euler number, then they are isomorphic. This was contained in the work of Dold and Whitney when the base is a four-manifold [**5**]. In general, this follows from observing in the diagram

$$\begin{array}{ccl} & & S^{2n} \hookrightarrow \mathrm{SO}/\mathrm{SO}(2n) \\ & \nearrow & \downarrow \\ M^{2n} & \rightrightarrows & \mathrm{BSO}(2n) \\ & \searrow & \downarrow \\ & & \mathrm{BSO} \end{array}$$

that the fiber $\mathrm{SO}/\mathrm{SO}(2n)$ of $\mathrm{BSO}(2n) \to \mathrm{BSO}$ is $(2n-1)$-connected, that $[M^{2n}, S^{2n}] \simeq \mathbb{Z}$ where the homotopy type is detected by the degree, and that the pullback of the Euler class to $S^{2n}$ is nontrivial (since $S^{2n} \to \mathrm{BSO}(2n)$ is the classifying map for $TS^{2n}$).

Consider the following operation on rank $m$ real vector bundles over $m$-dimensional manifolds. If $E$ is such a bundle and $D^m$ is a small disk in the base manifold $M$, let $E\sharp TS^m$ be the bundle obtained by gluing $E|_{M\setminus \mathrm{Int}\, D^m}$ to the trivial bundle $\mathbb{R}^m$ over $D^m$ by the characteristic map of $TS^m$, i.e., by the map $S^{m-1} \to \mathrm{SO}(m)$ which characterizes the tangent bundle of $S^m$ as the gluing over the equator of northern and southern trivial bundles [**27**, Section 18.1]. For an integer $k$, the bundle $E\sharp kTS^m$ is built analogously by taking the $k$th power of the characteristic map of $S^m$. By counting with orientations the vanishing points of a section transverse to zero, we see that $E\sharp kTS^m$ has Euler characteristic $\chi(E) + 2k$. We conclude that

$$i^*H \simeq {}^{Z_0}TM\sharp kTS^{2n}.$$

For $k$ positive, let $Z$ be the union of $Z_0$ with $k$ homologically trivial spheres $S^n$ contained in the negative part of $M \setminus Z_0$ with respect to the given coorientation. For $k$ negative, define $Z$ similarly but with the spheres in the positive part of $M \setminus Z_0$. It follows from the computations in [**6**, Section 3.9] that $i^*H$ and ${}^Z TM$ have the same Euler number, and hence are isomorphic. It is possible to start from the empty hypersurface, in which case a coorientation is not defined. Yet the same argument holds by taking $Z$ to be a union of spheres (as many as half of the absolute value of the difference of the Euler numbers of $TM$ and of $i^*H$) whose coorientation is determined by the sign of $k$ above. We have thus proved the following:

**Lemma 4.3.** *Let $H$ be a coorientable hyperplane field in $M \times \mathbb{R}$ and $i : M \hookrightarrow M \times \mathbb{R}$ the inclusion at level zero. The restriction $i^*H$ is isomorphic to ${}^Z TM$, where $Z$ is a separating hypersurface as described in the previous paragraph.*

## 5. Second instance of the h-principle

Throughout this section, let $M$ be an $m$-dimensional manifold with a hypersurface $Z$, and let $N$ be an $(m+1)$-dimensional manifold with a one-dimensional foliation $\mathcal{L}$. The following notions are due to Eliashberg [**7**].

**Definition.** A map $f : M \to N$ is a *$Z$-immersion relative to $\mathcal{L}$*, if near any point $p \in M \setminus Z$ there are coordinates $y_1, \dots, y_{m+1}$ in $N$ adapted to the foliation (i.e. each leaf is a level set of the first $m$ coordinates) where the induced map to each level set of $y_{m+1}$ is regular, and if near any $p \in Z$ and near its image there are coordinates centered at those points and adapted to the foliation where $f$ becomes

$$(x_1, x_2, \dots, x_m) \longmapsto (x_1^2, x_2, \dots, x_m, 0).$$

In the adapted coordinates $x_i$, the hypersurface $Z$ is given by $x_1 = 0$. Loosely speaking, a $Z$-immersion relative to $\mathcal{L}$ is a $Z$-immersion to the leaf space of $\mathcal{L}$. The definition extends to higher-dimensional foliations whose codimension is equal to the dimension of $M$.

