CHAPTER 3

Symplectic Geometry

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Introduction

This is an overview of symplectic geometry[1]—the geometry of *symplectic manifolds*. From a language for classical mechanics in the XVIII century, symplectic geometry has matured since the 1960's to a rich and central branch of differential geometry and topology. A current survey can thus only aspire to give a partial flavor on this exciting field. The following six topics have been chosen for this handbook:

1. *Symplectic manifolds* are manifolds equipped with *symplectic forms*. A symplectic form is a closed nondegenerate 2-form. The algebraic condition (nondegeneracy) says that the top exterior power of a symplectic form is a volume form, therefore symplectic manifolds are necessarily even-dimensional and orientable. The analytical condition (closedness) is a natural differential equation that forces all symplectic manifolds to being locally indistinguishable: they all locally look like an even-dimensional Euclidean space equipped with the $\sum dx_i \wedge dy_i$ symplectic form. All cotangent bundles admit canonical symplectic forms, a fact relevant for analysis of differential operators, dynamical systems, classical mechanics, etc. Basic properties, major classical examples, equivalence notions, local normal forms of symplectic manifolds and symplectic submanifolds are discussed in Section 1.

2. *Lagrangian submanifolds*[2] are submanifolds of symplectic manifolds of half dimension and where the restriction of the symplectic form vanishes identically. By the *Lagrangian creed* [137], everything is a Lagrangian submanifold, starting with closed 1-forms, real functions modulo constants and symplectomorphisms (diffeomorphisms that respect the symplectic forms). Section 2 also describes normal neighborhoods of Lagrangian submanifolds with applications.

3. *Complex structures* or almost complex structures abound in symplectic geometry: any symplectic manifold possesses almost complex structures, and even so in a *compatible* sense. This is the point of departure for the modern technique of studying pseudoholomorphic curves, as first proposed by Gromov [64]. Kähler geometry lies at the intersection of complex, Riemannian and symplectic geometries, and plays a central role in these three fields. Section 3 includes the local normal form for Kähler manifolds and a summary of Hodge theory for Kähler manifolds.

4. *Symplectic geography* is concerned with existence and uniqueness of symplectic forms on a given manifold. Important results from Kähler geometry remain true in the more general symplectic category, as shown using pseudoholomorphic methods. This viewpoint was more recently continued with work on the existence of certain symplectic

[1]The word *symplectic* in mathematics was coined in the late 1930's by Weyl [142, p. 165] who substituted the Latin root in *complex* by the corresponding Greek root in order to label the symplectic group (first studied by Abel). An English dictionary is likely to list *symplectic* as the name for a bone in a fish's head.

[2]The name *Lagrangian manifold* was introduced by Maslov [93] in the 1960's, followed by *Lagrangian plane*, etc., introduced by Arnold [2].

submanifolds, in the context of Seiberg–Witten invariants, and with topological descriptions in terms of Lefschetz pencils. Both of these directions are particularly relevant to 4-dimensional topology and to mathematical physics, where symplectic manifolds occur as building blocks or as key examples. Section 4 treats constructions of symplectic manifolds and invariants to distinguish them.

5. *Hamiltonian geometry* is the geometry of symplectic manifolds equipped with a *moment map*, that is, with a collection of quantities conserved by symmetries. With roots in Hamiltonian mechanics, moment maps became a consequential tool in geometry and topology. The notion of a moment map arises from the fact that, to any real function on a symplectic manifold, is associated a vector field whose flow preserves the symplectic form and the given function; this is called the *Hamiltonian vector field* of that (Hamiltonian) function. The Arnold conjecture in the 60's regarding Hamiltonian dynamics was a major driving force up to the establishment of Floer homology in the 80's. Section 5 deals mostly with the geometry of moment maps, including the classical Legendre transform, integrable systems and convexity.

6. *Symplectic reduction* is at the heart of many symplectic arguments. There are infinite-dimensional analogues with amazing consequences for differential geometry, as illustrated in a symplectic approach to Yang–Mills theory. Symplectic toric manifolds provide examples of extremely symmetric symplectic manifolds that arise from symplectic reduction using just the data of a polytope. All properties of a symplectic toric manifold may be read from the corresponding polytope. There are interesting interactions with algebraic geometry, representation theory and geometric combinatorics. The variation of reduced spaces is also addressed in Section 6.

1. Symplectic manifolds

1.1. *Symplectic linear algebra*

Let V be a vector space over $\mathbb{R}$, and let $\Omega : V \times V \to \mathbb{R}$ be a skew-symmetric bilinear map. By a skew-symmetric version of the Gram–Schmidt process,[3] there is a basis $u_1, \dots, u_k, e_1, \dots, e_n, f_1, \dots, f_n$ of V for which $\Omega(u_i, v) = \Omega(e_i, e_j) = \Omega(f_i, f_j) = 0$ and $\Omega(e_i, f_j) = \delta_{ij}$ for all i, j and all $v \in V$. Although such a basis is not unique, it is commonly referred to as a *canonical basis*. The dimension k of the subspace $U = \{u \in V \mid \Omega(u, v) = 0 \text{ for all } v \in V\}$ is an invariant of the pair (V, Ω). Since $k + 2n = \dim V$, the even number $2n$ is also an invariant of (V, Ω), called the *rank* of Ω. We denote by $\tilde{\Omega} : V \to V^*$ the linear map defined by $\tilde{\Omega}(v)(u) := \Omega(v, u)$. We say that Ω is *symplectic* (or *nondegenerate*) if the associated $\tilde{\Omega}$ is bijective (i.e., the kernel U of $\tilde{\Omega}$ is the trivial space $\{0\}$). In that case, the map Ω is called a *linear symplectic structure* on V, and the

[3] Let $u_1, \dots, u_k$ be a basis of $U := \{u \in V \mid \Omega(u, v) = 0 \text{ for all } v \in V\}$, and W a complementary subspace such that $V = U \oplus W$. Take any nonzero $e_1 \in W$. There is $f_1 \in W$ with $\Omega(e_1, f_1) = 1$. Let W_1 be the span of e_1, f_1 and $W_1^{\Omega} := \{v \in V \mid \Omega(v, u) = 0\ \forall u \in W_1\}$. Then $W = W_1 \oplus W_1^{\Omega}$. Take any nonzero $e_2 \in W_1^{\Omega}$. There is $f_2 \in W_1^{\Omega}$ for which $\Omega(e_2, f_2) = 1$. Let W_2 be the span of e_2, f_2, and so on.

pair (V, Ω) is called a *symplectic vector space*. A linear symplectic structure Ω expresses a *duality* by the bijection $\tilde{\Omega} : V \xrightarrow{\simeq} V^*$, similar to the (symmetric) case of an inner product. By considering a canonical basis, we see that the dimension of a symplectic vector space (V, Ω) *must be even*, $\dim V = 2n$, and that V admits a basis $e_1, \dots, e_n, f_1, \dots, f_n$ satisfying $\Omega(e_i, f_j) = \delta_{ij}$ and $\Omega(e_i, e_j) = 0 = \Omega(f_i, f_j)$. Such a basis is then called a *symplectic basis* of (V, Ω), and, in terms of exterior algebra, $\Omega = e_1^* \wedge f_1^* + \cdots + e_n^* \wedge f_n^*$, where $e_1^*, \dots, e_n^*, f_1^*, \dots, f_n^*$ is the dual basis. With respect to a symplectic basis, the map Ω is represented by the matrix

$$\begin{bmatrix} 0 & \mathrm{Id} \\ -\mathrm{Id} & 0 \end{bmatrix}.$$

EXAMPLES.

1. The *prototype of a symplectic vector space* is $(\mathbb{R}^{2n}, \Omega_0)$ with Ω_0 such that the canonical basis $e_1 = (1, 0, \dots, 0), \dots, e_n, f_1, \dots, f_n = (0, \dots, 0, 1)$ is a symplectic basis. Bilinearity then determines Ω_0 on other vectors.
2. For any real vector space E, the direct sum $V = E \oplus E^*$ has a *canonical symplectic structure* determined by the formula $\Omega_0(u \oplus \alpha, v \oplus \beta) = \beta(u) - \alpha(v)$. If $e_1, \dots, e_n$ is a basis of E, and $f_1, \dots, f_n$ is the dual basis, then $e_1 \oplus 0, \dots, e_n \oplus 0, 0 \oplus f_1, \dots, 0 \oplus f_n$ is a symplectic basis for V.

Given a linear subspace W of a symplectic vector space (V, Ω), its *symplectic orthogonal* is the subspace $W^\Omega := \{v \in V \mid \Omega(v, u) = 0 \text{ for all } u \in W\}$. By nondegeneracy, we have $\dim W + \dim W^\Omega = \dim V$ and $(W^\Omega)^\Omega = W$. For subspaces W and Y, we have $(W \cap Y)^\Omega = W^\Omega + Y^\Omega$, and if $W \subseteq Y$ then $Y^\Omega \subseteq W^\Omega$.

There are special types of linear subspaces of a symplectic vector space (V, Ω). A subspace W is a *symplectic subspace* if the restriction $\Omega|_W$ is nondegenerate, that is, $W \cap W^\Omega = \{0\}$, or equivalently $V = W \oplus W^\Omega$. A subspace W is an *isotropic subspace* if $\Omega|_W \equiv 0$, that is, $W \subseteq W^\Omega$. A subspace W is a *coisotropic subspace* if $W^\Omega \subseteq W$. A subspace W is a *Lagrangian subspace* if it is both isotropic and coisotropic, or equivalently, if it is an isotropic subspace with $\dim W = \frac{1}{2} \dim V$. A basis $e_1, \dots, e_n$ of a Lagrangian subspace can be extended to a symplectic basis: choose f_1 in the symplectic orthogonal to the linear span of $\{e_2, \dots, e_n\}$, etc.

EXAMPLES.

1. For a symplectic basis as above, the span of e_1, f_1 is symplectic, that of e_1, e_2 isotropic, that of $e_1, \dots, e_n, f_1$ coisotropic, and that of $e_1, \dots, e_n$ Lagrangian.
2. The graph of a linear map $A : E \to E^*$ is a Lagrangian subspace of $E \oplus E^*$ with the canonical symplectic structure if and only if A is symmetric (i.e., $(Au)v = (Av)u$). Therefore, the Grassmannian of all Lagrangian subspaces in a $2n$-dimensional symplectic vector space has dimension $\frac{n(n+1)}{2}$.

A *symplectomorphism* φ between symplectic vector spaces (V, Ω) and (V', Ω') is a linear isomorphism $\varphi : V \xrightarrow{\simeq} V'$ such that $\varphi^* \Omega' = \Omega$.[4] If a symplectomorphism exists,

[4]By definition, $(\varphi^* \Omega')(u, v) = \Omega'(\varphi(u), \varphi(v))$.

(V, Ω) and (V', Ω') are said to be *symplectomorphic*. Being symplectomorphic is clearly an equivalence relation in the set of all even-dimensional vector spaces. The existence of canonical bases shows that every $2n$-dimensional symplectic vector space (V, Ω) is symplectomorphic to the prototype $(\mathbb{R}^{2n}, \Omega_0)$; a choice of a symplectic basis for (V, Ω) yields a symplectomorphism to $(\mathbb{R}^{2n}, \Omega_0)$. Hence, nonnegative even integers classify equivalence classes for the relation of being symplectomorphic.

Let $\Omega(V)$ be the space of all linear symplectic structures on the vector space V. Take a $\Omega \in \Omega(V)$, and let $\mathrm{Sp}(V, \Omega)$ be the *group of symplectomorphisms* of (V, Ω). The group $\mathrm{GL}(V)$ of all isomorphisms of V acts *transitively* on $\Omega(V)$ by pullback (i.e., all symplectic structures are related by a linear isomorphism), and $\mathrm{Sp}(V, \Omega)$ is the stabilizer of the given Ω. Hence, $\Omega(V) \simeq \mathrm{GL}(V)/\mathrm{Sp}(V, \Omega)$.

1.2. *Symplectic forms*

Let ω be a de Rham 2-form on a manifold[5] M. For each point $p \in M$, the map $\omega_p : T_pM \times T_pM \to \mathbb{R}$ is skew-symmetric and bilinear on the tangent space to M at p, and ω_p varies smoothly in p.

DEFINITION 1.1. The 2-form ω is *symplectic* if ω is closed (i.e., its exterior derivative $d\omega$ is zero) and ω_p is symplectic for all $p \in M$. A *symplectic manifold* is a pair (M, ω) where M is a manifold and ω is a symplectic form.

Symplectic manifolds must be *even-dimensional*. Moreover, the nth exterior power ω^n of a symplectic form ω on a $2n$-dimensional manifold is a *volume form*.[6] Hence, any symplectic manifold (M, ω) is *canonically oriented*. The form $\frac{\omega^n}{n!}$ is called the *symplectic volume* or *Liouville volume* of (M, ω). When (M, ω) is a *compact* $2n$-dimensional symplectic manifold, the de Rham cohomology class $[\omega^n] \in H^{2n}(M; \mathbb{R})$ must be nonzero by Stokes theorem. Therefore, the class $[\omega]$ must be nonzero, as well as its powers $[\omega]^k = [\omega^k] \neq 0$. *Exact symplectic forms* can only exist on noncompact manifolds. Compact manifolds with a trivial even cohomology group $H^{2k}(M; \mathbb{R})$, $k = 0, 1, \ldots, n$, such as spheres S^{2n} with $n > 1$, can thus never be symplectic. On a manifold of dimension greater than 2, a function multiple $f\omega$ of a symplectic form ω is symplectic if and only if f is a nonzero locally constant function (this follows from the existence of a symplectic basis).

EXAMPLES.

1. Let $M = \mathbb{R}^{2n}$ with linear coordinates $x_1, \ldots, x_n, y_1, \ldots, y_n$. The form

$$\omega_0 = \sum_{i=1}^{n} dx_i \wedge dy_i$$

[5] Unless otherwise indicated, all vector spaces are real and finite-dimensional, all maps are smooth (i.e., C^∞) and all manifolds are smooth, Hausdorff and second countable.

[6] A *volume form* is a nonvanishing form of top degree. If Ω is a symplectic structure on a vector space V of dimension $2n$, its nth exterior power $\Omega^n = \Omega \wedge \cdots \wedge \Omega$ does not vanish. Actually, a skew-symmetric bilinear map Ω is symplectic if and only if $\Omega^n \neq 0$.

is symplectic, and the vectors $(\frac{\partial}{\partial x_1})_p, \dots, (\frac{\partial}{\partial x_n} t)_p$, $(\frac{\partial}{\partial y_1})_p, \dots, (\frac{\partial}{\partial y_n})_p$ constitute a symplectic basis of T_pM.

2. Let $M = \mathbb{C}^n$ with coordinates $z_1, \dots, z_n$. The form $\omega_0 = \frac{i}{2} \sum dz_k \wedge d\bar{z}_k$ is symplectic. In fact, this form coincides with that of the previous example under the identification $\mathbb{C}^n \simeq \mathbb{R}^{2n}$, $z_k = x_k + i y_k$.
3. The 2-sphere S^2, regarded as the set of unit vectors in $\mathbb{R}^3$, has tangent vectors at p identified with vectors orthogonal to p. The standard symplectic form on S^2 is induced by the standard inner (dot) and exterior (vector) products: $\omega_p(u, v) := \langle p, u \times v \rangle$, for $u, v \in T_pS^2 = \{p\}^\perp$. This is the standard area form on S^2 with total area 4π. In terms of cylindrical polar coordinates $0 \leqslant \theta < 2\pi$ and $-1 \leqslant z \leqslant 1$ away from the poles, it is written $\omega = d\theta \wedge dz$.
4. On any Riemann surface, regarded as a 2-dimensional oriented manifold, any area form, that is, any never vanishing 2-form, is a symplectic form.
5. Products of symplectic manifolds are naturally symplectic by taking the sum of the pullbacks of the symplectic forms from the factors.
6. If a $(2n+1)$-dimensional manifold X admits a *contact form*, that is, a 1-form α such that $\alpha \wedge (d\alpha)^n$ is never vanishing, then the 2-form $d(e^t \alpha)$ is symplectic on $X \times \mathbb{R}$, and the symplectic manifold $(X \times \mathbb{R}, d(e^t\alpha))$ is called the *symplectization* of the *contact manifold* (X, α). For more on *contact geometry*, see for instance the corresponding contribution in this volume.

DEFINITION 1.2. Let (M_1, ω_1) and (M_2, ω_2) be symplectic manifolds. A (smooth) map $\psi : M_1 \to M_2$ is *symplectic* if $\psi^* \omega_2 = \omega_1$.[7] A symplectic diffeomorphism $\varphi : M_1 \to M_2$ is a *symplectomorphism*. (M_1, ω_1) and (M_2, ω_2) are said to be *symplectomorphic* when there exists a symplectomorphism between them.

The classification of symplectic manifolds up to symplectomorphism is an open problem in symplectic geometry. However, the local classification is taken care of by the *Darboux theorem* (Theorem 1.9): the dimension is the only local invariant of symplectic manifolds up to symplectomorphisms. That is, just as any n-dimensional manifold is locally diffeomorphic to $\mathbb{R}^n$, any symplectic manifold (M^{2n}, ω) is locally symplectomorphic to $(\mathbb{R}^{2n}, \omega_0)$. As a consequence, if we prove for $(\mathbb{R}^{2n}, \omega_0)$ a local assertion that is invariant under symplectomorphisms, then that assertion holds for any symplectic manifold. We will hence refer to $\mathbb{R}^{2n}$, with linear coordinates $(x_1, \dots, x_n, y_1, \dots, y_n)$, and with symplectic form $\omega_0 = \sum_{i=1}^n dx_i \wedge dy_i$, as the *prototype of a local piece of a $2n$-dimensional symplectic manifold.*

1.3. *Cotangent bundles*

Cotangent bundles are major examples of symplectic manifolds. Let $(\mathcal{U}, x_1, \dots, x_n)$ be a coordinate chart for a manifold X, with associated cotangent coordinates $(T^*\mathcal{U}, x_1, \dots, x_n,$

[7]By definition of *pullback*, we have $(\psi^*\omega_2)_p(u, v) = (\omega_2)_{\psi(p)}(d\psi_p(u), d\psi_p(v))$, at tangent vectors $u, v \in T_pM_1$.

$\xi_1, \ldots, \xi_n)$.[8] Define a symplectic form on $T^*\mathcal{U}$ by

$$\omega = \sum_{i=1}^{n} dx_i \wedge d\xi_i .$$

One can check that this ω is intrinsically defined by considering the 1-form on $T^*\mathcal{U}$,

$$\alpha = \sum_{i=1}^{n} \xi_i \, dx_i ,$$

which satisfies $\omega = -d\alpha$ and is coordinate-independent: in terms of the natural projection $\pi : M \to X$, $p = (x, \xi) \mapsto x$, the form α may be equivalently defined pointwise without coordinates by

$$\alpha_p = (d\pi_p)^* \xi \in T_p^* M,$$

where $(d\pi_p)^* : T_x^* X \to T_p^* M$ is the transpose of $d\pi_p$, that is, $\alpha_p(v) = \xi((d\pi_p)v)$ for $v \in T_p M$. Or yet, the form α is uniquely characterized by the property that $\mu^*\alpha = \mu$ for every 1-form $\mu : X \to T^*X$ (see Proposition 2.2). The 1-form α is the *tautological form* (or the *Liouville* 1*-form*) and the 2-form ω is the *canonical symplectic form* on T^*X. When referring to a cotangent bundle as a symplectic manifold, the symplectic structure is meant to be given by this canonical ω.

Let X_1 and X_2 be n-dimensional manifolds with cotangent bundles $M_1 = T^*X_1$ and $M_2 = T^*X_2$, and tautological 1-forms α_1 and α_2. Suppose that $f : X_1 \to X_2$ is a diffeomorphism. Then there is a natural diffeomorphism $f_\sharp : M_1 \to M_2$ which *lifts* f; namely, for $p_1 = (x_1, \xi_1) \in M_1$ we define

$$f_\sharp(p_1) = p_2 = (x_2, \xi_2), \quad \text{with} \begin{cases} x_2 = f(x_1) \in X_2 & \text{and} \\ \xi_1 = (df_{x_1})^* \xi_2 \in T_{x_1}^* X_1, \end{cases}$$

where $(df_{x_1})^* : T_{x_2}^* X_2 \xrightarrow{\simeq} T_{x_1}^* X_1$, so $f_\sharp|_{T_{x_1}^*}$ is the inverse map of $(df_{x_1})^*$.

PROPOSITION 1.3. *The lift $f_\sharp$ of a diffeomorphism $f : X_1 \to X_2$ pulls the tautological form on T^*X_2 back to the tautological form on T^*X_1, i.e., $(f_\sharp)^*\alpha_2 = \alpha_1$.*

[8] If an n-dimensional manifold X is described by coordinate charts $(\mathcal{U}, x_1, \ldots, x_n)$ with $x_i : \mathcal{U} \to \mathbb{R}$, then, at any $x \in \mathcal{U}$, the differentials $(dx_i)_x$ form a basis of $T_x^* X$, inducing a map

$$\begin{aligned} T^*\mathcal{U} &\longrightarrow \mathbb{R}^{2n}, \\ (x, \xi) &\longmapsto (x_1, \ldots, x_n, \xi_1, \ldots, \xi_n), \end{aligned}$$

where $\xi_1, \ldots, \xi_n \in \mathbb{R}$ are the corresponding coordinates of $\xi \in T_x^* X$: $\xi = \sum_{i=1}^n \xi_i (dx_i)_x$. Then $(T^*\mathcal{U}, x_1, \ldots, x_n, \xi_1, \ldots, \xi_n)$ is a coordinate chart for the cotangent bundle T^*X; the coordinates $x_1, \ldots, x_n, \xi_1, \ldots, \xi_n$ are called the *cotangent coordinates* associated to the coordinates $x_1, \ldots, x_n$ on $\mathcal{U}$. One verifies that the transition functions on the overlaps are smooth, so T^*X is a $2n$-dimensional manifold.

PROOF. At $p_1 = (x_1, \xi_1) \in M_1$, the claimed identity says $(df_\sharp)^*_{p_1}(\alpha_2)_{p_2} = (\alpha_1)_{p_1}$, where $p_2 = f_\sharp(p_1)$, that is, $p_2 = (x_2, \xi_2)$ where $x_2 = f(x_1)$ and $(df_{x_1})^*\xi_2 = \xi_1$. This can be proved as follows:

$$\begin{aligned}
(df_\sharp)^*_{p_1}(\alpha_2)_{p_2} &= (df_\sharp)^*_{p_1}(d\pi_2)^*_{p_2}\xi_2 && \text{by definition of } \alpha_2\\
&= \big(d(\pi_2 \circ f_\sharp)\big)^*_{p_1}\xi_2 && \text{by the chain rule}\\
&= \big(d(f \circ \pi_1)\big)^*_{p_1}\xi_2 && \text{because } \pi_2 \circ f_\sharp = f \circ \pi_1\\
&= (d\pi_1)^*_{p_1}(df)^*_{x_1}\xi_2 && \text{by the chain rule}\\
&= (d\pi_1)^*_{p_1}\xi_1 && \text{by definition of } f_\sharp\\
&= (\alpha_1)_{p_1} && \text{by definition of } \alpha_1.
\end{aligned}$$

□

As a consequence of this naturality for the tautological form, a diffeomorphism of manifolds induces a canonical symplectomorphism of cotangent bundles:

COROLLARY 1.4. *The lift $f_\sharp : T^*X_1 \to T^*X_2$ of a diffeomorphism $f : X_1 \to X_2$ is a symplectomorphism for the canonical symplectic forms, i.e., $(f_\sharp)^*\omega_2 = \omega_1$.*

In terms of the group (under composition) of diffeomorphisms $\mathrm{Diff}(X)$ of a manifold X, and the *group of symplectomorphisms* $\mathrm{Sympl}(T^*X, \omega)$ of its cotangent bundle, we see that the injection $\mathrm{Diff}(X) \to \mathrm{Sympl}(T^*X, \omega)$, $f \mapsto f_\sharp$ is a group homomorphism. Clearly this is not surjective: for instance, consider the symplectomorphism $T^*X \to T^*X$ given by translation along cotangent fibers.

EXAMPLE. Let $X_1 = X_2 = S^1$. Then T^*S^1 is a cylinder $S^1 \times \mathbb{R}$. The canonical form is the area form $\omega = d\theta \wedge d\xi$. If $f : S^1 \to S^1$ is any diffeomorphism, then $f_\sharp : S^1 \times \mathbb{R} \to S^1 \times \mathbb{R}$ is a symplectomorphism, i.e., is an area-preserving diffeomorphism of the cylinder. Translation along the $\mathbb{R}$ direction is area-preserving but is not induced by a diffeomorphism of the base manifold S^1.

There is a criterion for which cotangent symplectomorphisms arise as lifts of diffeomorphisms in terms of the tautological form. First note the following feature of symplectic manifolds with *exact symplectic forms*. Let α be a 1-form on a manifold M such that $\omega = -d\alpha$ is symplectic. There exists a unique vector field v whose interior product with ω is α, i.e., $\iota_v\omega = -\alpha$. If $g : M \to M$ is a symplectomorphism that preserves α (that is, $g^*\alpha = \alpha$), then g commutes with the flow[9] of v, i.e., $(\exp tv) \circ g = g \circ (\exp tv)$. When

[9]For $p \in M$, $(\exp tv)(p)$ is the unique curve in M solving the initial value problem

$$\frac{d}{dt}\big(\exp tv(p)\big) = v\big(\exp tv(p)\big),$$
$$(\exp tv)(p)|_{t=0} = p$$

for t in some neighborhood of 0. The one-parameter group of diffeomorphisms $\exp tv$ is called the *flow* of the vector field v.

$M = T^*X$ is the cotangent bundle of an arbitrary n-dimensional manifold X, and α is the tautological 1-form on M, the vector field v is just $\sum \xi_i \frac{\partial}{\partial \xi_i}$ with respect to a cotangent coordinate chart $(T^*\mathcal{U}, x_1, \ldots, x_n, \xi_1, \ldots, \xi_n)$. The flow $\exp tv$, $-\infty < t < \infty$, satisfies $(\exp tv)(x, \xi) = (x, e^t \xi)$, for every (x, ξ) in M.

THEOREM 1.5. *A symplectomorphism $g : T^*X \to T^*X$ is a lift of a diffeomorphism $f : X \to X$ if and only if it preserves the tautological form*: $g^*\alpha = \alpha$.

PROOF. By Proposition 1.3, a lift $f_\sharp : T^*X \to T^*X$ of a diffeomorphism $f : X \to X$ preserves the tautological form. Conversely, if g is a symplectomorphism of M that preserves α, then g preserves the cotangent fibration: by the observation above, $g(x, \xi) = (y, \eta) \Rightarrow g(x, \lambda\xi) = (y, \lambda\eta)$ for all $(x, \xi) \in M$ and $\lambda > 0$, and this must hold also for $\lambda \leqslant 0$ by the differentiability of g at $(x, 0)$. Therefore, there exists a diffeomorphism $f : X \to X$ such that $\pi \circ g = f \circ \pi$, where $\pi : M \to X$ is the projection map $\pi(x, \xi) = x$, and $g = f_\#$. $\square$

The canonical form is natural also in the following way. Given a smooth function $h : X \to \mathbb{R}$, the diffeomorphism τ_h of $M = T^*X$ defined by $\tau_h(x, \xi) = (x, \xi + dh_x)$ turns out to be always a symplectomorphism. Indeed, if $\pi : M \to X$, $\pi(x, \xi) = x$, is the projection, we have $\tau_h^*\alpha = \alpha + \pi^* dh$, so that $\tau_h^*\omega = \omega$.

1.4. *Moser's trick*

There are other relevant notions of equivalence for symplectic manifolds[10] besides being symplectomorphic. Let M be a manifold with two symplectic forms ω_0, ω_1.

DEFINITION 1.6. The symplectic manifolds (M, ω_0) and (M, ω_1) are *strongly isotopic* if there is an isotopy $\rho_t : M \to M$ such that $\rho_1^*\omega_1 = \omega_0$. (M, ω_0) and (M, ω_1) are *deformation-equivalent* if there is a smooth family ω_t of symplectic forms joining ω_0 to ω_1. (M, ω_0) and (M, ω_1) are *isotopic* if they are deformation-equivalent and the de Rham cohomology class $[\omega_t]$ is independent of t.

Hence, being strongly isotopic implies being symplectomorphic, and being isotopic implies being deformation-equivalent. We also have that being strongly isotopic implies being isotopic, because, if $\rho_t : M \to M$ is an isotopy such that $\rho_1^*\omega_1 = \omega_0$, then $\omega_t := \rho_t^*\omega_1$ is a smooth family of symplectic forms joining ω_1 to ω_0 and $[\omega_t] = [\omega_1]$, $\forall t$, by the homotopy invariance of de Rham cohomology.

Moser [105] proved that, on a compact manifold, being isotopic implies being strongly isotopic (Theorem 1.7). McDuff showed that deformation-equivalence is indeed a necessary hypothesis: even if $[\omega_0] = [\omega_1] \in H^2(M; \mathbb{R})$, there are compact examples where (M, ω_0) and (M, ω_1) are not strongly isotopic; see Example 7.23 in [99]. In other words,

[10] Understanding these notions and the normal forms requires tools, such as isotopies (by *isotopy* we mean a smooth one-parameter family of diffeomorphisms starting at the identity, like the flow of a vector field), Lie derivative, tubular neighborhoods and the homotopy formula in de Rham theory, covered in differential geometry or differential topology texts.

fix $c \in H^2(M)$ and define S_c as the set of symplectic forms ω in M with $[\omega] = c$. On a compact manifold, all symplectic forms in the same path-connected component of S_c are symplectomorphic according to the Moser theorem, though there might be symplectic forms in different components of S_c that are not symplectomorphic.

THEOREM 1.7 (Moser). *Let M be a compact manifold with symplectic forms ω_0 and ω_1. Suppose that ω_t, $0 \leqslant t \leqslant 1$, is a smooth family of symplectic forms joining ω_0 to ω_1 with cohomology class $[\omega_t]$ independent of t. Then there exists an isotopy $\rho : M \times \mathbb{R} \to M$ such that $\rho_t^* \omega_t = \omega_0$, $0 \leqslant t \leqslant 1$.*

Moser applied an extremely useful argument, known as *Moser's trick*, starting with the following observation. If there existed an isotopy $\rho : M \times \mathbb{R} \to M$ such that $\rho_t^* \omega_t = \omega_0$, $0 \leqslant t \leqslant 1$, in terms of the associated time-dependent vector field

$$v_t := \frac{d\rho_t}{dt} \circ \rho_t^{-1}, \quad t \in \mathbb{R},$$

we would then have for all $0 \leqslant t \leqslant 1$ that

$$0 = \frac{d}{dt}(\rho_t^* \omega_t) = \rho_t^* \left(\mathcal{L}_{v_t} \omega_t + \frac{d\omega_t}{dt} \right) \quad \Longleftrightarrow \quad \mathcal{L}_{v_t} \omega_t + \frac{d\omega_t}{dt} = 0.$$

Conversely, the existence of a smooth time-dependent vector field v_t, $t \in \mathbb{R}$, satisfying the last equation is enough to produce by integration (since M is compact) the desired isotopy $\rho : M \times \mathbb{R} \to M$ satisfying $\rho_t^* \omega_t = \rho_0^* \omega_0 = \omega_0$, for all t. So everything boils down to solving the equation $\mathcal{L}_{v_t} \omega_t + \frac{d\omega_t}{dt} = 0$ for v_t.

PROOF. By the cohomology assumption that $[\frac{d}{dt}\omega_t] = 0$, there exists a *smooth* family of 1-forms μ_t such that

$$\frac{d\omega_t}{dt} = d\mu_t, \quad 0 \leqslant t \leqslant 1.$$

The argument involves the Poincaré lemma for compactly-supported forms, together with the Mayer–Vietoris sequence in order to use induction on the number of charts in a good cover of M; for a sketch, see page 95 in [99]. In the simplest case where $\omega_t = (1-t)\omega_0 + t\omega_1$ with $[\omega_0] = [\omega_1]$, we have that $\frac{d\omega_t}{dt} = \omega_1 - \omega_0 = d\mu$ is exact.

The nondegeneracy assumption on ω_t, guarantees that we can pointwise solve the equation, known as *Moser's equation*,

$$\imath_{v_t} \omega_t + \mu_t = 0$$

to obtain a unique smooth family of vector fields v_t, $0 \leqslant t \leqslant 1$. Extend v_t to all $t \in \mathbb{R}$. Thanks to the compactness of M, the vector fields v_t generate an isotopy ρ satisfying

$\frac{d\rho_t}{dt} = v_t \circ \rho_t$. Then we indeed have

$$\frac{d}{dt}(\rho_t^*\omega_t) = \rho_t^*\left(\mathcal{L}_{v_t}\omega_t + \frac{d\omega_t}{dt}\right) = \rho_t^*(d\imath_{v_t}\omega_t + d\mu_t) = \rho_t^* d(\imath_{v_t}\omega_t + \mu_t) = 0,$$

where we used Cartan's magic formula in $\mathcal{L}_{v_t}\omega_t = d\imath_{v_t}\omega_t + \imath_{v_t} d\omega_t$. □

EXAMPLE. On a compact oriented 2-dimensional manifold M, a symplectic form is just an area form. Let ω_0 and ω_1 be two area forms on M. If $[\omega_0] = [\omega_1]$, i.e., ω_0 and ω_1 give the same total area, then any convex combination of them is symplectic (because they induce the same orientation), and there is an isotopy $\varphi_t : M \to M$, $t \in [0, 1]$, such that $\varphi_1^*\omega_0 = \omega_1$. Therefore, up to strong isotopy, there is a unique symplectic representative in each nonzero 2-cohomology class of M.

On a *noncompact* manifold, given v_t, we would need to check the existence for $0 \leqslant t \leqslant 1$ of an isotopy ρ_t solving the differential equation $\frac{d\rho_t}{dt} = v_t \circ \rho_t$.

1.5. *Darboux and Moser theorems*

By a *submanifold* of a manifold M we mean either a manifold X with a *closed embedding*[11] $i : X \hookrightarrow M$, or an *open submanifold* (i.e., an open subset of M).

Given a $2n$-dimensional manifold M, a k-dimensional submanifold X, neighborhoods $\mathcal{U}_0, \mathcal{U}_1$ of X, and symplectic forms ω_0, ω_1 on $\mathcal{U}_0, \mathcal{U}_1$, we would like to know whether there exists a *local symplectomorphism preserving* X, i.e., a diffeomorphism $\varphi : \mathcal{U}_0 \to \mathcal{U}_1$ with $\varphi^*\omega_1 = \omega_0$ and $\varphi(X) = X$. Moser's Theorem 1.7 addresses the case where $X = M$. At the other extreme, when X is just one point, there is the classical Darboux theorem (Theorem 1.9). In general, we have:

THEOREM 1.8 (Moser theorem—relative version). *Let ω_0 and ω_1 be symplectic forms on a manifold M, and X a compact submanifold of M. Suppose that the forms coincide, $\omega_0|_p = \omega_1|_p$, at all points $p \in X$. Then there exist neighborhoods $\mathcal{U}_0$ and $\mathcal{U}_1$ of X in M, and a diffeomorphism $\varphi : \mathcal{U}_0 \to \mathcal{U}_1$ such that $\varphi^*\omega_1 = \omega_0$ and φ restricted to X is the identity map.*

PROOF. Pick a tubular neighborhood $\mathcal{U}_0$ of X. The 2-form $\omega_1 - \omega_0$ is closed on $\mathcal{U}_0$, and satisfies $(\omega_1 - \omega_0)_p = 0$ at all $p \in X$. By the homotopy formula on the tubular neighborhood, there exists a 1-form μ on $\mathcal{U}_0$ such that $\omega_1 - \omega_0 = d\mu$ and $\mu_p = 0$ at all $p \in X$. Consider the family $\omega_t = (1-t)\omega_0 + t\omega_1 = \omega_0 + t\, d\mu$ of closed 2-forms on $\mathcal{U}_0$. Shrinking $\mathcal{U}_0$ if necessary, we can assume that ω_t is symplectic for $t \in [0, 1]$, as nondegeneracy is an open property. Solve Moser's equation, $\imath_{v_t}\omega_t = -\mu$, for v_t By integration, shrinking $\mathcal{U}_0$ again if necessary, there exists a local isotopy $\rho : \mathcal{U}_0 \times [0, 1] \to M$ with $\rho_t^*\omega_t = \omega_0$, for all $t \in [0, 1]$. Since $v_t|_X = 0$, we have $\rho_t|_X = \mathrm{id}_X$. Set $\varphi = \rho_1$, $\mathcal{U}_1 = \rho_1(\mathcal{U}_0)$. □

[11] A *closed embedding* is a *proper* injective immersion. A map is *proper* when its preimage of a compact set is always compact.

THEOREM 1.9 (Darboux). *Let (M, ω) be a symplectic manifold, and let p be any point in M. Then we can find a chart $(\mathcal{U}, x_1, \dots, x_n, y_1, \dots y_n)$ centered at p where*

$$\omega = \sum_{i=1}^{n} dx_i \wedge dy_i .$$

Such a coordinate chart $(\mathcal{U}, x_1, \dots, x_n, y_1, \dots, y_n)$ is called a *Darboux chart*, and the corresponding coordinates are called *Darboux coordinates*.

The classical proof of Darboux's theorem is by induction on the dimension of the manifold [2], in the spirit of the argument for a symplectic basis (Section 1.1). The proof below, using Moser's theorem, was first provided by Weinstein [136].

PROOF. Apply Moser's relative theorem to $X = \{p\}$. More precisely, use any symplectic basis for (T_pM, ω_p) to construct coordinates $(x'_1, \dots, x'_n, y'_1, \dots y'_n)$ centered at p and valid on some neighborhood $\mathcal{U}'$, so that $\omega_p = \sum dx'_i \wedge dy'_i|_p$. There are two symplectic forms on $\mathcal{U}'$: the given $\omega_0 = \omega$ and $\omega_1 = \sum dx'_i \wedge dy'_i$. By Theorem 1.8, there are neighborhoods $\mathcal{U}_0$ and $\mathcal{U}_1$ of p, and a diffeomorphism $\varphi : \mathcal{U}_0 \to \mathcal{U}_1$ such that $\varphi(p) = p$ and $\varphi^*(\sum dx'_i \wedge dy'_i) = \omega$. Since $\varphi^*(\sum dx'_i \wedge dy'_i) = \sum d(x'_i \circ \varphi) \wedge d(y'_i \circ \varphi)$, we simply set new coordinates $x_i = x'_i \circ \varphi$, $y_i = y'_i \circ \varphi$. □

Darboux's theorem is easy in the 2-dimensional case. Being closed ω is locally exact, $\omega = d\alpha$. Every nonvanishing 1-form on a surface can be written locally as $\alpha = g\,dh$ for suitable functions g, h, where h is a coordinate on the local leaf space of the kernel foliation of α. The form $\omega = dg \wedge dh$ is nondegenerate if and only if (g, h) is a local diffeomorphism. By the way, transversality shows that the normal form for a *generic*[12] 2-form is $x\,dx \wedge dy$ near a point where it is degenerate.

1.6. *Symplectic submanifolds*

Moser's argument permeates many other proofs, including those of the next two results regarding *symplectic submanifolds*. Let (M, ω) be a symplectic manifold.

DEFINITION 1.10. A *symplectic submanifold* of (M, ω) is a submanifold X of M where, at each $p \in X$, the space T_pX is a symplectic subspace of (T_pM, ω_p).

If $i : X \hookrightarrow M$ is the inclusion of a symplectic submanifold X, then the restriction of ω to X is a symplectic form, so that $(X, i^*\omega)$ is itself a symplectic manifold.

Let X be a symplectic submanifold of (M, ω). At each $p \in X$, we have $T_pM = T_pX \oplus (T_pX)^{\omega_p}$ (Section 1.1), so the map $(T_pX)^{\omega_p} \to T_pM/T_pX$ is an isomorphism. This canonical identification of the *normal space* of X at p, $N_pX := T_pM/T_pX$, with the symplectic orthogonal $(T_pX)^{\omega_p}$, yields a canonical identification of the *normal bundle* NX

[12] *Generic* here means that the subset of those 2-forms having this behavior is open, dense and invariant under diffeomorphisms of the manifold.

with the *symplectic vector bundle* $(TX)^{\omega}$. A *symplectic vector bundle* is a vector bundle $E \to X$ equipped with a smooth[13] field Ω of fiberwise nondegenerate skew-symmetric bilinear maps $\Omega_p : E_p \times E_p \to \mathbb{R}$. The *symplectic normal bundle* is the normal bundle of a symplectic submanifold, with the symplectic structure induced by orthogonals. The next theorem, due to Weinstein [136], states that a neighborhood of a symplectic submanifold X is determined by X and (the isomorphism class of) its symplectic normal bundle.

THEOREM 1.11 (Symplectic neighborhood theorem). *Let (M_0, ω_0), (M_1, ω_1) be symplectic manifolds with diffeomorphic compact symplectic submanifolds X_0, X_1. Let $i_0 : X_0 \hookrightarrow M_0$, $i_1 : X_1 \hookrightarrow M_1$ be their inclusions. Suppose there is an isomorphism $\tilde{\phi} : NX_0 \to NX_1$ of the corresponding symplectic normal bundles covering a symplectomorphism $\phi : (X_0, i_0^*\omega_0) \to (X_1, i_1^*\omega_1)$. Then there exist neighborhoods $\mathcal{U}_0 \subset M_0$, $\mathcal{U}_1 \subset M_1$ of X_0, X_1 and a symplectomorphism $\varphi : \mathcal{U}_0 \to \mathcal{U}_1$ extending ϕ such that the restriction of $d\varphi$ to the normal bundle NX_0 is $\tilde{\phi}$.*

As first noted by Thurston [131], the form $\Omega + \pi^*\omega_X$ is symplectic in some neighborhood of the zero section in NX, where $\pi : NX \to X$ is the bundle projection and ω_X is the restriction of ω to X. Therefore, *a compact symplectic submanifold X always admits a tubular neighborhood in the ambient (M, ω) symplectomorphic to a tubular neighborhood of the zero section in the symplectic normal bundle NX.*

PROOF. By the Whitney extension theorem[14] there exist neighborhoods $\mathcal{U}_0 \subset M_0$ and $\mathcal{U}_1 \subset M_1$ of X_0 and X_1, and a diffeomorphism $h : \mathcal{U}_0 \to \mathcal{U}_1$ such that $h \circ i_0 = i_1 \circ \phi$ and the restriction of dh to the normal bundle NX_0 is the given $\tilde{\phi}$. Hence ω_0 and $h^*\omega_1$ are two symplectic forms on $\mathcal{U}_0$ which coincide at all points $p \in X_0$. The result now follows from Moser's relative theorem (Theorem 1.8). □

Carefully combining Moser's argument with the existence of an ambient isotopy that produces a given deformation of a compact submanifold, we can show:

THEOREM 1.12. *Let X_t, $t \in [0, 1]$, be a (smooth) family of compact symplectic submanifolds of a compact symplectic manifold (M, ω). Then there exists an isotopy $\rho : M \times \mathbb{R} \to M$ such that for all $t \in [0, 1]$ we have $\rho_t^*\omega = \omega$ and $\rho_t(X_0) = X_t$.*

Inspired by complex geometry, Donaldson [32] proved the following theorem on the existence of symplectic submanifolds. A major consequence is the characterization of symplectic manifolds in terms of *Lefschetz pencils*; see Section 4.6.

[13] Smoothness means that, for any pair of (smooth) sections u and v of E, the real-valued function $\Omega(u, v) : X \to \mathbb{R}$ given by evaluation at each point is smooth.

[14] **Whitney extension theorem.** *Let M be a manifold and X a submanifold of M. Suppose that at each $p \in X$ we are given a linear isomorphism $L_p : T_pM \xrightarrow{\simeq} T_pM$ such that $L_p|_{T_pX} = \mathrm{Id}_{T_pX}$ and L_p depends smoothly on p. Then there exists an embedding $h : \mathcal{N} \to M$ of some neighborhood $\mathcal{N}$ of X in M such that $h|_X = \mathrm{id}_X$ and $dh_p = L_p$ for all $p \in X$.* A proof relies on a tubular neighborhood model.

THEOREM 1.13 (Donaldson). *Let (M, ω) be a compact symplectic manifold. Assume that the cohomology class $[\omega]$ is integral, i.e., lies in $H^2(M;\mathbb{Z})$. Then, for every sufficiently large integer k, there exists a connected codimension-2 symplectic submanifold X representing the Poincaré dual of the integral cohomology class $k[\omega]$.*

Under the same hypotheses, Auroux extended this result to show that given $\alpha \in H_{2m}(M;\mathbb{Z})$ there exist positive $k, \ell \in \mathbb{Z}$ such that $k\mathrm{PD}[\omega^{n-m}] + \ell\alpha$ is realized by a $2m$-dimensional symplectic submanifold.

2. Lagrangian submanifolds

2.1. *First Lagrangian submanifolds*

Let (M, ω) be a symplectic manifold.

DEFINITION 2.1. A submanifold X of (M, ω) is *Lagrangian* (respectively, *isotropic* and *coisotropic*) if, at each $p \in X$, the space T_pX is a Lagrangian (respectively, isotropic and coisotropic) subspace of (T_pM, ω_p).

If $i : X \hookrightarrow M$ is the inclusion map, then X is a *Lagrangian submanifold* if and only if $i^*\omega = 0$ and $\dim X = \frac{1}{2}\dim M$.