**Lemma 5.4.** *Let $\widetilde{\omega}$ be a closed two-form of maximal rank in $N$ whose kernel is the tangent space to the leaves of $\mathcal{L}$. If $f : M \to N$ is a $Z$-immersion relative to $\mathcal{L}$, then $f^*\widetilde{\omega}$ is a folded symplectic form on $M$ with folding hypersurface $Z$.*

The reason is simply that the form $\widetilde{\omega}$ induces a symplectic form in the local leaf spaces and that the composition of $f$ with the local quotient maps is a $Z$-immersion.

*Proof.* Let $p \in M$. There is a neighborhood $\mathcal{U}$ of $f(p)$ where we have a trivialization $\mathcal{U} \simeq \mathcal{F}_{\mathcal{U}} \times \mathcal{L}_{\mathcal{U}}$, given in local coordinates centered at $f(p)$ by $(x_1, \dots, x_{m+1}) \mapsto ((x_1, \dots, x_m), x_{m+1})$, the set $\mathcal{F}_{\mathcal{U}}$ being a leaf space (say the level zero of $x_{m+1}$), and $\mathcal{L}_{\mathcal{U}}$ a typical leaf (say the level zero of $(x_1, \dots, x_m)$). The restriction of $\widetilde{\omega}$ to $\mathcal{F}_{\mathcal{U}}$ is a symplectic form, $\omega_{\mathcal{U}}$. The composition $g_{\mathcal{U}} : f^{-1}(\mathcal{U}) \to \mathcal{F}_{\mathcal{U}}$ of $f$ with the projection to $\mathcal{F}_{\mathcal{U}}$ is a $(Z \cap \mathcal{U})$-immersion, so that $g_{\mathcal{U}}^* \omega_{\mathcal{U}}$ is a folded symplectic form with folding hypersurface $Z \cap \mathcal{U}$. The result follows from the fact that $f^* \widetilde{\omega}$ on $f^{-1}(\mathcal{U})$ coincides with $g_{\mathcal{U}}^* \omega_{\mathcal{U}}$. □

We now turn to the formal analog of a $Z$-immersion.

**Definition.** A bundle map $F : TM \to TN$ is a *$Z$-monomorphism relative to $\mathcal{L}$*, if $F|_{T(M \setminus Z)}$ is transverse to $\mathcal{L}$, and if each $p \in Z$ admits a neighborhood $\mathcal{U}$ where $F|_{T\mathcal{U}}$ is the differential of some $(Z \cap \mathcal{U})$-immersion relative to $\mathcal{L}$.

The following lemma is a direct consequence of Eliashberg's result in [**7**, Section 6.3], where he extends to the case of foliations the result described in the introduction.

**Lemma 5.5.** *Let $N = M \times \mathbb{R}$ be equipped with a decomposition $TN \simeq H \oplus L$, where $L$ is a line field, and let $\mathcal{L}$ be the corresponding one-dimensional foliation. Let the hypersurface $Z$ be such that every connected component of $M \setminus Z$ is open. Then, for every $Z$-monomorphism $F : TM \to TN$ relative to $\mathcal{L}$, there exists a $Z$-immersion $f : M \to N$ relative to $\mathcal{L}$ whose differential $df$ is homotopic to $F$ through $Z$-monomorphisms relative to $\mathcal{L}$.*

Part of the work to prove Theorem A consists in showing a (general) procedure to deform by homotopy a weaker bundle map into a $Z$-monomorphism relative to $\mathcal{L}$. The weaker map is of the following type:

**Definition.** A bundle map $F : TM \to TN \simeq H \oplus L$ is a *$Z$-monomorphism relative to $L$*, if $\pi_L \circ F|_{T(M \setminus Z)}$ and $\pi_L \circ F|_{TZ}$ are fiberwise injective, $\pi_L : TN \to H$ being the projection along $L$, and if there is a tubular neighborhood $\mathcal{T}$ of $Z$ in $M$, with a fiber involution $\tau : \mathcal{T} \to \mathcal{T}$ whose set of fixed points is $Z$, where $F \circ d\tau = F$.