The problem of embedding[15] a compact manifold as a Lagrangian submanifold of a given symplectic manifold is often global. For instance, Gromov [64] proved that there can be no Lagrangian spheres in $(\mathbb{C}^n, \omega_0)$, except for the circle in $\mathbb{C}^2$, and more generally no compact *exact Lagrangian* submanifolds, in the sense that $\alpha_0 = \sum y_j\, dx_j$ restricts to an exact 1-form. The argument uses *pseudoholomorphic curves* (Section 3.6). Yet there are *immersed* Lagrangian spheres (Section 2.7). More recently were found topological and geometrical constraints on manifolds that admit Lagrangian embeddings into *compact* symplectic manifolds; see, for instance, [16,17,115].

EXAMPLES.

1. Any 1-dimensional submanifold of a symplectic surface is Lagrangian (because a 1-dimensional subspace of a symplectic vector space is always isotropic).

   Therefore, any product of n embedded curves arises as a Lagrangian submanifold of (a neighborhood of zero in) the prototype $(\mathbb{R}^{2n}, \omega_0)$. In particular, a *torus* $\mathbb{T}^n = S^1 \times \cdots \times S^1$ can be embedded as a Lagrangian submanifold of any $2n$-dimensional symplectic manifold, by Darboux's theorem (Theorem 1.9).
2. Let $M = T^*X$ be the cotangent bundle of a manifold X. With respect to a cotangent coordinate chart $(T^*U, x_1, \ldots, x_n, \xi_1, \ldots, \xi_n)$, the tautological form is $\alpha = \sum \xi_i\, dx_i$ and the canonical form is $\omega = -d\alpha = \sum dx_i \wedge d\xi_i$.

   The *zero section* $X_0 := \{(x, \xi) \in T^*X \mid \xi = 0 \text{ in } T_x^*X\}$ is an n-dimensional submanifold of T^*X whose intersection with T^*U is given by the equations $\xi_1 = \cdots =$

[15] An *embedding* is an immersion that is a homeomorphism onto its image.

$\xi_n = 0$. Clearly α vanishes on $X_0 \cap T^*U$. Hence, if $i_0 : X_0 \hookrightarrow T^*X$ is the inclusion map, we have $i_0^*\omega = i_0^* d\alpha = 0$, and so X_0 is Lagrangian.

A *cotangent fiber* $T^*_{x_0}X$ is an n-dimensional submanifold of T^*X given by the equations $x_i = (x_0)_i$, $i = 1, \dots, n$, on T^*U. Since the x_i's are constant, the form α vanishes identically, and $T^*_{x_0}X$ is a Lagrangian submanifold.

Let X_μ be (the image of) an arbitrary section, that is, an n-dimensional submanifold of T^*X of the form $X_\mu = \{(x, \mu_x) \mid x \in X,\ \mu_x \in T_x^*X\}$, where the covector μ_x depends smoothly on x, so $\mu : X \to T^*X$ is a de Rham 1-form. We will investigate when such an X_μ is Lagrangian. Relative to the inclusion $i : X_\mu \hookrightarrow T^*X$ and the cotangent projection $\pi : T^*X \to X$, these X_μ's are exactly the submanifolds for which $\pi \circ i : X_\mu \to X$ is a diffeomorphism.

PROPOSITION 2.2. *The tautological* 1*-form* α *on* T^*X *satisfies* $\mu^*\alpha = \mu$, *for any* 1*-form* $\mu : X \to T^*X$.

PROOF. Denote by $s_\mu : X \to T^*X$, $x \mapsto (x, \mu_x)$, the 1-form μ regarded exclusively as a map. From the definition, $\alpha_p = (d\pi_p)^*\xi$ at $p = (x, \xi) \in M$. For $p = s_\mu(x) = (x, \mu_x)$, we have $\alpha_p = (d\pi_p)^*\mu_x$. Then, since $\pi \circ s_\mu = \mathrm{id}_X$, we have

$$\left(s_\mu^*\alpha\right)_x = (ds_\mu)_x^*\alpha_p = (ds_\mu)_x^*(d\pi_p)^*\mu_x = \left(d(\pi \circ s_\mu)\right)_x^*\mu_x = \mu_x. \qquad \square$$

The map $s_\mu : X \to T^*X$, $s_\mu(x) = (x, \mu_x)$ is an embedding with image the section X_μ. The diffeomorphism $\tau : X \to X_\mu$, $\tau(x) := (x, \mu_x)$, satisfies $i \circ \tau = s_\mu$.

PROPOSITION 2.3. *The sections of* T^*X *that are Lagrangian are those corresponding to closed* 1*-forms on* X.

PROOF. Using the previous notation, the condition of X_μ being Lagrangian becomes: $i^*\, d\alpha = 0 \Leftrightarrow \tau^* i^*\, d\alpha = 0 \Leftrightarrow s_\mu^*\, d\alpha = 0 \Leftrightarrow d(s_\mu^*\alpha) = 0 \Leftrightarrow d\mu = 0$. $\square$

When $\mu = dh$ for some $h \in C^\infty(X)$, such a primitive h is called a *generating function* for the Lagrangian submanifold X_μ. Two functions generate the same Lagrangian submanifold if and only if they differ by a locally constant function. When X is simply connected, or at least $H^1_{\mathrm{deRham}}(X) = 0$, every Lagrangian X_μ admits a generating function.

Besides the cotangent fibers, there are lots of Lagrangian submanifolds of T^*X not covered by the description in terms of closed 1-forms. Let S be any submanifold of an n-dimensional manifold X. The *conormal space* of S at $x \in S$ is

$$N_x^*S = \left\{\xi \in T_x^*X \mid \xi(v) = 0 \text{ for all } v \in T_xS\right\}.$$

The *conormal bundle* of S is $N^*S = \{(x, \xi) \in T^*X \mid x \in S,\ \xi \in N_x^*S\}$. This is an n-dimensional submanifold of T^*X. In particular, taking $S = \{x\}$ to be one point, the conormal bundle is the corresponding cotangent fiber T_x^*X. Taking $S = X$, the conormal bundle is the zero section X_0 of T^*X.

PROPOSITION 2.4. *If $i: N^*S \hookrightarrow T^*X$ is the inclusion of the conormal bundle of a submanifold $S \subset X$, and α is the tautological 1-form on T^*X, then $i^*\alpha = 0$.*

PROOF. Let $(\mathcal{U}, x_1, \dots, x_n)$ be a coordinate chart on X adapted to S, so that $\mathcal{U} \cap S$ is described by $x_{k+1} = \cdots = x_n = 0$. Let $(T^*\mathcal{U}, x_1, \dots, x_n, \xi_1, \dots, \xi_n)$ be the associated cotangent coordinate chart. The submanifold $N^*S \cap T^*\mathcal{U}$ is described by $x_{k+1} = \cdots = x_n = 0$ and $\xi_1 = \cdots = \xi_k = 0$. Since $\alpha = \sum \xi_i \, dx_i$ on $T^*\mathcal{U}$, we conclude that, at $p \in N^*S$,

$$(i^*\alpha)_p = \alpha_p|_{T_p(N^*S)} = \sum_{i>k} \xi_i \, dx_i \bigg|_{\mathrm{span}\{\frac{\partial}{\partial x_i},\, i \leqslant k\}} = 0.$$ □

COROLLARY 2.5. *For any submanifold S of X, the conormal bundle N^*S is a Lagrangian submanifold of T^*X.*

2.2. *Lagrangian neighborhood theorem*

Weinstein [136] proved that, if a compact submanifold X is Lagrangian with respect to two symplectic forms ω_0 and ω_1, then the conclusion of the Moser relative theorem (Theorem 1.8) still holds. We need some algebra for the Weinstein theorem.

Suppose that U, W are n-dimensional vector spaces, and $\Omega : U \times W \to \mathbb{R}$ is a bilinear pairing; the map Ω gives rise to a linear map $\tilde{\Omega} : U \to W^*$, $\tilde{\Omega}(u) = \Omega(u, \cdot)$. Then Ω is nondegenerate if and only if $\tilde{\Omega}$ is bijective.

PROPOSITION 2.6. *Let (V, Ω) be a symplectic vector space, U a Lagrangian subspace of (V, Ω), and W any vector space complement to U, not necessarily Lagrangian. Then from W we can canonically build a Lagrangian complement to U.*

PROOF. From Ω we get a nondegenerate pairing $\Omega' : U \times W \to \mathbb{R}$, so $\tilde{\Omega}' : U \to W^*$ is bijective. We look for a Lagrangian complement to U of the form $W' = \{w + Aw \mid w \in W\}$ for some linear map $A : W \to U$. For W' to be Lagrangian we need that $\Omega(w_1, w_2) = \tilde{\Omega}'(Aw_2)(w_1) - \tilde{\Omega}'(Aw_1)(w_2)$. Let $A' = \tilde{\Omega}' \circ A$, and look for A' such that $\Omega(w_1, w_2) = A'(w_2)(w_1) - A'(w_1)(w_2)$ for all $w_1, w_2 \in W$. The canonical choice is $A'(w) = -\frac{1}{2}\Omega(w, \cdot)$. Set $A = (\tilde{\Omega}')^{-1} \circ A'$. □

PROPOSITION 2.7. *Let V be a vector space, let Ω_0 and Ω_1 be symplectic forms on V, let U be a subspace of V Lagrangian for Ω_0 and Ω_1, and let W be any complement to U in V. Then from W we can canonically construct a linear isomorphism $L : V \xrightarrow{\simeq} V$ such that $L|_U = \mathrm{Id}_U$ and $L^*\Omega_1 = \Omega_0$.*

PROOF. By Proposition 2.6, from W we canonically obtain complements W_0 and W_1 to U in V such that W_0 is Lagrangian for Ω_0 and W_1 is Lagrangian for Ω_1. The nondegenerate bilinear pairings $\Omega_i : W_i \times U \to \mathbb{R}$, $i = 0, 1$, give isomorphisms $\tilde{\Omega}_i : W_i \xrightarrow{\simeq} U^*$, $i = 0, 1$, respectively. Let $B : W_0 \to W_1$ be the linear map satisfying $\tilde{\Omega}_1 \circ B = \tilde{\Omega}_0$, i.e., $\Omega_0(w_0, u) =$

$\Omega_1(Bw_0, u)$, $\forall w_0 \in W_0$, $\forall u \in U$. Let $L := \mathrm{Id}_U \oplus B : U \oplus W_0 \to U \oplus W_1$ be the extension of B to the rest of V by setting it to be the identity on U. It satisfies:

$$\begin{aligned}(L^*\Omega_1)(u \oplus w_0, u' \oplus w_0') &= \Omega_1(u \oplus Bw_0, u' \oplus Bw_0')\\ &= \Omega_1(u, Bw_0') + \Omega_1(Bw_0, u')\\ &= \Omega_0(u, w_0') + \Omega_0(w_0, u')\\ &= \Omega_0(u \oplus w_0, u' \oplus w_0'). \end{aligned}$$ □

THEOREM 2.8 (Weinstein Lagrangian neighborhood theorem). *Let M be a $2n$-dimensional manifold, X a compact n-dimensional submanifold, $i : X \hookrightarrow M$ the inclusion map, and ω_0 and ω_1 symplectic forms on M such that $i^*\omega_0 = i^*\omega_1 = 0$, i.e., X is a Lagrangian submanifold of both (M, ω_0) and (M, ω_1). Then there exist neighborhoods $\mathcal{U}_0$ and $\mathcal{U}_1$ of X in M and a diffeomorphism $\varphi : \mathcal{U}_0 \to \mathcal{U}_1$ such that $\varphi^*\omega_1 = \omega_0$ and φ is the identity on X, i.e., $\varphi(p) = p$, $\forall p \in X$.*

PROOF. Put a Riemannian metric g on M. Fix $p \in X$, and let $V = T_pM$, $U = T_pX$ and $W = U^\perp$, the orthocomplement of U in V relative to the inner product $g_p(\cdot,\cdot)$. Since $i^*\omega_0 = i^*\omega_1 = 0$, the subspace U is Lagrangian for both $(V, \omega_0|_p)$ and $(V, \omega_1|_p)$. By Proposition 2.7, we canonically get from $U^\perp$ a linear isomorphism $L_p : T_pM \to T_pM$ depending smoothly on p, such that $L_p|_{T_pX} = \mathrm{Id}_{T_pX}$ and $L_p^*\omega_1|_p = \omega_0|_p$. By the Whitney extension theorem (Section 1.5), there exist a neighborhood $\mathcal{N}$ of X and an embedding $h : \mathcal{N} \hookrightarrow M$ with $h|_X = \mathrm{id}_X$ and $dh_p = L_p$ for $p \in X$. Hence, at any $p \in X$, we have $(h^*\omega_1)_p = (dh_p)^*\omega_1|_p = L_p^*\omega_1|_p = \omega_0|_p$. Applying the Moser relative theorem (Theorem 1.8) to ω_0 and $h^*\omega_1$, we find a neighborhood $\mathcal{U}_0$ of X and an embedding $f : \mathcal{U}_0 \to \mathcal{N}$ such that $f|_X = \mathrm{id}_X$ and $f^*(h^*\omega_1) = \omega_0$ on $\mathcal{U}_o$. Set $\varphi = h \circ f$ and $\mathcal{U}_1 = \varphi(\mathcal{U}_0)$. □

Theorem 2.8 has the following generalization. For a proof see, for instance, either of [61,70,139].

THEOREM 2.9 (Coisotropic embedding theorem). *Let M be a manifold of dimension $2n$, X a submanifold of dimension $k \geqslant n$, $i : X \hookrightarrow M$ the inclusion, and ω_0 and ω_1 symplectic forms on M, such that $i^*\omega_0 = i^*\omega_1$ and X is coisotropic for both (M, ω_0) and (M, ω_1). Then there exist neighborhoods $\mathcal{U}_0$ and $\mathcal{U}_1$ of X in M and a diffeomorphism $\varphi : \mathcal{U}_0 \to \mathcal{U}_1$ such that $\varphi^*\omega_1 = \omega_0$ and $\varphi|_X = \mathrm{id}_X$.*

2.3. *Weinstein tubular neighborhood theorem*

Let (V, Ω) be a symplectic linear space, and let U be a Lagrangian subspace. Then there is a canonical nondegenerate bilinear pairing $\Omega' : V/U \times U \to \mathbb{R}$ defined by $\Omega'([v], u) = \Omega(v, u)$ where $[v]$ is the equivalence class of v in V/U. Consequently, we get a canonical isomorphism $\tilde{\Omega}' : V/U \to U^*$, $\tilde{\Omega}'([v]) = \Omega'([v], \cdot)$.

In particular, if (M,ω) is a symplectic manifold, and X is a Lagrangian submanifold, then T_pX is a Lagrangian subspace of (T_pM,ω_p) for each $p \in X$ and there is a canonical identification of the *normal space* of X at p, $N_pX := T_pM/T_pX$, with the cotangent fiber T_p^*X. Consequently the normal bundle NX and the cotangent bundle T^*X are canonically identified.

THEOREM 2.10 (Weinstein tubular neighborhood theorem). *Let (M,ω) be a symplectic manifold, X a compact Lagrangian submanifold, ω_0 the canonical symplectic form on T^*X, $i_0 : X \hookrightarrow T^*X$ the Lagrangian embedding as the zero section, and $i : X \hookrightarrow M$ the Lagrangian embedding given by inclusion. Then there are neighborhoods $\mathcal{U}_0$ of X in T^*X, $\mathcal{U}$ of X in M, and a diffeomorphism $\varphi : \mathcal{U}_0 \to \mathcal{U}$ such that $\varphi^*\omega = \omega_0$ and $\varphi \circ i_0 = i$.*

PROOF. By the standard tubular neighborhood theorem[16] and since $NX \simeq T^*X$ are canonically identified, we can find a neighborhood $\mathcal{N}_0$ of X in T^*X, a neighborhood $\mathcal{N}$ of X in M, and a diffeomorphism $\psi : \mathcal{N}_0 \to \mathcal{N}$ such that $\psi \circ i_0 = i$. Let ω_0 be the canonical form on T^*X and $\omega_1 = \psi^*\omega$. The submanifold X is Lagrangian for both of these symplectic forms on $\mathcal{N}_0$. By the Weinstein Lagrangian neighborhood theorem (Theorem 2.8), there exist neighborhoods $\mathcal{U}_0$ and $\mathcal{U}_1$ of X in $\mathcal{N}_0$ and a diffeomorphism $\theta : \mathcal{U}_0 \to \mathcal{U}_1$ such that $\theta^*\omega_1 = \omega_0$ and $\theta \circ i_0 = i_0$. Take $\varphi = \psi \circ \theta$ and $\mathcal{U} = \varphi(\mathcal{U}_0)$. Then $\varphi^*\omega = \theta^*\psi^*\omega = \theta^*\omega_1 = \omega_0$. □

Theorem 2.10 classifies compact Lagrangian embeddings: up to local symplectomorphism, the set of Lagrangian embeddings is the set of embeddings of manifolds into their cotangent bundles as zero sections.

The classification of compact *isotropic* embeddings is also due to Weinstein in [137, 139]. An *isotropic embedding* of a manifold X into a symplectic manifold (M,ω) is a closed embedding $i : X \hookrightarrow M$ such that $i^*\omega = 0$. Weinstein showed that neighborhood equivalence of isotropic embeddings is in one-to-one correspondence with isomorphism classes of symplectic vector bundles.

The classification of compact *coisotropic* embeddings is due to Gotay [61]. A *coisotropic embedding* of a manifold X carrying a closed 2-form α of constant rank into a symplectic manifold (M,ω) is an embedding $i : X \hookrightarrow M$ such that $i^*\omega = \alpha$ and $i(X)$ is coisotropic as a submanifold of M. Let E be the *characteristic distribution* of a closed form α of constant rank on X, i.e., E_p is the kernel of α_p at $p \in X$. Gotay showed that then the total space E^* carries a symplectic structure in a neighborhood of the zero section, such that X embeds coisotropically onto this zero section and, moreover, every coisotropic embedding is equivalent to this in some neighborhood of the zero section.

[16] **Tubular neighborhood theorem.** *Let M be a manifold, X a submanifold, NX the normal bundle of X in M, $i_0 : X \hookrightarrow NX$ the zero section, and $i : X \hookrightarrow M$ the inclusion. Then there are neighborhoods $\mathcal{U}_0$ of X in NX, $\mathcal{U}$ of X in M and a diffeomorphism $\psi : \mathcal{U}_0 \to \mathcal{U}$ such that $\psi \circ i_0 = i$.* This theorem can be proved with the exponential map using a Riemannian metric; see, for instance, [120].

2.4. *Application to symplectomorphisms*

Let (M_1, ω_1) and (M_2, ω_2) be two $2n$-dimensional symplectic manifolds. Given a diffeomorphism $f : M_1 \xrightarrow{\simeq} M_2$, there is a way to express the condition of f being a symplectomorphism in terms of a certain submanifold being Lagrangian. Consider the two projection maps $\mathrm{pr}_i : M_1 \times M_2 \to M_i$, $(p_1, p_2) \mapsto p_i$, $i = 1, 2$. The *twisted product form* on $M_1 \times M_2$ is the symplectic[17] form

$$\tilde{\omega} = (\mathrm{pr}_1)^* \omega_1 - (\mathrm{pr}_2)^* \omega_2.$$

PROPOSITION 2.11. *A diffeomorphism $f : M_1 \xrightarrow{\simeq} M_2$ is a symplectomorphism if and only if the graph of f is a Lagrangian submanifold of $(M_1 \times M_2, \tilde{\omega})$.*

PROOF. The graph of f is the $2n$-dimensional submanifold $\mathrm{Graph}\, f = \{(p, f(p)) \mid p \in M_1\} \subseteq M_1 \times M_2$, which is the image of the embedding $\gamma : M_1 \to M_1 \times M_2$, $p \mapsto (p, f(p))$. We have $\gamma^* \tilde{\omega} = \gamma^* \mathrm{pr}_1^* \omega_1 - \gamma^* \mathrm{pr}_2^* \omega_2 = (\mathrm{pr}_1 \circ \gamma)^* \omega_1 - (\mathrm{pr}_2 \circ \gamma)^* \omega_2$, and $\mathrm{pr}_1 \circ \gamma$ is the identity map on M_1 whereas $\mathrm{pr}_2 \circ \gamma = f$. So $\mathrm{Graph}\, f$ is Lagrangian, i.e., $\gamma^* \tilde{\omega} = 0$, if and only if $f^* \omega_2 = \omega_1$, i.e., f is a symplectomorphism. $\square$

Lagrangian submanifolds of $(M_1 \times M_2, \tilde{\omega})$ are called *canonical relations*, when viewed as morphisms between (M_1, ω_1) and (M_2, ω_2), even if $\dim M_1 \neq \dim M_2$. Under a reasonable assumption, there is a notion of composition [137].

Take $M_1 = M_2 = M$ and suppose that (M, ω) is a *compact* symplectic manifold and $f \in \mathrm{Sympl}(M, \omega)$. The graphs $\mathrm{Graph}\, f$ and Δ, of f and of the identity map $\mathrm{id} : M \to M$, are Lagrangian submanifolds of $M \times M$ with $\tilde{\omega} = \mathrm{pr}_1^* \omega - \mathrm{pr}_2^* \omega$. By the Weinstein tubular neighborhood theorem, there exist a neighborhood $\mathcal{U}$ of Δ in $(M \times M, \tilde{\omega})$ and a neighborhood $\mathcal{U}_0$ of M in (T^*M, ω_0) with a symplectomorphism $\varphi : \mathcal{U} \to \mathcal{U}_0$ satisfying $\varphi(p, p) = (p, 0)$, $\forall p \in M$.

Suppose that f is sufficiently C^1-*close*[18] to id, i.e., f is in some sufficiently small neighborhood of the identity id in the C^1-topology. Hence we can assume that $\mathrm{Graph}\, f \subseteq \mathcal{U}$. Let $j : M \hookrightarrow \mathcal{U}$, $j(p) = (p, f(p))$, be the embedding as $\mathrm{Graph}\, f$, and $i : M \hookrightarrow \mathcal{U}$, $i(p) = (p, p)$, be the embedding as $\Delta = \mathrm{Graph}\, \mathrm{id}$. The map j is sufficiently C^1-close to i. These maps induce embeddings $\varphi \circ j = j_0 : M \hookrightarrow \mathcal{U}_0$ and $\varphi \circ i = i_0 : M \hookrightarrow \mathcal{U}_0$ as 0-section, respectively. Since the map j_0 is sufficiently C^1-close to i_0, the image set $j_0(M)$ intersects each fiber T_p^*M at one point μ_p depending smoothly on p. Therefore, the image of j_0 is the image of a smooth section $\mu : M \to T^*M$, that is, a 1-form $\mu = j_0 \circ (\pi \circ j_0)^{-1}$. We conclude that $\mathrm{Graph}\, f \simeq \{(p, \mu_p) \mid p \in M,\ \mu_p \in T_p^*M\}$. Conversely, if μ is a 1-form sufficiently C^1-close to the zero 1-form, then $\{(p, \mu_p) \mid p \in M,\ \mu_p \in T_p^*M\} \simeq \mathrm{Graph}\, f$, for some diffeomorphism $f : M \to M$.

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[17] More generally, $\lambda_1 (\mathrm{pr}_1)^* \omega_1 + \lambda_2 (\mathrm{pr}_2)^* \omega_2$ is symplectic for all $\lambda_1, \lambda_2 \in \mathbb{R} \setminus \{0\}$.

[18] Let X and Y be manifolds. A sequence of maps $f_i : X \to Y$ *converges in the C^0-topology* (a.k.a. the *compact-open topology*) to $f : X \to Y$ if and only if f_i converges uniformly on compact sets. A sequence of C^1 maps $f_i : X \to Y$ *converges in the C^1-topology* to $f : X \to Y$ if and only if it and the sequence of derivatives $df_i : TX \to TY$ converge uniformly on compact sets.

By Proposition 2.3, Graph f is Lagrangian if and only if μ is closed. A small C^1-neighborhood of id in $\mathrm{Sympl}(M,\omega)$ is thus homeomorphic to a C^1-neighborhood of zero in the vector space of closed 1-forms on M. So we obtain the model:

$$T_{\mathrm{id}}\big(\mathrm{Sympl}(M,\omega)\big) \simeq \big\{\mu \in \Omega^1(M) \mid d\mu = 0\big\}.$$

In particular, $T_{\mathrm{id}}(\mathrm{Sympl}(M,\omega))$ contains the space of exact 1-forms that correspond to generating functions, $C^\infty(M)/\{\text{locally constant functions}\}$.

THEOREM 2.12. *Let (M,ω) be a compact symplectic manifold (and not just one point) with $H^1_{\mathrm{deRham}}(M) = 0$. Then any symplectomorphism of M that is sufficiently C^1-close to the identity has at least two fixed points.*

PROOF. If $f \in \mathrm{Sympl}(M,\omega)$ is sufficiently C^1-close to id, then its graph corresponds to a closed 1-form μ on M. As $H^1_{\mathrm{deRham}}(M) = 0$, we have that $\mu = dh$ for some $h \in C^\infty(M)$. But h must have at least two critical points because M is compact. A point p where $\mu_p = dh_p = 0$ corresponds to a point in the intersection of the graph of f with the diagonal, that is, a fixed point of f. □

This result has the following analogue in terms of *Lagrangian intersections*: *if X is a compact Lagrangian submanifold of a symplectic manifold (M,ω) with $H^1_{\mathrm{deRham}}(X) = 0$, then every Lagrangian submanifold of M that is C^1-close*[19] *to X intersects X in at least two points.*

2.5. *Generating functions*

We focus on symplectomorphisms between the cotangent bundles $M_1 = T^*X_1$, $M_2 = T^*X_2$ of two n-dimensional manifolds X_1, X_2. Let α_1, α_2 and ω_1, ω_2 be the corresponding tautological and canonical forms. Under the natural identification

$$M_1 \times M_2 = T^*X_1 \times T^*X_2 \simeq T^*(X_1 \times X_2),$$

the tautological 1-form on $T^*(X_1 \times X_2)$ is $\alpha = \mathrm{pr}_1^*\alpha_1 + \mathrm{pr}_2^*\alpha_2$, the canonical 2-form on $T^*(X_1 \times X_2)$ is $\omega = -d\alpha = \mathrm{pr}_1^*\omega_1 + \mathrm{pr}_2^*\omega_2$, and the twisted product form is $\tilde\omega = \mathrm{pr}_1^*\omega_1 - \mathrm{pr}_2^*\omega_2$. We define the involution $\sigma_2 : M_2 \to M_2$, $(x_2,\xi_2) \mapsto (x_2, -\xi_2)$, which yields $\sigma_2^*\alpha_2 = -\alpha_2$. Let $\sigma = \mathrm{id}_{M_1} \times \sigma_2 : M_1 \times M_2 \to M_1 \times M_2$. Then $\sigma^*\tilde\omega = \mathrm{pr}_1^*\omega_1 + \mathrm{pr}_2^*\omega_2 = \omega$. If L is a Lagrangian submanifold of $(M_1 \times M_2, \omega)$, then its *twist* $L^\sigma := \sigma(L)$ is a Lagrangian submanifold of $(M_1 \times M_2, \tilde\omega)$.

For producing a symplectomorphism $M_1 = T^*X_1 \to M_2 = T^*X_2$ we can start with a Lagrangian submanifold L of $(M_1 \times M_2, \omega)$, twist it to obtain a Lagrangian submanifold L^σ of $(M_1 \times M_2, \tilde\omega)$, and, if L^σ happens to be the graph of some diffeomorphism $\varphi : M_1 \to M_2$, then φ is a symplectomorphism.

[19] We say that a submanifold Y of M is *C^1-close* to another submanifold X when there is a diffeomorphism $X \to Y$ that is, as a map into M, C^1-close to the inclusion $X \hookrightarrow M$.

A method to obtain Lagrangian submanifolds of $M_1 \times M_2 \simeq T^*(X_1 \times X_2)$ relies on generating functions. For any $f \in C^\infty(X_1 \times X_2)$, df is a closed 1-form on $X_1 \times X_2$. The *Lagrangian submanifold generated by* f is $L_f := \{((x, y), (df)_{(x,y)}) \mid (x, y) \in X_1 \times X_2\}$ (cf. Section 2.1). We adopt the loose notation

$$\begin{aligned} d_x f &:= d_x f(x, y) := (df)_{(x,y)} \text{ projected to } T_x^* X_1 \times \{0\}, \\ d_y f &:= d_y f(x, y) := (df)_{(x,y)} \text{ projected to } \{0\} \times T_y^* X_2, \end{aligned}$$

which enables us to write $L_f = \{(x, y, d_x f, d_y f) \mid (x, y) \in X_1 \times X_2\}$ and

$$L_f^\sigma = \{(x, y, d_x f, -d_y f) \mid (x, y) \in X_1 \times X_2\}.$$

When L_f^σ is in fact the graph of a diffeomorphism $\varphi : M_1 = T^*X_1 \to M_2 = T^*X_2$, we call φ the *symplectomorphism generated by* f, and call f the *generating function* of φ. The issue now is to determine whether a given L_f^σ is the graph of a diffeomorphism $\varphi : M_1 \to M_2$. Let $(\mathcal{U}_1, x_1, \dots, x_n)$, $(\mathcal{U}_2, y_1, \dots, y_n)$ be coordinate charts for X_1, X_2, with associated charts $(T^*\mathcal{U}_1, x_1, \dots, x_n, \xi_1, \dots, \xi_n)$, $(T^*\mathcal{U}_2, y_1, \dots, y_n, \eta_1, \dots, \eta_n)$ for M_1, M_2. The set L_f^σ is the graph of $\varphi : M_1 \to M_2$ exactly when, for any $(x, \xi) \in M_1$ and $(y, \eta) \in M_2$, we have $\varphi(x, \xi) = (y, \eta) \Leftrightarrow \xi = d_x f$ and $\eta = -d_y f$. Therefore, given a point $(x, \xi) \in M_1$, to find its image $(y, \eta) = \varphi(x, \xi)$ we must solve the *Hamilton look-alike equations*

$$\begin{cases} \xi_i = \frac{\partial f}{\partial x_i}(x, y), \\ \eta_i = -\frac{\partial f}{\partial y_i}(x, y). \end{cases}$$

If there is a solution $y = \varphi_1(x, \xi)$ of the first equation, we may feed it to the second thus obtaining $\eta = \varphi_2(x, \xi)$, so that $\varphi(x, \xi) = (\varphi_1(x, \xi), \varphi_2(x, \xi))$. By the implicit function theorem, in order to solve the first equation locally and smoothly for y in terms of x and ξ, we need the condition

$$\det\left[\frac{\partial}{\partial y_j}\left(\frac{\partial f}{\partial x_i}\right)\right]_{i,j=1}^n \neq 0.$$

This is a necessary condition for f to generate a symplectomorphism φ. Locally this is also sufficient, but globally there is the usual bijectivity issue.

EXAMPLE. Let $X_1 = X_2 = \mathbb{R}^n$, and $f(x, y) = -\frac{|x-y|^2}{2}$, the square of Euclidean distance up to a constant. In this case, the Hamilton equations are

$$\begin{cases} \xi_i = \frac{\partial f}{\partial x_i} = y_i - x_i, \\ \eta_i = -\frac{\partial f}{\partial y_i} = y_i - x_i, \end{cases} \iff \begin{cases} y_i = x_i + \xi_i, \\ \eta_i = \xi_i. \end{cases}$$

The symplectomorphism generated by f is $\varphi(x, \xi) = (x + \xi, \xi)$. If we use the Euclidean inner product to identify $T^*\mathbb{R}^n$ with $T\mathbb{R}^n$, and hence regard φ as $\tilde{\varphi} : T\mathbb{R}^n \to T\mathbb{R}^n$

and interpret ξ as the velocity vector, then the symplectomorphism φ corresponds to free translational motion in Euclidean space.

The previous example can be generalized to the *geodesic flow on a Riemannian manifold.*[20] Let (X, g) be a geodesically convex Riemannian manifold, where $d(x, y)$ is the Riemannian distance between points x and y. Consider the function

$$f : X \times X \longrightarrow \mathbb{R}, \quad f(x, y) = -\frac{d(x, y)^2}{2}.$$

We want to investigate if f generates a symplectomorphism $\varphi : T^*X \to T^*X$. Using the identification $\tilde{g}_x : T_xX \xrightarrow{\simeq} T_x^*X$, $v \mapsto g_x(v, \cdot)$, induced by the metric, we translate φ into a map $\tilde{\varphi} : TX \to TX$. We need to solve

$$\begin{cases} \tilde{g}_x(v) = \xi = d_x f(x, y), \\ \tilde{g}_y(w) = \eta = -d_y f(x, y) \end{cases} \tag{1}$$

for (y, η) in terms of (x, ξ) in order to find φ, or, equivalently, for (y, w) in terms (x, v) in order to find $\tilde{\varphi}$. Assume that (X, g) is *geodesically complete*, that is, every geodesic can be extended indefinitely.

PROPOSITION 2.13. *Under the identification $T_xX \simeq T_x^*X$ given by the metric, the symplectomorphism generated by f corresponds to the map*

$$\begin{aligned} &\tilde{\varphi} : TX \longrightarrow TX, \\ &(x, v) \longmapsto \left(\gamma(1), \frac{d\gamma}{dt}(1)\right), \end{aligned}$$

where γ is the geodesic with initial conditions $\gamma(0) = x$ and $\frac{d\gamma}{dt}(0) = v$.

[20] A *Riemannian metric* on a manifold X is a smooth function g that assigns to each point $x \in X$ an *inner product* g_x on T_xX, that is, a symmetric positive-definite bilinear map $g_x : T_xX \times T_xX \to \mathbb{R}$. Smoothness means that for every (smooth) vector field $v : X \to TX$ the real-valued function $x \mapsto g_x(v_x, v_x)$ is smooth. A *Riemannian manifold* is a pair (X, g) where g is a Riemannian metric on the manifold X. The *arc-length* of a piecewise smooth curve $\gamma : [a, b] \to X$ on a Riemannian (X, g) is $\int_a^b \|\frac{d\gamma}{dt}\| \, dt$, where $\frac{d\gamma}{dt}(t) = d\gamma_t(1) \in T_{\gamma(t)}X$ and $\|\frac{d\gamma}{dt}\| = \sqrt{g_{\gamma(t)}(\frac{d\gamma}{dt}, \frac{d\gamma}{dt})}$ is the *velocity* of γ. A *reparametrization* of a curve $\gamma : [a, b] \to X$ is a curve of the form $\gamma \circ \tau : [c, d] \to X$ for some $\tau : [c, d] \to [a, b]$. By the change of variable formula for the integral, we see that the arc-length of γ is invariant by reparametrization. The *Riemannian distance* between two points x and y of a connected Riemannian manifold (X, g) is the infimum $d(x, y)$ of the set of all arc-lengths for piecewise smooth curves joining x to y. A *geodesic* is a curve that locally minimizes distance and whose velocity is constant. Given any curve $\gamma : [a, b] \to X$ with $\frac{d\gamma}{dt}$ never vanishing, there is a reparametrization $\gamma \circ \tau : [a, b] \to X$ of constant velocity. A *minimizing geodesic* from x to y is a geodesic joining x to y whose arc-length is the Riemannian distance $d(x, y)$. A Riemannian manifold (X, g) is *geodesically convex* if every point x is joined to every other point y by a unique (up to reparametrization) minimizing geodesic. For instance, $(\mathbb{R}^n, \langle \cdot, \cdot \rangle)$ is a geodesically convex Riemannian manifold (where $g_x(v, w) = \langle v, w \rangle$ is the Euclidean inner product on $T\mathbb{R}^n \simeq \mathbb{R}^n \times \mathbb{R}^n$), for which the Riemannian distance is the usual Euclidean distance $d(x, y) = |x - y|$.

This map $\tilde{\varphi}$ is called the *geodesic flow* on (X, g).

PROOF. Given $(x, v) \in TX$, let $\exp(x, v) : \mathbb{R} \to X$ be the unique geodesic with initial conditions $\exp(x, v)(0) = x$ and $\frac{d \exp(x,v)}{dt}(0) = v$. In this notation, we need to show that the unique solution of the system of equations (1) is $\tilde{\varphi}(x, v) = (\exp(x, v)(1), d\frac{\exp(x,v)}{dt}(1))$.

The Gauss lemma in Riemannian geometry (see, for instance, [120]) asserts that geodesics are orthogonal to the level sets of the distance function. To solve the first equation for $y = \exp(x, u)(1)$ for some $u \in T_x X$, evaluate both sides at v and at vectors $v' \in T_x X$ orthogonal to v,

$$|v|^2 = \frac{d}{dt}\left[\frac{-d(\exp(x, v)(t), y)^2}{2}\right]_{t=0} \quad \text{and} \quad 0 = \frac{d}{dt}\left[\frac{-d(\exp(x, v')(t), y)^2}{2}\right]_{t=0}$$

to conclude that $u = v$, and thus $y = \exp(x, v)(1)$.

We have $-d_y f(x, y)(w') = 0$ at vectors $w' \in T_y X$ orthogonal to $W := \frac{d \exp(x,v)}{dt}(1)$, because $f(x, y)$ is essentially the arc-length of a minimizing geodesic. Hence $w = kW$ must be proportional to W, and $k = 1$ since

$$k|v|^2 = g_y(kW, W) = -\frac{d}{dt}\left[\frac{-d(x, \exp(x, v)(1-t))^2}{2}\right]_{t=0} = |v|^2. \qquad \square$$

2.6. *Fixed points*

Let X be an n-dimensional manifold, and $M = T^*X$ its cotangent bundle equipped with the canonical symplectic form ω. Let $f : X \times X \to \mathbb{R}$ be a smooth function generating a symplectomorphism $\varphi : M \to M$, $\varphi(x, d_x f) = (y, -d_y f)$, with the notation of Section 2.5. To describe the fixed points of φ, we introduce the function $\psi : X \to \mathbb{R}$, $\psi(x) = f(x, x)$.

PROPOSITION 2.14. *There is a one-to-one correspondence between the fixed points of the symplectomorphism φ and the critical points of ψ.*

PROOF. At $x_0 \in X$, $d_{x_0}\psi = (d_x f + d_y f)|_{(x,y)=(x_0,x_0)}$. Let $\xi = d_x f|_{(x,y)=(x_0,x_0)}$. Recalling that L_f^σ is the graph of φ, we have that x_0 is a critical point of ψ, i.e., $d_{x_0}\psi = 0$, if and only if $d_y f|_{(x,y)=(x_0,x_0)} = -\xi$, which happens if and only if the point in L_f^σ corresponding to $(x, y) = (x_0, x_0)$ is (x_0, x_0, ξ, ξ), i.e., $\varphi(x_0, \xi) = (x_0, \xi)$ is a fixed point. $\square$

Consider the iterates $\varphi^N = \varphi \circ \varphi \circ \cdots \circ \varphi$, $N = 1, 2, \ldots$, given by N successive applications of φ. According to the previous proposition, if the symplectomorphism $\varphi^N : M \to M$ is generated by some function $f^{(N)}$, then there is a one-to-one correspondence between the set of fixed points of φ^N and the set of critical points of $\psi^{(N)} : X \to \mathbb{R}$, $\psi^{(N)}(x) = f^{(N)}(x, x)$. It remains to know whether φ^N admits a generating function. We will see that to a certain extent it does.

For each pair $x, y \in X$, define a map $X \to \mathbb{R}$, $z \mapsto f(x, z) + f(z, y)$. Suppose that this map has a unique critical point z_0 and that z_0 is nondegenerate. As z_0 is determined for each (x, y) implicitly by the equation $d_y f(x, z_0) + d_x f(z_0, y) = 0$, by nondegeneracy, the implicit function theorem assures that $z_0 = z_0(x, y)$ is a smooth function. Hence, the function

$$f^{(2)} : X \times X \longrightarrow \mathbb{R}, \quad f^{(2)}(x, y) := f(x, z_0) + f(z_0, y)$$

is smooth. Since φ is generated by f, and z_0 is critical, we have

$$\begin{aligned}\varphi^2\big(x, d_x f^{(2)}(x, y)\big) &= \varphi\big(\varphi(x, d_x f(x, z_0))\big) = \varphi\big(z_0, -d_y f(x, z_0)\big)\\ &= \varphi\big(z_0, d_x f(z_0, y)\big) = \big(y, -d_y f(z_0, y)\big)\\ &= \big(y, -d_y f^{(2)}(x, y)\big).\end{aligned}$$

We conclude that the function $f^{(2)}$ is a generating function for φ^2, as long as, for each $\xi \in T_x^* X$, there is a unique $y \in X$ for which $d_x f^{(2)}(x, y)$ equals ξ.

There are similar partial recipes for generating functions of higher iterates. In the case of φ^3, suppose that the function $X \times X \to \mathbb{R}$, $(z, u) \mapsto f(x, z) + f(z, u) + f(u, y)$, has a unique critical point (z_0, u_0) and that it is a nondegenerate critical point. A generating function would be $f^{(3)}(x, y) = f(x, z_0) + f(z_0, u_0) + f(u_0, y)$.

When the generating functions $f, f^{(2)}, f^{(3)}, \ldots, f^{(N)}$ exist given by these formulas, the *N-periodic points* of φ, i.e., the fixed points of φ^N, are in one-to-one correspondence with the critical points of

$$(x_1, \ldots, x_N) \longmapsto f(x_1, x_2) + f(x_2, x_3) + \cdots + f(x_{N-1}, x_N) + f(x_N, x_1).$$

EXAMPLE. Let $\chi : \mathbb{R} \to \mathbb{R}^2$ be a smooth plane curve that is 1-periodic, i.e., $\chi(s + 1) = \chi(s)$, and parametrized by arc-length, i.e., $|\frac{d\chi}{ds}| = 1$. Assume that the region Y enclosed by the image of χ is *convex*, i.e., for any $s \in \mathbb{R}$, the tangent line $\{\chi(s) + t\frac{d\chi}{ds} \mid t \in \mathbb{R}\}$ intersects the image $X := \partial Y$ of χ only at the point $\chi(s)$.

Suppose that a ball is thrown into a billiard table of shape Y rolling with constant velocity and bouncing off the boundary subject to the usual law of reflection. The map describing successive points on the orbit of the ball is

$$\begin{aligned}\varphi : \mathbb{R}/\mathbb{Z} \times (-1, 1) &\longrightarrow \mathbb{R}/\mathbb{Z} \times (-1, 1),\\ (x, v) &\longmapsto (y, w),\end{aligned}$$

saying that when the ball bounces off $\chi(x)$ with angle $\theta = \arccos v$, it will next collide with $\chi(y)$ and bounce off with angle $\nu = \arccos w$. Then the function $f : \mathbb{R}/\mathbb{Z} \times \mathbb{R}/\mathbb{Z} \to \mathbb{R}$ defined by $f(x, y) = -|\chi(x) - \chi(y)|$ is smooth off the diagonal, and for $\varphi(x, v) = (y, w)$ satisfies

$$\begin{cases}\frac{\partial f}{\partial x}(x, y) = \frac{\chi(y) - \chi(x)}{|\chi(x) - \chi(y)|} \cdot \frac{d\chi}{ds}\Big|_{s=x} = \cos\theta = v,\\ \frac{\partial f}{\partial y}(x, y) = \frac{\chi(x) - \chi(y)}{|\chi(x) - \chi(y)|} \cdot \frac{d\chi}{ds}\Big|_{s=y} = -\cos\nu = -w.\end{cases}$$

We conclude that f is a generating function for φ. Similar approaches work for higher-dimensional billiard problems. Periodic points are obtained by finding critical points of real functions of N variables in X,

$$(x_1, \dots, x_N) \longmapsto |\chi(x_1) - \chi(x_2)| + \cdots + |\chi(x_{N-1}) - \chi(x_N)| + |\chi(x_N) - \chi(x_1)|,$$

that is, by finding the N-sided (generalized) polygons inscribed in X of critical perimeter. Notice that $\mathbb{R}/\mathbb{Z} \times (-1, 1) \simeq \{(x, v) \mid x \in X,\ v \in T_x X,\ |v| < 1\}$ is the open unit tangent ball bundle of a circle X, which is an open annulus A, and the map $\varphi: A \to A$ is area-preserving, as in the next two theorems.

While studying *Poincaré return maps* in dynamical systems, Poincaré arrived at the following results.

THEOREM 2.15 (Poincaré recurrence theorem). *Let $\varphi: A \to A$ be a volume-preserving diffeomorphism of a finite-volume manifold A, and $\mathcal{U}$ a nonempty open set in A. Then there is $q \in \mathcal{U}$ and a positive integer N such that $\varphi^N(q) \in \mathcal{U}$.*

Hence, under iteration, a mechanical system governed by φ will eventually return arbitrarily close to the initial state.

PROOF. Let $\mathcal{U}_0 = \mathcal{U}$, $\mathcal{U}_1 = \varphi(\mathcal{U})$, $\mathcal{U}_2 = \varphi^2(\mathcal{U}), \dots$. If all of these sets were disjoint, then, since $\mathrm{Volume}(\mathcal{U}_i) = \mathrm{Volume}(\mathcal{U}) > 0$ for all i, the volume of A would be greater or equal to $\sum_i \mathrm{Volume}(\mathcal{U}_i) = \infty$. To avoid this contradiction we must have $\varphi^k(\mathcal{U}) \cap \varphi^\ell(\mathcal{U}) \neq \emptyset$ for some $k > \ell$, which implies $\varphi^{k-\ell}(\mathcal{U}) \cap \mathcal{U} \neq \emptyset$. □

THEOREM 2.16 (Poincaré's last geometric theorem). *Suppose that $\varphi: A \to A$ is an area-preserving diffeomorphism of the closed annulus $A = \mathbb{R}/\mathbb{Z} \times [-1, 1]$ that preserves the two components of the boundary and twists them in opposite directions. Then φ has at least two fixed points.*

This theorem was proved in 1913 by Birkhoff [18], and hence is also called the *Poincaré–Birkhoff theorem*. It has important applications to dynamical systems and celestial mechanics. The *Arnold conjecture* on the existence of fixed points for symplectomorphisms of compact manifolds (see Section 5.2) may be regarded as a generalization of the Poincaré–Birkhoff theorem. This conjecture has motivated a significant amount of research involving a more general notion of generating function; see, for instance, [41,55].