## 6. Conclusion of the proof

Let $M$ be a compact $2n$-dimensional manifold with a stable almost complex structure $J$. Then $J$ is representable by a structure of complex vector bundle on $TM \oplus \mathbb{R}^2$, and any two such representatives are isomorphic, by Bott periodicity [**2**]. Let $N = M \times \mathbb{R}$ and denote still by $J$ an induced structure of complex vector bundle on $TN \oplus \mathbb{R}$ as in Section 3.

By Lemma 3.2, there exists on $N$, in any degree 2 cohomology class, a closed two-form $\widetilde{\omega}$ of maximal rank compatible with $J$. Let $\widetilde{\omega}$ be such a form and let $L$ be the line field given by its kernel, with associated foliation $\mathcal{L}$.

By Lemma 5.4, the existence of a folded symplectic form on $M$ with some folding hypersurface $Z$ is guaranteed by the existence of a $Z$-immersion $f : M \to N$ relative to $\mathcal{L}$. We will seek such a $Z$-immersion which is homotopic to the embedding at level zero $i : M \hookrightarrow N$, so that $f^* = i^*$ in cohomology. If $M$ is connected and $Z$ is nonempty, then $M \setminus Z$ is open.

By Lemma 5.5, in order to produce a $Z$-immersion $f$ relative to $\mathcal{L}$ for $M \setminus Z$ open, it suffices to show that there exists a $Z$-monomorphism $F : TM \to TN$ relative to $\mathcal{L}$. So that $f$ is homotopic to $i$, we search for an $F$ covering a map $M \to N$ homotopic to $i$.

By Lemma 4.3, we have a vector bundle isomorphism $F_0 : {}^Z TM \to i^* H$ for some hypersurface $Z$, which may be chosen so that each connected component of $M \setminus Z$ is open.

The map $F_0$ may be translated into a fiberwise injective bundle map $F_1 : {}^Z TM \to H$ covering the immersion $i : M \to N$. This map guarantees the existence of a (canonically unique up to homotopy) almost $Z$-monomorphism $F_2 : TM \to H \oplus L$ relative to $L$, still covering $i$, defined by the following recipe:

Choose a trivial line bundle $V$ over $Z$ spanned by a vector field on $M$ transverse to $Z$ pointing from $M^-$ to $M^+$. The quotient ${}^Z TM/V$ is isomorphic to $TZ$, so that ${}^Z TM|_Z \simeq TZ \oplus V$. We obtain $TM$ by gluing ${}^Z TM|_{M \setminus M^-}$ to ${}^Z TM|_{M \setminus M^+}$ by the constant diagonal map $\mathrm{Id} \oplus (-1) : Z \to \mathrm{GL}(TZ \oplus V)$. Using this recovery of $TM$ from ${}^Z TM$, we may define $F_2$ equal to $F_1 \oplus 0$ outside a tubular neighborhood $\mathcal{T}$ of $Z$ in $M$, and on $\mathcal{T}$ set

$$F_2(u \oplus v) = F_1(u \oplus \psi v) \oplus 0,$$

with respect to the decomposition ${}^Z TM|_{\mathcal{T}} \simeq \pi^*(TZ) \oplus \pi^* V$, where $\pi : \mathcal{T} \to Z$ is the tubular projection, and $\psi : \mathcal{T} \to [0,1]$ is equal to 1 outside a narrower tubular neighborhood of $Z$ and vanishes exactly over $Z$. By choosing $\psi$ symmetric with respect to an involution $\tau : \mathcal{T} \to \mathcal{T}$ whose set of fixed points is $Z$, we obtain $F_2$ invariant under $\tau$.