2.7. *Lagrangians and special Lagrangians in $\mathbb{C}^n$*

The standard *Hermitian inner product* $h(\cdot, \cdot)$ on $\mathbb{C}^n$ has real and imaginary parts given by the Euclidean inner product $\langle \cdot, \cdot \rangle$ and (minus) the symplectic form ω_0, respectively: for

$v = (x_1 + iy_1, \ldots, x_n + iy_n)$, $u = (a_1 + ib_1, \ldots, a_n + ib_n) \in \mathbb{C}^n$,

$$\begin{aligned} h(v, u) &= \sum_{k=1}^{n} (x_k + iy_k)(a_k - ib_k) = \sum_{k=1}^{n} (x_k a_k + y_k b_k) - i \sum_{k=1}^{n} (x_k b_k - y_k a_k) \\ &= \langle v, u \rangle - i\omega_0(v, u). \end{aligned}$$

LEMMA 2.17. *Let W be a subspace of $(\mathbb{C}^n, \omega_0)$ and $e_1, \ldots, e_n$ vectors in $\mathbb{C}^n$. Then:*
(a) *W is Lagrangian if and only if $W^\perp = iW$;*
(b) *$(e_1, \ldots, e_n)$ is an orthonormal basis of a Lagrangian subspace if and only if $(e_1, \ldots, e_n)$ is a unitary basis of $\mathbb{C}^n$.*

PROOF. (a) We always have $\omega_0(v, u) = -\operatorname{im} h(v, u) = \operatorname{re} h(iv, u) = \langle iv, u \rangle$. It follows that, if W is Lagrangian, so that $\omega_0(v, u) = 0$ for all $v, u \in W$, then $iW \subseteq W^\perp$. These spaces must be equal because they have the same dimension. Reciprocally, when $\langle iv, u \rangle = 0$ for all $v, u \in W$, the equality above shows that W must be isotropic. Since $\dim W = \dim iW = \dim W^\perp = 2n - \dim W$, the dimension of W must be n.

(b) If $(e_1, \ldots, e_n)$ is an orthonormal basis of a Lagrangian subspace W, then, by the previous part, $(e_1, \ldots, e_n, ie_1, \ldots, ie_n)$ is an orthonormal basis of $\mathbb{C}^n$ as a real vector space. Hence $(e_1, \ldots, e_n)$ must be a complex basis of $\mathbb{C}^n$ and it is unitary because $h(e_j, e_k) = \langle e_j, e_k \rangle - i\omega_0(e_j, e_k) = \delta_{jk}$. Conversely, if $(e_1, \ldots, e_n)$ is a unitary basis of $\mathbb{C}^n$, then the real span of these vectors is Lagrangian ($\omega_0(e_j, e_k) = -\operatorname{im} h(e_j, e_k) = 0$) and they are orthonormal ($\langle e_j, e_k \rangle = \operatorname{re} h(e_j, e_k) = \delta_{jk}$). □

The *Lagrangian Grassmannian* Λ_n is the set of all Lagrangian subspaces of $\mathbb{C}^n$. It follows from part (b) of Lemma 2.17 that Λ_n is the set of all subspaces of $\mathbb{C}^n$ admitting an orthonormal basis that is a unitary basis of $\mathbb{C}^n$. Therefore, we have

$$\Lambda_n \simeq \mathrm{U}(n)/\mathrm{O}(n).$$

Indeed $\mathrm{U}(n)$ acts transitively on Λ_n: given $W, W' \in \Lambda_n$ with orthonormal bases $(e_1, \ldots, e_n)$, $(e'_1, \ldots, e'_n)$, respectively, there is a unitary transformation of $\mathbb{C}^n$ that maps $(e_1, \ldots, e_n)$ to $(e'_1, \ldots, e'_n)$ as unitary bases of $\mathbb{C}^n$. And the stabilizer of $\mathbb{R}^n \in \Lambda_n$ is the subgroup of those unitary transformations that preserve this Lagrangian subspace, namely $\mathrm{O}(n)$. It follows that Λ_n is a compact connected manifold of dimension $\frac{n(n+1)}{2}$; cf. the last example of Section 1.1.

The Lagrangian Grassmannian comes with a *tautological vector bundle*

$$\tau_n := \{(W, v) \in \Lambda_n \times \mathbb{C}^n \mid v \in W\},$$

whose fiber over $W \in \Lambda_n$ is the n-dimensional real space W. It is a consequence of part (a) of Lemma 2.17 that the following map gives a well-defined global isomorphism of the complexification $\tau_n \otimes_{\mathbb{R}} \mathbb{C}$ with the trivial bundle $\underline{\mathbb{C}^n}$ over Λ_n (i.e., *a global trivialization*): $(W, v \otimes c) \mapsto (W, cv)$, for $W \in \Lambda_n$, $v \in W$, $c \in \mathbb{C}$.

DEFINITION 2.18. A *Lagrangian immersion* of a manifold X is an immersion $f: X \to \mathbb{C}^n$ such that $df_p(T_pX)$ is a Lagrangian subspace of $(\mathbb{C}^n, \omega_0)$, for every $p \in X$.

EXAMPLE. The graph of a map $h: \mathbb{R}^n \to i\mathbb{R}^n$ is an embedded n-dimensional submanifold X of $\mathbb{C}^n$. Its tangent space at $(p, h(p))$ is $\{v + dh_p(v) \mid v \in \mathbb{R}^n\}$. Let $e_1, \ldots, e_n$ be the standard basis of $\mathbb{R}^n$. Since $\omega_0(e_k + dh_p(e_k), e_j + dh_p(e_j)) = \langle e_k, -i\, dh_p(e_j)\rangle + \langle e_j, i\, dh_p(e_k)\rangle$, we see that X is Lagrangian if and only if $\frac{\partial h_k}{\partial x_j} = \frac{\partial h_j}{\partial x_k}$, $\forall j, k$, which in $\mathbb{R}^n$ is if and only if h is the gradient of some $H: \mathbb{R}^n \to i\mathbb{R}$.

If $f: X \to \mathbb{C}^n$ is a Lagrangian immersion, we can define a *Gauss map*

$$\begin{aligned} \lambda_f : X &\longrightarrow \Lambda_n, \\ p &\longmapsto df_p(T_pX). \end{aligned}$$

Since $\lambda_f^* \tau_n = TX$ and $\tau_n \otimes \mathbb{C} \simeq \underline{\mathbb{C}^n}$, we see that a necessary condition for an immersion $X \to \mathbb{C}^n$ to exist is that the complexification of TX be trivializable. Using the h-principle (Section 3.2), Gromov [65] showed that this is also sufficient: *an n-dimensional manifold X admits a Lagrangian immersion into $\mathbb{C}^n$ if and only if the complexification of its tangent bundle is trivializable.*

EXAMPLE. For the unit sphere $S^n = \{(t, x) \in \mathbb{R} \times \mathbb{R}^n\colon t^2 + |x|^2 = 1\}$, the *Whitney sphere immersion* is the map

$$\begin{aligned} f : S^n &\longrightarrow \mathbb{C}^n, \\ (t, x) &\longmapsto x + itx. \end{aligned}$$

The only self-intersection is at the origin where $f(-1, 0, \ldots, 0) = f(1, 0, \ldots, 0)$. Since $T_{(t,x)}S^n = (t, x)^\perp$, the differential $df_{(t,x)}: (u, v) \mapsto v + i(tv + ux)$ is always injective: $v + i(tv + ux) = 0 \Leftrightarrow v = 0$ and $ux = 0$, but when $x = 0$ it is $t = \pm 1$ and $T_{(\pm 1, 0)}S^n = \{0\} \times \mathbb{R}^n$, so it must be $u = 0$. We conclude that f is an immersion. By computing ω_0 at two vectors of the form $v + i(tv + ux)$, we find that the image $df_p(T_pS^n)$ is an n-dimensional isotropic subspace of $\mathbb{C}^n$. Therefore, f is a Lagrangian immersion of S^n, and the complexification $TS^n \otimes \mathbb{C}$ must be always trivializable, though the tangent bundle TS^n is only trivializable in dimensions $n = 0, 1, 3, 7$.

The *special Lagrangian Grassmannian* $S\Lambda_n$ is the set of all *oriented* subspaces of $\mathbb{C}^n$ admitting a *positive* orthonormal basis $(e_1, \ldots, e_n)$ that is a *special* unitary basis of $\mathbb{C}^n$. By the characterization of Lagrangian in the part (b) of Lemma 2.17, it follows that the elements of $S\Lambda_n$ are indeed Lagrangian submanifolds. Similarly to the case of the Lagrangian Grassmannian, we have that

$$S\Lambda_n \simeq \mathrm{SU}(n)/\mathrm{SO}(n)$$

is a compact connected manifold of dimension $\frac{n(n+1)}{2} - 1$.

We can single out the *special* Lagrangian subspaces by expressing the condition on the determinant in terms of the real n-form in $\mathbb{C}^n$,

$$\beta := \operatorname{im} \Omega, \quad \text{where } \Omega := dz_1 \wedge \cdots \wedge dz_n.$$

Since for $A \in \mathrm{SO}(n)$, we have $\det A = 1$ and $\Omega(e_1, \ldots, e_n) = \Omega(Ae_1, \ldots, Ae_n)$, we see that, for an oriented real n-dimensional subspace $W \subset \mathbb{C}^n$, the number $\Omega(e_1, \ldots, e_n)$ does not depend on the choice of a positive orthonormal basis $(e_1, \ldots, e_n)$ of W, thus can be denoted $\Omega(W)$ and its imaginary part $\beta(W)$.

PROPOSITION 2.19. *A subspace W of $(\mathbb{C}^n, \omega_0)$ has an orientation for which it is a special Lagrangian if and only if W is Lagrangian and $\beta(W) = 0$.*

PROOF. Any orthonormal basis $(e_1, \ldots, e_n)$ of a Lagrangian subspace $W \subset \mathbb{C}^n$ is the image of the canonical basis of $\mathbb{C}^n$ by some $A \in \mathrm{U}(n)$, and $\Omega(W) = \det A \in S^1$. Therefore, W admits an orientation for which such a *positive* $(e_1, \ldots, e_n)$ is a *special* unitary basis of $\mathbb{C}^n$ if and only if $\det A = \pm 1$, i.e., $\beta(W) = 0$. $\square$

DEFINITION 2.20. A *special Lagrangian immersion* of an oriented manifold X is a Lagrangian immersion $f : X \to \mathbb{C}^n$ such that, at each $p \in X$, the space $df_p(T_pX)$ is a special Lagrangian subspace of $(\mathbb{C}^n, \omega_0)$.

For a special Lagrangian immersion f, the Gauss map λ_f takes values in SA_n.

By Proposition 2.19, the immersion f of an n-dimensional manifold X in $(\mathbb{C}^n, \omega_0)$ is *special Lagrangian* if and only if $f^*\omega_0 = 0$ and $f^*\beta = 0$.

EXAMPLE. In $\mathbb{C}^2$, writing $z_k = x_k + iy_k$, we have $\beta = dx_1 \wedge dy_2 + dy_1 \wedge dx_2$. We have seen that the graph of the gradient $i\nabla H$ is Lagrangian, for any function $H : \mathbb{R}^2 \to \mathbb{R}$. So $f(x_1, x_2) = (x_1, x_2, i\frac{\partial H}{\partial x_1}, i\frac{\partial H}{\partial x_2})$ is a Lagrangian immersion. For f to be a *special* Lagrangian immersion, we need the vanish of

$$f^*\beta = dx_1 \wedge d\left(\frac{\partial H}{\partial x_2}\right) + d\left(\frac{\partial H}{\partial x_1}\right) \wedge dx_2 = \left(\frac{\partial^2 H}{\partial x_1^2} + \frac{\partial^2 H}{\partial x_2^2}\right) dx_1 \wedge dx_2.$$

Hence the graph of ∇H is special Lagrangian if and only if H is *harmonic*.

If $f : X \to \mathbb{C}^n$ is a special Lagrangian immersion, then $f^*\Omega$ is an exact (real) volume form: $f^*\Omega = d\,\mathrm{re}(z_1\, dz_2 \wedge \cdots \wedge dz_n)$. We conclude, by Stokes theorem, that there can be no special Lagrangian immersion of a compact manifold in $\mathbb{C}^n$. *Calabi–Yau manifolds*[21] are more general manifolds where a definition of special Lagrangian submanifold makes sense and where the space of special Lagrangian embeddings of a compact manifold is interesting. Special Lagrangian geometry was introduced by Harvey and Lawson [71]. For a treatment of Lagrangian and special Lagrangian submanifolds with many examples; see, for instance, [9].

[21] *Calabi–Yau manifolds* are compact *Kähler manifolds* (Section 3.4) with vanishing first Chern class.

3. Complex structures

3.1. *Compatible linear structures*

A *complex structure* on a vector space V is a linear map $J : V \to V$ such that $J^2 = -\mathrm{Id}$. The pair (V, J) is then called a *complex vector space*.A complex structure J on V is equivalent to a structure of vector space over $\mathbb{C}$, the map J corresponding to multiplication by i. If (V, Ω) is a symplectic vector space, a complex structure J on V is said to be *compatible* (with Ω, or Ω-compatible) if the bilinear map $G_J : V \times V \to \mathbb{R}$ defined by $G_J(u, v) = \Omega(u, Jv)$ is an inner product on V. This condition comprises J being a symplectomorphism (i.e., $\Omega(Ju, Jv) = \Omega(u, v)$, $\forall u, v$) and the so-called *taming*: $\Omega(u, Ju) > 0, \forall u \neq 0$.

EXAMPLE. For the symplectic vector space $(\mathbb{R}^{2n}, \Omega_0)$ with symplectic basis $e_1 = (1, 0, \ldots, 0), \ldots, e_n, f_1, \ldots, f_n = (0, \ldots, 0, 1)$, there is a standard compatible complex structure J_0 determined by $J_0(e_j) = f_j$ and $J_0(f_j) = -e_j$ for all $j = 1, \ldots, n$. This corresponds to a standard identification of $\mathbb{R}^{2n}$ with $\mathbb{C}^n$, and $\Omega_0(u, J_0 v) = \langle u, v\rangle$ is the standard Euclidean inner product. With respect to the symplectic basis $e_1, \ldots, e_n, f_1, \ldots, f_n$, the map J_0 is represented by the matrix

$$\begin{bmatrix} 0 & -\mathrm{Id} \\ \mathrm{Id} & 0 \end{bmatrix}.$$

The *symplectic linear group*, $\mathrm{Sp}(2n) := \{A \in \mathrm{GL}(2n; \mathbb{R}) \mid \Omega_0(Au, Av) = \Omega_0(u, v)$ for all $u, v \in \mathbb{R}^{2n}\}$, is the group of all linear transformations of $\mathbb{R}^{2n}$ that preserve the standard symplectic structure. The *orthogonal group* $\mathrm{O}(2n)$ is the group formed by the linear transformations A that preserve the Euclidean inner product, $\langle Au, Av\rangle = \langle u, v\rangle$, for all $u, v \in \mathbb{R}^{2n}$. The *general complex group* $\mathrm{GL}(n; \mathbb{C})$ is the group of linear transformations $A : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ commuting with J_0, $A(J_0 v) = J_0(Av)$, for all $v \in \mathbb{R}^{2n}$.[22] The compatibility between the structures Ω_0, $\langle\cdot,\cdot\rangle$ and J_0 implies that the intersection of *any two* of these subgroups of $\mathrm{GL}(2n; \mathbb{R})$ is the same group, namely the *unitary group* $\mathrm{U}(n)$.

As $(\mathbb{R}^{2n}, \Omega_0)$ is the prototype of a $2n$-dimensional symplectic vector space, the preceding example shows that compatible complex structures always exist on symplectic vector spaces.[23] There is yet a way to produce a *canonical* compatible complex structure J after the choice of an inner product G on (V, Ω), though the starting $G(u, v)$ is usually different from $G_J(u, v) := \Omega(u, Jv)$.

PROPOSITION 3.1. *Let (V, Ω) be a symplectic vector space, with an inner product G. Then there is a canonical compatible complex structure J on V.*

[22] Identify the complex $n \times n$ matrix $X + iY$ with the real $2n \times 2n$ matrix $\begin{pmatrix} X & -Y \\ Y & X \end{pmatrix}$.

[23] Conversely, given (V, J), there is a symplectic Ω with which J is compatible: take $\Omega(u, v) = G(Ju, v)$ for an inner product G such that $J^t = -J$.

PROOF. By nondegeneracy of Ω and G, the maps $u \mapsto \Omega(u, \cdot)$ and $w \mapsto G(w, \cdot)$ are both isomorphisms between V and V^*. Hence, $\Omega(u, v) = G(Au, v)$ for some linear $A: V \to V$. The map A is skew-symmetric, and the product AA^t is symmetric[24] and positive: $G(AA^t u, u) = G(A^t u, A^t u) > 0$, for $u \neq 0$. By the spectral theorem, these properties imply that AA^t diagonalizes with positive eigenvalues λ_i, say $AA^t = B\,\mathrm{diag}(\lambda_1, \ldots, \lambda_{2n})\, B^{-1}$. We may hence define an arbitrary real power of AA^t by rescaling the eigenspaces, in particular,

$$\sqrt{AA^t} := B\,\mathrm{diag}(\sqrt{\lambda_1}, \ldots, \sqrt{\lambda_{2n}})B^{-1}.$$

The linear transformation $\sqrt{AA^t}$ is symmetric, positive-definite and does not depend on the choice of B nor of the ordering of the eigenvalues. It is completely determined by its effect on each eigenspace of AA^t: on the eigenspace corresponding to the eigenvalue λ_k, the map $\sqrt{AA^t}$ is defined to be multiplication by $\sqrt{\lambda_k}$.

Let $J := (\sqrt{AA^t})^{-1}A$. Since A and $\sqrt{AA^t}$ commute, J is orthogonal ($JJ^t = \mathrm{Id}$), as well as skew-symmetric ($J^t = -J$). It follows that J is a complex structure on V. Compatibility is easily checked:

$$\Omega(Ju, Jv) = G(AJu, Jv) = G(JAu, Jv) = G(Au, v) = \Omega(u, v)$$

and

$$\Omega(u, Ju) = G(Au, Ju) = G(-JAu, u) = G\big(\sqrt{AA^t}u, u\big) > 0, \quad \text{for } u \neq 0. \quad \square$$

The factorization $A = \sqrt{AA^t}\, J$ is called the *polar decomposition* of A.

REMARK. Being *canonical*, this construction may be *smoothly* performed: when (V_t, Ω_t) is a family of symplectic vector spaces with a family G_t of inner products, all depending smoothly on a parameter t, an adaptation of the previous proof shows that there is a smooth family J_t of compatible complex structures on (V_t, Ω_t).

Let (V, Ω) be a symplectic vector space of dimension $2n$, and let J be a complex structure on V. If J is Ω-compatible and L is a Lagrangian subspace of (V, Ω), then JL is also Lagrangian and $JL = L^\perp$, where $\perp$ indicates orthogonality with respect to the inner product $G_J(u, v) = \Omega(u, Jv)$. Therefore, a complex structure J is Ω-compatible *if and only if* there exists a symplectic basis for V of the form

$$e_1, e_2, \ldots, e_n, \quad f_1 = Je_1, \quad f_2 = Je_2, \quad \ldots, \quad f_n = Je_n.$$

Let $\mathcal{J}(V, \Omega)$ be the *set of all compatible complex structures in a symplectic vector space* (V, Ω).

[24] A map $B: V \to V$ is *symmetric*, respectively *skew-symmetric*, when $B^t = B$, respectively $B^t = -B$, where the transpose $B^t: V \to V$ is determined by $G(B^t u, v) = G(u, Bv)$.

PROPOSITION 3.2. *The set $\mathcal{J}(V, \Omega)$ is contractible.*[25]

PROOF. Pick a Lagrangian subspace L_0 of (V, Ω). Let $\mathcal{L}(V, \Omega, L_0)$ be the space of all Lagrangian subspaces of (V, Ω) that intersect L_0 transversally. Let $\mathcal{G}(L_0)$ be the space of all inner products on L_0. The map

$$\Psi : \mathcal{J}(V, \Omega) \longrightarrow \mathcal{L}(V, \Omega, L_0) \times \mathcal{G}(L_0),$$
$$J \longmapsto (JL_0, G_J|_{L_0})$$

is a homeomorphism, with inverse as follows. Take $(L, G) \in \mathcal{L}(V, \Omega, L_0) \times \mathcal{G}(L_0)$. For $v \in L_0$, $v^{\perp} = \{u \in L_0 \mid G(u, v) = 0\}$ is a $(n-1)$-dimensional space of L_0; its symplectic orthogonal $(v^{\perp})^{\Omega}$ is $(n+1)$-dimensional. Then $(v^{\perp})^{\Omega} \cap L$ is 1-dimensional. Let Jv be the unique vector in this line such that $\Omega(v, Jv) = 1$. If we take v's in some G-orthonormal basis of L_0, this defines an element $J \in \mathcal{J}(V, \Omega)$.

The set $\mathcal{L}(V, \Omega, L_0)$ can be identified with the vector space of all symmetric $n \times n$ matrices. In fact, any n-dimensional subspace L of V that is transverse to L_0 is the graph of a linear map $JL_0 \to L_0$, and the Lagrangian ones correspond to symmetric maps (cf. Section 1.1). Hence, $\mathcal{L}(V, \Omega, L_0)$ is contractible. Since $\mathcal{G}(L_0)$ is contractible (it is even convex), we conclude that $\mathcal{J}(V, \Omega)$ is contractible. □

3.2. *Compatible almost complex structures*

An *almost complex structure* on a manifold M is a smooth[26] field of complex structures on the tangent spaces, $J_p : T_pM \to T_pM$, $p \in M$. The pair (M, J) is then called an *almost complex manifold.*

DEFINITION 3.3. An almost complex structure J on a symplectic manifold (M, ω) is *compatible* (with ω or ω-compatible) if the map that assigns to each point $p \in M$ the bilinear pairing $g_p : T_pM \times T_pM \to \mathbb{R}$, $g_p(u, v) := \omega_p(u, J_p v)$ is a Riemannian metric on M. A triple (ω, g, J) of a symplectic form, a Riemannian metric and an almost complex structure on a manifold M is a *compatible triple* when $g(\cdot, \cdot) = \omega(\cdot, J\cdot)$.

If (ω, J, g) is a compatible triple, each of ω, J or g can be written in terms of the other two.

EXAMPLES.

1. If we identify $\mathbb{R}^{2n}$ with $\mathbb{C}^n$ using coordinates $z_j = x_j + iy_j$, multiplication by i induces a constant linear map J_0 on the tangent spaces such that $J_0^2 = -\mathrm{Id}$, known

[25] *Contractibility* of $\mathcal{J}(V, \Omega)$ means that there exists a homotopy $h_t : \mathcal{J}(V, \Omega) \to \mathcal{J}(V, \Omega)$, $0 \leqslant t \leqslant 1$, starting at the identity $h_0 = \mathrm{Id}$, finishing at a trivial map $h_1 : \mathcal{J}(V, \Omega) \to \{J_0\}$, and fixing J_0 (i.e., $h_t(J_0) = J_0$, $\forall t$) for some $J_0 \in \mathcal{J}(V, \Omega)$.

[26] *Smoothness* means that for any vector field v, the image Jv is a (smooth) vector field.

as the *standard almost complex structure* on $\mathbb{R}^{2n}$:

$$J_0\left(\frac{\partial}{\partial x_j}\right) = \frac{\partial}{\partial y_j}, \qquad J_0\left(\frac{\partial}{\partial y_j}\right) = -\frac{\partial}{\partial x_j}.$$

For the standard symplectic form $\omega_0 = \sum dx_j \wedge dy_j$ and the Euclidean inner product $g_0 = \langle \cdot, \cdot \rangle$, the compatibility relation holds: $\omega_0(u, v) = g_0(J_0(u), v)$.

2. Any oriented hypersurface $\Sigma \subset \mathbb{R}^3$ carries a natural symplectic form and a natural compatible almost complex structure induced by the standard inner (or dot) and exterior (or vector) products. They are given by the formulas $\omega_p(u, v) := \langle \nu_p, u \times v \rangle$ and $J_p(v) = \nu_p \times v$ for $v \in T_p\Sigma$, where ν_p is the outward-pointing unit normal vector at $p \in \Sigma$ (in other words, $\nu : \Sigma \to S^2$ is the *Gauss map*). Cf. Example 3 of Section 1.2. The corresponding Riemannian metric is the restriction to Σ of the standard Euclidean metric $\langle \cdot, \cdot \rangle$.
3. The previous example generalizes to the oriented hypersurfaces $M \subset \mathbb{R}^7$. Regarding $u, v \in \mathbb{R}^7$ as imaginary *octonions* (or *Cayley numbers*), the natural vector product $u \times v$ is the imaginary part of the product of u and v as octonions. This induces a natural almost complex structure on M given by $J_p(v) = \nu_p \times v$, where ν_p is the outward-pointing unit normal vector at $p \in M$. In the case of S^6, at least, this J is not compatible with any symplectic form, as S^6 cannot be a symplectic manifold.

As a consequence of the remark in Section 3.1, we have:

PROPOSITION 3.4. *On any symplectic manifold (M, ω) with a Riemannian metric g, there is a canonical compatible almost complex structure J.*

Since Riemannian metrics always exist, we conclude that *any symplectic manifold has compatible almost complex structures.* The metric $g_J(\cdot, \cdot) := \omega(\cdot, J\cdot)$ tends to be different from the given $g(\cdot, \cdot)$.

PROPOSITION 3.5. *Let (M, J) be an almost complex manifold where J is compatible with two symplectic forms ω_0, ω_1 Then ω_0 and ω_1 are deformation-equivalent.*

PROOF. Simply take the convex combinations $\omega_t = (1 - t)\omega_0 + t\omega_1$, $0 \leqslant t \leqslant 1$. □

A counterexample to the converse of this proposition is provided by the family $\omega_t = \cos \pi t \, dx_1 \wedge dy_1 + \sin \pi t \, dx_1 \wedge dy_2 + \sin \pi t \, dy_1 \wedge dx_2 + \cos \pi t \, dx_2 \wedge dy_2$ for $0 \leqslant t \leqslant 1$. There is no J in $\mathbb{R}^4$ compatible with both ω_0 and $\omega_1 = -\omega_0$.

A submanifold X of an almost complex manifold (M, J) is an *almost complex submanifold* when $J(TX) \subseteq TX$, i.e., we have $J_p v \in T_p X$, $\forall p \in X, v \in T_p X$.

PROPOSITION 3.6. *Let (M, ω) be a symplectic manifold equipped with a compatible almost complex structure J. Then any almost complex submanifold X of (M, J) is a symplectic submanifold of (M, ω).*

PROOF. Let $i : X \hookrightarrow M$ be the inclusion. Then $i^*\omega$ is a closed 2-form on X. Since $\omega_p(u, v) = g_p(J_p u, v)$, $\forall p \in X$, $\forall u, v \in T_pX$, and since $g_p|_{T_pX}$ is nondegenerate, so is $\omega_p|_{T_pX}$, and $i^*\omega$ is nondegenerate. $\square$

It is easy to see that the *set $\mathcal{J}(M, \omega)$ of all compatible almost complex structures on a symplectic manifold* (M, ω) is path-connected. From two almost complex structures J_0, J_1 compatible with ω, we get two Riemannian metrics $g_0(\cdot, \cdot) = \omega(\cdot, J_0 \cdot)$, $g_1(\cdot, \cdot) = \omega(\cdot, J_1 \cdot)$. Their convex combinations

$$g_t(\cdot, \cdot) = (1 - t)g_0(\cdot, \cdot) + t g_1(\cdot, \cdot), \quad 0 \leqslant t \leqslant 1,$$

form a smooth family of Riemannian metrics. Applying the polar decomposition to the family (ω, g_t), we obtain a smooth path of compatible almost complex structures J_t joining J_0 to J_1. The set $\mathcal{J}(M, \omega)$ is even *contractible* (this is important for defining invariants). The first ingredient is the contractibility of the set of compatible complex structures on a vector space (Proposition 3.2). Consider the fiber bundle $\mathcal{J} \to M$ with fiber over $p \in M$ being the space $\mathcal{J}_p := \mathcal{J}(T_pM, \omega_p)$ of compatible complex structures on the tangent space at p. A compatible almost complex structure on (M, ω) is a section of $\mathcal{J}$. The space of sections of $\mathcal{J}$ is contractible because the fibers are contractible.[27]

The *first Chern class* $c_1(M, \omega)$ of a symplectic manifold (M, ω) is the first Chern class of (TM, J) for any compatible J. The class $c_1(M, \omega) \in H^2(M; \mathbb{Z})$ is invariant under deformations of ω.

We never used the closedness of ω to obtain compatible almost complex structures. The construction holds for an *almost symplectic manifold* (M, ω), that is, a pair of a manifold M and a nondegenerate 2-form ω, not necessarily closed. We could further work with a *symplectic vector bundle*, that is, a vector bundle $E \to M$ equipped with a smooth field ω of fiberwise nondegenerate skew-symmetric bilinear maps (Section 1.6). The existence of such a field ω is equivalent to being able to reduce the structure group of the bundle from the general linear group to the linear symplectic group. As both $\mathrm{Sp}(2n)$ and $\mathrm{GL}(n; \mathbb{C})$ retract to their common maximal compact subgroup $\mathrm{U}(n)$, a symplectic vector bundle can be always endowed with a structure of complex vector bundle, and vice-versa.

Gromov showed in his thesis [63] that any *open*[28] almost complex manifold admits a symplectic form. The books [42, §10.2] and [99, §7.3] contain proofs of this statement using different techniques.

THEOREM 3.7 (Gromov). *For an open manifold the existence of an almost complex structure J implies that of a symplectic form ω in any given 2-cohomology class and such that J is homotopic to an almost complex structure compatible with ω.*

From an almost complex structure J and a metric g, one builds a nondegenerate 2-form $\omega(u, v) = g(Ju, v)$, which will not be closed in general. Closedness is a *differential re-*

[27] The base being a (second countable and Hausdorff) manifold, a contraction can be produced using a countable cover by trivializing neighborhoods whose closures are compact subsets of larger trivializing neighborhoods, and such that each $p \in M$ belongs to only a finite number of such neighborhoods.

[28] A manifold is *open* if it has no closed connected components, where *closed* means compact and without boundary.

lation, i.e., a condition imposed on the partial derivatives, encoded as a subset of *jet space*. One says that a differential relation satisfies the *h-principle*[29] if any *formal solution* (i.e., a solution for the associated algebraic problem, in the present case a nondegenerate 2-form) is homotopic to a *holonomic solution* (i.e., a genuine solution, in the present case a closed nondegenerate 2-form). Therefore, when the h-principle holds, one may concentrate on a purely topological question (such as the existence of an almost complex structure) in order to prove the existence of a differential solution. Gromov showed that, for an open differential relation on an open manifold, when the relation is invariant under the group of diffeomorphisms of the underlying manifold, the inclusion of the space of holonomic solutions into the space of formal solutions is a weak homotopy equivalence, i.e., induces isomorphisms of all homotopy groups. The previous theorem fits here as an application.

For *closed* manifolds there is no such theorem: as discussed in Section 1.2, the existence of a 2-cohomology class whose top power is nonzero is also necessary for the existence of a symplectic form and there are further restrictions coming from *Gromov–Witten theory* (see Section 4.5).

3.3. *Integrability*

Any *complex manifold*[30] has a canonical almost complex structure J. It is defined locally over the domain $\mathcal{U}$ of a complex chart $\varphi : \mathcal{U} \to \mathcal{V} \subseteq \mathbb{C}^n$, by $J_p(\frac{\partial}{\partial x_j}|_p) = \frac{\partial}{\partial y_j}|_p$ and $J_p(\frac{\partial}{\partial y_j}|_p) = -\frac{\partial}{\partial x_j}|_p$, where these are the tangent vectors induced by the real and imaginary parts of the coordinates of $\varphi = (z_1, \dots, z_n)$, $z_j = x_j + iy_j$. This yields a globally well-defined J, thanks to the *Cauchy–Riemann equations* satisfied by the components of the transition maps.

An almost complex structure J on a manifold M is called *integrable* when J is induced by some underlying structure of complex manifold on M as above. The question arises whether some compatible almost complex structure J on a symplectic manifold (M, ω) is integrable. To understand what is involved, we review Dolbeault theory and the Newlander–Nirenberg theorem.

Let (M, J) be a $2n$-dimensional almost complex manifold. The fibers of the complexified tangent bundle, $TM \otimes \mathbb{C}$, are $2n$-dimensional vector spaces over $\mathbb{C}$. We may extend J linearly to $TM \otimes \mathbb{C}$ by $J(v \otimes c) = Jv \otimes c$, $v \in TM$, $c \in \mathbb{C}$. Since $J^2 = -\mathrm{Id}$, on the complex vector space $(TM \otimes \mathbb{C})_p$ the linear map J_p has eigenvalues $\pm i$. The $(\pm i)$-eigenspaces of J are denoted $T_{1,0}$ and $T_{0,1}$, respectively, and called the spaces of *J-holomorphic* and of *J-anti-holomorphic tangent vectors*. We have an isomorphism

$$\begin{aligned} (\pi_{1,0}, \pi_{0,1}) : TM \otimes \mathbb{C} &\xrightarrow{\simeq} T_{1,0} \oplus T_{0,1}, \\ v &\longmapsto \frac{1}{2}(v - iJv, v + iJv), \end{aligned}$$

[29]There are in fact different h-principles depending on the different possible coincidences of homotopy groups for the spaces of formal solutions and of holonomic solutions.

[30]A *complex manifold* of (complex) dimension n is a set M with a complete complex atlas $\{(\mathcal{U}_\alpha, \mathcal{V}_\alpha, \varphi_\alpha),\ \alpha \in$ index set $I\}$ where $M = \bigcup_\alpha \mathcal{U}_\alpha$, the $\mathcal{V}_\alpha$'s are open subsets of $\mathbb{C}^n$, and the maps $\varphi_\alpha : \mathcal{U}_\alpha \to \mathcal{V}_\alpha$ are bijections such that the transition maps $\psi_{\alpha\beta} = \varphi_\beta \circ \varphi_\alpha^{-1} : \mathcal{V}_{\alpha\beta} \to \mathcal{V}_{\beta\alpha}$ are *biholomorphic* (i.e., bijective, holomorphic and with holomorphic inverse) as maps on open subsets of $\mathbb{C}^n$, $\mathcal{V}_{\alpha\beta} = \varphi_\alpha(\mathcal{U}_\alpha \cap \mathcal{U}_\beta)$.

where the maps to each summand satisfy $\pi_{1,0} \circ J = i\pi_{1,0}$ and $\pi_{0,1} \circ J = -i\pi_{0,1}$. Restricting $\pi_{1,0}$ to TM, we see that $(TM, J) \simeq T_{1,0} \simeq \overline{T_{0,1}}$, as complex vector bundles, where the multiplication by i is given by J in (TM, J) and where $\overline{T_{0,1}}$ denotes the complex conjugate bundle of $T_{0,1}$.

Similarly, J^* defined on $T^*M \otimes \mathbb{C}$ by $J^*\xi = \xi \circ J$ has $(\pm i)$-eigenspaces $T^{1,0} = (T_{1,0})^*$ and $T^{0,1} = (T_{0,1})^*$, respectively, called the spaces of *complex-linear* and of *complex-antilinear cotangent vectors*. Under the two natural projections $\pi^{1,0}, \pi^{0,1}$, the complexified cotangent bundle splits as

$$\begin{aligned}(\pi^{1,0}, \pi^{0,1}) : T^*M \otimes \mathbb{C} &\xrightarrow{\simeq} T^{1,0} \oplus T^{0,1}, \\ \xi &\longmapsto \frac{1}{2}(\xi - iJ^*\xi, \xi + iJ^*\xi).\end{aligned}$$

Let

$$\Lambda^k(T^*M \otimes \mathbb{C}) := \Lambda^k(T^{1,0} \oplus T^{0,1}) = \bigoplus_{\ell+m=k} \Lambda^{\ell,m},$$

where $\Lambda^{\ell,m} := (\Lambda^\ell T^{1,0}) \wedge (\Lambda^m T^{0,1})$, and let $\Omega^k(M; \mathbb{C})$ be the space of sections of $\Lambda^k(T^*M \otimes \mathbb{C})$, called *complex-valued k-forms on M*. The *differential forms of type* (ℓ, m) on (M, J) are the sections of $\Lambda^{\ell,m}$, and the space of these differential forms is denoted $\Omega^{\ell,m}$. The decomposition of forms by Dolbeault type is $\Omega^k(M; \mathbb{C}) = \bigoplus_{\ell+m=k} \Omega^{\ell,m}$. Let $\pi^{\ell,m} : \Lambda^k(T^*M \otimes \mathbb{C}) \to \Lambda^{\ell,m}$ be the projection map, where $\ell + m = k$. The usual exterior derivative d (extended linearly to smooth complex-valued forms) composed with two of these projections induces the *del* and *del-bar* differential operators, ∂ and $\bar{\partial}$, on forms of type (ℓ, m):

$$\partial := \pi^{\ell+1,m} \circ d : \Omega^{\ell,m} \longrightarrow \Omega^{\ell+1,m}$$

and

$$\bar{\partial} := \pi^{\ell,m+1} \circ d : \Omega^{\ell,m} \longrightarrow \Omega^{\ell,m+1}.$$

If $\beta \in \Omega^{\ell,m}(M)$, with $k = \ell + m$, then $d\beta \in \Omega^{k+1}(M; \mathbb{C})$:

$$d\beta = \sum_{r+s=k+1} \pi^{r,s} d\beta = \pi^{k+1,0} d\beta + \cdots + \partial\beta + \bar{\partial}\beta + \cdots + \pi^{0,k+1} d\beta.$$

In particular, on complex-valued functions we have $df = d(\operatorname{re} f) + i\, d(\operatorname{im} f)$ and $d = \partial + \bar{\partial}$, where $\partial = \pi^{1,0} \circ d$ and $\bar{\partial} = \pi^{0,1} \circ d$. A function $f : M \to \mathbb{C}$ is *J-holomorphic at* $p \in M$ if df_p is complex linear, i.e., $df_p \circ J_p = i\, df_p$ (or $df_p \in T_p^{1,0}$). A function f is *J-holomorphic* if it is holomorphic at all $p \in M$. A function $f : M \to \mathbb{C}$ is *J-anti-holomorphic at* $p \in M$ if df_p is complex antilinear, i.e., $df_p \circ J_p = -i\, df_p$ (or $df_p \in T_p^{0,1}$), that is, when the conjugate function $\bar{f}$ is holomorphic at $p \in M$. In terms of ∂ and $\bar{\partial}$, a function f is J-holomorphic if and only if $\bar{\partial} f = 0$, and f is J-anti-holomorphic if and only if $\partial f = 0$.

When M is a *complex manifold* and J is its canonical almost complex structure, the splitting $\Omega^k(M;\mathbb{C}) = \bigoplus_{\ell+m=k} \Omega^{\ell,m}$ is particularly interesting. Let $\mathcal{U} \subseteq M$ be the domain of a complex coordinate chart $\varphi = (z_1, \dots, z_n)$, where the corresponding real coordinates $x_1, y_1, \dots, x_n, y_n$ satisfy $z_j = x_j + iy_j$. In terms of

$$\frac{\partial}{\partial z_j} := \frac{1}{2}\left(\frac{\partial}{\partial x_j} - i\frac{\partial}{\partial y_j}\right) \quad \text{and} \quad \frac{\partial}{\partial \bar{z}_j} := \frac{1}{2}\left(\frac{\partial}{\partial x_j} + i\frac{\partial}{\partial y_j}\right),$$

the $(\pm i)$-eigenspaces of J_p $(p \in \mathcal{U})$ can be written

$$(T_{1,0})_p = \mathbb{C}\text{-span}\left\{ \left.\frac{\partial}{\partial z_j}\right|_p : j = 1, \dots, n \right\} \quad \text{and} \quad (T_{0,1})_p = \mathbb{C}\text{-span}\left\{ \left.\frac{\partial}{\partial \bar{z}_j}\right|_p \right\}.$$

Similarly, putting $dz_j = dx_j + i\,dy_j$ and $d\bar{z}_j = dx_j - i\,dy_j$, we obtain simple formulas for the differentials of a $b \in C^\infty(\mathcal{U};\mathbb{C})$, $\partial b = \sum \frac{\partial b}{\partial z_j} dz_j$ and $\bar{\partial} b = \sum \frac{\partial b}{\partial \bar{z}_j} d\bar{z}_j$, and we have $T^{1,0} = \mathbb{C}\text{-span}\{dz_j\colon j = 1, \dots, n\}$ and $T^{0,1} = \mathbb{C}\text{-span}\{d\bar{z}_j\colon j = 1, \dots, n\}$. If we use multi-index notation $J = (j_1, \dots, j_\ell)$ where $1 \leqslant j_1 < \dots < j_\ell \leqslant n$, $|J| = \ell$ and $dz_J = dz_{j_1} \wedge dz_{j_2} \wedge \dots \wedge dz_{j_\ell}$, then the set of (ℓ, m)-forms on $\mathcal{U}$ is

$$\Omega^{\ell,m} = \left\{ \sum_{|J|=\ell,\ |K|=m} b_{J,K}\, dz_J \wedge d\bar{z}_K \mid b_{J,K} \in C^\infty(\mathcal{U};\mathbb{C}) \right\}.$$

A form $\beta \in \Omega^k(M;\mathbb{C})$ may be written over $\mathcal{U}$ as

$$\beta = \sum_{\ell+m=k} \left(\sum_{|J|=\ell,\ |K|=m} b_{J,K}\, dz_J \wedge d\bar{z}_K \right).$$

Since $d = \partial + \bar{\partial}$ on functions, we get

$$\begin{aligned} d\beta &= \sum_{\ell+m=k} \left(\sum_{|J|=\ell,\ |K|=m} db_{J,K} \wedge dz_J \wedge d\bar{z}_K \right) \\ &= \sum_{\ell+m=k} \Bigg(\underbrace{\sum_{|J|=\ell,\ |K|=m} \partial b_{J,K} \wedge dz_J \wedge d\bar{z}_K}_{\in \Omega^{\ell+1,m}} \\ &\qquad + \underbrace{\sum_{|J|=\ell,\ |K|=m} \bar{\partial} b_{J,K} \wedge dz_J \wedge d\bar{z}_K}_{\in \Omega^{\ell,m+1}} \Bigg) \\ &= \partial\beta + \bar{\partial}\beta, \end{aligned}$$

and conclude that, *on a complex manifold,* $d = \partial + \bar\partial$ *on forms of any degree.* This cannot be proved for an almost complex manifold, because there are no coordinate functions z_j to give a suitable basis of 1-forms.

When $d = \partial + \bar\partial$, for any form $\beta \in \Omega^{\ell,m}$, we have

$$0 = d^2\beta = \underbrace{\partial^2\beta}_{\in\Omega^{\ell+2,m}} + \underbrace{\partial\bar\partial\beta + \bar\partial\partial\beta}_{\in\Omega^{\ell+1,m+1}} + \underbrace{\bar\partial^2\beta}_{\in\Omega^{\ell,m+2}} \quad \Longrightarrow \quad \begin{cases} \bar\partial^2 = 0, \\ \partial\bar\partial + \bar\partial\partial = 0, \\ \partial^2 = 0. \end{cases}$$

Since $\bar\partial^2 = 0$, the chain $0 \longrightarrow \Omega^{\ell,0} \xrightarrow{\bar\partial} \Omega^{\ell,1} \xrightarrow{\bar\partial} \Omega^{\ell,2} \xrightarrow{\bar\partial} \cdots$ is a differential complex. Its cohomology groups

$$H^{\ell,m}_{\mathrm{Dolbeault}}(M) := \frac{\ker\bar\partial : \Omega^{\ell,m} \to \Omega^{\ell,m+1}}{\operatorname{im}\bar\partial : \Omega^{\ell,m-1} \to \Omega^{\ell,m}}$$

are called the *Dolbeault cohomology* groups. The Dolbeault theorem states that for complex manifolds $H^{\ell,m}_{\mathrm{Dolbeault}}(M) \simeq H^m(M; \mathcal{O}(\Omega^{(\ell,0)}))$, where $\mathcal{O}(\Omega^{(\ell,0)})$ is the sheaf of forms of type $(\ell, 0)$ over M.

It is natural to ask whether the identity $d = \partial + \bar\partial$ could hold for manifolds other than complex manifolds. Newlander and Nirenberg [106] showed that the answer is no: for an almost complex manifold (M, J), the following are equivalent

$$\begin{aligned} M \text{ is a complex manifold} \quad &\Longleftrightarrow \quad \mathcal{N} \equiv 0 \quad \Longleftrightarrow \quad d = \partial + \bar\partial \\ &\Longleftrightarrow \quad \bar\partial^2 = 0, \end{aligned}$$

where $\mathcal{N}$ is the *Nijenhuis tensor*:

$$\mathcal{N}(X, Y) := [JX, JY] - J[JX, Y] - J[X, JY] - [X, Y],$$

for vector fields X and Y on M, $[\cdot,\cdot]$ being the usual bracket.[31] The Nijenhuis tensor can be thought of as a measure of the existence of J-holomorphic functions: if there exist n J-holomorphic functions, $f_1, \ldots, f_n$, on $\mathbb{R}^{2n}$, that are independent at some point p, i.e., the real and imaginary parts of $(df_1)_p, \ldots, (df_n)_p$ form a basis of $T^*_p\mathbb{R}^{2n}$, then $\mathcal{N}$ vanishes identically at p. More material related to Dolbeault theory or to the Newlander–Nirenberg theorem can be found in [23,37,62,76,141].

EXAMPLE. Out of all spheres, only S^2 and S^6 admit almost complex structures [121, §41.20]. As a complex manifold, S^2 if referred to as the *Riemann sphere* $\mathbb{CP}^1$. The almost complex structure on S^6 from Example 3 of Section 3.2 is not integrable, but it is not yet known whether S^6 admits a structure of complex manifold.

[31] The *bracket* of vector fields X and Y is the vector field $[X, Y]$ characterized by the property that $\mathcal{L}_{[X,Y]}f := \mathcal{L}_X(\mathcal{L}_Y f) - \mathcal{L}_Y(\mathcal{L}_X f)$, for $f \in C^\infty(M)$, where $\mathcal{L}_X f = df(X)$.

In the (real) 2-dimensional case $\mathcal{N}$ always vanishes simply because $\mathcal{N}$ is a tensor, i.e., $\mathcal{N}(fX, gY) = fg\mathcal{N}(X, Y)$ for any $f, g \in C^\infty(M)$, and $\mathcal{N}(X, JX) = 0$ for any vector field X. Combining this with the fact that any orientable surface is symplectic, we conclude that any orientable surface is a complex manifold, a result already known to Gauss. However, most almost complex structures on higher-dimensional manifolds are not integrable. In Section 3.5 we see that the existence of a complex structure compatible with a symplectic structure on a compact manifold imposes significant topological constraints.

3.4. *Kähler manifolds*

DEFINITION 3.8. A *Kähler manifold* is a symplectic manifold (M, ω) equipped with an integrable compatible almost complex structure J. The symplectic form ω is then called a *Kähler form*.

As a complex manifold, a Kähler manifold (M, ω, J) has Dolbeault cohomology. As it is also a symplectic manifold, it is interesting to understand where the symplectic form ω sits with respect to the Dolbeault type decomposition.

PROPOSITION 3.9. *A Kähler form ω is a ∂- and $\bar{\partial}$-closed $(1, 1)$-form that is given on a local complex chart $(\mathcal{U}, z_1, \dots, z_n)$ by*

$$\omega = \frac{i}{2} \sum_{j,k=1}^{n} h_{jk}\, dz_j \wedge d\bar{z}_k,$$

where, at every point $p \in \mathcal{U}$, $(h_{jk}(p))$ is a positive-definite Hermitian matrix.

In particular, ω defines a Dolbeault $(1, 1)$-cohomology class, $[\omega] \in H^{1,1}_{\text{Dolbeault}}(M)$.

PROOF. Being a form in $\Omega^2(M; \mathbb{C}) = \Omega^{2,0} \oplus \Omega^{1,1} \oplus \Omega^{0,2}$, with respect to a local complex chart, ω can be written

$$\omega = \sum a_{jk}\, dz_j \wedge dz_k + \sum b_{jk}\, dz_j \wedge d\bar{z}_k + \sum c_{jk}\, d\bar{z}_j \wedge d\bar{z}_k$$

for some $a_{jk}, b_{jk}, c_{jk} \in C^\infty(\mathcal{U}; \mathbb{C})$. By the compatibility of ω with the complex structure, J is a symplectomorphism, that is, $J^*\omega = \omega$ where $(J^*\omega)(u, v) := \omega(Ju, Jv)$. Since $J^* dz_j = dz_j \circ J = i\, dz_j$ and $J^* d\bar{z}_j = d\bar{z}_j \circ J = -i\, d\bar{z}_j$, we have $J^*\omega = \omega$ if and only if the coefficients a_{jk} and c_{jk} all vanish identically, that is, if and only if $\omega \in \Omega^{1,1}$. Since ω is closed, of type $(1, 1)$ and $d\omega = \partial\omega + \bar{\partial}\omega$, we must have $\partial\omega = 0$ and $\bar{\partial}\omega = 0$. Set $b_{jk} = \frac{i}{2} h_{jk}$. As ω is real-valued, i.e., $\omega = \frac{i}{2} \sum h_{jk}\, dz_j \wedge d\bar{z}_k$ and $\bar{\omega} = -\frac{i}{2} \sum \overline{h_{jk}}\, d\bar{z}_j \wedge dz_k$ coincide, we must have $h_{jk} = \overline{h_{kj}}$ for all j and k. In other words, at every point $p \in \mathcal{U}$, the $n \times n$ matrix $(h_{jk}(p))$ is Hermitian. The nondegeneracy amounts to the nonvanishing of

$$\omega^n = n! \left(\frac{i}{2}\right)^n \det(h_{jk})\, dz_1 \wedge d\bar{z}_1 \wedge \cdots \wedge dz_n \wedge d\bar{z}_n.$$

Therefore, at every $p \in M$, the matrix $(h_{jk}(p))$ must be nonsingular. Finally, the positivity condition $\omega(v, Jv) > 0$, $\forall v \neq 0$, from compatibility, implies that, at each $p \in \mathcal{U}$, the matrix $(h_{jk}(p))$ is positive-definite. $\square$

Consequently, if ω_0 and ω_1 are both Kähler forms on a compact manifold M with $[\omega_0] = [\omega_1] \in H^2_{\text{deRham}}(M)$, then (M, ω_0) and (M, ω_1) are strongly isotopic by Moser's Theorem 1.7. Indeed $\omega_t = (1-t)\omega_0 + t\omega_1$ is symplectic for $t \in [0,1]$, as convex combinations of positive-definite matrices are still positive-definite.

Another consequence is the following recipe for Kähler forms. A smooth real function ρ on a complex manifold M is *strictly plurisubharmonic* (*s.p.s.h.*) if, on each local complex chart $(\mathcal{U}, z_1, \ldots, z_n)$, the matrix $(\frac{\partial^2 \rho}{\partial z_j \partial \bar{z}_k}(p))$ is positive-definite at all $p \in \mathcal{U}$. If $\rho \in C^\infty(M; \mathbb{R})$ is s.p.s.h., then the form

$$\omega = \frac{i}{2} \partial \bar{\partial} \rho$$

is Kähler. The function ρ is then called a (global) *Kähler potential.*

EXAMPLE. Let $M = \mathbb{C}^n \simeq \mathbb{R}^{2n}$, with complex coordinates $(z_1, \ldots, z_n)$ and corresponding real coordinates $(x_1, y_1, \ldots, x_n, y_n)$ via $z_j = x_j + iy_j$. The function

$$\rho(x_1, y_1, \ldots, x_n, y_n) = \sum_{j=1}^{n} (x_j^2 + y_j^2) = \sum |z_j|^2 = \sum z_j \bar{z}_j$$

is s.p.s.h. and is a Kähler potential for the standard Kähler form:

$$\frac{i}{2} \partial \bar{\partial} \rho = \frac{i}{2} \sum_{j,k} \delta_{jk}\, dz_j \wedge d\bar{z}_k = \frac{i}{2} \sum_j dz_j \wedge d\bar{z}_j = \sum_j dx_j \wedge dy_j = \omega_0.$$

There is a local converse to the previous construction of Kähler forms.

PROPOSITION 3.10. *Let ω be a closed real-valued $(1,1)$-form on a complex manifold M and let $p \in M$. Then on a neighborhood $\mathcal{U}$ of p we have $\omega = \frac{i}{2} \partial \bar{\partial} \rho$ for some $\rho \in C^\infty(\mathcal{U}; \mathbb{R})$.*

The proof of this theorem requires holomorphic versions of Poincaré's lemma, namely, the local triviality of Dolbeault groups (the fact that any point in a complex manifold admits a neighborhood $\mathcal{U}$ such that $H^{\ell,m}_{\text{Dolbeault}}(\mathcal{U}) = 0$ for all $m > 0$) and the local triviality of the holomorphic de Rham groups; see [62].

For a Kähler ω, such a local function ρ is called a *local Kähler potential.*

PROPOSITION 3.11. *Let M be a complex manifold, $\rho \in C^\infty(M; \mathbb{R})$ s.p.s.h., X a complex submanifold, and $i : X \hookrightarrow M$ the inclusion map. Then $i^*\rho$ is s.p.s.h.*

PROOF. It suffices to verify this locally by considering a complex chart $(z_1, \dots, z_n)$ for M adapted to X so that X is given there by the equations $z_1 = \dots = z_m = 0$. Being a principal minor of the positive-definite matrix $(\frac{\partial^2}{\partial z_j \partial \bar{z}_k}(0, \dots, 0, z_{m+1}, \dots, z_n))$ the matrix $(\frac{\partial^2 \rho}{\partial z_{m+j} \partial \bar{z}_{m+k}}(0, \dots, 0, z_{m+1}, \dots, z_n))$ is also positive-definite. □

COROLLARY 3.12. *Any complex submanifold of a Kähler manifold is also Kähler.*

DEFINITION 3.13. Let (M, ω) be a Kähler manifold, X a complex submanifold, and $i : X \hookrightarrow M$ the inclusion. Then $(X, i^*\omega)$ is called a *Kähler submanifold.*

EXAMPLES.
1. Complex vector space $(\mathbb{C}^n, \omega_0)$ where $\omega_0 = \frac{i}{2} \sum dz_j \wedge d\bar{z}_j$ is Kähler. According to Corollary 3.12, every complex submanifold of $\mathbb{C}^n$ is Kähler.
2. In particular, *Stein manifolds* are Kähler. *Stein manifolds* are the properly embedded complex submanifolds of $\mathbb{C}^n$. They can be alternatively characterized as being the Kähler manifolds (M, ω) that admit a global proper Kähler potential, i.e., $\omega = \frac{i}{2}\partial\bar{\partial}\rho$ for some proper function $\rho : M \to \mathbb{R}$.
3. The function $z \mapsto \log(|z|^2 + 1)$ on $\mathbb{C}^n$ is strictly plurisubharmonic. Therefore the 2-form
$$\omega_{\mathrm{FS}} = \frac{i}{2}\partial\bar{\partial}\log(|z|^2 + 1)$$
is another Kähler form on $\mathbb{C}^n$ This is called the *Fubini–Study form* on $\mathbb{C}^n$.
4. Let $\{(\mathcal{U}_j, \mathbb{C}^n, \varphi_j), j = 0, \dots, n\}$ be the usual complex atlas for *complex projective space.*[32] The form ω_{FS} is preserved by the transition maps, hence $\varphi_j^*\omega_{\mathrm{FS}}$ and $\varphi_k^*\omega_{\mathrm{FS}}$ agree on the overlap $\mathcal{U}_j \cap \mathcal{U}_k$. The *Fubini–Study form* on $\mathbb{CP}^n$ is the Kähler form obtained by gluing together the $\varphi_j^*\omega_{\mathrm{FS}}$, $j = 0, \dots, n$.
5. Consequently, all *nonsingular projective varieties* are Kähler submanifolds. Here by nonsingular we mean smooth, and by projective variety we mean the zero locus of some collection of homogeneous polynomials.
6. All *Riemann surfaces* are Kähler, since any compatible almost complex structure is integrable for dimension reasons (Section 3.3).

[32]The *complex projective space* $\mathbb{CP}^n$ is the complex n-dimensional manifold given by the space of complex lines in $\mathbb{C}^{n+1}$. It can be obtained from $\mathbb{C}^{n+1} \setminus \{0\}$ by making the identifications $(z_0, \dots, z_n) \sim (\lambda z_0, \dots, \lambda z_n)$ for all $\lambda \in \mathbb{C} \setminus \{0\}$. One denotes by $[z_0, \dots, z_n]$ the equivalence class of $(z_0, \dots, z_n)$, and calls $z_0, \dots, z_n$ the *homogeneous coordinates* of the point $p = [z_0, \dots, z_n]$. (Homogeneous coordinates are, of course, only determined up to multiplication by a nonzero complex number λ.) Let $\mathcal{U}_j$ be the subset of $\mathbb{CP}^n$ consisting of all points $p = [z_0, \dots, z_n]$ for which $z_j \neq 0$. Let $\varphi_j : \mathcal{U}_j \to \mathbb{C}^n$ be the map defined by
$$\varphi_j([z_0, \dots, z_n]) = \frac{z_0}{z_j}, \dots, \frac{z_{j-1}}{z_j}, \frac{z_{j+1}}{z_j}, \dots, \frac{z_n}{z_j}.$$
The collection $\{(\mathcal{U}_j, \mathbb{C}^n, \varphi_j), j = 0, \dots, n\}$ is the *usual complex atlas* for $\mathbb{CP}^n$. For instance, the transition map from $(\mathcal{U}_0, \mathbb{C}^n, \varphi_0)$ to $(\mathcal{U}_1, \mathbb{C}^n, \varphi_1)$ is $\varphi_{0,1}(z_1, \dots, z_n) = (\frac{1}{z_1}, \frac{z_2}{z_1}, \dots, \frac{z_n}{z_1})$ defined from the set $\{(z_1, \dots, z_n) \in \mathbb{C}^n \mid z_1 \neq 0\}$ to itself.

7. The Fubini–Study form on the chart $\mathcal{U}_0 = \{[z_0, z_1] \in \mathbb{CP}^1 \mid z_0 \neq 0\}$ of the *Riemann sphere* $\mathbb{CP}^1$ is given by the formula

$$\omega_{\mathrm{FS}} = \frac{dx \wedge dy}{(x^2 + y^2 + 1)^2},$$

where $\frac{z_1}{z_0} = z = x + iy$ is the usual coordinate on $\mathbb{C}$. The standard area form $\omega_{\mathrm{std}} = d\theta \wedge dh$ is induced by regarding $\mathbb{CP}^1$ as the unit sphere S^2 in $\mathbb{R}^3$ (Example 3 of Section 1.2). Stereographic projection shows that $\omega_{\mathrm{FS}} = \frac{1}{4}\omega_{\mathrm{std}}$.
8. *Complex tori* are Kähler. Complex tori look like quotients $\mathbb{C}^n/\mathbb{Z}^n$ where $\mathbb{Z}^n$ is a lattice in $\mathbb{C}^n$. The form $\omega = \sum dz_j \wedge d\bar{z}_j$ induced by the Euclidean structure is Kähler.
9. Just like products of symplectic manifolds are symplectic, also products of Kähler manifolds are Kähler.

3.5. *Hodge theory*

Hodge [73] identified the spaces of cohomology classes of forms with spaces of actual forms, by picking *the* representative from each class that solves a certain differential equation, namely the *harmonic* representative. We give a sketch of Hodge's idea. The first part makes up ordinary Hodge theory, which works for any compact oriented Riemannian manifold (M, g), not necessarily Kähler.

At a point $p \in M$, let $e_1, \ldots, e_n$ be a positively oriented orthonormal basis of the cotangent space T_p^*M, with respect to the induced inner product and orientation. The *Hodge star operator* is the linear operator on the exterior algebra of T_p^*M defined by

$$\begin{aligned} &*(1) = e_1 \wedge \cdots \wedge e_n, \\ &*(e_1 \wedge \cdots \wedge e_n) = 1, \\ &*(e_1 \wedge \cdots \wedge e_k) = e_{k+1} \wedge \cdots \wedge e_n. \end{aligned}$$

We see that $* : \Lambda^k(T_p^*M) \to \Lambda^{n-k}(T_p^*M)$ and satisfies $** = (-1)^{k(n-k)}$. The *codifferential* and the *Laplacian* are the operators defined by

$$\begin{aligned} &\delta = (-1)^{n(k+1)+1} * d* &&: \Omega^k(M) \to \Omega^{k-1}(M), \\ &\Delta = d\delta + \delta d &&: \Omega^k(M) \to \Omega^k(M). \end{aligned}$$

The operator Δ is also called the *Laplace–Beltrami operator* and satisfies $\Delta * = * \Delta$. On $\Omega^0(\mathbb{R}^n) = C^\infty(\mathbb{R}^n)$, it is simply the usual Laplacian $\Delta = -\sum_{i=1}^n \frac{\partial^2}{\partial x_i^2}$. The *inner product on forms* of any degree,

$$\langle \cdot, \cdot \rangle : \Omega^k(M) \times \Omega^k(M) \longrightarrow \mathbb{R}, \quad \langle \alpha, \beta \rangle := \int_M \alpha \wedge *\beta,$$

satisfies $\langle d\alpha, \beta\rangle = \langle \alpha, \delta\beta\rangle$, so the codifferential δ is often denoted by d^* and called the *adjoint*[33] *of* d. Also, Δ is self-adjoint (i.e., $\langle \Delta\alpha, \beta\rangle = \langle \alpha, \Delta\beta\rangle$), and $\langle \Delta\alpha, \alpha\rangle = |d\alpha|^2 + |\delta\alpha|^2 \geqslant 0$, where $|\cdot|$ is the norm with respect to this inner product. The *harmonic* k*-forms* are the elements of $\mathcal{H}^k := \{\alpha \in \Omega^k \mid \Delta\alpha = 0\}$. Note that $\Delta\alpha = 0$ if and only if $d\alpha = \delta\alpha = 0$. Since a harmonic form is d-closed, it defines a de Rham cohomology class.

THEOREM 3.14 (Hodge). *Every de Rham cohomology class on a compact oriented Riemannian manifold* (M, g) *possesses a unique harmonic representative, i.e., there is an isomorphism* $\mathcal{H}^k \simeq H^k_{\mathrm{deRham}}(M; \mathbb{R})$. *In particular, the spaces* $\mathcal{H}^k$ *are finite-dimensional. We also have the following orthogonal decomposition with respect to the inner product on forms*: $\Omega^k \simeq \mathcal{H}^k \oplus \Delta(\Omega^k(M)) \simeq \mathcal{H}^k \oplus d\Omega^{k-1} \oplus \delta\Omega^{k+1}$.

This decomposition is called the *Hodge decomposition on forms*. The proof of this and the next theorem involves functional analysis, elliptic differential operators, pseudodifferential operators and Fourier analysis; see for instance [62,83,141].

Here is where *complex Hodge theory* begins. When M is Kähler, the Laplacian satisfies $\Delta = 2(\bar\partial\bar\partial^* + \bar\partial^*\bar\partial)$ (see, for example, [62]) and preserves the decomposition according to type, $\Delta : \Omega^{\ell,m} \to \Omega^{\ell,m}$. Hence, harmonic forms are also bigraded

$$\mathcal{H}^k = \bigoplus_{\ell+m=k} \mathcal{H}^{\ell,m}$$

and satisfy a Künneth formula $\mathcal{H}^{\ell,m}(M \times N) \simeq \bigoplus_{p+r=\ell,\ q+s=m} \mathcal{H}^{p,q}(M) \otimes \mathcal{H}^{r,s}(N)$.

THEOREM 3.15 (Hodge). *Every Dolbeault cohomology class on a compact Kähler manifold* (M, ω) *possesses a unique harmonic representative, i.e., there is an isomorphism* $\mathcal{H}^{\ell,m} \simeq H^{\ell,m}_{\mathrm{Dolbeault}}(M)$.

Combining the two theorems of Hodge, we find the decomposition of cohomology groups for a compact Kähler manifold

$$H^k_{\mathrm{deRham}}(M; \mathbb{C}) \simeq \bigoplus_{\ell+m=k} H^{\ell,m}_{\mathrm{Dolbeault}}(M),$$

known as the *Hodge decomposition*. In particular, the Dolbeault cohomology groups $H^{\ell,m}_{\mathrm{Dolbeault}}$ are finite-dimensional and $H^{\ell,m} \simeq \overline{H^{m,\ell}}$.

Let $b^k(M) := \dim H^k_{\mathrm{deRham}}(M)$ be the usual *Betti numbers* of M, and let $h^{\ell,m}(M) := \dim H^{\ell,m}_{\mathrm{Dolbeault}}(M)$ be the *Hodge numbers* of M.

For an arbitrary compact symplectic manifold (M, ω), the even Betti numbers must be positive, because ω^k is closed but not exact ($k = 0, 1, \ldots, n$). In fact, if it were $\omega^k = d\alpha$, by Stokes' theorem we would have $\int_M \omega^n = \int_M d(\alpha \wedge \omega^{n-k}) = 0$, which contradicts ω^n being a volume form.

[33] When M is not compact, we still have a *formal adjoint* of d with respect to the nondegenerate bilinear pairing $\langle\cdot,\cdot\rangle : \Omega^k(M) \times \Omega^k_c(M) \to \mathbb{R}$ defined by a similar formula, where $\Omega^k_c(M)$ is the space of compactly supported k-forms.

For a compact Kähler manifold (M, ω), there are finer topological consequences coming from the Hodge theorems, as we must have $b^k = \sum_{\ell+m=k} h^{\ell,m}$ and $h^{\ell,m} = h^{m,\ell}$. The odd Betti numbers must be even because $b^{2k+1} = \sum_{\ell+m=2k+1} h^{\ell,m} = 2\sum_{\ell=0}^{k} h^{\ell,(2k+1-\ell)}$. The number $h^{1,0} = \frac{1}{2}b^1$ must be a topological invariant. The numbers $h^{\ell,\ell}$ are positive, because $0 \neq [\omega^\ell] \in H^{\ell,\ell}_{\text{Dolbeault}}(M)$. First of all, $[\omega^\ell]$ defines an element of $H^{\ell,\ell}_{\text{Dolbeault}}$ as $\omega \in \Omega^{1,1}$ implies that $\omega^\ell \in \Omega^{\ell,\ell}$, and the closedness of ω^ℓ implies that $\bar\partial\omega^\ell = 0$. If it were $\omega^\ell = \bar\partial\beta$ for some $\beta \in \Omega^{\ell-1,\ell}$, then $\omega^n = \omega^\ell \wedge \omega^{n-\ell} = \bar\partial(\beta \wedge \omega^{n-\ell})$ would be $\bar\partial$-exact. But $[\omega^n] \neq 0$ in $H^{2n}_{\text{deRham}}(M; \mathbb{C}) \simeq H^{n,n}_{\text{Dolbeault}}(M)$ since it is a volume form. A popular diagram to describe relations among Hodge numbers is the *Hodge diamond*:

$$
\begin{array}{ccccccc}
 & & & h^{n,n} & & & \\
 & & h^{n,n-1} & & h^{n-1,n} & & \\
 & h^{n,n-2} & & h^{n-1,n-1} & & h^{n-2,n} & \\
\cdots & & & \vdots & & & \cdots \\
 & h^{2,0} & & h^{1,1} & & h^{0,2} & \\
 & & h^{1,0} & & h^{0,1} & & \\
 & & & h^{0,0} & & &
\end{array}
$$

Complex conjugation gives symmetry with respect to the middle vertical, whereas the Hodge star operator induces symmetry about the center of the diamond. The middle vertical axis is all nonzero.

There are further symmetries and ongoing research on how to compute $H^{\ell,m}_{\text{Dolbeault}}$ for a compact Kähler manifold (M, ω). In particular, the *hard Lefschetz theorem* states isomorphisms $L^k : H^{n-k}_{\text{deRham}}(M) \xrightarrow{\simeq} H^{n+k}_{\text{deRham}}(M)$ given by wedging with ω^k at the level of forms and the *Lefschetz decompositions* $H^m_{\text{deRham}}(M) \simeq \bigoplus_k L^k(\ker L^{n-m+2k+1}|_{H^{m-2k}})$. The *Hodge conjecture* claims, for projective manifolds M (i.e., complex submanifolds of complex projective space), that the Poincaré duals of elements in $H^{\ell,\ell}_{\text{Dolbeault}}(M) \cap H^{2\ell}(M; \mathbb{Q})$ are rational linear combinations of classes of complex codimension ℓ subvarieties of M. This has been proved only for the $\ell = 1$ case (it is the Lefschetz theorem on $(1, 1)$-classes; see, for instance, [62]).

3.6. *Pseudoholomorphic curves*

Whereas an almost complex manifold (M, J) tends to have no J-holomorphic functions $M \to \mathbb{C}$ at all,[34] it has plenty of *J-holomorphic curves* $\mathbb{C} \to M$. Gromov first realized that *pseudoholomorphic curves* provide a powerful tool in symplectic topology in an extremely influential paper [64]. Fix a closed Riemann surface (Σ, j), that is, a compact complex 1-dimensional manifold Σ without boundary and equipped with the canonical almost complex structure j.

[34] However, the study of *asymptotically J-holomorphic functions* has been recently developed to obtain important results [32,34,13]; see Section 4.6.

DEFINITION 3.16. A parametrized *pseudoholomorphic curve* (or *J-holomorphic curve*) in (M, J) is a (smooth) map $u : \Sigma \to M$ whose differential intertwines j and J, that is, $du_p \circ j_p = J_p \circ du_p$, $\forall p \in \Sigma$.

In other words, the *Cauchy–Riemann equation* $du + J \circ du \circ j = 0$ holds.

Pseudoholomorphic curves are related to parametrized 2-dimensional symplectic submanifolds. If a pseudoholomorphic curve $u : \Sigma \to M$ is an embedding, then its image $S := u(\Sigma)$ is a 2-dimensional almost complex submanifold, hence a symplectic submanifold. Conversely, the inclusion $i : S \hookrightarrow M$ of a 2-dimensional symplectic submanifold can be seen as a pseudoholomorphic curve. An appropriate compatible almost complex structure J on (M, ω) can be constructed starting from S, such that TS is J-invariant. The restriction j of J to TS is necessarily integrable because S is 2-dimensional.

The group G of complex diffeomorphisms of (Σ, j) acts on (parametrized) pseudoholomorphic curves by reparametrization: $u \mapsto u \circ \gamma$, for $\gamma \in G$. This normally means that each curve u has a noncompact orbit under G. The orbit space $\mathcal{M}_g(A, J)$ is the set of unparametrized pseudoholomorphic curves in (M, J) whose domain Σ has genus g and whose image $u(\Sigma)$ has homology class $A \in H_2(M; \mathbb{Z})$. The space $\mathcal{M}_g(A, J)$ is called the *moduli space of unparametrized pseudoholomorphic curves* of genus g representing the class A. For generic J, Fredholm theory shows that pseudoholomorphic curves occur in finite-dimensional smooth families, so that the moduli spaces $\mathcal{M}_g(A, J)$ can be manifolds, after avoiding singularities given by *multiple coverings*.[35]

EXAMPLE. Usually Σ is the Riemann sphere $\mathbb{CP}^1$, whose complex diffeomorphisms are those given by *fractional linear transformations* (or *Möbius transformations*). So the 6-dimensional noncompact group of projective linear transformations PSL(2; $\mathbb{C}$) acts on *pseudoholomorphic spheres* by reparametrization $u \mapsto u \circ \gamma_A$, where $A = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in$ PSL(2; $\mathbb{C}$) acts by $\gamma_A : \mathbb{CP}^1 \to \mathbb{CP}^1$, $\gamma_A[z, 1] = [\frac{az+b}{cz+d}, 1]$.

When J is an almost complex structure *compatible* with a symplectic form ω, the area of the image of a pseudoholomorphic curve u (with respect to the metric $g_J(\cdot, \cdot) = \omega(\cdot, J\cdot)$) is determined by the class A that it represents. The number

$$E(u) := \omega(A) = \int_\Sigma u^*\omega = \text{area of the image of } u \text{ with respect to } g_J$$

is called the *energy* of the curve u and is a topological invariant: it only depends on $[\omega]$ and on the homotopy class of u. Gromov proved that the constant energy of all the pseudoholomorphic curves representing a homology class A ensured that the space $\mathcal{M}_g(A, J)$, though not necessarily compact, had natural *compactifications* $\overline{\mathcal{M}}_g(A, J)$ by including what he called *cusp-curves*.

THEOREM 3.17 (Gromov's compactness theorem). *If (M, ω) is a compact manifold equipped with a generic compatible almost complex structure J, and if u_j is a sequence*

[35] A curve $u : \Sigma \to M$ is a *multiple covering* if u factors as $u = u' \circ \sigma$ where $\sigma : \Sigma \to \Sigma'$ is a holomorphic map of degree greater than 1.

of pseudoholomorphic curves in $\mathcal{M}_g(A, J)$, then there is a subsequence that weakly converges to a cusp-curve in $\overline{\mathcal{M}}_g(A, J)$.

Hence the cobordism class of the compactified moduli space $\overline{\mathcal{M}}_g(A, J)$ might be a nice symplectic invariant of (M, ω), as long as it is not empty or null-cobordant. Actually a nontrivial regularity criterion for J ensures the existence of pseudoholomorphic curves. And even when $\overline{\mathcal{M}}_g(A, J)$ is null-cobordant, we can define an invariant to be the (signed) number of pseudoholomorphic curves of genus g in class A that intersect a specified set of representatives of homology classes in M [112,128,145]. For more on pseudoholomorphic curves; see, for instance, [100] (for a comprehensive discussion of the genus 0 case) or [11] (for higher genus). Here is a selection of applications of (developments from) pseudoholomorphic curves:

- Proof of the *nonsqueezing theorem* [64]: for $R > r$ there is no symplectic embedding of a ball B_R^{2n} of radius R into a cylinder $B_r^2 \times \mathbb{R}^{2n-2}$ of radius r, both in $(\mathbb{R}^{2n}, \omega_0)$.
- Proof that there are *no Lagrangian spheres* in $(\mathbb{C}^n, \omega_0)$, except for the circle in $\mathbb{C}^2$, and more generally *no compact exact Lagrangian submanifolds*, in the sense that the tautological 1-form α restricts to an exact form [64].
- Proof that if (M, ω) is a connected symplectic 4-manifold symplectomorphic to $(\mathbb{R}^4, \omega_0)$ outside a compact set and containing no symplectic S^2's, then (M, ω) symplectomorphic to $(\mathbb{R}^4, \omega_0)$ [64].
- Study questions of *symplectic packing* [15,98,134] such as: for a given $2n$-dimensional symplectic manifold (M, ω), what is the maximal radius R for which there is a symplectic embedding of N disjoint balls B_R^{2n} into (M, ω)?
- Study *groups of symplectomorphisms* of 4-manifolds (for a review see [97]). Gromov [64] showed that $\mathrm{Sympl}(\mathbb{CP}^2, \omega_{\mathrm{FS}})$ and $\mathrm{Sympl}(S^2 \times S^2, \mathrm{pr}_1^*\sigma \oplus \mathrm{pr}_2^*\sigma)$ deformation retract onto the corresponding groups of standard isometries.
- Development of *Gromov–Witten invariants* allowing to prove, for instance, the nonexistence of symplectic forms on $\mathbb{CP}^2 \# \mathbb{CP}^2 \# \mathbb{CP}^2$ or the classification of symplectic structures on *ruled surfaces* (Section 4.3).
- Development of *Floer homology* to prove the Arnold conjecture on the fixed points of symplectomorphisms of compact symplectic manifolds, or on the intersection of Lagrangian submanifolds (Section 5.2).
- Development of *symplectic field theory* introduced by Eliashberg, Givental and Hofer [40] extending Gromov–Witten theory, exhibiting a rich algebraic structure and also with applications to *contact geometry*.

4. Symplectic geography

4.1. *Existence of symplectic forms*

The utopian goal of symplectic classification addresses the standard questions:

- (*Existence*) Which manifolds carry symplectic forms?
- (*Uniqueness*) What are the distinct symplectic structures on a given manifold?

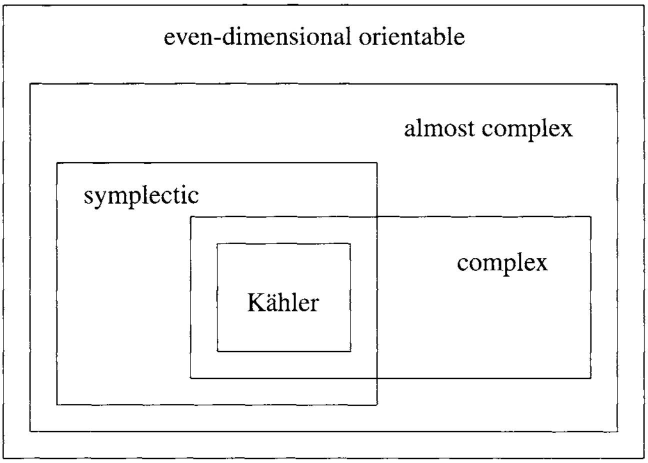

Fig. 1.

Existence is tackled through central examples in this subsection and symplectic constructions in the next two sections. Uniqueness is treated in the remainder of this subsection dealing with invariants that allow to distinguish symplectic manifolds.

A Kähler structure naturally yields both a symplectic form and a complex structure (compatible ones). Either a symplectic or a complex structure on a manifold implies the existence of an almost complex structure. Figure 1 represents the relations among these structures. In dimension 2, orientability trivially guarantees the existence of all other structures, so the picture collapses. In dimension 4, the first interesting dimension, the picture above is faithful—we will see that there are *closed* 4-dimensional examples in each region. *Closed* here means compact and without boundary.

Not all 4-dimensional manifolds are almost complex. A result of Wu [146] gives a necessary and sufficient condition in terms of the signature σ and the Euler characteristic χ of a 4-dimensional closed manifold M for the existence of an almost complex structure: $3\sigma + 2\chi = h^2$ *for some* $h \in H^2(M;\mathbb{Z})$ *congruent with the second Stiefel–Whitney class* $w_2(M)$ *modulo* 2. For example, S^4 and $(S^2 \times S^2)\,\#\,(S^2 \times S^2)$ are not almost complex. When an almost complex structure exists, the first Chern class of the tangent bundle (regarded as a complex vector bundle) satisfies the condition for h. The sufficiency of Wu's condition is the remarkable part.[36]

According to Kodaira's classification of closed complex surfaces [82], such a surface admits a Kähler structure if and only if its first Betti number b_1 is even. The necessity of this condition is a Hodge relation on the Betti numbers (Section 3.5). The complex projective plane $\mathbb{CP}^2$ with the Fubini–Study form (Section 3.4) might be called the simplest example of a closed Kähler 4-manifold.

The *Kodaira–Thurston example* [131] first demonstrated that a manifold that admits both a symplectic and a complex structure does not have to admit any Kähler structure.

[36]Moreover, such solutions h are in one-to-one correspondence with *isomorphism* classes of almost complex structures.

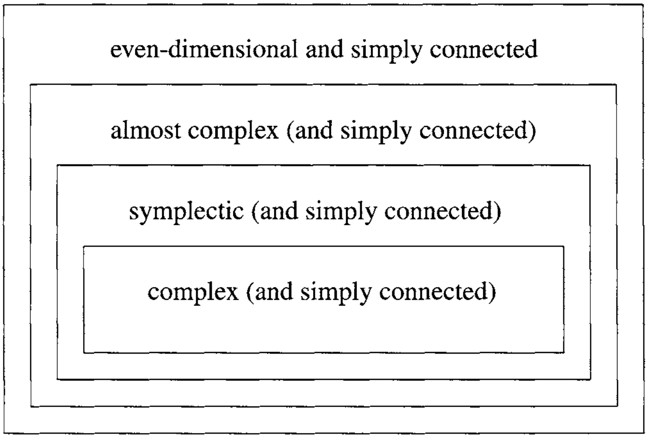

Fig. 2.

Take $\mathbb{R}^4$ with $dx_1 \wedge dy_1 + dx_2 \wedge dy_2$, and Γ the discrete group generated by the four symplectomorphisms:

$$\begin{aligned}
(x_1, x_2, y_1, y_2) &\longmapsto (x_1 + 1, x_2, y_1, y_2),\\
(x_1, x_2, y_1, y_2) &\longmapsto (x_1, x_2 + 1, y_1, y_2),\\
(x_1, x_2, y_1, y_2) &\longmapsto (x_1, x_2 + y_2, y_1 + 1, y_2),\\
(x_1, x_2, y_1, y_2) &\longmapsto (x_1, x_2, y_1, y_2 + 1).
\end{aligned}$$

Then $M = \mathbb{R}^4/\Gamma$ is a symplectic manifold that is a 2-torus bundle over a 2-torus. Kodaira's classification [82] shows that M has a complex structure. However, $\pi_1(M) = \Gamma$, hence $H_1(\mathbb{R}^4/\Gamma; \mathbb{Z}) = \Gamma/[\Gamma, \Gamma]$ has rank 3, so $b_1 = 3$ is *odd*.

Fernández–Gotay–Gray [44] first exhibited symplectic manifolds that do not admit any complex structure at all. Their examples are circle bundles over circle bundles (i.e., a *tower* of circle bundles) over a 2-torus.

The *Hopf surface* is the complex surface diffeomorphic to $S^1 \times S^3$ obtained as the quotient $\mathbb{C}^2 \setminus \{0\}/\Gamma$ where $\Gamma = \{2^n \mathrm{Id} \mid n \in \mathbb{Z}\}$ is a group of *complex* transformations, i.e., we factor $\mathbb{C}^2 \setminus \{0\}$ by the equivalence relation $(z_1, z_2) \sim (2z_1, 2z_2)$. The Hopf surface is not symplectic because $H^2(S^1 \times S^3) = 0$.

The manifold $\mathbb{CP}^2 \# \mathbb{CP}^2 \# \mathbb{CP}^2$ is almost complex but is neither complex (since it does not fit Kodaira's classification [82]), nor symplectic as shown by Taubes [126] using Seiberg–Witten invariants (Section 4.5).

We could go through the previous discussion restricting to closed 4-dimensional examples *with a specific fundamental group*. We will do this restricting to simply connected examples, where Figure 2 holds.

It is a consequence of Wu's result [146] that a simply connected manifold admits an almost complex structure if and only if b_2^+ is odd.[37] In particular, the connected sum

[37] The *intersection form* of an oriented *topological* closed 4-manifold M is the bilinear pairing $Q_M : H^2(M; \mathbb{Z}) \times H^2(M; \mathbb{Z}) \to \mathbb{Z}$, $Q_M(\alpha, \beta) := \langle \alpha \cup \beta, [M] \rangle$, where $\alpha \cup \beta$ is the *cup product* and $[M]$ is the *fundamental class*.

$\#_m \mathbb{CP}^2 \#_n \overline{\mathbb{CP}^2}$ (of m copies of $\mathbb{CP}^2$ with n copies of $\overline{\mathbb{CP}^2}$) has an almost complex structure if and only if m is odd.[38]

By Kodaira's classification [82], a simply connected complex surface always admits a compatible symplectic form (since $b^1 = 0$ is even), i.e., it is always Kähler.

Since they are simply connected, S^4, $\mathbb{CP}^2 \# \mathbb{CP}^2 \# \mathbb{CP}^2$ and $\mathbb{CP}^2$ live in three of the four regions in the picture for simply connected examples. All of $\mathbb{CP}^2 \#_m \overline{\mathbb{CP}^2}$ are also simply connected Kähler manifolds because they are *pointwise blow-ups* $\mathbb{CP}^2$ and the *blow-down map* is holomorphic; see Section 4.3.

There is a family of manifolds obtained from $\mathbb{CP}^2 \#_9 \overline{\mathbb{CP}^2} =: E(1)$ by a *knot surgery* [45] that were shown by Fintushel and Stern to be symplectic and confirmed not to admit a complex structure [109]. The first example of a closed simply connected symplectic manifold that cannot be Kähler, was a 10-dimensional manifold obtained by McDuff [94] as follows. The Kodaira–Thurston example $\mathbb{R}^4/\Gamma$ (not simply connected) embeds symplectically in $(\mathbb{CP}^5, \omega_{\mathrm{FS}})$ [65,132]. McDuff's example is a *blow-up* of $(\mathbb{CP}^5, \omega_{\mathrm{FS}})$ along the image of $\mathbb{R}^4/\Gamma$.

Geography problems are problems on the existence of simply connected closed oriented 4-dimensional manifolds with some additional structure (such as, a symplectic form or a complex structure) for each pair of *topological coordinates*. As a consequence of the work of Freedman [51] and Donaldson [30] in the 80's, it became known that the homeomorphism class of a connected simply connected closed oriented *smooth* 4-manifold is determined by the two integers—the second Betti number and the signature (b_2, σ)—and the *parity*[39] of the intersection form. Forgetting about the parity, the numbers (b_2, σ) can be treated as *topological coordinates*. For each pair (b_2, σ) there could well be infinite different (i.e., nondiffeomorphic) smooth manifolds. Using Riemannian geometry, Cheeger [22] showed that there are at most *countably many* different smooth types for closed 4-manifolds. There are no known finiteness results for the smooth types of a given topological 4-manifold, in contrast to other dimensions.

Traditionally, the numbers used are $(c_1^2, c_2) := (3\sigma + 2\chi, \chi) = (3\sigma + 4 + 2b_2, 2 + b_2)$, and frequently just the *slope* c_1^2/c_2 is considered. If M admits an almost complex structure J, then (TM, J) is a complex vector bundle, hence has Chern classes $c_1 = c_1(M, J)$ and $c_2 = c_2(M, J)$. Both $c_1^2 := c_1 \cup c_1$ and c_2 may be regarded as numbers since $H^4(M; \mathbb{Z}) \simeq \mathbb{Z}$. They satisfy $c_1^2 = 3\sigma + 2\chi$ (by Hirzebruch's signature formula) and $c_2 = \chi$ (because the top Chern class is always the Euler class), justifying the notation for the topological coordinates in this case.

Since Q_M always vanishes on torsion elements, descending to $H^2(M; \mathbb{Z})/\text{torsion}$ it can be represented by a matrix. When M is smooth and simply connected, this pairing is $Q_M(\alpha, \beta) := \int_M \alpha \wedge \beta$ since nontorsion elements are representable by 2-forms. As Q_M is symmetric (in the smooth case, the wedge product of 2-forms is symmetric) and unimodular (the determinant of a matrix representing Q_M is ± 1 by Poincaré duality), it is diagonalizable over $\mathbb{R}$ with eigenvalues ± 1. We denote by b_2^+ (respectively b_2^-) the number of positive (respectively negative) eigenvalues of Q_M counted with multiplicities, i.e., the dimension of a maximal subspace where Q_M is positive-definite (respectively negative-definite). The *signature* of M is the difference $\sigma := b_2^+ - b_2^-$, whereas the second Betti number is the sum $b_2 = b_2^+ + b_2^-$, i.e., the *rank* of Q_M. The *type* of an intersection form is *definite* if it is positive or negative definite (i.e., $|\sigma| = b_2$) and *indefinite* otherwise.

[38]The intersection form of a connected sum $M_0 \# M_1$ is (isomorphic to) $Q_{M_0} \oplus Q_{M_1}$.

[39]We say that the *parity* of an intersection form Q_M is *even* when $Q_M(\alpha, \alpha)$ is even for all $\alpha \in H^2(M; \mathbb{Z})$, and *odd* otherwise.

EXAMPLES. The manifold $\mathbb{CP}^2$ has $(b_2, \sigma) = (1, 1)$, i.e., $(c_1^2, c_2) = (9, 3)$. Reversing the orientation $\overline{\mathbb{CP}^2}$ has $(b_2, \sigma) = (1, -1)$, i.e., $(c_1^2, c_2) = (3, 3)$. Their connected sum $\mathbb{CP}^2 \# \overline{\mathbb{CP}^2}$ has $(b_2, \sigma) = (2, 0)$, i.e., $(c_1^2, c_2) = (8, 0)$. The product $S^2 \times S^2$ also has $(b_2, \sigma) = (2, 0)$, i.e., $(c_1^2, c_2) = (8, 4)$. But $\mathbb{CP}^2 \# \overline{\mathbb{CP}^2}$ has an *odd* intersection form whereas $S^2 \times S^2$ has an *even* intersection form: $\left[\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right]$ vs. $\left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right]$.

Symplectic geography [60,122] addresses the following question: What is the set of pairs of integers $(m, n) \in \mathbb{Z} \times \mathbb{Z}$ for which there exists a connected simply connected closed *symplectic* 4-manifold M having second Betti number $b_2(M) = m$ and signature $\sigma(M) = n$? This problem includes the usual geography of simply connected complex surfaces, since all such surfaces are Kähler according to Kodaira's classification [82]. Often, instead of the numbers (b_2, σ), the question is equivalently phrased in terms of the Chern numbers (c_1^2, c_2) for a compatible almost complex structure, which satisfy $c_1^2 = 3\sigma + 2\chi$ [146] and $c_2 = \chi$, where $\chi = b_2 + 2$ is the *Euler number*. Usually only *minimal* (Section 4.3) or *irreducible* manifolds are considered to avoid trivial examples. A manifold is *irreducible* when it is not a connected sum of other manifolds, except when one of the summands is a homotopy sphere.

It was speculated that perhaps any simply connected closed smooth 4-manifold other than S^4 is diffeomorphic to a connected sum of symplectic manifolds, where any orientation is allowed on each summand (the so-called *minimal conjecture* for smooth 4-manifolds). Szabó [124,125] provided counterexamples in a family of irreducible simply connected closed nonsymplectic smooth 4-manifolds.

All these problems could be posed for other fundamental groups. Gompf [57] used *symplectic sums* (Section 4.2) to prove the following theorem. He also proved that his surgery construction can be adapted to produce *non*-Kähler examples. Since finitely-presented groups are not classifiable, this shows that compact symplectic 4-manifold are not classifiable.

THEOREM 4.1 (Gompf). *Every finitely-presented group occurs as the fundamental group $\pi_1(M)$ of a compact symplectic 4-manifold (M, ω).*

4.2. *Fibrations and sums*

Products of symplectic manifolds are naturally symplectic. As we will see, special kinds of *twisted products*, i.e., fibrations,[40] are also symplectic.

[40] A *fibration* (or *fiber bundle*) is a manifold M (called the *total space*) with a submersion $\pi : M \to X$ to a manifold X (the *base*) that is locally trivial in the sense that there is an open covering of X, such that, to each set $\mathcal{U}$ in that covering corresponds a diffeomorphism of the form $\varphi_{\mathcal{U}} = (\pi, s_{\mathcal{U}}) : \pi^{-1}(\mathcal{U}) \to \mathcal{U} \times F$ (a *local trivialization*) where F is a fixed manifold (the *model fiber*). A collection of local trivializations such that the sets $\mathcal{U}$ cover X is called a *trivializing cover* for π. Given two local trivializations, the second entry of the composition $\varphi_{\mathcal{V}} \circ \varphi_{\mathcal{U}}^{-1} = (\mathrm{id}, \psi_{\mathcal{U}\mathcal{V}})$ on $(\mathcal{U} \cap \mathcal{V}) \times F$ gives the corresponding *transition function* $\psi_{\mathcal{U}\mathcal{V}}(x) : F \to F$ at each $x \in \mathcal{U} \cap \mathcal{V}$.