For each $p \in Z$, choose a connected neighborhood $\mathcal{U}$ whose image $i(\mathcal{U})$ is contained in a connected trivialization $\mathcal{N}_{\mathcal{U}} \simeq \mathcal{F}_{\mathcal{U}} \times \mathcal{L}_{\mathcal{U}}$ of the foliation $\mathcal{L}$, the set $\mathcal{F}_{\mathcal{U}}$ being a local leaf space and $\mathcal{L}_{\mathcal{U}}$ a leaf segment. Let $\pi_{\mathcal{U}} : \mathcal{N}_{\mathcal{U}} \to \mathcal{F}_{\mathcal{U}}$ be the projection to the first factor. The composition $F_{2,\mathcal{U}} = d\pi_{\mathcal{U}} \circ F_2|_{\mathcal{U}} : T\mathcal{U} \to T\mathcal{F}_{\mathcal{U}}$ is a $(Z \cap \mathcal{U})$-monomorphism.

$$\begin{array}{ccc} & & T\mathcal{N}_{\mathcal{U}} \\ & {\scriptstyle F_2} \nearrow & \downarrow {\scriptstyle d\pi_{\mathcal{U}}} \\ T\mathcal{U} & \xrightarrow{F_{2,\mathcal{U}}} & T\mathcal{F}_{\mathcal{U}} \end{array}$$

By Eliashberg [**6**, Section 2.2], the composition $F_{2,\mathcal{U}}$ is homotopic, through $(Z \cap \mathcal{U})$-monomorphisms, to the differential $dg_{\mathcal{U}}$ of a $Z$-immersion $g_{\mathcal{U}} : \mathcal{U} \to \mathcal{F}_{\mathcal{U}}$. Moreover, if over a closed subset $\mathcal{W} \subset \mathcal{U}$, the composition $F_{2,\mathcal{U}}$ was already the differential of a map, then there is a homotopy which is constant on $\mathcal{W}$. Let $G_t : T\mathcal{U} \to T\mathcal{F}_{\mathcal{U}}$, $1 \leq t \leq 2$, be a homotopy such that $G_1 = dg_{\mathcal{U}}$ and $G_2 = F_{2,\mathcal{U}}$.

Choose a $(Z \cap \mathcal{U})$-immersion $\widetilde{g}_{\mathcal{U}} : \mathcal{U} \to \mathcal{N}_{\mathcal{U}}$ relative to $\mathcal{L}$ such that $\pi_{\mathcal{U}} \circ \widetilde{g}_{\mathcal{U}} = g_{\mathcal{U}}$. We can always pick a $\widetilde{g}_{\mathcal{U}}$ extending a sensible preassigned lift over a closed subset $\mathcal{W}$ of $\mathcal{U}$.

By the covering homotopy property for the fibering $T\mathcal{N}_{\mathcal{U}} \to T\mathcal{F}_{\mathcal{U}}$, there is a lifted homotopy $\widetilde{G}_t : T\mathcal{U} \to T\mathcal{N}_{\mathcal{U}}$, $1 \leq t \leq 2$, through $Z$-monomorphisms relative to $L$ such that $\widetilde{G}_1 = d\widetilde{g}_{\mathcal{U}}$ and $d\pi_{\mathcal{U}} \circ \widetilde{G}_t = G_t$ for all $t$. If $G_t$ was constant on a closed subset $\mathcal{W}$, then we may choose $\widetilde{G}_t$ also constant on $\mathcal{W}$.

$$\begin{array}{ccc} & & T\mathcal{N}_{\mathcal{U}} \\ & \overset{\widetilde{G}_t}{\nearrow} & \downarrow_{d\pi_{\mathcal{U}}} \\ T\mathcal{U} & \xrightarrow{G_t} & T\mathcal{F}_{\mathcal{U}} \end{array}$$

Since $d\pi_{\mathcal{U}} \circ \widetilde{G}_2 = G_2 = F_{2,\mathcal{U}} = d\pi_{\mathcal{U}} \circ F_2$, the difference $\widetilde{G}_2 - F_2$ takes values in $L = \ker d\pi_{\mathcal{U}}$. By fiberwise homotopy, we may deform the vertical component of $\widetilde{G}_2$ to make it equal to $F_2$. Without loss of generality, we hence assume that $\widetilde{G}_t$ also satisfies $\widetilde{G}_2 = F_2$, and that all maps are invariant with respect to the same involution $\tau$.