DEFINITION 4.2. A *symplectic fibration* is a fibration $\pi : M \to X$ where the model fiber is a symplectic manifold (F, σ) and with a trivializing cover for which all the transition functions are symplectomorphisms $F \to F$.

In a symplectic fibration each *fiber* $\pi^{-1}(x)$ carries a *canonical symplectic form* σ_x defined by the restriction of $s_{\mathcal{U}}^*\sigma$, for any domain $\mathcal{U}$ of a trivialization covering x (i.e., $x \in \mathcal{U}$). A symplectic form ω on the total space M of a symplectic fibration is called *compatible* with the fibration if each fiber $(\pi^{-1}(x), \sigma_x)$ is a symplectic submanifold of (M, ω), i.e., σ_x is the restriction of ω to $\pi^{-1}(x)$.

EXAMPLES.

1. *Every compact oriented*[41] *fibration whose model fiber F is an oriented surface admits a structure of symplectic fibration* for the following reason. Let σ_0 be an area form on F. Each transition function $\psi_{\mathcal{UV}}(x) : F \to F$ pulls σ_0 back to a cohomologous area form σ_1 (depending on $\psi_{\mathcal{UV}}(x)$). Convex combinations $\sigma_t = (1-t)\sigma_0 + t\sigma_1$ give a path of area forms from σ_0 to σ_1 with constant class $[\sigma_t]$. By Moser's argument (Section 1.4), there exists a diffeomorphism $\rho(x) : F \to F$ isotopic to the identity, depending smoothly on $x \in \mathcal{U} \cap \mathcal{V}$, such that $\psi_{\mathcal{UV}}(x) \circ \rho(x)$ is a symplectomorphism of (F, σ_0). By successively adjusting local trivializations for a finite covering of the base, we can make all transition functions into symplectomorphisms.
2. *Every fibration with connected base and compact fibers having a symplectic form ω for which all fibers are symplectic submanifolds admits a structure of symplectic fibration compatible with ω*. Indeed, under trivializations, the restrictions of ω to the fibers give cohomologous symplectic forms in the model fiber F. So by Moser's Theorem 1.7, all fibers are strongly isotopic to (F, σ) where σ is the restriction of ω to a chosen fiber. These isotopies can be used to produce a trivializing cover where each $s_{\mathcal{U}}(x)$ is a symplectomorphism.

In the remainder of this subsection, assume that for a fibration $\pi : M \to X$ the total space is compact and the base is connected. For the existence of a compatible symplectic form on a symplectic fibration, a necessary condition is the existence of a cohomology class in M that restricts to the classes of the fiber symplectic forms. Thurston [131] showed that, when the base admits also a symplectic form, this condition is sufficient. Yet not all symplectic fibrations with a compatible symplectic form have a symplectic base [138].

THEOREM 4.3 (Thurston). *Let $\pi : M \to X$ be a compact symplectic fibration with connected symplectic base (X, α) and model fiber (F, σ). If there is a class $[\nu] \in H^2(M)$ pulling back to $[\sigma]$, then, for sufficiently large $k > 0$, there exists a symplectic form ω_k on M that is compatible with the fibration and is in $[\nu + k\pi^*\alpha]$.*

PROOF. We first find a form τ on M in the class $[\nu]$ that restricts to the canonical symplectic form on each fiber. Pick a trivializing cover $\{\varphi_i = (\pi, s_i) \mid i \in I\}$ with contractible

[41] An *oriented fibration* is a fibration whose model fiber is oriented and there is a trivializing cover for which all transition functions preserve orientation.

domains $\mathcal{U}_i$. Let ρ_i, $i \in I$, be a partition of unity subordinate to this covering and let $\tilde{\rho}_i := \rho_i \circ \pi : M \to \mathbb{R}$. Since $[\nu]$ always restricts to the class of the canonical symplectic form $[\sigma_x]$, and the $\mathcal{U}_i$'s are contractible, on each $\pi_i^{-1}(\mathcal{U}_i)$ the forms $s_i^*\sigma - \nu$ are exact. Choose 1-forms λ_i such that $s_i^*\sigma = \nu + d\lambda_i$, and set

$$\tau := \nu + \sum_{i \in I} d(\tilde{\rho}_i \lambda_i).$$

Since τ is nondegenerate on the (vertical) subbundle given by the kernel of $d\pi$, for $k > 0$ large enough the form $\tau + k\pi^*\alpha$ is nondegenerate on M. □

COROLLARY 4.4. *Let $\pi : M \to X$ be a compact oriented fibration with connected symplectic base (X, α) and model fiber an oriented surface F of genus $g(F) \neq 1$. Then π admits a compatible symplectic form.*

PROOF. By Example 1 above, $\pi : M \to X$ admits a structure of symplectic fibration with model fiber (F, σ). Since the fiber is not a torus ($g(F) \neq 1$), the Euler class of the tangent bundle TF (which coincides with $c_1(F, \sigma)$) is $\lambda[\sigma]$ for some $\lambda \neq 0$. Hence, the first Chern class $[c]$ of the *vertical* subbundle given by the kernel of $d\pi$ (assembling the tangent bundles to the fibers) restricts to $\lambda[\sigma_x]$ on the fiber over $x \in X$. We can apply Theorem 4.3 using the class $[\nu] = \lambda^{-1}[c]$. □

A pointwise connected sum $M_0 \# M_1$ of symplectic manifolds (M_0, ω_0) and (M_1, ω_1) tends to not admit a symplectic form, even if we only require the eventual symplectic form to be isotopic to ω_i on each M_i minus a ball. The reason [7] is that such a symplectic form on $M_0 \# M_1$ would allow to construct an almost complex structure on the sphere formed by the union of the two removed balls, which is known not to exist except on S^2 and S^6. Therefore:

PROPOSITION 4.5. *Let (M_0, ω_0) and (M_1, ω_1) be two compact symplectic manifolds of dimension not* 2 *nor* 6. *Then the connected sum $M_0 \# M_1$ does not admit any symplectic structure isotopic to ω_i on M_i minus a ball, $i = 1, 2$.*

For connected sums to work in the symplectic category, they should be done along codimension 2 symplectic submanifolds. The following construction, already mentioned in [65], was dramatically explored and popularized by Gompf [57] (he used it to prove Theorem 4.1). Let (M_0, ω_0) and (M_1, ω_1) be two $2n$-dimensional symplectic manifolds. Suppose that a compact symplectic manifold (X, α) of dimension $2n - 2$ admits symplectic embeddings to both $i_0 : X \hookrightarrow M_0$, $i_1 : X \hookrightarrow M_1$. For simplicity, assume that the corresponding normal bundles are trivial (in general, they need to have symmetric Euler classes). By the symplectic neighborhood theorem (Theorem 1.11), there exist symplectic embeddings $j_0 : X \times B_\varepsilon \to M_0$ and $j_1 : X \times B_\varepsilon \to M_1$ (called *framings*) where B_ε is a ball of radius ε and centered at the origin in $\mathbb{R}^2$ such that $j_k^*\omega_k = \alpha + dx \wedge dy$ and $j_k(p, 0) = i_k(p)$, $\forall p \in X$, $k = 0, 1$. Chose an area- and orientation-preserving diffeomorphism ϕ of the annulus $B_\varepsilon \setminus B_\delta$ for $0 < \delta < \varepsilon$ that interchanges the two boundary components. Let $\mathcal{U}_k = j_k(X \times B_\delta) \subset M_k$, $k = 0, 1$. A *symplectic sum* of M_0 and M_1 along X is defined to be

$$M_0 \#_X M_1 := (M_0 \setminus \mathcal{U}_0) \cup_\phi (M_1 \setminus \mathcal{U}_1),$$

where the symbol $\cup_\phi$ means that we identify $j_1(p,q)$ with $j_0(p,\phi(q))$ for all $p \in X$ and $\delta < |q| < \varepsilon$. As ω_0 and ω_1 agree on the regions under identification, they induce a symplectic form on $M_0 \#_X M_1$. The result depends on j_0, j_1, δ and ϕ.

Rational blow-down is a surgery on 4-manifolds that replaces a neighborhood of a chain of embedded S^2's with boundary a *lens space* $L(n^2, n-1)$ by a manifold with the same rational homology as a ball. This simplifies the homology possibly at the expense of complicating the fundamental group. Symington [123] showed that rational blow-down preserves a symplectic structure if the original spheres are symplectic surfaces in a symplectic 4-manifold.

4.3. *Symplectic blow-up*

Symplectic blow-up is the extension to the symplectic category of the blow-up operation in algebraic geometry. It is due to Gromov according to the first printed exposition of this operation in [94].

Let L be the *tautological line bundle* over $\mathbb{CP}^{n-1}$, that is,

$$L = \{([p], z) \mid p \in \mathbb{C}^n \setminus \{0\},\ z = \lambda p \text{ for some } \lambda \in \mathbb{C}\}$$

with projection to $\mathbb{CP}^{n-1}$ given by $\pi : ([p], z) \mapsto [p]$. The fiber of L over the point $[p] \in \mathbb{CP}^{n-1}$ is the complex line in $\mathbb{C}^n$ represented by that point. The *blow-up of* $\mathbb{C}^n$ *at the origin* is the total space of the bundle L, sometimes denoted $\tilde{\mathbb{C}}^n$. The corresponding *blow-down map* is the map $\beta : L \to \mathbb{C}^n$ defined by $\beta([p], z) = z$. The total space of L may be decomposed as the disjoint union of two sets: the zero section

$$E := \{([p], 0) \mid p \in \mathbb{C}^n \setminus \{0\}\}$$

and

$$S := \{([p], z) \mid p \in \mathbb{C}^n \setminus \{0\},\ z = \lambda p \text{ for some } \lambda \in \mathbb{C}^*\}.$$

The set E is called the *exceptional divisor*; it is diffeomorphic to $\mathbb{CP}^{n-1}$ and gets mapped to the origin by β. On the other hand, the restriction of β to the complementary set S is a diffeomorphism onto $\mathbb{C}^n \setminus \{0\}$. Hence, we may regard L as being obtained from $\mathbb{C}^n$ by smoothly replacing the origin by a copy of $\mathbb{CP}^{n-1}$. Every biholomorphic map $f : \mathbb{C}^n \to \mathbb{C}^n$ with $f(0) = 0$ lifts uniquely to a biholomorphic map $\tilde{f} : L \to L$ with $\tilde{f}(E) = E$. The lift is given by the formula

$$\tilde{f}([p], z) = \begin{cases} ([f(z)], f(z)) & \text{if } z \neq 0, \\ ([p], 0) & \text{if } z = 0. \end{cases}$$

There are actions of the unitary group $U(n)$ on L, E and S induced by the standard linear action on $\mathbb{C}^n$, and the map β is $U(n)$-equivariant. For instance, $\beta^*\omega_0 + \pi^*\omega_{FS}$ is a $U(n)$-invariant Kähler form on L.

DEFINITION 4.6. A *blow-up symplectic form* on the tautological line bundle L is a $U(n)$-invariant symplectic form ω such that the difference $\omega - \beta^*\omega_0$ is compactly supported, where $\omega_0 = \frac{i}{2}\sum_{k=1}^n dz_k \wedge d\bar{z}_k$ is the standard symplectic form on $\mathbb{C}^n$.

Two blow-up symplectic forms are *equivalent* if one is the pullback of the other by a $U(n)$-equivariant diffeomorphism of L. Guillemin and Sternberg [69] showed that two blow-up symplectic forms are equivalent if and only if they have equal restrictions to the exceptional divisor $E \subset L$. Let Ω^ε ($\varepsilon > 0$) be the set of all blow-up symplectic forms on L whose restriction to the exceptional divisor $E \simeq \mathbb{CP}^{n-1}$ is $\varepsilon\omega_{FS}$, where ω_{FS} is the Fubini–Study form (Section 3.4). An *ε-blow-up* of $\mathbb{C}^n$ at the origin is a pair (L, ω) with $\omega \in \Omega^\varepsilon$.

Let (M, ω) be a $2n$-dimensional symplectic manifold. It is a consequence of Darboux's theorem (Theorem 1.9) that, for each point $p \in M$, there exists a complex chart $(\mathcal{U}, z_1, \ldots, z_n)$ centered at p and with image in $\mathbb{C}^n$ where $\omega|_{\mathcal{U}} = \frac{i}{2}\sum_{k=1}^n dz_k \wedge d\bar{z}_k$. It is shown in [69] that, for ε small enough, we can perform an ε-blow-up of M at p modeled on $\mathbb{C}^n$ at the origin, without changing the symplectic structure outside of a small neighborhood of p. The resulting manifold is called an *ε-blow-up of M at p*. As a manifold, the blow-up of M at a point is diffeomorphic to the *connected sum*[42] $M \# \overline{\mathbb{CP}^n}$, where $\overline{\mathbb{CP}^n}$ is the manifold $\mathbb{CP}^n$ equipped with the orientation opposite to the natural complex one.

EXAMPLE. Let $\mathbb{P}(L \oplus \mathbb{C})$ be the $\mathbb{CP}^1$-bundle over $\mathbb{CP}^{n-1}$ obtained by projectivizing the direct sum of the tautological line bundle L with a trivial complex line bundle. Consider the map

$$\begin{aligned} \beta : \mathbb{CP}(L \oplus \mathbb{C}) &\longrightarrow \mathbb{CP}^n, \\ \big([p], [\lambda p : w]\big) &\longmapsto [\lambda p : w], \end{aligned}$$

where $[\lambda p : w]$ on the right represents a line in $\mathbb{C}^{n+1}$, forgetting that, for each $[p] \in \mathbb{CP}^{n-1}$, that line sits in the 2-complex-dimensional subspace $L_{[p]} \oplus \mathbb{C} \subset \mathbb{C}^n \oplus \mathbb{C}$. Notice that β maps the *exceptional divisor*

$$E := \big\{\big([p], [0 : \ldots : 0 : 1]\big) \mid [p] \in \mathbb{CP}^{n-1}\big\} \simeq \mathbb{CP}^{n-1}$$

to the point $[0 : \ldots : 0 : 1] \in \mathbb{CP}^n$, and β is a diffeomorphism on the complement

$$S := \big\{\big([p], [\lambda p : w]\big) \mid [p] \in \mathbb{CP}^{n-1},\ \lambda \in \mathbb{C}^*,\ w \in \mathbb{C}\big\} \simeq \mathbb{CP}^n \setminus \big\{[0 : \ldots : 0 : 1]\big\}.$$

[42]The *connected sum* of two oriented m-dimensional manifolds M_0 and M_1 is the manifold, denoted $M_0 \# M_1$, obtained from the union of those manifolds each with a small ball removed $M_i \setminus B_i$ by identifying the boundaries via a (smooth) map $\phi : \partial B_1 \to \partial B_2$ that extends to an orientation-preserving diffeomorphism of neighborhoods of ∂B_1 and ∂B_2 (interchanging the inner and outer boundaries of the annuli).

Therefore, we may regard $\mathbb{CP}(L \oplus \mathbb{C})$ as being obtained from $\mathbb{CP}^n$ by smoothly replacing the point $[0:\ldots:0:1]$ by a copy of $\mathbb{CP}^{n-1}$. The space $\mathbb{CP}(L \oplus \mathbb{C})$ is the blow-up of $\mathbb{CP}^n$ at the point $[0:\ldots:0:1]$, and β is the corresponding blow-down map. The manifold $\mathbb{CP}(L \oplus \mathbb{C})$ for $n = 2$ is a *Hirzebruch surface.*

When $(\mathbb{CP}^{n-1}, \omega_{\mathrm{FS}})$ is symplectically embedded in a symplectic manifold (M, ω) with image X and normal bundle isomorphic to the tautological bundle L, it can be subject to a *blow-down* operation. By the symplectic neighborhood theorem (Theorem 1.11), some neighborhood $\mathcal{U} \subset M$ of the image X is symplectomorphic to a neighborhood $\mathcal{U}_0 \subset L$ of the zero section. It turns out that some neighborhood of $\partial\mathcal{U}_0$ in L is symplectomorphic to a spherical shell in $(\mathbb{C}^n, \omega_0)$. The *blow-down of M along X* is a manifold obtained from the union of $M \setminus \mathcal{U}$ with a ball in $\mathbb{C}^n$. For more details, see [99, §7.1].

Following algebraic geometry, we call *minimal* a $2n$-dimensional symplectic manifold (M, ω) without any symplectically embedded $(\mathbb{CP}^{n-1}, \omega_{\mathrm{FS}})$, so that (M, ω) is not the blow-up at a point of another symplectic manifold. In dimension 4, a manifold is minimal if it does not contain any embedded sphere S^2 with self-intersection -1. Indeed, by the work of Taubes [126,129], if such a sphere S exists, then either the homology class $[S]$ or its symmetric $-[S]$ can be represented by a *symplectically* embedded sphere with self-intersection -1.

For a symplectic manifold (M, ω), let $i : X \hookrightarrow M$ be the inclusion of a symplectic submanifold. The normal bundle NX to X in M admits a structure of complex vector bundle (as it is a symplectic vector bundle). Let $\mathbb{P}(NX) \to X$ be the projectivization of the bundle $NX \to X$, let Z be the zero section of NX, let $L(NX)$ be the corresponding *tautological line bundle* (given by assembling the tautological line bundles over each fiber) and let $\beta : L(NX) \to NX$ be the blow-down map. On the *exceptional divisor*

$$E := \{([p], 0) \in L(NX) \mid p \in NX \setminus Z\} \simeq \mathbb{P}(NX)$$

the map β is just projection to the zero section Z. The restriction of β to the complement $L(NX) \setminus E$ is a diffeomorphism to $NX \setminus Z$. Hence, $L(NX)$ may be viewed as being obtained from NX by smoothly replacing each point of the zero section by the projectivization of its normal space. We symplectically identify some tubular neighborhood $\mathcal{U}$ of X in M with a tubular neighborhood $\mathcal{U}_0$ of the zero section Z in NX. A *blow-up of the symplectic manifold (M, ω) along the symplectic submanifold X* is the manifold obtained from the union of $M \setminus \mathcal{U}$ and $\beta^{-1}(\mathcal{U}_0)$ by identifying neighborhoods of $\partial\mathcal{U}$, and equipped with a symplectic form that restricts to ω on $M \setminus \mathcal{U}$ [94]. When X is one point, this construction reduces to the previous symplectic blow-up at a point.

Often symplectic geography concentrates on minimal examples. McDuff [95] showed that a minimal symplectic 4-manifold with a symplectically embedded S^2 with nonnegative self-intersection is symplectomorphic either to $\mathbb{CP}^2$ or to an S^2-bundle over a surface. Using Seiberg–Witten theory it was proved:

THEOREM 4.7. *Let (M, ω) be a minimal closed symplectic 4-manifold.*

(a) (Taubes [129]) *If $b_2^+ > 1$, then $c_1^2 \geqslant 0$.*

(b) (Liu [89]) *If* $b_2^+ = 1$ *and* $c_1^2 < 0$, *then* M *is the total space of an* S^2*-fibration over a surface of genus* g *where* ω *is nondegenerate on the fibers, and* $(c_1^2, c_2) = (8 - 8g, 4 - 4g)$, *i.e.*, (M, ω) *is a* symplectic ruled surface.

A *symplectic ruled surface*[43] is a symplectic 4-manifold (M, ω) that is the total space of an S^2-fibration where ω is nondegenerate on the fibers.

A *symplectic rational surface* is a symplectic 4-manifold (M, ω) that can be obtained from the standard $(\mathbb{CP}^2, \omega_{\mathrm{FS}})$ by blowing up and blowing down.

With $b_2^+ = 1$ and $c_1^2 = 0$, we have symplectic manifolds $\mathbb{CP}^2 \#_9 \overline{\mathbb{CP}^2} =: E(1)$, the *Dolgachev surfaces* $E(1, p, q)$, the results $E(1)_K$ of surgery on a fibered knot $K \subset S^3$, etc. With $b_2^+ = 1$ and $c_1^2 > 0$, we have symplectic manifolds $\mathbb{CP}^2$, $S^2 \times S^2$, $\mathbb{CP}^2 \#_n \overline{\mathbb{CP}^2}$ for $n \leqslant 8$ and the *Barlow surface*. For $b_2^+ = 1$ and $c_1^2 \geqslant 0$, Park [109] gave a criterion for a symplectic 4-manifold to be rational or ruled in terms of Seiberg–Witten theory.

4.4. *Uniqueness of symplectic forms*

Besides the notions listed in Section 1.4, the following equivalence relation for symplectic manifolds is considered. As it allows the cleanest statements about uniqueness, this relation is simply called *equivalence*.

DEFINITION 4.8. Symplectic manifolds (M, ω_0) and (M, ω_1) are *equivalent* if they are related by a combination of deformation-equivalences and symplectomorphisms.

Recall that (M, ω_0) and (M, ω_1) are *deformation-equivalent* when there is a smooth family ω_t of symplectic forms joining ω_0 to ω_1 (Section 1.4), and they are *symplectomorphic* when there is a diffeomorphism $\varphi : M \to M$ such that $\varphi^*\omega_1 = \omega_0$ (Section 1.2). Hence, equivalence is the relation generated by deformations and diffeomorphisms. The corresponding equivalence classes can be viewed as the connected components of the moduli space of symplectic forms up to diffeomorphism. This is a useful notion when focusing on topological properties.

EXAMPLES.

1. The complex projective plane $\mathbb{CP}^2$ has a unique symplectic structure up to symplectomorphism and scaling. This was shown by Taubes [128] relating Seiberg–Witten invariants (Section 4.5) to pseudoholomorphic curves to prove the existence of a pseudoholomorphic sphere. Previous work of Gromov [64] and McDuff [96] showed that the existence of a pseudoholomorphic sphere implies that the symplectic form is standard.

   Lalonde and McDuff [85] concluded similar classifications for symplectic ruled surfaces and for symplectic rational surfaces (Section 4.3). The symplectic form on

[43] A (rational) *ruled surface* is a complex (Kähler) surface that is the total space of a holomorphic fibration over a Riemann surface with fiber $\mathbb{CP}^1$. When the base is also a sphere, these are the *Hirzebruch surfaces* $\mathbb{P}(L \oplus \mathbb{C})$ where L is a holomorphic line bundle over $\mathbb{CP}^1$.

a symplectic ruled surface is unique up to symplectomorphism in its cohomology class, and is isotopic to a standard Kähler form. In particular, any symplectic form on $S^2 \times S^2$ is symplectomorphic to $a\pi_1^*\sigma + b\pi_2^*\sigma$ for some $a, b > 0$ where σ is the standard area form on S^2.

Li–Liu [88] showed that the symplectic structure on $\mathbb{CP}^2 \#_n \overline{\mathbb{CP}^2}$ for $2 \leqslant n \leqslant 9$ is unique up to equivalence.

2. McMullen and Taubes [101] first exhibited simply connected closed 4-manifolds admitting inequivalent symplectic structures. Their examples were constructed using 3-dimensional topology, and distinguished by analyzing the structure of Seiberg–Witten invariants to show that the first Chern classes (Section 3.2) of the two symplectic structures lie in disjoint orbits of the diffeomorphism group. In higher dimensions there were previously examples of manifolds with inequivalent symplectic forms; see, for instance, [111].

   With symplectic techniques and avoiding gauge theory, Smith [117] showed that, for each $n \geqslant 2$, there is a simply connected closed 4-manifold that admits at least n inequivalent symplectic forms, also distinguished via the first Chern classes. It is not yet known whether there exist inequivalent symplectic forms on a 4-manifold with the same first Chern class.

4.5. *Invariants for* 4*-manifolds*

Very little was known about 4-dimensional manifolds until 1981, when Freedman [51] provided a complete classification of closed simply connected *topological* 4-manifolds, and shortly thereafter Donaldson [30] showed that the panorama for *smooth* 4-manifolds was much wilder.[44] Freedman showed that, modulo homeomorphism, such topological manifolds are essentially classified by their intersection forms (for an *even* intersection form there is exactly one class, whereas for an *odd* intersection form there are exactly two classes distinguished by the *Kirby–Siebenmann invariant KS*, at most one of which admits smooth representatives—smoothness requires $KS = 0$). Donaldson showed that, whereas the existence of a smooth structure imposes strong constraints on the topological type of a manifold, for the same topological manifold there can be infinite different smooth structures.[45] In other words, by far not all intersection forms can occur for smooth 4-manifolds and the same intersection form may correspond to nondiffeomorphic manifolds.

Donaldson's key tool was a set of gauge-theoretic invariants, defined by counting with signs the equivalence classes (modulo gauge equivalence) of connections on SU(2)- (or SO(3)-) bundles over M whose curvature has vanishing self-dual part. For a dozen years there was hard work on the invariants discovered by Donaldson but limited advancement on the understanding of smooth 4-manifolds.

[44] It had been proved by Rokhlin in 1952 that if such a smooth manifold M has even intersection form Q_M (i.e., $w_2 = 0$), then the signature of Q_M must be a multiple of 16. It had been proved by Whitehead and Milnor that two such topological manifolds are homotopy equivalent if and only if they have the same intersection form.

[45] It is known that in dimensions $\leqslant 3$, each topological manifold has exactly one smooth structure, and in dimensions $\geqslant 5$ each topological manifold has at most finitely many smooth structures. For instance, whereas each topological $\mathbb{R}^n$, $n \neq 4$, admits a unique smooth structure, the topological $\mathbb{R}^4$ admits uncountably many smooth structures.

EXAMPLES. Finding *exotic*[46] smooth structures on closed simply connected manifolds with small b_2 has long been an interesting problem, especially in view of the smooth Poincaré conjecture for 4-manifolds. The first exotic smooth structures on a rational surface $\mathbb{CP}^2 \#_n \overline{\mathbb{CP}^2}$ were found in the late 80's for $n = 9$ by Donaldson [31] and for $n = 8$ by Kotschick [84]. There was no progress until the recent work of Park [110] constructing a symplectic exotic $\mathbb{CP}^2 \#_7 \overline{\mathbb{CP}^2}$ and using this to exhibit a third distinct smooth structure $\mathbb{CP}^2 \#_8 \overline{\mathbb{CP}^2}$, thus illustrating how the existence of symplectic forms is tied to the existence of different smooth structures. This stimulated research by Fintushel, Ozsváth, Park, Stern, Stipsicz and Szabó, which together shows that there are infinitely many exotic smooth structures on $\mathbb{CP}^2 \#_n \overline{\mathbb{CP}^2}$ for $n = 5, 6, 7, 8$ (the case $n = 9$ had been shown in the late 80's by Friedman–Morgan and by Okonek–Van de Ven).

In 1994 Witten brought about a revolution in Donaldson theory by introducing a new set of invariants—the *Seiberg–Witten invariants*—which are much simpler to calculate and to apply. This new viewpoint was inspired by developments due to Seiberg and Witten in the understanding of $N = 2$ *supersymmetric Yang–Mills*.

Let M be a smooth oriented closed 4-dimensional manifold with $b_2^+(M) > 1$ (there is a version for $b_2^+(M) = 1$). All such 4-manifolds M (with any $b_2^+(M)$) admit a spin-c structure, i.e., a $\mathrm{Spin}^c(4)$-bundle over M with an isomorphism of the associated SO(4)-bundle to the bundle of oriented frames on the tangent bundle for some chosen Riemannian metric. Let $\mathcal{C}_M = \{a \in H^2(M;\mathbb{Z}) \mid a \equiv w_2(TM)(2)\}$ be the set of characteristic elements, and let $\mathrm{Spin}^c(M)$ be the set of spin-c structures on M. For simplicity, assume that M is simply connected (or at least that $H_1(M;\mathbb{Z})$ has no 2-torsion), so that $\mathrm{Spin}^c(M)$ is isomorphic to $\mathcal{C}_M$ with isomorphism given by the first Chern class of the *determinant line bundle* (the *determinant line bundle* is the line bundle associated by a natural group homomorphism $\mathrm{Spin}^c(4) \to \mathrm{U}(1)$). Fix an orientation of a maximal-dimensional positive-definite subspace $H_+^2(M;\mathbb{R}) \subset H^2(M;\mathbb{R})$. The Seiberg–Witten invariant is the function

$$\mathrm{SW}_M : \mathcal{C}_M \longrightarrow \mathbb{Z}$$

defined as follows. Given a spin-c structure $\alpha \in \mathrm{Spin}^c(M) \simeq \mathcal{C}_M$, the image $\mathrm{SW}_M(\alpha) = [\mathcal{M}] \in H_d(\mathcal{B}^*;\mathbb{Z})$ is the homology class of the moduli space $\mathcal{M}$ of solutions (called *monopoles*) of the Seiberg–Witten (SW) equations modulo gauge equivalence. The SW equations are nonlinear differential equations on a pair of a connection A on the determinant line bundle of α and of a section φ of an associated U(2)-bundle, called the positive (half) spinor bundle:

$$F_A^+ = iq(\varphi) \quad \text{and} \quad D_A\varphi = 0,$$

where F_A^+ is the self-dual part of the (imaginary) curvature of A, q is a squaring operation taking sections of the positive spinor bundle to self-dual 2-forms, and D_A is the corresponding Dirac operator. For a generic perturbation of the equations (replacing the first equation by $F_A^+ = iq(\varphi) + i\nu$, where ν is a self-dual 2-form) and of the Riemannian

[46] A manifold homeomorphic but not diffeomorphic to a smooth manifold M is called an *exotic* M.

metric, a transversality argument shows that the moduli space $\mathcal{M}$ is well-behaved and actually inside the space $\mathcal{B}^*$ of gauge-equivalence classes of irreducible pairs (those (A, φ) for which $\varphi \neq 0$), which is homotopy-equivalent to $\mathbb{CP}^\infty$ and hence has even-degree homology groups $H_d(\mathcal{B}^*; \mathbb{Z}) \simeq \mathbb{Z}$. When the dimension d of $\mathcal{M}$ is odd or when $\mathcal{M}$ is empty, the invariant $\mathrm{SW}_M(\alpha)$ is set to be zero. The *basic classes* are the classes $\alpha \in \mathcal{C}_M$ for which $\mathrm{SW}_M(\alpha) \neq 0$. The set of basic classes is always finite, and if α is a basic class then so is $-\alpha$. The main results are that the Seiberg–Witten invariants are invariants of the diffeomorphism type of the 4-manifold M and satisfy vanishing and nonvanishing theorems, which allowed to answer an array of questions about specific manifolds.

Taubes [128] discovered an equivalence between Seiberg–Witten and Gromov invariants (using pseudoholomorphic curves) for symplectic 4-manifolds, by proving the existence of pseudoholomorphic curves from solutions of the Seiberg–Witten equations and vice-versa. As a consequence, he proved:

THEOREM 4.9 (Taubes). *Let (M, ω) be a compact symplectic 4-manifold.*
If $b_2^+ > 1$, then $c_1(M, \omega)$ admits a smooth pseudoholomorphic representative.
If $M = M_1 \# M_2$, then one of the M_i's has negative definite intersection form.

There are results also for $b_2^+ = 1$, and follow-ups describe the set of basic classes of a connected sum $M \# N$ in terms of the set of basic classes of M when N is a manifold with negative definite intersection form (starting with $\overline{\mathbb{CP}^2}$).

In an attempt to understand other 4-manifolds via Seiberg–Witten and Gromov invariants, some analysis of pseudoholomorphic curves has been extended to nonsymplectic 4-manifolds by equipping these with a *nearly nondegenerate closed 2-form*. In particular, Taubes [130] has related Seiberg–Witten invariants to pseudoholomorphic curves for compact oriented 4-manifolds with $b_2^+ > 0$. Any compact oriented 4-manifold M with $b_2^+ > 0$ admits a closed 2-form that vanishes along a union of circles and is symplectic elsewhere [54,75]. In fact, for a generic metric on M, there is a self-dual harmonic form ω which is transverse to zero as a section of $\Lambda^2 T^*M$. The vanishing locus of ω is the union of a finite number of embedded circles, and ω is symplectic elsewhere.

The generic behavior of closed 2-forms on orientable 4-manifolds is partially understood [3, pp. 23–24]. Here is a summary. Let ω be a generic closed 2-form on a 4-manifold M. At the points of some hypersurface Z, the form ω has rank 2. At a generic point of M, ω is nondegenerate; in particular, has the Darboux normal form $dx_1 \wedge dy_1 + dx_2 \wedge dy_2$. There is a codimension-1 submanifold Z where ω has rank 2, and there are no points where ω vanishes. At a generic point of Z, the kernel of $\tilde{\omega}$ is transverse to Z; the normal form near such a point is $x_1\, dx_1 \wedge dy_1 + dx_2 \wedge dy_2$. There is a curve C where the kernel of $\tilde{\omega}$ is not transverse to Z, hence sits in TZ. At a generic point of C, the kernel of $\tilde{\omega}$ is transverse to C; there are two possible normal forms near such points, called *elliptic* and *hyperbolic*, $d(x - \frac{z^2}{2}) \wedge dy + d(xz \pm ty - \frac{z^3}{3}) \wedge dt$. The hyperbolic and elliptic sections of C are separated by *parabolic* points, where the kernel is tangent to C. It is known that there exists at least one continuous family of inequivalent degeneracies in a parabolic neighborhood [56].

4.6. *Lefschetz pencils*

Lefschetz pencils in symplectic geometry imitate linear systems in complex geometry. Whereas holomorphic functions on a projective surface must be constant, there are interesting functions on the complement of a finite set, and generic such functions have only quadratic singularities. A Lefschetz pencil can be viewed as a complex Morse function or as a very singular fibration, in the sense that, not only some fibers are singular (have ordinary double points) but all *fibers* go through some points.

DEFINITION 4.10. A *Lefschetz pencil* on an oriented 4-manifold M is a map $f : M \setminus \{b_1, \ldots, b_n\} \to \mathbb{CP}^1$ defined on the complement of a finite set in M, called the *base locus*, that is a submersion away from a finite set $\{p_1, \ldots, p_{n+1}\}$, and obeying local models $(z_1, z_2) \mapsto z_1/z_2$ near the b_j's and $(z_1, z_2) \mapsto z_1 z_2$ near the p_j's, where (z_1, z_2) are oriented local complex coordinates.

Usually it is also required that each fiber contains at most one singular point. By blowing up M at the b_j's, we obtain a map to $\mathbb{CP}^1$ on the whole manifold, called a *Lefschetz fibration*. Lefschetz pencils and Lefschetz fibrations can be defined on higher-dimensional manifolds where the b_j's are replaced by codimension 4 submanifolds. By working on the Lefschetz fibration, Gompf [59,58] proved that a structure of Lefschetz pencil (with a nontrivial base locus) gives rise to a symplectic form, canonical up to isotopy, such that the fibers are symplectic.

Using asymptotically holomorphic techniques [12,32], Donaldson [34] proved that symplectic 4-manifolds admit Lefschetz pencils. More precisely:

THEOREM 4.11 (Donaldson). *Let J be a compatible almost complex structure on a compact symplectic 4-manifold (M, ω) where the class $[\omega]/2\pi$ is integral. Then J can be deformed through almost complex structures to an almost complex structure J' such that M admits a Lefschetz pencil with J'-holomorphic fibers.*

The closure of a smooth fiber of the Lefschetz pencil is a symplectic submanifold Poincaré dual to $k[\omega]/2\pi$; cf. Theorem 1.13. Other perspectives on Lefschetz pencils have been explored, including in terms of representations of the free group $\pi_1(\mathbb{CP}^1 \setminus \{p_1, \ldots, p_{n+1}\})$ in the mapping class group Γ_g of the generic fiber surface [118].

Similar techniques were used by Auroux [13] to realize symplectic 4-manifolds as *branched covers* of $\mathbb{CP}^2$, and thus reduce the classification of symplectic 4-manifolds to a (hard) algebraic question about factorization in the braid group. Let M and N be compact oriented 4-manifolds, and let ν be a symplectic form on N.

DEFINITION 4.12. A map $f : M \to N$ is a *symplectic branched cover* if for any $p \in M$ there are complex charts centered at p and $f(p)$ such that ν is positive on each complex line and where f is given by: a local diffeomorphism $(x, y) \to (x, y)$, or a simple branching $(x, y) \to (x^2, y)$, or an ordinary cusp $(x, y) \to (x^3 - xy, y)$.

THEOREM 4.13 (Auroux). *Let (M, ω) be a compact symplectic 4-manifold where the class $[\omega]$ is integral, and let k be a sufficiently large integer. Then there is a symplectic* *branched cover $f_k: (M, k\omega) \to \mathbb{CP}^2$, that is canonical up to isotopy for k large enough. Conversely, given a symplectic branched cover $f: M \to N$, the domain M inherits a symplectic form canonical up to isotopy in the class $f^*[\nu]$.*

5. Hamiltonian geometry

5.1. *Symplectic and Hamiltonian vector fields*

Let (M, ω) be a symplectic manifold and let $H: M \to \mathbb{R}$ be a smooth function. By nondegeneracy, there is a unique vector field X_H on M such that $\imath_{X_H}\omega = dH$. Supposing that X_H is complete (this is always the case when M is compact), let $\rho_t: M \to M$, $t \in \mathbb{R}$, be its flow (cf. Section 1.3). Each diffeomorphism ρ_t preserves ω, i.e., $\rho_t^*\omega = \omega$, because $\frac{d}{dt}\rho_t^*\omega = \rho_t^*\mathcal{L}_{X_H}\omega = \rho_t^*(d\imath_{X_H}\omega + \imath_{X_H} d\omega) = 0$. Therefore, every function on (M, ω) produces a family of symplectomorphisms. Notice how this feature involves both the *nondegeneracy* and the *closedness* of ω.

DEFINITION 5.1. A vector field X_H such that $\imath_{X_H}\omega = dH$ for some $H \in C^\infty(M)$ is a *Hamiltonian vector field* with *Hamiltonian function* H.

Hamiltonian vector fields preserve their Hamiltonian functions ($\mathcal{L}_{X_H}H = \imath_{X_H}dH = \imath_{X_H}\imath_{X_H}\omega = 0$), so each integral curve $\{\rho_t(x) \mid t \in \mathbb{R}\}$ of a Hamiltonian vector field X_H must be contained in a level set of the Hamiltonian function H. In $(\mathbb{R}^{2n}, \omega_0 = \sum dx_j \wedge dy_j)$, the *symplectic gradient* $X_H = \sum(\frac{\partial H}{\partial y_j}\frac{\partial}{\partial x_j} - \frac{\partial H}{\partial x_j}\frac{\partial}{\partial y_j})$ and the usual (Euclidean) gradient $\nabla H = \sum_j(\frac{\partial H}{\partial x_j}\frac{\partial}{\partial x_j} + \frac{\partial H}{\partial y_j}\frac{\partial}{\partial y_j})$ of a function H are related by $JX_H = \nabla H$, where J is the standard almost complex structure.

EXAMPLES.

1. For the height function $H(\theta, h) = h$ on the sphere $(M, \omega) = (S^2, d\theta \wedge dh)$, from $\imath_{X_H}(d\theta \wedge dh) = dh$ we get $X_H = \frac{\partial}{\partial\theta}$. Thus, $\rho_t(\theta, h) = (\theta + t, h)$, which is rotation about the vertical axis, preserving the height H.
2. Let X be any vector field on a manifold W. There is a unique vector field $X_\sharp$ on the cotangent bundle T^*W whose flow is the lift of the flow of X. Let α be the tautological form and $\omega = -d\alpha$ the canonical symplectic form on T^*W. The vector field $X_\sharp$ is Hamiltonian with Hamiltonian function $H := \imath_{X_\sharp}\alpha$.
3. Consider Euclidean space $\mathbb{R}^{2n}$ with coordinates $(q_1, \dots, q_n, p_1, \dots, p_n)$ and $\omega_0 = \sum dq_j \wedge dp_j$. The curve $\rho_t = (q(t), p(t))$ is an integral curve for a Hamiltonian vector field X_H exactly when it satisfies the *Hamilton equations*:

$$\begin{cases} \frac{dq_i}{dt}(t) = \frac{\partial H}{\partial p_i}, \\ \frac{dp_i}{dt}(t) = -\frac{\partial H}{\partial q_i}. \end{cases}$$

4. *Newton's second law* states that a particle of mass m moving in *configuration space* $\mathbb{R}^3$ with coordinates $q = (q_1, q_2, q_3)$ under a potential $V(q)$ moves along a curve $q(t)$ such that

$$m\frac{d^2q}{dt^2} = -\nabla V(q).$$

Introduce the *momenta* $p_i = m\frac{dq_i}{dt}$ for $i = 1, 2, 3$, and *energy* function $H(q, p) = \frac{1}{2m}|p|^2 + V(q)$ on the *phase space*[47] $\mathbb{R}^6 = T^*\mathbb{R}^3$ with coordinates $(q_1, q_2, q_3, p_1, p_2, p_3)$. The energy H is conserved by the motion and Newton's second law in $\mathbb{R}^3$ is then equivalent to the Hamilton equations in $\mathbb{R}^6$:

$$\begin{cases} \frac{dq_i}{dt} = \frac{1}{m}p_i = \frac{\partial H}{\partial p_i}, \\ \frac{dp_i}{dt} = m\frac{d^2q_i}{dt^2} = -\frac{\partial V}{\partial q_i} = -\frac{\partial H}{\partial q_i}. \end{cases}$$

DEFINITION 5.2. A vector field X on M preserving ω (i.e., such that $\mathcal{L}_X\omega = 0$) is a *symplectic vector field.*

Hence, a vector field X on (M, ω) is called *symplectic* when $\imath_X\omega$ is closed, and *Hamiltonian* when $\imath_X\omega$ is exact. In the latter case, a *primitive* H of $\imath_X\omega$ is called a *Hamiltonian function* of X. On a contractible open set every symplectic vector field is Hamiltonian. Globally, the group $H^1_{\mathrm{deRham}}(M)$ measures the obstruction for symplectic vector fields to be Hamiltonian. For instance, the vector field $X_1 = \frac{\partial}{\partial\theta_1}$ on the 2-torus $(M, \omega) = (\mathbb{T}^2, d\theta_1 \wedge d\theta_2)$ is symplectic but not Hamiltonian.

A vector field X is a differential operator on functions: $X \cdot f := \mathcal{L}_X f = df(X)$ for $f \in C^\infty(M)$. As such, the bracket $W = [X, Y]$ is the commutator: $\mathcal{L}_W = [\mathcal{L}_X, \mathcal{L}_Y] = \mathcal{L}_X\mathcal{L}_Y - \mathcal{L}_Y\mathcal{L}_X$ (cf. Section 3.3). This endows the set $\chi(M)$ of vector fields on a manifold M with a structure of *Lie algebra.*[48] For a symplectic manifold (M, ω), using $\imath_{[X,Y]} = [\mathcal{L}_X, \imath_Y]$ and Cartan's magic formula, we find that $\imath_{[X,Y]}\omega = d\imath_X\imath_Y\omega + \imath_X d\imath_Y\omega - \imath_Y d\imath_X\omega - \imath_Y\imath_X\, d\omega = d(\omega(Y, X))$. Therefore:

PROPOSITION 5.3. *If X and Y are symplectic vector fields on a symplectic manifold (M, ω), then $[X, Y]$ is Hamiltonian with Hamiltonian function $\omega(Y, X)$.*

Hence, Hamiltonian vector fields and symplectic vector fields form Lie subalgebras for the Lie bracket $[\cdot, \cdot]$.

DEFINITION 5.4. The *Poisson bracket* of two functions $f, g \in C^\infty(M)$ is the function $\{f, g\} := \omega(X_f, X_g) = \mathcal{L}_{X_g} f$.

[47] The *phase space* of a system of n particles is the space parametrizing the position and momenta of the particles. The mathematical model for a phase space is a symplectic manifold.

[48] A (real) *Lie algebra* is a (real) vector space $\mathfrak{g}$ together with a *Lie bracket* $[\cdot, \cdot]$, i.e., a bilinear map $[\cdot, \cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ satisfying *antisymmetry*, $[x, y] = -[y, x]$, $\forall x, y \in \mathfrak{g}$, and the *Jacobi identity*, $[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$, $\forall x, y, z \in \mathfrak{g}$.

By Proposition 5.3 we have $X_{\{f,g\}} = -[X_f, X_g]$. Moreover, the bracket $\{\cdot,\cdot\}$ satisfies the *Jacobi identity*, $\{f,\{g,h\}\} + \{g,\{h,f\}\} + \{h,\{f,g\}\} = 0$, and the *Leibniz rule*, $\{f, gh\} = \{f,g\}h + g\{f,h\}$.

DEFINITION 5.5. A *Poisson algebra* $(\mathcal{P}, \{\cdot,\cdot\})$ is a commutative associative algebra $\mathcal{P}$ with a Lie bracket $\{\cdot,\cdot\}$ satisfying the Leibniz rule.

When (M,ω) is a symplectic manifold, $(C^\infty(M), \{\cdot,\cdot\})$ is a Poisson algebra, and the map $C^\infty(M) \to \chi(M)$, $H \mapsto X_H$ is a Lie algebra anti-homomorphism.

EXAMPLES.

1. For the prototype $(\mathbb{R}^{2n}, \sum dx_i \wedge dy_i)$, we have $X_{x_i} = -\frac{\partial}{\partial y_i}$ and $X_{y_i} = \frac{\partial}{\partial x_i}$, so that $\{x_i, x_j\} = \{y_i, y_j\} = 0$ and $\{x_i, y_j\} = \delta_{ij}$ for all i, j. Arbitrary functions $f, g \in C^\infty(\mathbb{R}^{2n})$ have the *classical Poisson bracket*

$$\{f,g\} = \sum_{i=1}^{n} \left(\frac{\partial f}{\partial x_i}\frac{\partial g}{\partial y_i} - \frac{\partial f}{\partial y_i}\frac{\partial g}{\partial x_i} \right).$$

2. Let G be a Lie group,[49] $\mathfrak{g}$ its Lie algebra and $\mathfrak{g}^*$ the dual vector space of $\mathfrak{g}$. The vector field ${}^{\mathfrak{g}}X^\#$ generated by $X \in \mathfrak{g}$ for the adjoint action[50] of G on $\mathfrak{g}$ has value $[X, Y]$ at $Y \in \mathfrak{g}$. The vector field $X^\#$ generated by $X \in \mathfrak{g}$ for the coadjoint action of G on $\mathfrak{g}^*$ is $\langle X^\#_\xi, Y\rangle = \langle \xi, [Y, X]\rangle$, $\forall \xi \in \mathfrak{g}^*$, $Y \in \mathfrak{g}$. The skew-symmetric pairing ω on $\mathfrak{g}$ defined at $\xi \in \mathfrak{g}^*$ by

$$\omega_\xi(X, Y) := \langle \xi, [X, Y] \rangle$$

has kernel at ξ the Lie algebra $\mathfrak{g}_\xi$ of the stabilizer of ξ for the coadjoint action. Therefore, ω restricts to a nondegenerate 2-form on the tangent spaces to the orbits of the coadjoint action. As the tangent spaces to an orbit are generated by the vector fields $X^\#$, the Jacobi identity in $\mathfrak{g}$ implies that this form is closed. It is called the *canonical symplectic form* (or the *Lie–Poisson* or *Kirillov–Kostant–Souriau symplectic structure*) on the *coadjoint orbits*. The corresponding Poisson structure on $\mathfrak{g}^*$ is the canonical one induced by the Lie bracket:

$$\{f, g\}(\xi) = \langle \xi, [df_\xi, dg_\xi] \rangle$$

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[49] A *Lie group* is a manifold G equipped with a group structure where the group operation $G \times G \to G$ and inversion $G \to G$ are smooth maps. An *action* of a Lie group G on a manifold M is a group homomorphism $G \to \mathrm{Diff}(M)$, $g \mapsto \psi_g$, where the *evaluation map* $M \times G \to M$, $(p, g) \mapsto \psi_g(p)$ is a smooth map. The *orbit* of G through $p \in M$ is $\{\psi_g(p) \mid g \in G\}$. The *stabilizer* (or *isotropy*) of $p \in M$ is $G_p := \{g \in G \mid \psi_g(p) = p\}$.

[50] Any Lie group G acts on itself by *conjugation*: $g \in G \mapsto \psi_g \in \mathrm{Diff}(G)$, $\psi_g(a) = g \cdot a \cdot g^{-1}$. Let $\mathrm{Ad}_g : \mathfrak{g} \to \mathfrak{g}$ be the derivative at the identity of $\psi_g : G \to G$. We identify the Lie algebra $\mathfrak{g}$ with the tangent space T_eG. For matrix groups, $\mathrm{Ad}_g X = gXg^{-1}$. Letting g vary, we obtain the *adjoint action* of G on its Lie algebra $\mathrm{Ad}: G \to \mathrm{GL}(\mathfrak{g})$. Let $\langle\cdot,\cdot\rangle : \mathfrak{g}^* \times \mathfrak{g} \to \mathbb{R}$ be the natural pairing $\langle \xi, X\rangle = \xi(X)$. Given $\xi \in \mathfrak{g}^*$, we define $\mathrm{Ad}^*_g \xi$ by $\langle \mathrm{Ad}^*_g \xi, X\rangle = \langle \xi, \mathrm{Ad}_{g^{-1}} X\rangle$, for any $X \in \mathfrak{g}$. The collection of maps Ad^*_g forms the *coadjoint action* of G on the dual of its Lie algebra $\mathrm{Ad}^* : G \to \mathrm{GL}(\mathfrak{g}^*)$. These satisfy $\mathrm{Ad}_g \circ \mathrm{Ad}_h = \mathrm{Ad}_{gh}$ and $\mathrm{Ad}^*_g \circ \mathrm{Ad}^*_h = \mathrm{Ad}^*_{gh}$.

for $f, g \in C^\infty(\mathfrak{g}^*)$ and $\xi \in \mathfrak{g}^*$. The differential $df_\xi : T_\xi \mathfrak{g}^* \simeq \mathfrak{g}^* \to \mathbb{R}$ is identified with an element of $\mathfrak{g} \simeq \mathfrak{g}^{**}$.

5.2. *Arnold conjecture and Floer homology*

There is an important generalization of Poincaré's last geometric theorem (Theorem 2.16) conjectured by Arnold starting around 1966. Let (M, ω) be a compact symplectic manifold, and $h_t : M \to \mathbb{R}$ a 1-periodic (i.e., $h_t = h_{t+1}$) smooth family of functions. Let $\rho : M \times \mathbb{R} \to M$ be the isotopy generated by the time-dependent Hamiltonian vector field v_t defined by the equation $\omega(v_t, \cdot) = dh_t$. The symplectomorphism $\varphi = \rho_1$ is then said to be *exactly homotopic to the identity*. In other words, a symplectomorphism exactly homotopic to the identity is the time-1 map of the isotopy generated by some time-dependent 1-periodic Hamiltonian function. There is a one-to-one correspondence between the fixed points of φ and the period-1 orbits of ρ. When all the fixed points of such φ are nondegenerate (generic case), we call φ *nondegenerate*. The *Arnold conjecture* [2, Appendix 9] predicted that

$$\#\{\text{fixed points of a nondegenerate } \varphi\} \geqslant \sum_{i=0}^{2n} \dim H^i(M; \mathbb{R})$$

(or even that the number of fixed points of a nondegenerate φ is at least the minimal number of critical points of a Morse function[51]). When the Hamiltonian $h : M \to \mathbb{R}$ is independent of t, this relation is trivial: a point p is critical for h if and only if $dh_p = 0$, if and only if $v_p = 0$, if and only if $\rho(t, p) = p$, $\forall t \in \mathbb{R}$, which implies that p is a fixed point of $\rho_1 = \varphi$, so the Arnold conjecture reduces to a Morse inequality. Notice that, according to the Lefschetz fixed point theorem, the Euler characteristic of M, i.e., the *alternating* sum of the Betti numbers, $\sum(-1)^i \dim H^i(M; \mathbb{R})$, is a (weaker) lower bound for the number of fixed points of φ.

The Arnold conjecture was gradually proved from the late 70's to the late 90's by Eliashberg [39], Conley–Zehnder [24], Floer [49], Sikorav [116], Weinstein [140], Hofer–Salamon [74], Ono [108], culminating with independent proofs by Fukaya–Ono [52] and Liu–Tian [90]. There are open conjectures for sharper bounds on the number of fixed points. The breakthrough tool for establishing the Arnold conjecture was *Floer homology*—an ∞-dimensional analogue of Morse theory. Floer homology was defined by Floer [46–50] and developed through the work of numerous people after Floer's death. It combines the variational approach of Conley and Zehnder [25], with Witten's Morse–Smale complex [144], and with Gromov's compactness theorem for pseudoholomorphic curves [64].

Floer theory starts from a symplectic action functional on the space of loops $\mathcal{L}M$ of a symplectic manifold (M, ω) whose zeros of the differential $dF : T(\mathcal{L}M) \to \mathbb{R}$ are the period-1 orbits of the isotopy ρ above. The tangent bundle $T(\mathcal{L}M)$ is the space of loops with vector fields over them: pairs (ℓ, v), where $\ell : S^1 \to M$ and $v : S^1 \to \ell^*(TM)$ is a

[51] A *Morse function* is a smooth function $f : M \to \mathbb{R}$ all of whose critical points are nondegenerate, i.e., at any critical point the Hessian matrix is nondegenerate.

section. Then $df(\ell, v) = \int_0^1 \omega(\dot{\ell}(t) - X_{h_t}(\ell(t), v(t))\,dt$. The *Floer complex*[52] is the chain complex freely generated by the critical points of F (corresponding to the fixed points of φ), with *relative grading* index(x, y) given by the difference in the number of positive eigenvalues from the spectral flow. The Floer differential is given by counting the number $n(x, y)$ of pseudoholomorphic surfaces (the *gradient flow lines* joining two fixed points):

$$C_* = \bigoplus_{x \in \mathrm{Crit}(F)} \mathbb{Z}\langle x\rangle \quad \text{and} \quad \partial\langle x\rangle = \sum_{\substack{y \in \mathrm{Crit}(F) \\ \mathrm{index}(x,y)=1}} n(x, y)\langle y\rangle.$$

Pondering transversality, compactness and orientation, Floer's theorem states that the homology of (C_*, ∂) is isomorphic to the ordinary homology of M. In particular, the sum of the Betti numbers is a lower bound for the number of fixed points of φ.

From the above *symplectic Floer homology*, Floer theory has branched out to tackle other differential geometric problems in symplectic geometry and 3- and 4-dimensional topology. It provides a rigorous definition of invariants viewed as homology groups of infinite-dimensional Morse-type theories, with relations to gauge theory and quantum field theory. There is *Lagrangian Floer homology* (for the case of Lagrangian intersections, i.e., intersection of a Lagrangian submanifold with a Hamiltonian deformation of itself), *instanton Floer homology* (for invariants of 3-manifolds), *Seiberg–Witten Floer homology*, *Heegaard Floer homology* and *knot Floer homology*. For more on Floer homology; see, for instance, [35,113].

5.3. *Euler–Lagrange equations*

The equations of motion in classical mechanics arise from *variational principles*. The physical path of a general mechanical system of n particles is the path that *minimizes* a quantity called the *action*. When dealing with systems with constraints, such as the simple

[52]The *Morse complex* for a Morse function on a compact manifold, $f : M \to \mathbb{R}$, is the chain complex freely generated by the critical points of f, graded by the *Morse index* ι and with differential given by counting the number $n(x, y)$ of flow lines of the negative gradient $-\nabla f$ (for a metric on X) from the point x to the point y whose indices differ by 1:

$$C_* = \bigoplus_{x \in \mathrm{Crit}(f)} \mathbb{Z}\langle x\rangle \quad \text{and} \quad \partial\langle x\rangle = \sum_{\substack{y \in \mathrm{Crit}(f) \\ \iota(y)=\iota(x)-1}} n(x, y)\langle y\rangle.$$

The coefficient $n(x, y)$ is thus the number of solutions (modulo $\mathbb{R}$-reparametrization) $u : \mathbb{R} \to X$ of the ordinary differential equation $\frac{d}{dt}u(t) = -\nabla f(u(t))$ with conditions $\lim_{t\to-\infty} u(t) = x$, $\lim_{t\to+\infty} u(t) = y$. The *Morse index* of a critical point of f is the dimension of its unstable manifold, i.e., the number of negative eigenvalues of the Hessian of f at that point. For a generic metric, the unstable manifold of a critical point $W^u(x)$ intersects transversally with the stable manifold of another critical point $W^s(y)$. When $\iota(x) - \iota(y) = 1$, the intersection $W^u(x) \cap W^s(y)$ has dimension 1, so when we quotient out by the $\mathbb{R}$-reparametrization (to count actual image curves) we get a discrete set, which is finite by compactness. That (C_*, ∂) is indeed a complex, i.e., $\partial^2 = 0$, follows from counting broken flow lines between points whose indices differ by 2. Morse's theorem states that the homology of the Morse complex coincides with the ordinary homology of M. In particular, the sum of all the Betti numbers $\sum \dim H^i(M; \mathbb{R})$ is a lower bound for the number of critical points of a Morse function.

pendulum, or two point masses attached by a rigid rod, or a rigid body, the language of variational principles becomes more appropriate than the explicit analogues of Newton's second laws. Variational principles are due mostly to D'Alembert, Maupertius, Euler and Lagrange.

Let M be an n-dimensional manifold, and let $F : TM \to \mathbb{R}$ be a function on its tangent bundle. If $\gamma : [a,b] \to M$ is a curve on M, the *lift of* γ *to* TM is the curve on TM given by $\tilde{\gamma} : [a,b] \to TM$, $t \mapsto (\gamma(t), \frac{d\gamma}{dt}(t))$. The *action* of γ is

$$\mathcal{A}_\gamma := \int_a^b (\tilde{\gamma}^* F)(t)\,dt = \int_a^b F\left(\gamma(t), \frac{d\gamma}{dt}(t)\right) dt.$$

For fixed p, q, let $\mathcal{P}(a,b,p,q) = \{\gamma : [a,b] \to M \text{ smooth} \mid \gamma(a) = p,\ \gamma(b) = q\}$. The goal is to find, among all $\gamma \in \mathcal{P}(a,b,p,q)$, the curve that *locally minimizes* $\mathcal{A}_\gamma$. (Minimizing curves are always locally minimizing.) Assume that p, q and the image of γ lie in a coordinate neighborhood $(\mathcal{U}, x_1, \dots, x_n)$. On $T\mathcal{U}$ we have coordinates $(x_1, \dots, x_n, v_1, \dots, v_n)$ associated with a trivialization of $T\mathcal{U}$ by $\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}$. Using this trivialization, a curve $\gamma : [a,b] \to \mathcal{U}$, $\gamma(t) = (\gamma_1(t), \dots, \gamma_n(t))$ lifts to

$$\tilde{\gamma} : [a,b] \longrightarrow T\mathcal{U}, \quad \tilde{\gamma}(t) = \left(\gamma_1(t), \dots, \gamma_n(t), \frac{d\gamma_1}{dt}(t), \dots, \frac{d\gamma_n}{dt}(t)\right).$$

Consider infinitesimal variations of γ. Let $c_1, \dots, c_n \in C^\infty([a,b])$ be such that $c_k(a) = c_k(b) = 0$. For ε small, let $\gamma_\varepsilon : [a,b] \to \mathcal{U}$ be the curve $\gamma_\varepsilon(t) = (\gamma_1(t) + \varepsilon c_1(t), \dots, \gamma_n(t) + \varepsilon c_n(t))$. Let $\mathcal{A}_\varepsilon := \mathcal{A}_{\gamma_\varepsilon}$. A necessary condition for $\gamma = \gamma_0 \in \mathcal{P}(a,b,p,q)$ to minimize the action is that $\varepsilon = 0$ be a critical point of $\mathcal{A}_\varepsilon$. By the Leibniz rule and integration by parts, we have that

$$\begin{aligned}\frac{d\mathcal{A}_\varepsilon}{d\varepsilon}(0) &= \int_a^b \sum_k \left[\frac{\partial F}{\partial x_k}\left(\gamma_0(t), \frac{d\gamma_0}{dt}(t)\right) c_k(t) + \frac{\partial F}{\partial v_k}\left(\gamma_0, \frac{d\gamma_0}{dt}\right) \frac{dc_k}{dt}(t)\right] dt \\ &= \int_a^b \sum_k \left[\frac{\partial F}{\partial x_k}(\dots) - \frac{d}{dt}\frac{\partial F}{\partial v_k}(\dots)\right] c_k(t)\,dt.\end{aligned}$$

For $\frac{d\mathcal{A}_\varepsilon}{d\varepsilon}(0)$ to vanish for all c_k's satisfying boundary conditions $c_k(a) = c_k(b) = 0$, the path γ_0 must satisfy the *Euler–Lagrange equations*:

$$\frac{\partial F}{\partial x_k}\left(\gamma_0(t), \frac{d\gamma_0}{dt}(t)\right) = \frac{d}{dt}\frac{\partial F}{\partial v_k}\left(\gamma_0(t), \frac{d\gamma_0}{dt}(t)\right), \quad k = 1, \dots, n.$$

EXAMPLES.

1. Let (M, g) be a Riemannian manifold. Let $F : TM \to \mathbb{R}$ be the function whose restriction to each tangent space is the quadratic form defined by the Riemannian

metric. On a coordinate chart $F(x, v) = |v|^2 = \sum g_{ij}(x)v^i v^j$. Let $p, q \in M$ and $\gamma : [a, b] \to M$ a curve joining p to q. The *action* of γ is

$$\mathcal{A}_\gamma = \int_a^b \left|\frac{d\gamma}{dt}\right|^2 dt.$$

The Euler–Lagrange equations become the *Christoffel equations* for a geodesic

$$\frac{d^2\gamma^k}{dt^2} + \sum (\Gamma_{ij}^k \circ \gamma)\frac{d\gamma^i}{dt}\frac{d\gamma^j}{dt} = 0,$$

where the *Christoffel symbols* Γ_{ij}^k's are defined in terms of the coefficients of the Riemannian metric (g^{ij} is the matrix inverse to g_{ij}) by

$$\Gamma_{ij}^k = \frac{1}{2}\sum_\ell g^{\ell k}\left(\frac{\partial g_{\ell i}}{\partial x_j} + \frac{\partial g_{\ell j}}{\partial x_i} - \frac{\partial g_{ij}}{\partial x_\ell}\right).$$

2. Consider a point-particle of mass m moving in $\mathbb{R}^3$ under a *force field* G. The *work* of G on a path $\gamma : [a, b] \to \mathbb{R}^3$ is $W_\gamma := \int_a^b G(\gamma(t)) \cdot \frac{d\gamma}{dt}(t)\, dt$. Suppose that G is *conservative*, i.e., W_γ depends only on the initial and final points, $p = \gamma(a)$ and $q = \gamma(b)$. We can define the *potential energy* as $V : \mathbb{R}^3 \to \mathbb{R}$, $V(q) := W_\gamma$, where γ is a path joining a fixed base point $p_0 \in \mathbb{R}^3$ to q. Let $\mathcal{P}$ be the set of all paths going from p to q over time $t \in [a, b]$. By the *principle of least action*, the physical path is the path $\gamma \in \mathcal{P}$ that minimizes a kind of mean value of kinetic minus potential energy, known as the *action*:

$$\mathcal{A}_\gamma := \int_a^b \left(\frac{m}{2}\left|\frac{d\gamma}{dt}(t)\right|^2 - V(\gamma(t))\right) dt.$$

The Euler–Lagrange equations are then equivalent to *Newton's second law*:

$$m\frac{d^2x}{dt^2}(t) - \frac{\partial V}{\partial x}(x(t)) = 0 \quad \Longleftrightarrow \quad m\frac{d^2x}{dt^2}(t) = G(x(t)).$$

In the case of the earth moving about the sun, both regarded as point-masses and assuming that the sun to be stationary at the origin, the *gravitational potential*$V(x) = \frac{\text{const}}{|x|}$ yields the *inverse square law* for the motion.

3. Consider now n point-particles of masses $m_1, \dots, m_n$ moving in $\mathbb{R}^3$ under a conservative force corresponding to a potential energy $V \in C^\infty(\mathbb{R}^{3n})$. At any instant t, the configuration of this system is described by a vector $x = (x_1, \dots, x_n)$ in configuration space $\mathbb{R}^{3n}$, where $x_k \in \mathbb{R}^3$ is the position of the kth particle. For fixed $p, q \in \mathbb{R}^{3n}$, let

$\mathcal{P}$ be the set of all paths $\gamma = (\gamma_1, \dots, \gamma_n) : [a, b] \to \mathbb{R}^{3n}$ from p to q. The *action* of a path $\gamma \in \mathcal{P}$ is

$$\mathcal{A}_\gamma := \int_a^b \left(\sum_{k=1}^n \frac{m_k}{2} \left| \frac{d\gamma_k}{dt}(t) \right|^2 - V\big(\gamma(t)\big) \right) dt.$$

The Euler–Lagrange equations reduce to Newton's law for each particle. Suppose that the particles are restricted to move on a submanifold M of $\mathbb{R}^{3n}$ called the *constraint set*. By the *principle of least action for a constrained system*, the physical path has minimal action among all paths satisfying the rigid constraints. I.e., we single out the actual physical path as the one that minimizes $\mathcal{A}_\gamma$ among all $\gamma : [a, b] \to M$ with $\gamma(a) = p$ and $\gamma(b) = q$.

In the case where $F(x, v)$ does not depend on v, the Euler–Lagrange equations are simply $\frac{\partial F}{\partial x_i}(\gamma_0(t), \frac{d\gamma_0}{dt}(t)) = 0$. These are satisfied if and only if the curve γ_0 sits on the critical set of F. For generic F, the critical points are isolated, hence $\gamma_0(t)$ must be a constant curve. In the case where $F(x, v)$ depends affinely on v, $F(x, v) = F_0(x) + \sum_{j=1}^n F_j(x) v_j$, the Euler–Lagrange equations become

$$\frac{\partial F_0}{\partial x_i}\big(\gamma(t)\big) = \sum_{j=1}^n \left(\frac{\partial F_i}{\partial x_j} - \frac{\partial F_j}{\partial x_i} \right) \big(\gamma(t)\big) \frac{d\gamma_j}{dt}(t).$$

If the $n \times n$ matrix $(\frac{\partial F_i}{\partial x_j} - \frac{\partial F_j}{\partial x_i})$ has an inverse $G_{ij}(x)$, we obtain the system of first order ordinary differential equations $\frac{d\gamma_j}{dt}(t) = \sum G_{ji}(\gamma(t)) \frac{\partial F_0}{\partial x_i}(\gamma(t))$. Locally it has a unique solution through each point p. If q is not on this curve, there is no solution at all to the Euler–Lagrange equations belonging to $\mathcal{P}(a, b, p, q)$.

Therefore, we need nonlinear dependence of F on the v variables in order to have appropriate solutions. From now on, assume the *Legendre condition*:

$$\det\left(\frac{\partial^2 F}{\partial v_i \partial v_j} \right) \neq 0.$$

Letting $G_{ij}(x, v) = (\frac{\partial^2 F}{\partial v_i \partial v_j}(x, v))^{-1}$, the Euler–Lagrange equations become

$$\frac{d^2\gamma_j}{dt^2} = \sum_i G_{ji} \frac{\partial F}{\partial x_i}\left(\gamma, \frac{d\gamma}{dt}\right) - \sum_{i,k} G_{ji} \frac{\partial^2 F}{\partial v_i \partial x_k}\left(\gamma, \frac{d\gamma}{dt}\right) \frac{d\gamma_k}{dt}.$$

This second order ordinary differential equation has a unique solution given initial conditions $\gamma(a) = p$ and $\frac{d\gamma}{dt}(a) = v$. Assume that $(\frac{\partial^2 F}{\partial v_i \partial v_j}(x, v)) \gg 0$, $\forall (x, v)$, i.e., with the x variable frozen, the function $v \mapsto F(x, v)$ is *strictly convex*. Then the path $\gamma_0 \in \mathcal{P}(a, b, p, q)$ satisfying the above Euler–Lagrange equations does indeed locally minimize $\mathcal{A}_\gamma$ (globally it is only critical):

PROPOSITION 5.6. *For every sufficiently small subinterval* $[a_1, b_1]$ *of* $[a, b]$, $\gamma_0|_{[a_1,b_1]}$ *is locally minimizing in* $\mathcal{P}(a_1, b_1, p_1, q_1)$ *where* $p_1 = \gamma_0(a_1)$, $q_1 = \gamma_0(b_1)$.

PROOF. Take $c = (c_1, \ldots, c_n)$ with $c_i \in C^\infty([a, b])$, $c_i(a) = c_i(b) = 0$. Let $\gamma_\varepsilon = \gamma_0 + \varepsilon c \in \mathcal{P}(a, b, p, q)$, and let $\mathcal{A}_\varepsilon = \mathcal{A}_{\gamma_\varepsilon}$. Suppose that $\gamma_0 : [a, b] \to \mathcal{U}$ satisfies the Euler–Lagrange equations, i.e., $\frac{d\mathcal{A}_\varepsilon}{d\varepsilon}(0) = 0$. Then

$$\frac{d^2\mathcal{A}_\varepsilon}{d\varepsilon^2}(0) = \int_a^b \sum_{i,j} \frac{\partial^2 F}{\partial x_i \partial x_j}\left(\gamma_0, \frac{d\gamma_0}{dt}\right) c_i c_j \, dt \tag{A}$$

$$+ 2\int_a^b \sum_{i,j} \frac{\partial^2 F}{\partial x_i \partial v_j}\left(\gamma_0, \frac{d\gamma_0}{dt}\right) c_i \frac{dc_j}{dt} \, dt \tag{B}$$

$$+ \int_a^b \sum_{i,j} \frac{\partial^2 F}{\partial v_i \partial v_j}\left(\gamma_0, \frac{d\gamma_0}{dt}\right) \frac{dc_i}{dt} \frac{dc_j}{dt} \, dt. \tag{C}$$

Since $(\frac{\partial^2 F}{\partial v_i \partial v_j}(x, v)) \gg 0$ at all x, v, we have

$$|(\mathrm{A})| \leqslant K_{\mathrm{A}} |c|^2_{L^2[a,b]}, \qquad |(\mathrm{B})| \leqslant K_{\mathrm{B}} |c|_{L^2[a,b]} \left|\frac{dc}{dt}\right|_{L^2[a,b]}$$

and

$$(\mathrm{C}) \geqslant K_{\mathrm{C}} \left|\frac{dc}{dt}\right|^2_{L^2[a,b]},$$

where $K_{\mathrm{A}}, K_{\mathrm{B}}, K_{\mathrm{C}}$ are positive constants. By the Wirtinger inequality[53], if $b - a$ is very small, then $(\mathrm{C}) > |(\mathrm{A})| + |(\mathrm{B})|$ when $c \not\equiv 0$. Hence, γ_0 is a local minimum. □

In Section 5.1 we saw that solving Newton's second law in *configuration space* $\mathbb{R}^3$ is equivalent to solving in *phase space* for the integral curve in $T^*\mathbb{R}^3 = \mathbb{R}^6$ of the Hamiltonian vector field with Hamiltonian function H. In the next subsection we will see how this correspondence extends to more general Euler–Lagrange equations.

5.4. *Legendre transform*

The Legendre transform gives the relation between the variational (Euler–Lagrange) and the symplectic (Hamilton–Jacobi) formulations of the equations of motion.

[53]The *Wirtinger inequality* states that, for $f \in C^1([a, b])$ with $f(a) = f(b) = 0$, we have

$$\int_a^b \left|\frac{df}{dt}\right|^2 dt \geqslant \frac{\pi^2}{(b-a)^2} \int_a^b |f|^2 dt.$$

This can be proved with Fourier series.

Let V be an n-dimensional vector space, with $e_1, \dots, e_n$ a basis of V and $v_1, \dots, v_n$ the associated coordinates. Let $F : V \to \mathbb{R}$, $F = F(v_1, \dots, v_n)$, be a smooth function. The function F is *strictly convex* if and only if for every pair of elements $p, v \in V$, $v \neq 0$, the restriction of F to the line $\{p + xv \mid x \in \mathbb{R}\}$ is strictly convex.[54] It follows from the case of real functions on $\mathbb{R}$ that, for a strictly convex function F on V, the following are equivalent:[55]

(a) F has a critical point, i.e., a point where $dF_p = 0$;
(b) F has a local minimum at some point;
(c) F has a unique critical point (global minimum); and
(d) F is proper, that is, $F(p) \to +\infty$ as $p \to \infty$ in V.

A strictly convex function F is *stable* when it satisfies conditions (a)–(d) above.

DEFINITION 5.7. The *Legendre transform* associated to $F \in C^\infty(V)$ is the map

$$\begin{aligned} L_F : V &\longrightarrow V^*, \\ p &\longmapsto dF_p \in T_p^* V \simeq V^*, \end{aligned}$$

where $T_p^* V \simeq V^*$ is the canonical identification for a vector space V.

From now on, assume that F is a strictly convex function on V. Then, for every point $p \in V$, L_F maps a neighborhood of p diffeomorphically onto a neighborhood of $L_F(p)$. Given $\ell \in V^*$, let

$$F_\ell : V \longrightarrow \mathbb{R}, \quad F_\ell(v) = F(v) - \ell(v).$$

Since $(d^2 F)_p = (d^2 F_\ell)_p$, F is strictly convex if and only if F_ℓ is strictly convex. The *stability set* of F is

$$S_F = \{\ell \in V^* \mid F_\ell \text{ is stable}\}.$$

The set S_F is open and convex, and L_F maps V diffeomorphically onto S_F. (A way to ensure that $S_F = V^*$ and hence that L_F maps V diffeomorphically onto V^*, is to assume that a strictly convex function F has *quadratic growth at infinity*, i.e., there exists a positive-definite quadratic form Q on V and a constant K such that $F(p) \geqslant Q(p) - K$, for all p.) The inverse to L_F is the map $L_F^{-1} : S_F \to V$ described as follows: for $\ell \in S_F$,

[54] A function $F : V \to \mathbb{R}$ is *strictly convex* if at every $p \in V$ the *Hessian* $d^2 F_p$ is positive definite. Let $u = \sum_{i=1}^n u_i e_i \in V$. The *Hessian* of F at p is the quadratic function on V,

$$(d^2 F)_p(u) := \sum_{i,j} \frac{\partial^2 F}{\partial v_i \partial v_j}(p) u_i u_j = \frac{d^2}{dt^2} F(p + tu) \Big|_{t=0}.$$

[55] A smooth function $f : \mathbb{R} \to \mathbb{R}$ is *strictly convex* if $f''(x) > 0$ for all $x \in \mathbb{R}$. Assuming that f is strictly convex, the following four conditions are equivalent: $f'(x) = 0$ at some point, f has a local minimum, f has a unique (global) minimum, and $f(x) \to +\infty$ as $x \to \pm\infty$. The function f is *stable* if it satisfies one (and hence all) of these conditions. For instance, $e^x + ax$ is strictly convex for any $a \in \mathbb{R}$, but it is stable only for $a < 0$. The function $x^2 + ax$ is strictly convex and stable for any $a \in \mathbb{R}$.

the value $L_F^{-1}(\ell)$ is the unique minimum point $p_\ell \in V$ of F_ℓ. Indeed p is the minimum of $F(v) - dF_p(v)$.

DEFINITION 5.8. The *dual function* F^* to F is

$$F^* : S_F \longrightarrow \mathbb{R}, \qquad F^*(\ell) = -\min_{p \in V} F_\ell(p).$$

The dual function F^* is smooth and, for all $p \in V$ and all $\ell \in S_F$, satisfies the *Young inequality* $F(p) + F^*(\ell) \geqslant \ell(p)$.

On one hand we have $V \times V^* \simeq T^*V$, and on the other hand, since $V = V^{**}$, we have $V \times V^* \simeq V^* \times V \simeq T^*V^*$. Let α_1 be the tautological 1-form on T^*V and α_2 be the tautological 1-form on T^*V^*. Via the identifications above, we can think of both of these forms as living on $V \times V^*$. Since $\alpha_1 = d\beta - \alpha_2$, where $\beta : V \times V^* \to \mathbb{R}$ is the function $\beta(p, \ell) = \ell(p)$, we conclude that the forms $\omega_1 = -d\alpha_1$ and $\omega_2 = -d\alpha_2$ satisfy $\omega_1 = -\omega_2$.

THEOREM 5.9. *For a strictly convex function F we have that $L_F^{-1} = L_{F^*}$.*

PROOF. The graph Λ_F of the Legendre transform L_F is a Lagrangian submanifold of $V \times V^*$ with respect to the symplectic form ω_1. Hence, Λ_F is also Lagrangian for ω_2. Let $\mathrm{pr}_1 : \Lambda_F \to V$ and $\mathrm{pr}_2 : \Lambda_F \to V^*$ be the restrictions of the projection maps $V \times V^* \to V$ and $V \times V^* \to V^*$, and let $i : \Lambda_F \hookrightarrow V \times V^*$ be the inclusion map. Then $i^*\alpha_1 = d(\mathrm{pr}_1)^*F$ as both sides have value dF_p at $(p, dF_p) \in \Lambda_F$. It follows that $i^*\alpha_2 = d(i^*\beta - (\mathrm{pr}_1)^*F) = d(\mathrm{pr}_2)^*F^*$, which shows that Λ_F is the graph of the inverse of L_{F^*}. From this we conclude that the inverse of the Legendre transform associated with F is the Legendre transform associated with F^*. □

Let M be a manifold and $F : TM \to \mathbb{R}$. We return to the Euler–Lagrange equations for minimizing the action $\mathcal{A}_\gamma = \int \tilde{\gamma}^*F$. At $p \in M$, let $F_p := F|_{T_pM} : T_pM \to \mathbb{R}$. Assume that F_p is strictly convex for all $p \in M$. To simplify notation, assume also that $S_{F_p} = T_p^*M$. The Legendre transform on each tangent space $L_{F_p} : T_pM \xrightarrow{\simeq} T_p^*M$ is essentially given by the first derivatives of F in the v directions. Collect these and the dual functions $F_p^* : T_p^*M \to \mathbb{R}$ into maps

$$\mathcal{L} : TM \longrightarrow T^*M, \quad \mathcal{L}|_{T_pM} = L_{F_p} \quad \text{and} \quad H : T^*M \longrightarrow \mathbb{R}, \quad H|_{T_p^*M} = F_p^*.$$

The maps H and $\mathcal{L}$ are smooth, and $\mathcal{L}$ is a diffeomorphism.

THEOREM 5.10. *Let $\gamma : [a, b] \to M$ be a curve, and $\tilde{\gamma} : [a, b] \to TM$ its lift. Then γ satisfies the Euler–Lagrange equations on every coordinate chart if and only if $\mathcal{L} \circ \tilde{\gamma} : [a, b] \to T^*M$ is an integral curve of the Hamiltonian vector field X_H.*

PROOF. Let $(\mathcal{U}, x_1, \ldots, x_n)$ be a coordinate chart in M, with associated tangent $(T\mathcal{U}, x_1, \ldots, x_n, v_1, \ldots, v_n)$ and cotangent $(T^*\mathcal{U}, x_1, \ldots, x_n, \xi_1, \ldots, \xi_n)$ coordinates. On $T\mathcal{U}$ we have $F = F(x, v)$, on $T^*\mathcal{U}$ we have $H = H(x, \xi)$, and

$$\begin{aligned} \mathcal{L}: T\mathcal{U} &\longrightarrow T^*\mathcal{U}, & H: T^*\mathcal{U} &\longrightarrow \mathbb{R}, \\ (x, v) &\longmapsto (x, \xi), & (x, \xi) &\longmapsto F_x^*(\xi) = \xi \cdot v - F(x, v), \end{aligned}$$

where $\xi := L_{F_x}(v) = \frac{\partial F}{\partial v}(x, v)$ is called the *momentum*. Integral curves $(x(t), \xi(t))$ of X_H satisfy the Hamilton equations:

$$\text{(H)} \quad \begin{cases} \frac{dx}{dt} = \frac{\partial H}{\partial \xi}(x, \xi), \\ \frac{d\xi}{dt} = -\frac{\partial H}{\partial x}(x, \xi), \end{cases}$$

whereas the physical path $x(t)$ satisfies the Euler–Lagrange equations:

$$\text{(E–L)} \quad \frac{\partial F}{\partial x}\left(x, \frac{dx}{dt}\right) = \frac{d}{dt}\frac{\partial F}{\partial v}\left(x, \frac{dx}{dt}\right).$$

Let $(x(t), \xi(t)) = \mathcal{L}(x(t), \frac{dx}{dt}(t))$. For an arbitrary curve $x(t)$, we want to prove that $t \mapsto (x(t), \xi(t))$ satisfies (H) if and only if $t \mapsto (x(t), \frac{dx}{dt}(t))$ satisfies (E–L). The first line of (H) comes automatically from the definition of ξ:

$$\xi = L_{F_x}\left(\frac{dx}{dt}\right) \iff \frac{dx}{dt} = L_{F_x}^{-1}(\xi) = L_{F_x^*}(\xi) = \frac{\partial H}{\partial \xi}(x, \xi).$$

If $(x, \xi) = \mathcal{L}(x, v)$, by differentiating both sides of $H(x, \xi) = \xi \cdot v - F(x, v)$ with respect to x, where $\xi = L_{F_x}(v) = \xi(x, v)$ and $v = \frac{\partial H}{\partial \xi}$, we obtain

$$\frac{\partial H}{\partial x} + \frac{\partial H}{\partial \xi}\frac{\partial \xi}{\partial x} = \frac{\partial \xi}{\partial x} \cdot v - \frac{\partial F}{\partial x} \iff \frac{\partial F}{\partial x}(x, v) = -\frac{\partial H}{\partial x}(x, \xi).$$

Using the last equation and the definition of ξ, the second line of (H) becomes (E–L):

$$\frac{d\xi}{dt} = -\frac{\partial H}{\partial x}(x, \xi) \iff \frac{d}{dt}\frac{\partial F}{\partial v}(x, v) = \frac{\partial F}{\partial x}(x, v).$$ □

5.5. *Integrable systems*

DEFINITION 5.11. A *Hamiltonian system* is a triple (M, ω, H), where (M, ω) is a symplectic manifold and $H \in C^\infty(M)$ is the *Hamiltonian function*.

PROPOSITION 5.12. *For a function f on a symplectic manifold (M, ω) we have that $\{f, H\} = 0$ if and only if f is constant along integral curves of X_H.*

PROOF. Let ρ_t be the flow of X_H. Then

$$\frac{d}{dt}(f \circ \rho_t) = \rho_t^* \mathcal{L}_{X_H} f = \rho_t^* \imath_{X_H} df = \rho_t^* \imath_{X_H} \imath_{X_f} \omega = \rho_t^* \omega(X_f, X_H)$$
$$= \rho_t^* \{f, H\}. \qquad \square$$

A function f as in Proposition 5.12 is called an *integral of motion* (or a *first integral* or a *constant of motion*). In general, Hamiltonian systems do not admit integrals of motion that are *independent* of the Hamiltonian function. Functions $f_1, \dots, f_n$ are said to be *independent* if their differentials $(df_1)_p, \dots, (df_n)_p$ are linearly independent at all points p in some dense subset of M. Loosely speaking, a Hamiltonian system is (*completely*) *integrable* if it has as many *commuting* integrals of motion as possible. *Commutativity* is with respect to the Poisson bracket. If $f_1, \dots, f_n$ are commuting integrals of motion for a Hamiltonian system (M, ω, H), then $\omega(X_{f_i}, X_{f_j}) = \{f_i, f_j\} = 0$, so at each $p \in M$ the Hamiltonian vector fields generate an isotropic subspace of T_pM. When $f_1, \dots, f_n$ are independent, by symplectic linear algebra n can be at most half the dimension of M.

DEFINITION 5.13. A Hamiltonian system (M, ω, H) where M is a $2n$-dimensional manifold is (*completely*) *integrable* if it possesses n independent commuting integrals of motion, $f_1 = H, f_2, \dots, f_n$.

Any 2-dimensional Hamiltonian system (where the set of nonfixed points is dense) is trivially integrable. Basic examples are the simple pendulum and the harmonic oscillator. A Hamiltonian system (M, ω, H) where M is 4-dimensional is integrable if there is an integral of motion independent of H (the commutativity condition is automatically satisfied). A basic example is the spherical pendulum. Sophisticated examples of integrable systems can be found in [8,72].

EXAMPLES.

1. The *simple pendulum* is a mechanical system consisting of a massless rigid rod of length ℓ, fixed at one end, whereas the other end has a bob of mass m, which may oscillate in the vertical plane. We assume that the force of gravity is constant pointing vertically downwards and the only external force acting on this system. Let θ be the oriented angle between the rod and the vertical direction. Let ξ be the coordinate along the fibers of T^*S^1 induced by the standard angle coordinate on S^1. The energy function $H : T^*S^1 \to \mathbb{R}$, $H(\theta, \xi) = \frac{\xi^2}{2m\ell^2} + m\ell(1 - \cos\theta)$, is an appropriate Hamiltonian function to describe the simple pendulum. Gravity is responsible for the potential energy $V(\theta) = m\ell(1 - \cos\theta)$, and the kinetic energy is given by $K(\theta, \xi) = \frac{1}{2m\ell^2}\xi^2$.
2. The *spherical pendulum* consists of a massless rigid rod of length ℓ, fixed at one end, whereas the other end has a bob of mass m, which may oscillate *freely in all directions*. For simplicity let $m = \ell = 1$. Again assume that gravity is the only external force. Let φ, θ $(0 < \varphi < \pi,\ 0 < \theta < 2\pi)$ be spherical coordinates for the bob, inducing coordinates η, ξ along the fibers of T^*S^2. An appropriate Hamiltonian function for this system is the energy function $H : T^*S^2 \to \mathbb{R}$, $H(\varphi, \theta, \eta, \xi) =$

$\frac{1}{2}(\eta^2 + \frac{\xi^2}{(\sin\varphi)^2}) + \cos\varphi$. The function $J(\varphi, \theta, \eta, \xi) = \xi$ is an independent integral of motion corresponding to the group of symmetries given by rotations about the vertical axis (Section 5.6). The points $p \in T^*S^2$ where dH_p and dJ_p are linearly dependent are:

- the two critical points of H (where both dH and dJ vanish);
- if $x \in S^2$ is in the southern hemisphere ($x_3 < 0$), then there exist exactly two points, $p_+ = (x, \eta, \xi)$ and $p_- = (x, -\eta, -\xi)$, in the cotangent fiber above x where dH_p and dJ_p are linearly dependent;
- since dH_p and dJ_p are linearly dependent along the trajectory of the Hamiltonian vector field of H through p_+, this trajectory is also a trajectory of the Hamiltonian vector field of J and hence its projection onto S^2 is a latitudinal (or horizontal) circle. The projection of the trajectory through p_- is the same latitudinal circle traced in the opposite direction.

Let (M, ω, H) be an integrable system of dimension $2n$ with integrals of motion $f_1 = H, f_2, \ldots, f_n$. Let $c \in \mathbb{R}^n$ be a regular value of $f := (f_1, \ldots, f_n)$. The corresponding level set $f^{-1}(c)$ is a Lagrangian submanifold, as it is n-dimensional and its tangent bundle is isotropic. If the flows are complete on $f^{-1}(c)$, by following them we obtain global coordinates. Any compact component of $f^{-1}(c)$ must hence be a torus. These components, when they exist, are called *Liouville tori.* A way to ensure that compact components exist is to have one of the f_i's proper.

THEOREM 5.14 (Arnold–Liouville [2]). *Let (M, ω, H) be an integrable system of dimension $2n$ with integrals of motion $f_1 = H, f_2, \ldots, f_n$. Let $c \in \mathbb{R}^n$ be a regular value of $f := (f_1, \ldots, f_n)$. The level $f^{-1}(c)$ is a Lagrangian submanifold of M.*

(a) *If the flows of the Hamiltonian vector fields $X_{f_1}, \ldots, X_{f_n}$ starting at a point $p \in f^{-1}(c)$ are complete, then the connected component of $f^{-1}(c)$ containing p is a homogeneous space for $\mathbb{R}^n$, i.e., is of the form $\mathbb{R}^{n-k} \times \mathbb{T}^k$ for some k, $0 \leqslant k \leqslant n$, where $\mathbb{T}^k$ is a k-dimensional torus.. With respect to this affine structure, that component has coordinates $\varphi_1, \ldots, \varphi_n$, known as* angle coordinates, *in which the flows of $X_{f_1}, \ldots, X_{f_n}$ are linear.*

(b) *There are coordinates $\psi_1, \ldots, \psi_n$, known as* action coordinates, *complementary to the angle coordinates, such that the ψ_i's are integrals of motion and $\varphi_1, \ldots, \varphi_n$, $\psi_1, \ldots, \psi_n$ form a Darboux chart.*

Therefore, the dynamics of an integrable system has a simple explicit solution in action-angle coordinates. The proof of part (a)—the easy part of the theorem—is sketched above. For the proof of part (b) see, for instance, [2,36]. Geometrically, regular levels being Lagrangian submanifolds implies that, in a neighborhood of a regular value, the map $f : M \to \mathbb{R}^n$ collecting the given integrals of motion is a *Lagrangian fibration*, i.e., it is locally trivial and its fibers are Lagrangian submanifolds. Part (a) states that there are coordinates along the fibers, the angle coordinates,[56] in which the flows of $X_{f_1}, \ldots, X_{f_n}$ are linear. Part (b) guarantees the existence of coordinates on $\mathbb{R}^n$, the action coordinates,

[56]The name *angle coordinates* is used even if the fibers are not tori.

$\psi_1, \ldots, \psi_n$, complementary to the angle coordinates, that (Poisson) commute among themselves and satisfy $\{\varphi_i, \psi_j\} = \delta_{ij}$. The action coordinates are generally not the given integrals of motion because $\varphi_1, \ldots, \varphi_n, f_1, \ldots, f_n$ do not form a Darboux chart.

5.6. *Symplectic and Hamiltonian actions*

Let (M, ω) be a symplectic manifold, and G a Lie group.

DEFINITION 5.15. An action[57] $\psi : G \to \mathrm{Diff}(M)$, $g \mapsto \psi_g$, is a *symplectic action* if each ψ_g is a symplectomorphism, i.e., $\psi : G \to \mathrm{Sympl}(M, \omega) \subset \mathrm{Diff}(M)$.

In particular, symplectic actions of $\mathbb{R}$ on (M, ω) are in one-to-one correspondence with complete symplectic vector fields on M:

$$\psi = \exp tX \quad \longleftrightarrow \quad X_p = \left.\frac{d\psi_t(p)}{dt}\right|_{t=0}, \quad p \in M.$$

We may define a symplectic action ψ of S^1 or $\mathbb{R}$ on (M, ω) to be *Hamiltonian* if the vector field X generated by ψ is Hamiltonian, that is, when there is $H : M \to \mathbb{R}$ with $dH = \imath_X \omega$. An action of S^1 may be viewed as a periodic action of $\mathbb{R}$.

EXAMPLES.