Take a riemannian metric symmetric with respect to $\tau$. For a point $p \in Z$, choose spherical neighborhoods $\mathcal{U}_1$ and $\mathcal{U}_2$ in $\mathcal{T}$, consisting of points at a riemannian distance less than $\varepsilon$ and $4\varepsilon$ from $p$, with $\varepsilon > 0$ small enough for the exponential map to be injective and for the closure of $\mathcal{U}_2$ to be contained in the neighborhood $\mathcal{U}$ above. Choose a smooth function $\rho : \mathcal{U}_2 \to [1, 2]$ satisfying $\rho(q) = 2$ if the distance from $p$ to $q$ is greater than $3\varepsilon$, and $\rho(q) = 1$ if the distance from $p$ to $q$ is less than $2\varepsilon$. Define $F_3 : TM \to TN$ by

$$F_3 = \begin{cases} F_2 & \text{on } M \setminus \mathcal{U}_2, \\ \widetilde{G}_{\rho(q)} & \text{over points } q \in \mathcal{U}_2 \setminus \mathcal{U}_1, \\ d\widetilde{g}_{\mathcal{U}} & \text{on } \mathcal{U}_1. \end{cases}$$

Then $F_3$ is a $Z$-monomorphism with respect to $L$ whose restriction to $\mathcal{U}_1$ is the differential of a $(Z \cap \mathcal{U}_1)$-immersion relative to $\mathcal{L}$.

Since $Z$ is compact, take a subcover of $Z$ in $M$ by a finite number of the $\mathcal{U}_1$'s. Apply iteratively the construction of the previous paragraph to an ordering of the $\mathcal{U}_1$'s, starting first from $F_2$ and then from its replacements $F_3$, etc. At each stage, the homotopy should be taken constant over the closure $\mathcal{W}$ of the previous $\mathcal{U}_1$'s.

We have thus concluded the proof of Theorem A in the compact case by showing the existence of a $Z$-monomorphism relative to $\mathcal{L}$ covering a map homotopic to $i$.

**Remark.** If $M$ is a compact oriented two-dimensional manifold, folded symplectic forms on $M$ are generic two-forms. The cohomology class of a two-form is determined by its total integral. The isomorphism classes of complex structures on $TM \oplus \mathbb{R}^2$ are determined by the Euler number, which is an even integer. By changing $Z$ as in Section 4, any even number may be obtained as Euler number for ${}^Z TM$, thus fitting any given stable complex structure. Let $\omega$ be a two-form which vanishes transversally on an appropriate $Z$. By changing the values of $\omega$ over $M \setminus Z$, any real number may be obtained as total integral of $\omega$. Hence, Theorem A holds easily (and not interestingly) for compact two-manifolds.

For the noncompact case, a statement stronger than Theorem A is true. If a $2n$-dimensional manifold $M$ is orientable, connected, not compact and $TM \oplus \mathbb{R}^2$ has a complex structure, then $M$ has an almost complex structure because it retracts to a $(2n-1)$-dimensional cell complex [**22**, Theorem 8.1] and $\Pi_q(\mathrm{SO}(2n)/U(n)) \simeq \Pi_q(\mathrm{SO}(2n+2)/U(n+1))$ for $q \leq 2n-2$. By Gromov's theorem [**12**], $M$ admits a compatible symplectic form in any degree 2 cohomology class.

Let $E$ be a rank $2m$ oriented real bundle over $M$. The condition $W_3(E) = 0$ ensures the existence over the three-skeleton of $M$ of a section for the associated $(\mathrm{SO}(2m)/U(m))$-bundle. By Bott's periodicity, $\Pi_q(\mathrm{SO}(6)/U(3)) = 0$ for $q < 5$. Therefore, the Hirzebruch–Hopf fact [**15**] that $W_3(M) = 0$ for any orientable four-manifold asserts the existence of a stable complex structure on any such manifold.

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