1. On $(\mathbb{R}^{2n}, \omega_0)$, the orbits of the action generated by $X = -\frac{\partial}{\partial y_1}$ are lines parallel to the y_1-axis, $\{(x_1, y_1 - t, x_2, y_2, \ldots, x_n, y_n) \mid t \in \mathbb{R}\}$. Since X is Hamiltonian with Hamiltonian function x_1, this is a *Hamiltonian action* of $\mathbb{R}$.
2. On the 2-sphere $(S^2, d\theta \wedge dh)$ in cylindrical coordinates, the one-parameter group of diffeomorphisms given by rotation around the vertical axis, $\psi_t(\theta, h) = (\theta + t, h)$ $(t \in \mathbb{R})$ is a symplectic action of the group $S^1 \simeq \mathbb{R}/\langle 2\pi \rangle$, as it preserves the area form $d\theta \wedge dh$. Since the vector field corresponding to ψ is Hamiltonian with Hamiltonian function h, this is a *Hamiltonian action* of S^1.

When G is a product of S^1's or $\mathbb{R}$'s, an action $\psi : G \to \mathrm{Sympl}(M, \omega)$ is called *Hamiltonian* when the restriction to each 1-dimensional factor is Hamiltonian in the previous sense *with Hamiltonian function preserved by the action of the rest of G*.

For an arbitrary Lie group G, we use an upgraded Hamiltonian function μ, known as a *moment map*, determined up to an additive local constant by coordinate functions μ_i indexed by a basis of the Lie algebra of G. We require that the constant be such that μ is *equivariant*, i.e., μ intertwines the action of G on M and the coadjoint action of G on the dual of its Lie algebra. (If M is compact, equivariance can be achieved by adjusting the constant so that $\int_M \mu\omega^n = 0$. Similarly when there is a fixed point p (on each component of M) by imposing $\mu(p) = 0$.)

Let G be a Lie group, $\mathfrak{g}$ the Lie algebra of G, and $\mathfrak{g}^*$ the dual vector space of $\mathfrak{g}$.

[57] A (smooth) *action* of G on M is a group homomorphism $G \to \mathrm{Diff}(M)$, $g \mapsto \psi_g$, whose evaluation map $M \times G \to M$, $(p, g) \mapsto \psi_g(p)$, is smooth.

DEFINITION 5.16. An action $\psi : G \to \mathrm{Diff}(M)$ on a symplectic manifold (M, ω) is a *Hamiltonian action* if there exists a map $\mu : M \to \mathfrak{g}^*$ satisfying:

- For each $X \in \mathfrak{g}$, we have $d\mu^X = \imath_{X^\#}\omega$, i.e., μ^X is a Hamiltonian function for the vector field $X^\#$, where
  - $\mu^X : M \to \mathbb{R}$, $\mu^X(p) := \langle \mu(p), X \rangle$, is the component of μ along X,
  - $X^\#$ is the vector field on M generated by the one-parameter subgroup $\{\exp tX \mid t \in \mathbb{R}\} \subseteq G$.
- The map μ is *equivariant* with respect to the given action ψ on M and the coadjoint action: $\mu \circ \psi_g = \mathrm{Ad}_g^* \circ \mu$, for all $g \in G$.

Then (M, ω, G, μ) is a *Hamiltonian G-space* and μ is a *moment map.*

This definition matches the previous one when G is an Abelian group $\mathbb{R}$, S^1 or $\mathbb{T}^n$, for which equivariance becomes invariance since the coadjoint action is trivial.

EXAMPLES.

1. Let $\mathbb{T}^n = \{(t_1, \ldots, t_n) \in \mathbb{C}^n \colon |t_j| = 1, \text{ for all } j\}$ be a torus acting on $\mathbb{C}^n$ by $(t_1, \ldots, t_n) \cdot (z_1, \ldots, z_n) = (t_1^{k_1} z_1, \ldots, t_n^{k_n} z_n)$, where $k_1, \ldots, k_n \in \mathbb{Z}$ are fixed. This action is Hamiltonian with a moment map $\mu : \mathbb{C}^n \to (\mathfrak{t}^n)^* \simeq \mathbb{R}^n$, $\mu(z_1, \ldots, z_n) = -\frac{1}{2}(k_1|z_1|^2, \ldots, k_n|z_n|^2)$.
2. When a Lie group G acts on two symplectic manifolds (M_j, ω_j), $j = 1, 2$, with moment maps $\mu_j : M_j \to \mathfrak{g}^*$, the diagonal action of G on $M_1 \times M_2$ has moment map $\mu : M_1 \times M_2 \to \mathfrak{g}^*$, $\mu(p_1, p_2) = \mu_1(p_1) + \mu_2(p_2)$.
3. Equip the coadjoint orbits of a Lie group G with the canonical symplectic form (Section 5.1). Then, for each $\xi \in \mathfrak{g}^*$, the coadjoint action on the orbit $G \cdot \xi$ is Hamiltonian with moment map simply the inclusion map $\mu : G \cdot \xi \hookrightarrow \mathfrak{g}^*$.
4. Identify the Lie algebra of the unitary group $\mathrm{U}(n)$ with its dual via the inner product $\langle A, B \rangle = \mathrm{trace}(A^* B)$. The natural action of $\mathrm{U}(n)$ on $(\mathbb{C}^n, \omega_0)$ is Hamiltonian with moment map $\mu : \mathbb{C}^n \to \mathfrak{u}(n)$ given by $\mu(z) = \frac{i}{2} z z^*$. Similarly, a moment map for the natural action of $\mathrm{U}(k)$ on the space $(\mathbb{C}^{k \times n}, \omega_0)$ of complex $(k \times n)$-matrices is given by $\mu(A) = \frac{i}{2} A A^*$ for $A \in \mathbb{C}^{k \times n}$. Thus the $\mathrm{U}(n)$-action by conjugation on the space $(\mathbb{C}^{n^2}, \omega_0)$ of complex $(n \times n)$-matrices is Hamiltonian, with moment map given by $\mu(A) = \frac{i}{2}[A, A^*]$.
5. For the spherical pendulum (Section 5.5), the *energy-momentum map* $(H, J) : T^*S^2 \to \mathbb{R}^2$ is a moment map for the $\mathbb{R} \times S^1$ action given by time flow and rotation about the vertical axis.
6. Suppose that a compact Lie group acts on a symplectic manifold (M, ω) in a Hamiltonian way, and that $q \in M$ is a fixed point for the G-action. Then, by an equivariant version of Darboux's theorem,[58] there exists a Darboux chart $(\mathcal{U}, z_1, \ldots, z_n)$ centered at q that is G-equivariant with respect to a linear action of G on $\mathbb{C}^n$. Consider an ε-blow-up of M relative to this chart, for ε sufficiently small. Then G acts on the blow-up in a Hamiltonian way.

[58]**Equivariant Darboux theorem** [136]. *Let (M, ω) be a $2n$-dimensional symplectic manifold equipped with a symplectic action of a compact Lie group G, and let q be a fixed point. Then there exists a G-invariant chart*

The concept of a moment map was introduced by Souriau [119] under the French name *application moment*; besides the more standard English translation to *moment map*, the alternative *momentum map* is also used, and recently James Stasheff has proposed the short unifying new word *momap*. The name comes from being the generalization of *linear and angular momenta* in classical mechanics.

Let $\mathbb{R}^3$ act on $(\mathbb{R}^6 \simeq T^*\mathbb{R}^3, \omega_0 = \sum dx_i \wedge dy_i)$ by *translations*:

$$a \in \mathbb{R}^3 \longmapsto \psi_a \in \mathrm{Sympl}(\mathbb{R}^6, \omega_0), \quad \psi_a(x, y) = (x + a, y).$$

The vector field generated by $X = a = (a_1, a_2, a_3)$ is $X^\# = a_1 \frac{\partial}{\partial x_1} + a_2 \frac{\partial}{\partial x_2} + a_3 \frac{\partial}{\partial x_3}$, and the *linear momentum* map

$$\mu : \mathbb{R}^6 \longrightarrow \mathbb{R}^3, \quad \mu(x, y) = y$$

is a moment map, with $\mu^a(x, y) = \langle \mu(x, y), a\rangle = y \cdot a$. Classically, y is called the *momentum vector* corresponding to the *position vector* x.

The SO(3)-action on $\mathbb{R}^3$ by *rotations* lifts to a symplectic action ψ on the cotangent bundle $\mathbb{R}^6$. The infinitesimal version of this action is[59]

$$a \in \mathbb{R}^3 \longmapsto d\psi(a) \in \chi^{\mathrm{sympl}}(\mathbb{R}^6), \quad d\psi(a)(x, y) = (a \times x, a \times y).$$

Then the *angular momentum* map

$$\mu : \mathbb{R}^6 \longrightarrow \mathbb{R}^3, \quad \mu(x, y) = x \times y$$

is a moment map, with $\mu^a(x, y) = \langle \mu(x, y), a\rangle = (x \times y) \cdot a$.

The notion of a moment map associated to a group action on a symplectic manifold formalizes the *Noether principle*, which asserts that there is a one-to-one correspondence between *symmetries* (or one-parameter group actions) and *integrals of motion* (or conserved quantities) for a mechanical system.

$(\mathcal{U}, x_1, \dots, x_n, y_1, \dots, y_n)$ centered at q and G-equivariant with respect to a linear action of G on $\mathbb{R}^{2n}$ such that

$$\omega|_{\mathcal{U}} = \sum_{k=1}^{n} dx_k \wedge dy_k.$$

A suitable linear action on $\mathbb{R}^{2n}$ is equivalent to the induced action of G on $T_q M$. The proof relies on an equivariant version of the Moser trick and may be found in [70].

[59] The Lie group $\mathrm{SO}(3) = \{A \in \mathrm{GL}(3; \mathbb{R}) \mid A^t A = \mathrm{Id} \text{ and } \det A = 1\}$, has Lie algebra, $\mathfrak{g} = \{A \in \mathfrak{gl}(3; \mathbb{R}) \mid A + A^t = 0\}$, the space of 3×3 skew-symmetric matrices. The standard identification of $\mathfrak{g}$ with $\mathbb{R}^3$ carries the Lie bracket to the exterior product:

$$A = \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix} \longmapsto a = (a_1, a_2, a_3),$$

$$[A, B] = AB - BA \longmapsto a \times b.$$

DEFINITION 5.17. An *integral of motion* of a Hamiltonian G-space (M, ω, G, μ) is a G-invariant function $f : M \to \mathbb{R}$. When μ is constant on the trajectories of a Hamiltonian vector field X_f, the corresponding flow $\{\exp tX_f \mid t \in \mathbb{R}\}$ (regarded as an $\mathbb{R}$-action) is a *symmetry* of the Hamiltonian G-space (M, ω, G, μ).

THEOREM 5.18 (Noether). *Let (M, ω, G, μ) be a Hamiltonian G-space where G is connected. If f is an integral of motion, the flow of its Hamiltonian vector field X_f is a symmetry. If the flow of some Hamiltonian vector field X_f is a symmetry, then a corresponding Hamiltonian function f is an integral of motion.*

PROOF. Let $\mu^X = \langle \mu, X\rangle : M \to \mathbb{R}$ for $X \in \mathfrak{g}$. We have $\mathcal{L}_{X_f}\mu^X = \imath_{X_f} d\mu^X = \imath_{X_f}\imath_{X^\#}\omega = -\imath_{X^\#}\imath_{X_f}\omega = -\imath_{X^\#}df = -\mathcal{L}_{X^\#}f$. So μ is invariant over the flow of X_f if and only if f is invariant under the infinitesimal G-action. $\square$

We now turn to the questions of existence and uniqueness of moment maps.

Let $\mathfrak{g}$ be a Lie algebra, and let $C^k := \Lambda^k\mathfrak{g}^*$ be the set of *k-cochains* on $\mathfrak{g}$, that is, of alternating k-linear maps $\mathfrak{g} \times \cdots \times \mathfrak{g} \to \mathbb{R}$. The linear operator $\delta : C^k \to C^{k+1}$ defined by $\delta c(X_0, \ldots, X_k) = \sum_{i<j}(-1)^{i+j}c([X_i, X_j], X_0, \ldots, \hat{X}_i, \ldots, \hat{X}_j, \ldots, X_k)$ satisfies $\delta^2 = 0$. The *Lie algebra cohomology groups* (or *Chevalley cohomology groups*) of $\mathfrak{g}$ are the cohomology groups of the complex $0 \xrightarrow{\delta} C^0 \xrightarrow{\delta} C^1 \xrightarrow{\delta} \cdots$:

$$H^k(\mathfrak{g}; \mathbb{R}) := \frac{\ker\delta : C^k \to C^{k+1}}{\operatorname{im}\delta : C^{k-1} \to C^k}.$$

It is always $H^0(\mathfrak{g}; \mathbb{R}) = \mathbb{R}$. If $c \in C^1 = \mathfrak{g}^*$, then $\delta c(X, Y) = -c([X, Y])$. The *commutator ideal* $[\mathfrak{g}, \mathfrak{g}]$ is the subspace of $\mathfrak{g}$ spanned by $\{[X, Y] \mid X, Y \in \mathfrak{g}\}$. Since $\delta c = 0$ if and only if c vanishes on $[\mathfrak{g}, \mathfrak{g}]$, we conclude that $H^1(\mathfrak{g}; \mathbb{R}) = [\mathfrak{g}, \mathfrak{g}]^0$, where $[\mathfrak{g}, \mathfrak{g}]^0 \subseteq \mathfrak{g}^*$ is the *annihilator* of $[\mathfrak{g}, \mathfrak{g}]$. An element of C^2 is an alternating bilinear map $c : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$, and $\delta c(X, Y, Z) = -c([X, Y], Z) + c([X, Z], Y) - c([Y, Z], X)$. If $c = \delta b$ for some $b \in C^1$, then $c(X, Y) = (\delta b)(X, Y) = -b([X, Y])$.

If $\mathfrak{g}$ is the Lie algebra of a compact connected Lie group G, then by averaging one can show that the de Rham cohomology may be computed from the subcomplex of G-invariant forms, and hence $H^k(\mathfrak{g}; \mathbb{R}) = H^k_{\mathrm{deRham}}(G)$.

PROPOSITION 5.19. *If $H^1(\mathfrak{g}; \mathbb{R}) = H^2(\mathfrak{g}, \mathbb{R}) = 0$, then any symplectic G-action is Hamiltonian.*

PROOF. Let $\psi : G \to \mathrm{Sympl}(M, \omega)$ be a symplectic action of G on a symplectic manifold (M, ω). Since $H^1(\mathfrak{g}; \mathbb{R}) = 0$ means that $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$, and since commutators of symplectic vector fields are Hamiltonian, we have $d\psi : \mathfrak{g} = [\mathfrak{g}, \mathfrak{g}] \to \chi^{\mathrm{ham}}(M)$. The action ψ is Hamiltonian if and only if there is a Lie algebra homomorphism $\mu^* : \mathfrak{g} \to C^\infty(M)$ such that the Hamiltonian vector field of $\mu^*(\xi)$ is $d\psi(\xi)$. We first take an arbitrary vector space lift $\tau : \mathfrak{g} \to C^\infty(M)$ with this property, i.e., for each basis vector $X \in \mathfrak{g}$, we choose $\tau(X) = \tau^X \in C^\infty(M)$ such that $v_{(\tau^X)} = d\psi(X)$. The map $X \mapsto \tau^X$ may not be a Lie algebra homomorphism. By construction, $\tau^{[X,Y]}$ is a Hamiltonian function for $[X, Y]^\#$, and (as computed in Section 5.5) $\{\tau^X, \tau^Y\}$ is a Hamiltonian function for $-[X^\#, Y^\#]$. Since $[X, Y]^\# = -[X^\#, Y^\#]$, the corresponding Hamiltonian functions must differ by a constant:

$$\tau^{[X,Y]} - \left\{\tau^X, \tau^Y\right\} = c(X, Y) \in \mathbb{R}.$$

By the Jacobi identity, $\delta c = 0$. Since $H^2(\mathfrak{g}; \mathbb{R}) = 0$, there is $b \in \mathfrak{g}^*$ satisfying $c = \delta b$, $c(X, Y) = -b([X, Y])$. We define

$$\begin{aligned} \mu^* : \mathfrak{g} &\longrightarrow C^\infty(M), \\ X &\longmapsto \mu^*(X) = \tau^X + b(X) = \mu^X. \end{aligned}$$

Now μ^* is a Lie algebra homomorphism: $\mu^*([X, Y]) = \{\tau^X, \tau^Y\} = \{\mu^X, \mu^Y\}$. □

By the Whitehead lemmas (see, for instance, [77, pp. 93–95]) a semisimple Lie group G has $H^1(\mathfrak{g}; \mathbb{R}) = H^2(\mathfrak{g}; \mathbb{R}) = 0$. As a corollary, *when G is semisimple, any symplectic G-action is Hamiltonian.*[60]

PROPOSITION 5.20. *For a connected Lie group G, if $H^1(\mathfrak{g}; \mathbb{R}) = 0$, then moment maps for Hamiltonian G-actions are unique.*

PROOF. Suppose that μ_1 and μ_2 are two moment maps for an action ψ. For each $X \in \mathfrak{g}$, μ_1^X and μ_2^X are both Hamiltonian functions for $X^\#$, thus $\mu_1^X - \mu_2^X = c(X)$ is locally constant. This defines $c \in \mathfrak{g}^*$, $X \mapsto c(X)$. Since the corresponding $\mu_i^* : \mathfrak{g} \to C^\infty(M)$ are Lie algebra homomorphisms, we have $c([X, Y]) = 0$, $\forall X, Y \in \mathfrak{g}$, i.e., $c \in [\mathfrak{g}, \mathfrak{g}]^0 = \{0\}$. Hence, $\mu_1 = \mu_2$. □

In general, if $\mu : M \to \mathfrak{g}^*$ is a moment map, then given any $c \in [\mathfrak{g}, \mathfrak{g}]^0$, $\mu_1 = \mu + c$ is another moment map. In other words, moment maps are unique up to elements of the dual of the Lie algebra that annihilate the commutator ideal.

The two extreme cases are when

- G is semisimple: any symplectic action is Hamiltonian, moment maps are unique;
- G is Abelian: symplectic actions may not be Hamiltonian, moment maps are unique up to a constant $c \in \mathfrak{g}^*$.

[60] A compact Lie group G has $H^1(\mathfrak{g}; \mathbb{R}) = H^2(\mathfrak{g}; \mathbb{R}) = 0$ if and only if it is semisimple. In fact, a compact Lie group G is semisimple when $\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}]$. The unitary group $\mathrm{U}(n)$ is not semisimple because the multiples of the identity, $S^1 \cdot \mathrm{Id}$, form a nontrivial center; at the level of the Lie algebra, this corresponds to the subspace $\mathbb{R} \cdot \mathrm{Id}$ of scalar matrices, which are not commutators since they are not traceless. Any Abelian Lie group is *not* semisimple. Any direct product of the other compact classical groups $\mathrm{SU}(n)$, $\mathrm{SO}(n)$ and $\mathrm{Sp}(n)$ is semisimple. An arbitrary compact Lie group admits a finite cover by a direct product of tori and semisimple Lie groups.

5.7. *Convexity*

Atiyah, Guillemin and Sternberg [4,68] showed that the image of the moment map for a Hamiltonian torus action on a compact connected symplectic manifold is always a polytope.[61] A proof of this theorem can also be found in [99].

THEOREM 5.21 (Atiyah, Guillemin–Sternberg). *Let (M, ω) be a compact connected symplectic manifold. Suppose that $\psi : \mathbb{T}^m \to \mathrm{Sympl}(M, \omega)$ is a Hamiltonian action of an m-torus with moment map $\mu : M \to \mathbb{R}^m$. Then:*

(a) *the levels $\mu^{-1}(c)$ are connected ($c \in \mathbb{R}^m$);*
(b) *the image $\mu(M)$ is convex;*
(c) *$\mu(M)$ is the convex hull of the images of the fixed points of the action.*

The image $\mu(M)$ of the moment map is called the *moment polytope*.

EXAMPLES.

1. Suppose that $\mathbb{T}^m$ acts linearly on $(\mathbb{C}^n, \omega_0)$. Let $\lambda^{(1)}, \ldots, \lambda^{(n)} \in \mathbb{Z}^m$ be the *weights* appearing in the corresponding weight space decomposition, that is,

$$\mathbb{C}^n \simeq \bigoplus_{k=1}^{n} V_{\lambda^{(k)}},$$

where, for $\lambda^{(k)} = (\lambda_1^{(k)}, \ldots, \lambda_m^{(k)})$, the torus $\mathbb{T}^m$ acts on the complex line $V_{\lambda^{(k)}}$ by $(e^{it_1}, \ldots, e^{it_m}) \cdot v = e^{i \sum_j \lambda_j^{(k)} t_j} v$. If the action is effective[62], then $m \leqslant n$ and the weights $\lambda^{(1)}, \ldots, \lambda^{(n)}$ are part of a $\mathbb{Z}$-basis of $\mathbb{Z}^m$. If the action is symplectic (hence Hamiltonian in this case), then the weight spaces $V_{\lambda^{(k)}}$ are symplectic subspaces. In this case, a moment map is given by

$$\mu(v) = -\frac{1}{2} \sum_{k=1}^{n} \lambda^{(k)} |v_{\lambda^{(k)}}|^2,$$

where $|\cdot|$ is the standard norm[63] and $v = v_{\lambda^{(1)}} + \cdots + v_{\lambda^{(n)}}$ is the weight space decomposition of v. We conclude that, if $\mathbb{T}^n$ acts on $\mathbb{C}^n$ in a linear, effective and Hamiltonian way, then any moment map μ is a submersion, i.e., each differential $d\mu_v : \mathbb{C}^n \to \mathbb{R}^n$ ($v \in \mathbb{C}^n$) is surjective.

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[61] A *polytope* in $\mathbb{R}^n$ is the convex hull of a finite number of points in $\mathbb{R}^n$. A *convex polyhedron* is a subset of $\mathbb{R}^n$ that is the intersection of a finite number of affine half-spaces. Hence, polytopes coincide with bounded convex polyhedra.

[62] An action of a group G on a manifold M is called *effective* if each group element $g \neq e$ moves at least one point $p \in M$, that is, $\bigcap_{p \in M} G_p = \{e\}$, where $G_p = \{g \in G \mid g \cdot p = p\}$ is the stabilizer of p.

[63] The standard inner product satisfies $\langle v, w \rangle = \omega_0(v, Jv)$ where $J\frac{\partial}{\partial z} = i\frac{\partial}{\partial z}$ and $J\frac{\partial}{\partial \bar{z}} = -i\frac{\partial}{\partial \bar{z}}$. In particular, the standard norm is invariant for a symplectic complex-linear action.

2. Consider a coadjoint orbit $\mathcal{O}_\lambda$ for the unitary group $\mathrm{U}(n)$. Multiplying by i, the orbit $\mathcal{O}_\lambda$ can be viewed as the set of Hermitian matrices with a given eigenvalue spectrum $\lambda = (\lambda_1 \geqslant \cdots \geqslant \lambda_n)$. The restriction of the coadjoint action to the maximal torus $\mathbb{T}^n$ of diagonal unitary matrices is Hamiltonian with moment map $\mu : \mathcal{O}_\lambda \to \mathbb{R}^n$ taking a matrix to the vector of its diagonal entries. Then the moment polytope $\mu(\mathcal{O}_\lambda)$ is the convex hull C of the points given by all the permutations of $(\lambda_1, \ldots, \lambda_n)$. This is a rephrasing of the classical theorem of Schur ($\mu(\mathcal{O}_\lambda) \subseteq C$) and Horn ($C \subseteq \mu(\mathcal{O}_\lambda)$).

Example 1 is related to the universal local picture for a moment map near a fixed point of a Hamiltonian torus action:

THEOREM 5.22. *Let $(M^{2n}, \omega, \mathbb{T}^m, \mu)$ be a Hamiltonian $\mathbb{T}^m$-space, where q is a fixed point. Then there exists a chart $(\mathcal{U}, x_1, \ldots, x_n, y_1, \ldots, y_n)$ centered at q and weights $\lambda^{(1)}, \ldots, \lambda^{(n)} \in \mathbb{Z}^m$ such that*

$$\omega|_{\mathcal{U}} = \sum_{k=1}^{n} dx_k \wedge dy_k \quad \textit{and} \quad \mu|_{\mathcal{U}} = \mu(q) - \frac{1}{2}\sum_{k=1}^{n} \lambda^{(k)}\bigl(x_k^2 + y_k^2\bigr).$$

The following two results use the crucial fact that any effective action of an m-torus on a manifold has orbits of dimension m; a proof may be found in [19].

COROLLARY 5.23. *Under the conditions of the convexity theorem, if the $\mathbb{T}^m$-action is effective, then there must be at least $m+1$ fixed points.*

PROOF. At a point p of an m-dimensional orbit the moment map is a submersion, i.e., $(d\mu_1)_p, \ldots, (d\mu_m)_p$ are linearly independent. Hence, $\mu(p)$ is an interior point of $\mu(M)$, and $\mu(M)$ is a nondegenerate polytope. A nondegenerate polytope in $\mathbb{R}^m$ has at least $m+1$ vertices. The vertices of $\mu(M)$ are images of fixed points. $\square$

PROPOSITION 5.24. *Let $(M, \omega, \mathbb{T}^m, \mu)$ be a Hamiltonian $\mathbb{T}^m$-space. If the $\mathbb{T}^m$-action is effective, then* $\dim M \geqslant 2m$.

PROOF. Since the moment map is constant on an orbit $\mathcal{O}$, for $p \in \mathcal{O}$ the differential $d\mu_p : T_pM \to \mathfrak{g}^*$ maps $T_p\mathcal{O}$ to 0. Thus $T_p\mathcal{O} \subseteq \ker d\mu_p = (T_p\mathcal{O})^\omega$, where $(T_p\mathcal{O})^\omega$ is the symplectic orthogonal of $T_p\mathcal{O}$. This shows that orbits $\mathcal{O}$ of a Hamiltonian torus action are isotropic submanifolds of M. In particular, by symplectic linear algebra we have that $\dim \mathcal{O} \leqslant \frac{1}{2} \dim M$. Now consider an m-dimensional orbit. $\square$

For a Hamiltonian action of an arbitrary compact Lie group G on a compact symplectic manifold (M, ω), the following *non-Abelian* convexity theorem was proved by Kirwan [81]: if $\mu : M \to \mathfrak{g}^*$ is a moment map, then the intersection $\mu(M) \cap \mathfrak{t}_+^*$ of the image of μ with a Weyl chamber for a Cartan subalgebra $\mathfrak{t} \subseteq \mathfrak{g}$ is a convex polytope. This had been conjectured by Guillemin and Sternberg and proved by them in particular cases.

6. Symplectic reduction

6.1. *Marsden–Weinstein–Meyer theorem*

Classical physicists realized that, whenever there is a symmetry group of dimension k acting on a mechanical system, the number of degrees of freedom for the position and momenta of the particles may be reduced by $2k$. Symplectic reduction formulates this process mathematically.

THEOREM 6.1 (Marsden–Weinstein, Meyer [92,102]). *Let (M,ω,G,μ) be a Hamiltonian G-space (Section 5.6) for a compact Lie group G. Let $i:\mu^{-1}(0)\hookrightarrow M$ be the inclusion map. Assume that G acts freely on $\mu^{-1}(0)$. Then*

(a) *the orbit space $M_{\text{red}}=\mu^{-1}(0)/G$ is a manifold,*

(b) *$\pi:\mu^{-1}(0)\to M_{\text{red}}$ is a principal G-bundle, and*

(c) *there is a symplectic form ω_{red} on M_{red} satisfying $i^*\omega=\pi^*\omega_{\text{red}}$.*

DEFINITION 6.2. The symplectic manifold $(M_{\text{red}},\omega_{\text{red}})$ is the *reduction* (or *reduced space*, or *symplectic quotient*) of (M,ω) with respect to G,μ.

When M is Kähler and the action of G preserves the complex structure, we can show that the symplectic reduction has a natural Kähler structure.

Let (M,ω,G,μ) be a Hamiltonian G-space for a compact Lie group G. To reduce at a level $\xi\in\mathfrak{g}^*$ of μ, we need $\mu^{-1}(\xi)$ to be preserved by G, or else take the G-orbit of $\mu^{-1}(\xi)$, or else take the quotient by the maximal subgroup of G that preserves $\mu^{-1}(\xi)$. Since μ is equivariant, G preserves $\mu^{-1}(\xi)$ if and only if $\mathrm{Ad}_g^*\xi=\xi,\forall g\in G$. Of course, the level 0 is always preserved. Also, when G is a torus, any level is preserved and *reduction at ξ* for the moment map μ, is equivalent to reduction at 0 for a shifted moment map $\phi:M\to\mathfrak{g}^*$, $\phi(p):=\mu(p)-\xi$. In general, let $\mathcal{O}$ be a coadjoint orbit in $\mathfrak{g}^*$ equipped with the *canonical symplectic form* $\omega_{\mathcal{O}}$ (defined in Section 5.1). Let $\mathcal{O}^-$ be the orbit $\mathcal{O}$ equipped with $-\omega_{\mathcal{O}}$. The natural product action of G on $M\times\mathcal{O}^-$ is Hamiltonian with moment map $\mu_{\mathcal{O}}(p,\xi)=\mu(p)-\xi$. If the hypothesis of Theorem 6.1 is satisfied for $M\times\mathcal{O}^-$, then one obtains a *reduced space with respect to the coadjoint orbit $\mathcal{O}$*.

EXAMPLES.

1. The standard symplectic form on $\mathbb{C}^n$ is $\omega_0=\frac{i}{2}\sum dz_i\wedge d\bar z_i=\sum dx_i\wedge dy_i=\sum r_i\,dr_i\wedge d\theta_i$ in polar coordinates. The S^1-action on $(\mathbb{C}^n,\omega_0)$ where $e^{it}\in S^1$ acts as multiplication by e^{it} has vector field $X^\#=\frac{\partial}{\partial\theta_1}+\frac{\partial}{\partial\theta_2}+\cdots+\frac{\partial}{\partial\theta_n}$. This action is Hamiltonian with moment map $\mu:\mathbb{C}^n\to\mathbb{R}$, $\mu(z)=-\frac{|z|^2}{2}$, since $\imath_{X^\#}\omega=\sum r_i dr_i=-\frac12\sum dr_i^2=d\mu$. The level $\mu^{-1}(-\frac12)$ is the unit sphere S^{2n-1}, whose orbit space is the projective space,
$$\mu^{-1}\left(-\frac{1}{2}\right)/S^1=S^{2n-1}/S^1=\mathbb{CP}^{n-1}.$$

The reduced symplectic form at level $-\frac{1}{2}$ is $\omega_{\text{red}} = \omega_{\text{FS}}$ the Fubini–Study symplectic form. Indeed, if $\text{pr}: \mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{CP}^n$ is the standard projection, the forms $\text{pr}^*\omega_{\text{FS}} = \frac{i}{2}\partial\bar{\partial}\log(|z|^2)$ and ω_0 have the same restriction to S^{2n+1}.

2. Consider the natural action of $\mathrm{U}(k)$ on $\mathbb{C}^{k\times n}$ with moment map $\mu(A) = \frac{i}{2}AA^* + \frac{\mathrm{Id}}{2i}$ for $A \in \mathbb{C}^{k\times n}$ (Section 5.6). Since $\mu^{-1}(0) = \{A \in \mathbb{C}^{k\times n} \mid AA^* = \mathrm{Id}\}$, the reduced manifold is the Grassmannian of k-planes in $\mathbb{C}^n$:
$$\mu^{-1}(0)/\mathrm{U}(k) = \mathbb{G}(k, n).$$

For the case where $G = S^1$ and $\dim M = 4$, here is a glimpse of reduction. Let $\mu: M \to \mathbb{R}$ be the moment map and $p \in \mu^{-1}(0)$. Choose local coordinates near p: θ along the orbit through p, μ given by the moment map, and η_1, η_2 the pullback of coordinates on $M_{\text{red}} = \mu^{-1}(0)/S^1$. Then the symplectic form can be written

$$\omega = A\,d\theta \wedge d\mu + \sum B_j\,d\theta \wedge d\eta_j + \sum C_j\,d\mu \wedge d\eta_j + D\,d\eta_1 \wedge d\eta_2.$$

As $d\mu = \imath(\frac{\partial}{\partial\theta})\omega$, we must have $A = 1$, $B_j = 0$. Since ω is symplectic, it must be $D \neq 0$. Hence, $i^*\omega = D\,d\eta_1 \wedge d\eta_2$ is the pullback of a symplectic form on M_{red}.

The actual proof of Theorem 6.1 requires some preliminary ingredients.

Let $\mu: M \to \mathfrak{g}^*$ be the moment map for an (Hamiltonian) action of a Lie group G on a symplectic manifold (M, ω). Let $\mathfrak{g}_p$ be the Lie algebra of the stabilizer of a point $p \in M$, let $\mathfrak{g}_p^0 = \{\xi \in \mathfrak{g}^* \mid \langle \xi, X\rangle = 0,\ \forall X \in \mathfrak{g}_p\}$ be the annihilator of $\mathfrak{g}_p$, and let $\mathcal{O}_p$ be the G-orbit through p. Since $\omega_p(X_p^\#, v) = \langle d\mu_p(v), X\rangle$, for all $v \in T_pM$ and all $X \in \mathfrak{g}$, the differential $d\mu_p: T_pM \to \mathfrak{g}^*$ has

$$\ker d\mu_p = (T_p\mathcal{O}_p)^{\omega_p} \quad \text{and} \quad \operatorname{im} d\mu_p = \mathfrak{g}_p^0.$$

Consequently, the action is locally free[64] at p if and only if p is a regular point of μ (i.e., $d\mu_p$ is surjective), and we obtain:

LEMMA 6.3. *If G acts freely on $\mu^{-1}(0)$, then 0 is a regular value of μ, the level $\mu^{-1}(0)$ is a submanifold of M of codimension* $\dim G$, *and, for $p \in \mu^{-1}(0)$, the tangent space $T_p\mu^{-1}(0) = \ker d\mu_p$ is the symplectic orthogonal to $T_p\mathcal{O}_p$ in T_pM.*

In particular, *orbits in $\mu^{-1}(0)$ are isotropic*. Since any tangent vector to the orbit is the value of a vector field generated by the group, we can show this directly by computing, for any $X, Y \in \mathfrak{g}$ and $p \in \mu^{-1}(0)$, the Hamiltonian function for $[Y^\#, X^\#] = [Y, X]^\#$ at that point: $\omega_p(X_p^\#, Y_p^\#) = \mu^{[Y,X]}(p) = 0$.

LEMMA 6.4. *Let (V, Ω) be a symplectic vector space, and I an isotropic subspace. Then Ω induces a canonical symplectic structure Ω_{red} on I^Ω/I.*

[64]The action is *locally free* at p when $\mathfrak{g}_p = \{0\}$, i.e., the stabilizer of p is a discrete group. The action is *free* at p when the stabilizer of p is trivial, i.e., $G_p = \{e\}$.

PROOF. Let $[u], [v]$ be the classes in I^Ω/I of $u, v \in I^\Omega$. We have $\Omega(u+i, v+j) = \Omega(u, v)$, $\forall i, j \in I$, because $\Omega(u, j) = \Omega(i, v) = \Omega(i, j) = 0$. Hence, we can define $\Omega_{\text{red}}([u],[v]) := \Omega(u, v)$. This is nondegenerate: if $u \in I^\Omega$ has $\Omega(u, v) = 0$, for all $v \in I^\Omega$, then $u \in (I^\Omega)^\Omega = I$, i.e., $[u] = 0$. □

PROPOSITION 6.5. *If a compact Lie group G acts freely on a manifold M, then M/G is a manifold and the map $\pi : M \to M/G$ is a principal G-bundle.*

PROOF. We first show that, for any $p \in M$, the G-orbit through p is a compact submanifold of M diffeomorphic to G.[65] The G-orbit through p is the image of the smooth injective map $\mathrm{ev}_p : G \to M$, $\mathrm{ev}_p(g) = g \cdot p$. The map ev_p is proper because, if A is a compact, hence closed, subset of M, then its inverse image $(\mathrm{ev}_p)^{-1}(A)$, being a closed subset of the compact Lie group G, is also compact. The differential $d(\mathrm{ev}_p)_e$ is injective because $d(\mathrm{ev}_p)_e(X) = 0 \Leftrightarrow X_p^\# = 0 \Leftrightarrow X = 0$, $\forall X \in T_eG$, as the action is free. At any other point $g \in G$, for $X \in T_gG$ we have $d(\mathrm{ev}_p)_g(X) = 0 \Leftrightarrow d(\mathrm{ev}_p \circ R_g)_e \circ (dR_{g^{-1}})_g(X) = 0$, where $R_g : G \to G$, $h \mapsto hg$, is right multiplication by g. But $\mathrm{ev}_p \circ R_g = \mathrm{ev}_{g \cdot p}$ has an injective differential at e, and $(dR_{g^{-1}})_g$ is an isomorphism. It follows that $d(\mathrm{ev}_p)_g$ is always injective, so ev_p is an immersion. We conclude that ev_p is a closed embedding.

We now apply the slice theorem[66] which is an equivariant tubular neighborhood theorem. For $p \in M$, let $q = \pi(p) \in M/G$. Choose a G-invariant neighborhood $\mathcal{U}$ of p as in the slice theorem, so that $\mathcal{U} \simeq G \times S$ where S is an appropriate slice. Then $\pi(\mathcal{U}) = \mathcal{U}/G =: \mathcal{V}$ is a neighborhood of q in M/G homeomorphic[67] to S. Such neighborhoods $\mathcal{V}$ are used as charts on M/G. To show that the associated transition maps are smooth, consider two G-invariant open sets $\mathcal{U}_1, \mathcal{U}_2$ in M and corresponding slices S_1, S_2. Then $S_{12} = S_1 \cap \mathcal{U}_2$, $S_{21} = S_2 \cap \mathcal{U}_1$ are both slices for the G-action on $\mathcal{U}_1 \cap \mathcal{U}_2$. To compute the transition map $S_{12} \to S_{21}$, consider the sequence $S_{12} \xrightarrow{\simeq} \{e\} \times S_{12} \hookrightarrow G \times S_{12} \xrightarrow{\simeq} \mathcal{U}_1 \cap \mathcal{U}_2$ and similarly for S_{21}. The composition $S_{12} \hookrightarrow \mathcal{U}_1 \cap \mathcal{U}_2 \xrightarrow{\simeq} G \times S_{21} \xrightarrow{\mathrm{pr}} S_{21}$ is smooth.

Finally, we show that $\pi : M \to M/G$ is a principal G-bundle. For $p \in M$, $q = \pi(p)$, choose a G-invariant neighborhood $\mathcal{U}$ of p of the form $\eta : G \times S \xrightarrow{\simeq} \mathcal{U}$. Then $\mathcal{V} = \mathcal{U}/G \simeq S$ is the corresponding neighborhood of q in M/G:

$$\begin{array}{ccccc} M \supseteq \mathcal{U} & \overset{\eta}{\simeq} & G \times S & \simeq G \times \mathcal{V} \\ \downarrow \pi & & & \downarrow \\ M/G \supseteq \mathcal{V} & & = & \mathcal{V} \end{array}$$

[65] Even if the action is not free, the orbit through p is a compact submanifold of M. In that case, the orbit of a point p is diffeomorphic to the quotient G/G_p of G by the stabilizer of p.

[66] **Slice theorem.** *Let G be a compact Lie group acting on a manifold M such that G acts freely at $p \in M$. Let S be a transverse section to $\mathcal{O}_p$ at p (this is called a* slice*). Choose a coordinate chart $x_1, \dots, x_n$ centered at p such that $\mathcal{O}_p \simeq G$ is given by $x_1 = \cdots = x_k = 0$ and S by $x_{k+1} = \cdots = x_n = 0$. Let $S_\varepsilon = S \cap B_\varepsilon$ where B_ε is the ball of radius ε centered at* 0 *with respect to these coordinates. Let $\eta : G \times S \to M$, $\eta(g, s) = g \cdot s$. Then, for sufficiently small ε, the map $\eta : G \times S_\varepsilon \to M$ takes $G \times S_\varepsilon$ diffeomorphically onto a G-invariant neighborhood $\mathcal{U}$ of the G-orbit through p.* In particular, if the action of G is free at p, then the action is free on $\mathcal{U}$, so the set of points where G acts freely is open.

[67] We equip the orbit space M/G with the *quotient topology*, i.e., $\mathcal{V} \subseteq M/G$ is open if and only if $\pi^{-1}(\mathcal{V})$ is open in M.

Since the projection on the right is smooth, π is smooth. By considering the overlap of two trivializations $\phi_1 : \mathcal{U}_1 \to G \times \mathcal{V}_1$ and $\phi_2 : \mathcal{U}_2 \to G \times \mathcal{V}_2$, we check that the transition map $\phi_2 \circ \phi_1^{-1} = (\sigma_{12}, \mathrm{id}) : G \times (\mathcal{V}_1 \cap \mathcal{V}_2) \to G \times (\mathcal{V}_1 \cap \mathcal{V}_2)$ is smooth. □

PROOF OF THEOREM 6.1. Since G acts freely on $\mu^{-1}(0)$, by Lemma 6.3 the level $\mu^{-1}(0)$ is a submanifold. Applying Proposition 6.5 to the free action of G on the manifold $\mu^{-1}(0)$, we conclude the assertions (a) and (b).

At $p \in \mu^{-1}(0)$ the tangent space to the orbit $T_p\mathcal{O}_p$ is an isotropic subspace of the symplectic vector space (T_pM, ω_p). By Lemma 6.4 there is a canonical symplectic structure on the quotient $T_p\mu^{-1}(0)/T_p\mathcal{O}_p$. The point $[p] \in M_{\mathrm{red}} = \mu^{-1}(0)/G$ has tangent space $T_{[p]}M_{\mathrm{red}} \simeq T_p\mu^{-1}(0)/T_p\mathcal{O}_p$. This gives a well-defined nondegenerate 2-form ω_{red} on M_{red} because ω is G-invariant. By construction $i^*\omega = \pi^*\omega_{\mathrm{red}}$ where

$$\begin{array}{ccc} \mu^{-1}(0) & \overset{i}{\hookrightarrow} & M \\ \downarrow \pi & & \\ M_{\mathrm{red}} & & \end{array}$$

The injectivity of π^* yields closedness: $\pi^* d\omega_{\mathrm{red}} = d\pi^*\omega_{\mathrm{red}} = d\iota^*\omega = \iota^* d\omega = 0$. □

6.2. *Applications and generalizations*

Let (M, ω, G, μ) be a Hamiltonian G-space for a compact Lie group G. Suppose that another Lie group H acts on (M, ω) in a Hamiltonian way with moment map $\phi : M \to \mathfrak{h}^*$. Suppose that the H-action commutes with the G-action, that ϕ is G-invariant and that μ is H-invariant. Assuming that G acts freely on $\mu^{-1}(0)$, let $(M_{\mathrm{red}}, \omega_{\mathrm{red}})$ be the corresponding reduced space. Since the action of H preserves $\mu^{-1}(0)$ and ω and commutes with the G-action, the reduced space $(M_{\mathrm{red}}, \omega_{\mathrm{red}})$ inherits a symplectic action of H. Since ϕ is preserved by the G-action, the restriction of this moment map to $\mu^{-1}(0)$ descends to a moment map $\phi_{\mathrm{red}} : M_{\mathrm{red}} \to \mathfrak{h}^*$ satisfying $\phi_{\mathrm{red}} \circ \pi = \phi \circ i$, where $\pi : \mu^{-1}(0) \to M_{\mathrm{red}}$ and $i : \mu^{-1}(0) \hookrightarrow M$. Therefore, $(M_{\mathrm{red}}, \omega_{\mathrm{red}}, H, \phi_{\mathrm{red}})$ is a Hamiltonian H-space.

Consider now the action of a *product group* $G = G_1 \times G_2$, where G_1 and G_2 are compact connected Lie groups. We have $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$ and $\mathfrak{g}^* = \mathfrak{g}_1^* \oplus \mathfrak{g}_2^*$. Suppose that (M, ω, G, ψ) is a Hamiltonian G-space with moment map

$$\psi = (\psi_1, \psi_2) : M \longrightarrow \mathfrak{g}_1^* \oplus \mathfrak{g}_2^*,$$

where $\psi_i : M \to \mathfrak{g}_i^*$ for $i = 1, 2$. The fact that ψ is equivariant implies that ψ_1 is invariant under G_2 and ψ_2 is invariant under G_1. Assume that G_1 acts freely on $Z_1 := \psi_1^{-1}(0)$. Let $(M_1 = Z_1/G_1, \omega_1)$ be the reduction of (M, ω) with respect to G_1, ψ_1. From the observation above, (M_1, ω_1) inherits a Hamiltonian G_2-action with moment map $\mu_2 : M_1 \to \mathfrak{g}_2^*$ such that $\mu_2 \circ \pi = \psi_2 \circ i$, where $\pi : Z_1 \to M_1$ and $i : Z_1 \hookrightarrow M$. If G acts freely on $\psi^{-1}(0, 0)$, then G_2 acts freely on $\mu_2^{-1}(0)$, and there is a natural symplectomorphism

$$\mu_2^{-1}(0)/G_2 \simeq \psi^{-1}(0, 0)/G.$$

This technique of performing reduction with respect to one factor of a product group at a time is called *reduction in stages*. It may be extended to reduction by a normal subgroup $H \subset G$ and by the corresponding quotient group G/H.

EXAMPLE. Finding symmetries for a mechanical problem may reduce degrees of freedom by two at a time: an integral of motion f for a $2n$-dimensional Hamiltonian system (M, ω, H) may allow to understand the trajectories of this system in terms of the trajectories of a $(2n-2)$-dimensional Hamiltonian system $(M_{\mathrm{red}}, \omega_{\mathrm{red}}, H_{\mathrm{red}})$. Locally this process goes as follows. Let $(\mathcal{U}, x_1, \ldots, x_n, \xi_1, \ldots, \xi_n)$ be a Darboux chart for M such that $f = \xi_n$.[68] Since ξ_n is an integral of motion, $0 = \{\xi_n, H\} = -\frac{\partial H}{\partial x_n}$, the trajectories of the Hamiltonian vector field X_H lie on a constant level $\xi_n = c$ (Proposition 5.12), and H does not depend on x_n. The *reduced space* is $\mathcal{U}_{\mathrm{red}} = \{(x_1, \ldots, x_{n-1}, \xi_1, \ldots, \xi_{n-1}) \mid \exists a\colon (x_1, \ldots, x_{n-1}, a, \xi_1, \ldots, \xi_{n-1}, c) \in \mathcal{U}\}$ and the *reduced Hamiltonian* is $H_{\mathrm{red}}\colon \mathcal{U}_{\mathrm{red}} \to \mathbb{R}$, $H_{\mathrm{red}}(x_1, \ldots, x_{n-1}, \xi_1, \ldots, \xi_{n-1}) = H(x_1, \ldots, x_{n-1}, a, \xi_1, \ldots, \xi_{n-1}, c)$ for some a. In order to find the trajectories of the original system on the hypersurface $\xi_n = c$, we look for the trajectories $(x_1(t), \ldots, x_{n-1}(t), \xi_1(t), \ldots, \xi_{n-1}(t))$ of the reduced system on $\mathcal{U}_{\mathrm{red}}$, and integrate the equation $\frac{dx_n}{dt}(t) = \frac{\partial H}{\partial \xi_n}$ to obtain the original trajectories where

$$\begin{cases} x_n(t) = x_n(0) + \int_0^t \frac{\partial H}{\partial \xi_n}(x_1(t), \ldots, x_{n-1}(t), \xi_1(t), \ldots, \xi_{n-1}(t), c)\, dt, \\ \xi_n(t) = c. \end{cases}$$

By Sard's theorem, the singular values of a moment map $\mu\colon M \to \mathfrak{g}^*$ form a set of measure zero. So, perturbing if necessary, we may assume that a level of μ is regular hence, when G is compact, that any point p of that level has finite stabilizer G_p. Let $\mathcal{O}_p$ be the orbit of p. By the slice theorem for the case of orbifolds, near $\mathcal{O}_p$ the orbit space of the level is modeled by S/G_p, where S is a G_p-invariant disk in the level and transverse to $\mathcal{O}_p$ (a *slice*). Thus, the orbit space is an *orbifold*.[69] This implies that, when $G = \mathbb{T}^n$ is an n-torus, for most levels reduction goes through, however the quotient space is not necessarily a manifold but an orbifold. Roughly speaking, orbifolds are singular manifolds where each singularity is locally modeled on $\mathbb{R}^m/\Gamma$, for some finite group $\Gamma \subset \mathrm{GL}(m; \mathbb{R})$. The differential-geometric notions of vector fields, differential forms, exterior

[68] To obtain such a chart, in the proof of Darboux's Theorem 1.9 start with coordinates $(x'_1, \ldots, x'_n, y'_1, \ldots y'_n)$ such that $y'_n = f$ and $\frac{\partial}{\partial x'_n} = X_f$.

[69] Let $|M|$ be a Hausdorff topological space satisfying the second axiom of countability. An *orbifold chart* on $|M|$ is a triple $(\mathcal{V}, \Gamma, \varphi)$, where $\mathcal{V}$ is a connected open subset of some Euclidean space $\mathbb{R}^m$, Γ is a finite group that acts linearly on $\mathcal{V}$ so that the set of points where the action is not free has codimension at least two, and $\varphi\colon \mathcal{V} \to |M|$ is a Γ-invariant map inducing a homeomorphism from $\mathcal{V}/\Gamma$ onto its image $\mathcal{U} \subset |M|$. An *orbifold atlas* $\mathcal{A}$ for $|M|$ is a collection of orbifold charts on $|M|$ such that: the collection of images $\mathcal{U}$ forms a basis of open sets in $|M|$, and the charts are compatible in the sense that, whenever two charts $(\mathcal{V}_1, \Gamma_1, \varphi_1)$ and $(\mathcal{V}_2, \Gamma_2, \varphi_2)$ satisfy $\mathcal{U}_1 \subseteq \mathcal{U}_2$, there exists an injective homomorphism $\lambda\colon \Gamma_1 \to \Gamma_2$ and a λ-equivariant open embedding $\psi\colon \mathcal{V}_1 \to \mathcal{V}_2$ such that $\varphi_2 \circ \psi = \varphi_1$. Two orbifold atlases are *equivalent* if their union is still an atlas. An m-dimensional *orbifold* M is a Hausdorff topological space $|M|$ satisfying the second axiom of countability, plus an equivalence class of orbifold atlases on $|M|$. We do not require the action of each group Γ to be effective. Given a point p on an orbifold M, let $(\mathcal{V}, \Gamma, \varphi)$ be an orbifold chart for a neighborhood $\mathcal{U}$ of p. The *orbifold structure group* of p, Γ_p, is (the isomorphism class of) the stabilizer of a preimage of p under ϕ. Orbifolds were introduced by Satake in [114].

differentiation, group actions, etc., extend naturally to orbifolds by gluing corresponding local Γ-invariant or Γ-equivariant objects. In particular, a *symplectic orbifold* is a pair (M, ω) where M is an orbifold and ω is a closed 2-form on M that is nondegenerate at every point.

EXAMPLES. The S^1-action on $\mathbb{C}^2$ given by $e^{i\theta} \cdot (z_1, z_2) = (e^{ik\theta} z_1, e^{i\ell\theta} z_2)$, for some integers k and ℓ, has moment map $\mu : \mathbb{C}^2 \to \mathbb{R}$, $(z_1, z_2) \mapsto -\frac{1}{2}(k|z_1|^2 + \ell|z_2|^2)$. Any $\xi < 0$ is a regular value and $\mu^{-1}(\xi)$ is a 3-dimensional ellipsoid.

When $\ell = 1$ and $k \geqslant 2$, the stabilizer of (z_1, z_2) is $\{1\}$ if $z_2 \neq 0$ and is $\mathbb{Z}_k = \{e^{i\frac{2\pi m}{k}} \mid m = 0, 1, \ldots, k-1\}$ if $z_2 = 0$. The reduced space $\mu^{-1}(\xi)/S^1$ is then called a *teardrop* orbifold or *conehead*; it has one *cone* (or *dunce cap*) singularity with cone angle $\frac{2\pi}{k}$, that is, a point with orbifold structure group $\mathbb{Z}_k$.

When $k, \ell \geqslant 2$ are relatively prime, for $z_1, z_2 \neq 0$ the stabilizer of $(z_1, 0)$ is $\mathbb{Z}_k$, of $(0, z_2)$ is $\mathbb{Z}_\ell$ and of (z_1, z_2) is $\{1\}$. The quotient $\mu^{-1}(\xi)/S^1$ is called a *football* orbifold: it has two cone singularities, with angles $\frac{2\pi}{k}$ and $\frac{2\pi}{\ell}$.

For S^1 acting on $\mathbb{C}^n$ by $e^{i\theta} \cdot (z_1, \ldots, z_n) = (e^{ik_1\theta} z_1, \ldots, e^{ik_n\theta} z_n)$ the reduced spaces are orbifolds called *weighted* (or *twisted*) *projective spaces*.

Let (M, ω) be a symplectic manifold where S^1 acts in a Hamiltonian way, $\rho : S^1 \to \mathrm{Diff}(M)$, with moment map $\mu : M \to \mathbb{R}$. Suppose that:

- M has a unique nondegenerate minimum at q where $\mu(q) = 0$, and
- for ε sufficiently small, S^1 acts freely on the level set $\mu^{-1}(\varepsilon)$.

Let $\mathbb{C}$ be equipped with the symplectic form $-i\, dz \wedge d\bar{z}$. Then the action of S^1 on the product $\psi : S^1 \to \mathrm{Diff}(M \times \mathbb{C})$, $\psi_t(p, z) = (\rho_t(p), t \cdot z)$, is Hamiltonian with moment map

$$\phi : M \times \mathbb{C} \longrightarrow \mathbb{R}, \quad \phi(p, z) = \mu(p) - |z|^2.$$

Observe that S^1 acts freely on the ε-level of ϕ for ε small enough:

$$\begin{aligned}
\phi^{-1}(\varepsilon) &= \{(p, z) \in M \times \mathbb{C} \mid \mu(p) - |z|^2 = \varepsilon\} \\
&= \{(p, 0) \in M \times \mathbb{C} \mid \mu(p) = \varepsilon\} \\
&\quad \cup \{(p, z) \in M \times \mathbb{C} \mid |z|^2 = \mu(p) - \varepsilon > 0\}.
\end{aligned}$$

The reduced space is hence

$$\phi^{-1}(\varepsilon)/S^1 \simeq \mu^{-1}(\varepsilon)/S^1 \cup \{p \in M \mid \mu(p) > \varepsilon\}.$$

The open submanifold of M given by $\{p \in M \mid \mu(p) > \varepsilon\}$ embeds as an open dense submanifold into $\phi^{-1}(\varepsilon)/S^1$. The reduced space $\phi^{-1}(\varepsilon)/S^1$ is the ε-blow-up of M at q (Section 5.6). This global description of blow-up for Hamiltonian S^1-spaces is due to Lerman [86], as a particular instance of his *cutting* technique. *Symplectic cutting* is the application of symplectic reduction to the product of a Hamiltonian S^1-space with the standard

$\mathbb{C}$ as above, in a way that the reduced space for the original Hamiltonian S^1-space embeds symplectically as a codimension 2 submanifold in a symplectic manifold. As it is a local construction, the cutting operation may be more generally performed at a local minimum (or maximum) of the moment map μ. There is a remaining S^1-action on the cut space $M_{\text{cut}}^{\geqslant \varepsilon} := \phi^{-1}(\varepsilon)/S^1$ induced by

$$\tau : S^1 \longrightarrow \mathrm{Diff}(M \times \mathbb{C}), \quad \tau_t(p, z) = \big(\rho_t(p), z\big).$$

In fact, τ is a Hamiltonian S^1-action on $M \times \mathbb{C}$ that commutes with ψ, thus descends to an action $\tilde{\tau} : S^1 \to \mathrm{Diff}(M_{\text{cut}}^{\geqslant \varepsilon})$, which is also Hamiltonian.

Loosely speaking, the cutting technique provides a Hamiltonian way to close the open manifold $\{p \in M \mid \mu(p) > \varepsilon\}$, by using the reduced space at level ε, $\mu^{-1}(\varepsilon)/S^1$. We may similarly close $\{p \in M \mid \mu(p) < \varepsilon\}$. The resulting Hamiltonian S^1-spaces are called *cut spaces*, and denoted $M_{\text{cut}}^{\geqslant \varepsilon}$ and $M_{\text{cut}}^{\leqslant \varepsilon}$. If another group G acts on M in a Hamiltonian way that commutes with the S^1-action, then the cut spaces are also Hamiltonian G-spaces.

6.3. *Moment map in gauge theory*

Let G be a Lie group and P a principal G-bundle over B.[70] If A is a connection (form)[71] on P, and if $a \in \Omega^1_{\text{horiz}} \otimes \mathfrak{g}$ is G-invariant for the product action, then $A + a$ is also a connection on P. Reciprocally, any two connections on P differ by an $a \in (\Omega^1_{\text{horiz}} \otimes \mathfrak{g})^G$.

[70] Let G be a Lie group and B a manifold. A *principal G-bundle over B* is a fibration $\pi : P \to B$ (Section 4.2) with a free action of G (the *structure group*) on the total space P, such that the base B is the orbit space, the map π is the point-orbit projection and the local trivializations are of the form $\varphi_{\mathcal{U}} = (\pi, s_{\mathcal{U}}) : \pi^{-1}(\mathcal{U}) \to \mathcal{U} \times G$ with $s_{\mathcal{U}}(g \cdot p) = g \cdot s_{\mathcal{U}}(p)$ for all $g \in G$ and all $p \in \pi^{-1}(\mathcal{U})$. A principal G-bundle is represented by a diagram

$$\begin{array}{ccc} G & \hookrightarrow & P \\ & & \downarrow \pi \\ & & B \end{array}$$

For instance, the *Hopf fibration* is a principal S^1-bundle over $S^2 (= \mathbb{CP}^1)$ with total space S^3 regarded as unit vectors in $\mathbb{C}^2$ where circle elements act by complex multiplication.

[71] An action $\psi : G \to \mathrm{Diff}(P)$ induces an infinitesimal action $d\psi : \mathfrak{g} \to \chi(P)$ mapping $X \in \mathfrak{g}$ to the vector field $X^\#$ generated by the one-parameter group $\{\exp tX(e) \mid t \in \mathbb{R}\} \subseteq G$. Fix a basis $X_1, \ldots, X_k$ of $\mathfrak{g}$. Let P be a principal G-bundle over B. Since the G-action is free, the vector fields $X_1^\#, \ldots, X_k^\#$ are linearly independent at each $p \in P$. The *vertical bundle* V is the rank k subbundle of TP generated by $X_1^\#, \ldots, X_k^\#$. Alternatively, V is the set of vectors tangent to P that lie in the kernel of the derivative of the bundle projection π, so V is indeed independent of the choice of basis for $\mathfrak{g}$. An (*Ehresmann*) *connection* on P is a choice of a splitting $TP = V \oplus H$, where H (called the *horizontal bundle*) is a G-invariant subbundle of TP complementary to the vertical bundle V. A *connection form* on P is a Lie-algebra-valued 1-form $A = \sum_{i=1}^k A_i \otimes X_i \in \Omega^1(P) \otimes \mathfrak{g}$ such that A is G-invariant, with respect to the product action of G on $\Omega^1(P)$ (induced by the action on P) and on $\mathfrak{g}$ (the adjoint action), and A is vertical, in the sense that $\imath_{X^\#} A = X$ for any $X \in \mathfrak{g}$. A connection $TP = V \oplus H$ determines a connection (form) A and vice-versa by the formula $H = \ker A = \{v \in TP \mid \imath_v A = 0\}$. Given a connection on P, the splitting $TP = V \oplus H$ induces splittings for bundles $T^*P = V^* \oplus H^*$, $\wedge^2 T^*P = (\wedge^2 V^*) \oplus (V^* \wedge H^*) \oplus (\wedge^2 H^*)$, etc., and for their sections: $\Omega^1(P) = \Omega^1_{\text{vert}} \oplus \Omega^1_{\text{horiz}}$, $\Omega^2(P) = \Omega^2_{\text{vert}} \oplus \Omega^2_{\text{mix}} \oplus \Omega^2_{\text{horiz}}$, etc. The corresponding connection form A is in $\Omega^1_{\text{vert}} \otimes \mathfrak{g}$.

We conclude that the *set $\mathcal{A}$ of all connections* on the principal G-bundle P is an affine space modeled on the linear space $\mathfrak{a} = (\Omega^1_{\text{horiz}} \otimes \mathfrak{g})^G$.

Now let P be a principal G-bundle over a compact Riemann surface. Suppose that the group G is compact or semisimple. Atiyah and Bott [6] noticed that the corresponding space $\mathcal{A}$ of all connections may be treated as an *infinite-dimensional symplectic manifold*. This requires choosing a G-invariant inner product $\langle\cdot,\cdot\rangle$ on $\mathfrak{g}$, which always exists, either by averaging any inner product when G is compact, or by using the *Killing form* on semisimple groups.

Since $\mathcal{A}$ is an affine space, its tangent space at any point A is identified with the model linear space $\mathfrak{a}$. With respect to a basis $X_1, \dots, X_k$ for the Lie algebra $\mathfrak{g}$, elements $a, b \in \mathfrak{a}$ are written

$$a = \sum a_i \otimes X_i \quad \text{and} \quad b = \sum b_i \otimes X_i .$$

If we wedge a and b, and then integrate over B, we obtain a real number:

$$\begin{aligned} \omega : \mathfrak{a} \times \mathfrak{a} &\longrightarrow \left(\Omega^2_{\text{horiz}}(P)\right)^G \simeq \Omega^2(B) \longrightarrow \mathbb{R}, \\ (a,b) &\longmapsto \sum_{i,j} a_i \wedge b_j \langle X_i, X_j\rangle \qquad \longmapsto \int_B \sum_{i,j} a_i \wedge b_j \langle X_i, X_j\rangle . \end{aligned}$$

We used that the pullback $\pi^* : \Omega^2(B) \to \Omega^2(P)$ is an isomorphism onto its image $(\Omega^2_{\text{horiz}}(P))^G$. When $\omega(a,b) = 0$ for all $b \in \mathfrak{a}$, then a must be zero. The map ω is nondegenerate, skew-symmetric, bilinear and constant in the sense that it does not depend on the base point A. Therefore, it has the right to be called a symplectic form on $\mathcal{A}$, so the pair $(\mathcal{A}, \omega)$ is an *infinite-dimensional symplectic manifold*.

A diffeomorphism $f : P \to P$ commuting with the G-action determines a diffeomorphism $f_{\text{basic}} : B \to B$ by projection. Such a diffeomorphism f is called a *gauge transformation* if the induced f_{basic} is the identity. The *gauge group* of P is the group $\mathcal{G}$ of all gauge transformations of P.

The derivative of an $f \in \mathcal{G}$ takes an *Ehresmann connection* $TP = V \oplus H$ to another connection $TP = V \oplus H_f$, and thus induces an action of $\mathcal{G}$ in the space $\mathcal{A}$ of all connections. Atiyah and Bott [6] noticed that the action of $\mathcal{G}$ on $(\mathcal{A}, \omega)$ is Hamiltonian, where the moment map (appropriately interpreted) is

$$\begin{aligned} \mu : \mathcal{A} &\longrightarrow \left(\Omega^2(P) \otimes \mathfrak{g}\right)^G, \\ A &\longmapsto \operatorname{curv} A, \end{aligned}$$

i.e., the moment map *is* the curvature.[72] The reduced space $\mathcal{M} = \mu^{-1}(0)/\mathcal{G}$ is the space of *flat connections* modulo gauge equivalence, known as the *moduli space of flat connections*, which is a finite-dimensional symplectic orbifold.

[72] The exterior derivative of a connection A decomposes into three components,

$$dA = (dA)_{\text{vert}} + (dA)_{\text{mix}} + (dA)_{\text{horiz}} \in \left(\Omega^2_{\text{vert}} \oplus \Omega^2_{\text{mix}} \oplus \Omega^2_{\text{horiz}}\right) \otimes \mathfrak{g}$$

EXAMPLE. We describe the Atiyah–Bott construction for the case of a circle bundle

$$\begin{array}{ccc} S^1 & \hookrightarrow & P \\ & & \downarrow \pi \\ & & B \end{array}$$

Let v be the generator of the S^1-action on P, corresponding to the basis 1 of $\mathfrak{g} \simeq \mathbb{R}$. A connection form on P is an ordinary 1-form $A \in \Omega^1(P)$ such that $\mathcal{L}_v A = 0$ and $\imath_v A = 1$. If we fix one particular connection A_0, then any other connection is of the form $A = A_0 + a$ for some $a \in \mathfrak{a} = (\Omega^1_{\text{horiz}}(P))^G = \Omega^1(B)$. The symplectic form on $\mathfrak{a} = \Omega^1(B)$ is simply

$$\begin{aligned} \omega : \mathfrak{a} \times \mathfrak{a} &\longrightarrow \Omega^2(B) \longrightarrow \mathbb{R}, \\ (a, b) &\longmapsto \ a \wedge b \ \longmapsto \int_B a \wedge b. \end{aligned}$$

The gauge group is $\mathcal{G} = \text{Maps}(B, S^1)$, because a gauge transformation is multiplication by some element of S^1 over each point in B encoded in a map $h : B \to S^1$. The action $\phi : \mathcal{G} \to \text{Diff}(P)$ takes $h \in \mathcal{G}$ to the diffeomorphism

$$\phi_h : p \longmapsto h\big(\pi(p)\big) \cdot p.$$

The Lie algebra of $\mathcal{G}$ is $\text{Lie}\,\mathcal{G} = \text{Maps}(B, \mathbb{R}) = C^\infty(B)$ with dual $(\text{Lie}\,\mathcal{G})^* = \Omega^2(B)$, where the (smooth) duality is provided by integration $C^\infty(B) \times \Omega^2(B) \to \mathbb{R}$, $(h, \beta) \mapsto \int_B h\beta$. The gauge group acts on the space of all connections by

$$\begin{aligned} \psi : \mathcal{G} &\longrightarrow \text{Diff}(\mathcal{A}), \\ \big(h : x \mapsto e^{i\theta(x)}\big) &\longmapsto (\psi_h : A \mapsto A - \pi^* d\theta). \end{aligned}$$

(In the case where $P = S^1 \times B$ is a trivial bundle, every connection can be written $A = dt + \beta$, with $\beta \in \Omega^1(B)$. A gauge transformation $h \in \mathcal{G}$ acts on P by $\phi_h : (t, x) \mapsto (t + \theta(x), x)$ and on $\mathcal{A}$ by $A \mapsto \phi^*_{h^{-1}}(A)$.) The infinitesimal action is

$$\begin{aligned} d\psi : \text{Lie}\,\mathcal{G} &\longrightarrow \chi(\mathcal{A}), \\ X &\longmapsto X^\# = \text{vector field described by } (A \mapsto A - dX), \end{aligned}$$

so that $X^\# = -dX$. It remains to check that

$$\begin{aligned} \mu : \mathcal{A} &\longrightarrow (\text{Lie}\,\mathcal{G})^* = \Omega^2(B), \\ A &\longmapsto \text{curv}\, A \end{aligned}$$

---

satisfying $(dA)_{\text{mix}} = 0$ and $(dA)_{\text{vert}}(X, Y) = [X, Y]$, i.e., $(dA)_{\text{vert}} = \frac{1}{2} \sum_{i,\ell,m} c^i_{\ell m} A_\ell \wedge A_m \otimes X_i$, where the $c^i_{\ell m}$'s are the *structure constants* of the Lie algebra with respect to the chosen basis, and defined by $[X_\ell, X_m] = \sum_{i,\ell,m} c^i_{\ell m} X_i$. So the relevance of dA may come only from its horizontal component, called the *curvature form* of the connection A, and denoted $\text{curv}\, A = (dA)_{\text{horiz}} \in \Omega^2_{\text{horiz}} \otimes \mathfrak{g}$. A connection is called *flat* if its curvature is zero.

is indeed a moment map for the action of the gauge group on $\mathcal{A}$. Since in this case curv $A = dA \in (\Omega^2_{\text{horiz}}(P))^G = \Omega^2(B)$, the action of $\mathcal{G}$ on $\Omega^2(B)$ is trivial and μ is $\mathcal{G}$-invariant, the equivariance condition is satisfied. Take any $X \in \text{Lie}\,\mathcal{G} = C^\infty(B)$. Since the map $\mu^X : A \mapsto \langle X, dA\rangle = \int_B X \cdot dA$ is linear in A, its differential is

$$\begin{aligned} d\mu^X : \mathfrak{a} &\longrightarrow \mathbb{R}, \\ a &\longmapsto \int_B X\, da. \end{aligned}$$

By definition of ω and the Stokes theorem, we have that

$$\omega(X^\#, a) = \int_B X^\# \cdot a = -\int_B dX \cdot a = \int_B X \cdot da = d\mu^X(a), \qquad \forall a \in \Omega^1(B),$$

so we are done in proving that μ is the moment map.

The function $\|\mu\|^2 : \mathcal{A} \to \mathbb{R}$ giving the square of the L^2 norm of the curvature is the *Yang–Mills functional*, whose Euler–Lagrange equations are the *Yang–Mills equations*. Atiyah and Bott [6] studied the topology of $\mathcal{A}$ by regarding $\|\mu\|^2$ as an equivariant Morse function. In general, it is a good idea to apply Morse theory to the norm square of a moment map [80].

6.4. *Symplectic toric manifolds*

Toric manifolds are smooth *toric varieties*.[73] When studying the symplectic features of these spaces, we refer to them as *symplectic toric manifolds*. Relations between the algebraic and symplectic viewpoints on toric manifolds are discussed in [21].

DEFINITION 6.6. A *symplectic toric manifold* is a compact connected symplectic manifold (M, ω) equipped with an effective Hamiltonian action of a torus $\mathbb{T}$ of dimension equal to half the dimension of the manifold, $\dim \mathbb{T} = \frac{1}{2}\dim M$, and with a choice of a corresponding moment map μ. Two symplectic toric manifolds, $(M_i, \omega_i, \mathbb{T}_i, \mu_i)$, $i = 1, 2$, are *equivalent* if there exists an isomorphism $\lambda : \mathbb{T}_1 \to \mathbb{T}_2$ and a λ-equivariant symplectomorphism $\varphi : M_1 \to M_2$ such that $\mu_1 = \mu_2 \circ \varphi$.

EXAMPLES.

1. The circle S^1 acts on the 2-sphere $(S^2, \omega_{\text{standard}} = d\theta \wedge dh)$ by rotations, $e^{i\nu} \cdot (\theta, h) = (\theta + \nu, h)$. with moment map $\mu = h$ equal to the height function and moment polytope $[-1, 1]$ (see Figure 3).

[73] Toric varieties were introduced by Demazure in [29]. There are many nice surveys of the theory of toric varieties in algebraic geometry; see, for instance, [27,53,79,107]. Toric geometry has recently become an important tool in physics in connection with mirror symmetry [26].

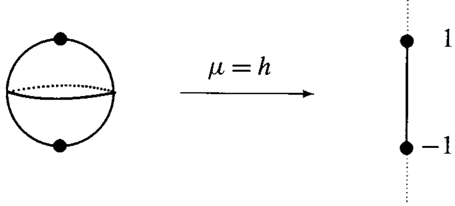

Fig. 3.

Analogously, S^1 acts on the Riemann sphere $\mathbb{CP}^1$ with the Fubini–Study form $\omega_{\text{FS}} = \frac{1}{4}\omega_{\text{standard}}$, by $e^{i\theta} \cdot [z_0, z_1] = [z_0, e^{i\theta} z_1]$. This is Hamiltonian with moment map $\mu[z_0, z_1] = -\frac{1}{2} \cdot \frac{|z_1|^2}{|z_0|^2+|z_1|^2}$, and moment polytope $[-\frac{1}{2}, 0]$.

2. For the $\mathbb{T}^n$-action on the product of n Riemann spheres $\mathbb{CP}^1 \times \cdots \times \mathbb{CP}^1$ by

$$\left(e^{i\theta_1}, \ldots, e^{i\theta_n}\right) \cdot \left([z_1, w_1], \ldots, [z_n, w_n]\right) = \left(\left[z_1, e^{i\theta_1} w_1\right], \ldots, \left[w_0, e^{i\theta_n} w_1\right]\right),$$

the moment polytope is an n-dimensional cube.

3. Let $(\mathbb{CP}^2, \omega_{\text{FS}})$ be 2-(complex-)dimensional complex projective space equipped with the Fubini–Study form defined in Section 3.4. The $\mathbb{T}^2$-action on $\mathbb{CP}^2$ by $(e^{i\theta_1}, e^{i\theta_2}) \cdot [z_0, z_1, z_2] = [z_0, e^{i\theta_1} z_1, e^{i\theta_2} z_2]$ has moment map

$$\mu[z_0, z_1, z_2] = -\frac{1}{2}\left(\frac{|z_1|^2}{|z_0|^2 + |z_1|^2 + |z_2|^2}, \frac{|z_2|^2}{|z_0|^2 + |z_1|^2 + |z_2|^2}\right).$$

The image is the isosceles triangle with vertices $(0, 0)$, $(-\frac{1}{2}, 0)$ and $(0, -\frac{1}{2})$.

4. For the $\mathbb{T}^n$-action on $(\mathbb{CP}^n, \omega_{\text{FS}})$ by

$$\left(e^{i\theta_1}, \ldots, e^{i\theta_n}\right) \cdot [z_0, z_1, \ldots, z_n] = \left[z_0, e^{i\theta_1} z_1, \ldots, e^{i\theta_n} z_n\right]$$

the moment polytope is an n-dimensional simplex.

Since the coordinates of the moment map are commuting integrals of motion, a symplectic toric manifold gives rise to a completely integrable system. By Proposition 5.24, symplectic toric manifolds are optimal Hamiltonian torus-spaces. By Theorem 5.21, they have an associated polytope. It turns out that the moment polytope contains enough information to sort all symplectic toric manifolds. We now define the class of polytopes that arise in the classification. For a symplectic toric manifold the weights $\lambda^{(1)}, \ldots, \lambda^{(n)}$ in Theorem 5.22 form a $\mathbb{Z}$-basis of $\mathbb{Z}^m$, hence the moment polytope is a *Delzant polytope*:

DEFINITION 6.7. A *Delzant polytope* in $\mathbb{R}^n$ is a polytope satisfying:

- *simplicity*, i.e., there are n edges meeting at each vertex;
- *rationality*, i.e., the edges meeting at the vertex p are rational in the sense that each edge is of the form $p + tu_i$, $t \geqslant 0$, where $u_i \in \mathbb{Z}^n$;

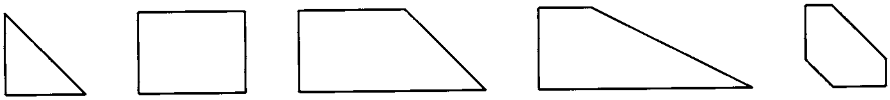

Fig. 4.

- *smoothness*, i.e., for each vertex, the corresponding $u_1, \ldots, u_n$ can be chosen to be a $\mathbb{Z}$-basis of $\mathbb{Z}^n$.

In $\mathbb{R}^2$ the simplicity condition is always satisfied (by nondegenerate polytopes). In $\mathbb{R}^3$, for instance, a square pyramid fails the simplicity condition.

EXAMPLES. Figure 4 represents Delzant polytopes in $\mathbb{R}^2$.

The following theorem classifies (equivalence classes of) symplectic toric manifolds in terms of the combinatorial data encoded by a Delzant polytope.

THEOREM 6.8 (Delzant [28]). *Toric manifolds are classified by Delzant polytopes, and their bijective correspondence is given by the moment map*:

$$\begin{aligned} &\{\textit{toric manifolds}\} \longleftrightarrow \{\textit{Delzant polytopes}\}, \\ &\left(M^{2n}, \omega, \mathbb{T}^n, \mu\right) \longmapsto \mu(M). \end{aligned}$$

Delzant's construction (Section 6.5) shows that for a toric manifold the moment map takes the fixed points bijectively to the vertices of the moment polytope and takes points with a k-dimensional stabilizer to the codimension k faces of the polytope. The moment polytope is exactly the orbit space, i.e., the preimage under μ of each point in the polytope is exactly one orbit. For instance, consider $(S^2, \omega = d\theta \wedge dh, S^1, \mu = h)$, where S^1 acts by rotation. The image of μ is the line segment $I = [-1, 1]$. The product $S^1 \times I$ is an open-ended cylinder. We can recover the 2-sphere by collapsing each end of the cylinder to a point. Similarly, we can build $\mathbb{CP}^2$ from $\mathbb{T}^2 \times \Delta$ where Δ is a rectangular isosceles triangle, and so on.

EXAMPLES.

1. By a linear transformation in $\mathrm{SL}(2; \mathbb{Z})$, we can make one of the angles in a Delzant triangle into a right angle. Out of the rectangular triangles, only the isosceles one satisfies the smoothness condition. Therefore, up to translation, change of scale and the action of $\mathrm{SL}(2; \mathbb{Z})$, there is just one 2-dimensional Delzant polytope with three vertices, namely an *isosceles triangle*. We conclude that the projective space $\mathbb{CP}^2$ is the only 4-dimensional toric manifold with three fixed points, up to choices of a constant in the moment map, of a multiple of ω_{FS} and of a lattice basis in the Lie algebra of $\mathbb{T}^2$.
2. Up to translation, change of scale and the action of $\mathrm{SL}(n; \mathbb{Z})$, the *standard n-simplex* Δ in $\mathbb{R}^n$ (spanned by the origin and the standard basis vectors $(1, 0, \ldots, 0), \ldots,$

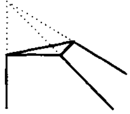

Fig. 5.

$(0, \ldots, 0, 1))$ is the only n-dimensional Delzant polytope with $n+1$ vertices. Hence, $M_\Delta = \mathbb{CP}^n$ is the only $2n$-dimensional toric manifold with $n+1$ fixed points, up to choices of a constant in the moment map, of a multiple of ω_{FS} and of a lattice basis in the Lie algebra of $\mathbb{T}^N$.

3. A transformation in $\mathrm{SL}(2;\mathbb{Z})$ makes one of the angles in a Delzant quadrilateral into a right angle. Automatically an adjacent angle also becomes $90°$. Smoothness imposes that the slope of the skew side be integral. Thus, up to translation, change of scale and $\mathrm{SL}(2;\mathbb{Z})$-action, the 2-dimensional Delzant polytopes with four vertices are trapezoids with vertices $(0,0)$, $(0,1)$, $(\ell,1)$ and $(\ell+n,0)$, for n a nonnegative integer and $\ell > 0$. Under Delzant's construction (that is, under symplectic reduction of $\mathbb{C}^4$ with respect to an action of $(S^1)^2$), these correspond to the so-called *Hirzebruch surfaces*—the only 4-dimensional symplectic toric manifolds that have four fixed points up to equivalence as before. Topologically, they are S^2-bundles over S^2, either the trivial bundle $S^2 \times S^2$ when n is even or the nontrivial bundle (given by the blow-up of $\mathbb{CP}^2$ at a point; see Section 4.3) when n is odd.

Let Δ be an n-dimensional Delzant polytope, and let $(M_\Delta, \omega_\Delta, \mathbb{T}^n, \mu_\Delta)$ be the associated symplectic toric manifold. The ε-blow-up of $(M_\Delta, \omega_\Delta)$ at a fixed point of the $\mathbb{T}^n$-action is a new symplectic toric manifold (Sections 4.3 and 5.6). Let q be a fixed point of the $\mathbb{T}^n$-action on $(M_\Delta, \omega_\Delta)$, and let $p = \mu_\Delta(q)$ be the corresponding vertex of Δ. Let $u_1, \ldots, u_n$ be the primitive (inward-pointing) edge vectors at p, so that the rays $p + tu_i$, $t \geqslant 0$, form the edges of Δ at p.

PROPOSITION 6.9. *The ε-blow-up of $(M_\Delta, \omega_\Delta)$ at a fixed point q is the symplectic toric manifold associated to the polytope Δ_ε obtained from Δ by replacing the vertex p by the n vertices $p + \varepsilon u_i$, $i = 1, \ldots, n$.*

In other words, the moment polytope for the blow-up of $(M_\Delta, \omega_\Delta)$ at q is obtained from Δ by chopping off the corner corresponding to q, thus substituting the original set of vertices by the same set with the vertex corresponding to q replaced by exactly n new vertices. The truncated polytope is Delzant. We may view the ε-blow-up of $(M_\Delta, \omega_\Delta)$ as being obtained from M_Δ by smoothly replacing q by $(\mathbb{CP}^{n-1}, \varepsilon\omega_{\mathrm{FS}})$ (whose moment polytope is an $(n-1)$-dimensional simplex). (See Figure 5.)

EXAMPLE. The moment polytope for the standard $\mathbb{T}^2$-action on $(\mathbb{CP}^2, \omega_{\mathrm{FS}})$ is a right isosceles triangle Δ. If we blow up $\mathbb{CP}^2$ at $[0:0:1]$ we obtain a symplectic toric manifold associated to the trapezoid below: a *Hirzebruch surface* (see Figure 6).

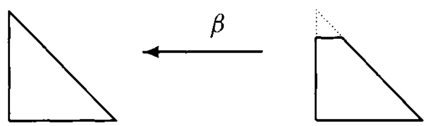

Fig. 6.

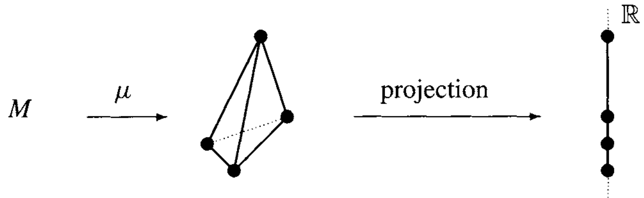

Fig. 7.

Let $(M, \omega, \mathbb{T}^n, \mu)$ be a $2n$-dimensional symplectic toric manifold. Choose a suitably generic direction in $\mathbb{R}^n$ by picking a vector X whose components are independent over $\mathbb{Q}$. This condition ensures that:

- the one-dimensional subgroup $\mathbb{T}^X$ generated by the vector X is dense in $\mathbb{T}^n$,
- X is not parallel to the facets of the moment polytope $\Delta := \mu(M)$, and
- the vertices of Δ have different projections along X.

Then the fixed points for the $\mathbb{T}^n$-action are exactly the fixed points of the action restricted to $\mathbb{T}^X$, that is, are the zeros of the vector field, $X^\#$ on M generated by X. The projection of μ along X, $\mu^X := \langle \mu, X\rangle : M \to \mathbb{R}$, is a Hamiltonian function for the vector field $X^\#$ generated by X. We conclude that the critical points of μ^X are precisely the fixed points of the $\mathbb{T}^n$-action (see Figure 7).

By Theorem 5.22, if q is a fixed point for the $\mathbb{T}^n$-action, then there exists a chart $(\mathcal{U}, x_1, \dots, x_n, y_1, \dots, y_n)$ centered at q and weights $\lambda^{(1)}, \dots, \lambda^{(n)} \in \mathbb{Z}^n$ such that

$$\mu^X|_{\mathcal{U}} = \langle \mu, X\rangle|_{\mathcal{U}} = \mu^X(q) - \frac{1}{2}\sum_{k=1}^{n} \langle \lambda^{(k)}, X\rangle \left(x_k^2 + y_k^2\right).$$

Since the components of X are independent over $\mathbb{Q}$, all coefficients $\langle \lambda^{(k)}, X\rangle$ are nonzero, so q is a *nondegenerate* critical point of μ^X. Moreover, the *index*[74] of q is twice the number of labels k such that $-\langle \lambda^{(k)}, X\rangle < 0$. But the $-\lambda^{(k)}$'s are precisely the edge vectors u_i which satisfy Delzant's conditions. Therefore, geometrically, the index of q can be read from the moment polytope Δ, by taking twice the number of edges whose inward-pointing

[74] A *Morse function* on an m-dimensional manifold M is a smooth function $f : M \to \mathbb{R}$ all of whose critical points (where df vanishes) are nondegenerate (i.e., the *Hessian matrix* is nonsingular). Let q be a nondegenerate critical point for $f : M \to \mathbb{R}$. The *index of f at q* is the index of the Hessian $H_q : \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}$ regarded as a symmetric bilinear function, that is, the maximal dimension of a subspace of $\mathbb{R}$ where H is negative definite.

edge vectors at $\mu(q)$ *point up relative to* X, that is, whose inner product with X is positive. In particular, μ^X is a *perfect Morse function*[75] and we have

PROPOSITION 6.10. *Let $X \in \mathbb{R}^n$ have components independent over $\mathbb{Q}$. The degree-2k homology group of the symplectic toric manifold $(M, \omega, \mathbb{T}, \mu)$ has dimension equal to the number of vertices of the moment polytope where there are exactly k (primitive inward-pointing) edge vectors that point up relative to the projection along the X. All odd-degree homology groups of M are zero.*

By Poincaré duality (or by taking $-X$ instead of X), the words *point up* may be replaced by *point down*. The Euler characteristic of a symplectic toric manifold is simply the number of vertices of the corresponding polytope. There is a combinatorial way of understanding the cohomology ring [53].

A *symplectic toric orbifold* is a compact connected symplectic orbifold (M, ω) equipped with an effective Hamiltonian action of a torus of dimension equal to half the dimension of the orbifold, and with a choice of a corresponding moment map. Symplectic toric orbifolds were classified by Lerman and Tolman [87] in a theorem that generalizes Delzant's: a symplectic toric orbifold is determined by its moment polytope plus a positive integer label attached to each of the polytope facets. The polytopes that occur are more general than the Delzant polytopes in the sense that only simplicity and rationality are required; the edge vectors $u_1, \ldots, u_n$ need only form a rational basis of $\mathbb{Z}^n$. When the integer labels are all equal to 1, the failure of the polytope smoothness accounts for all orbifold singularities.

6.5. *Delzant's construction*

Following [28,66], we prove the existence part (or surjectivity) in Delzant's theorem, by using symplectic reduction to associate to an n-dimensional Delzant polytope Δ a symplectic toric manifold $(M_\Delta, \omega_\Delta, \mathbb{T}^n, \mu_\Delta)$.

Let Δ be a Delzant polytope in $(\mathbb{R}^n)^*$[76] and with d facets.[77] We can algebraically describe Δ as an intersection of d halfspaces. Let $v_i \in \mathbb{Z}^n$, $i = 1, \ldots, d$, be the primitive[78] outward-pointing normal vectors to the facets of Δ. Then, for some $\lambda_i \in \mathbb{R}$, we can write $\Delta = \{x \in (\mathbb{R}^n)^* \mid \langle x, v_i\rangle \leqslant \lambda_i,\ i = 1, \ldots, d\}$.

[75] A *perfect Morse function* is a Morse function f for which the *Morse inequalities* [103,104] are equalities, i.e., $b_\lambda(M) = C_\lambda$ and $b_\lambda(M) - b_{\lambda-1}(M) + \cdots \pm b_0(M) = C_\lambda - C_{\lambda-1} + \cdots \pm C_0$ where $b_\lambda(M) = \dim H_\lambda(M)$ and C_λ be the number of critical points of f with index λ. If all critical points of a Morse function f have even index, then f is a perfect Morse function.

[76] Although we identify $\mathbb{R}^n$ with its dual via the Euclidean inner product, it may be more clear to see Δ in $(\mathbb{R}^n)^*$ for Delzant's construction.

[77] A *face* of a polytope Δ is a set of the form $F = P \cap \{x \in \mathbb{R}^n \mid f(x) = c\}$ where $c \in \mathbb{R}$ and $f \in (\mathbb{R}^n)^*$ satisfies $f(x) \geqslant c$, $\forall x \in P$. A *facet* of an n-dimensional polytope is an $(n-1)$-dimensional face.

[78] A lattice vector $v \in \mathbb{Z}^n$ is *primitive* if it cannot be written as $v = ku$ with $u \in \mathbb{Z}^n$, $k \in \mathbb{Z}$ and $|k| > 1$; for instance, $(1,1)$, $(4,3)$, $(1,0)$ are primitive, but $(2,2)$, $(3,6)$ are not.

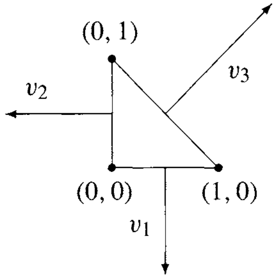

Fig. 8.

EXAMPLE. When Δ is the triangle shown in Figure 8, we have

$$\Delta = \{x \in (\mathbb{R}^2)^* \mid \langle x, (-1,0)\rangle \leqslant 0,\ \langle x, (0,-1)\rangle \leqslant 0,\ \langle x, (1,1)\rangle \leqslant 1\}.$$

For the standard basis $e_1 = (1, 0, \dots, 0), \dots, e_d = (0, \dots, 0, 1)$ of $\mathbb{R}^d$, consider

$$\begin{aligned} \pi : \mathbb{R}^d &\longrightarrow \mathbb{R}^n, \\ e_i &\longmapsto v_i. \end{aligned}$$

LEMMA 6.11. *The map π is onto and maps $\mathbb{Z}^d$ onto $\mathbb{Z}^n$.*

PROOF. We need to show that the set $\{v_1, \dots, v_d\}$ spans $\mathbb{Z}^n$. At a vertex p, the edge vectors $u_1, \dots, u_n \in (\mathbb{R}^n)^*$ form a basis for $(\mathbb{Z}^n)^*$ which, by a change of basis if necessary, we may assume is the standard basis. Then the corresponding primitive normal vectors to the facets meeting at p are $-u_1, \dots, -u_n$. $\square$

We still call π the induced surjective map $\mathbb{T}^d = \mathbb{R}^d/(2\pi\mathbb{Z}^d) \xrightarrow{\pi} \mathbb{T}^n = \mathbb{R}^n/(2\pi\mathbb{Z}^n)$. The kernel N of π is a $(d-n)$-dimensional Lie subgroup of $\mathbb{T}^d$ with inclusion $i : N \hookrightarrow \mathbb{T}^d$. Let $\mathfrak{n}$ be the Lie algebra of N. The exact sequence of tori

$$1 \longrightarrow N \xrightarrow{i} \mathbb{T}^d \xrightarrow{\pi} \mathbb{T}^n \longrightarrow 1$$

induces an exact sequence of Lie algebras

$$0 \longrightarrow \mathfrak{n} \xrightarrow{i} \mathbb{R}^d \xrightarrow{\pi} \mathbb{R}^n \longrightarrow 0$$

with dual exact sequence

$$0 \longrightarrow (\mathbb{R}^n)^* \xrightarrow{\pi^*} (\mathbb{R}^d)^* \xrightarrow{i^*} \mathfrak{n}^* \longrightarrow 0.$$

Consider $\mathbb{C}^d$ with symplectic form $\omega_0 = \frac{i}{2}\sum dz_k \wedge d\bar{z}_k$, and standard Hamiltonian action of $\mathbb{T}^d$ given by $(e^{it_1}, \dots, e^{it_d}) \cdot (z_1, \dots, z_d) = (e^{it_1}z_1, \dots, e^{it_d}z_d)$. A moment map is $\phi : \mathbb{C}^d \to (\mathbb{R}^d)^*$ defined by

$$\phi(z_1, \dots, z_d) = -\frac{1}{2}\big(|z_1|^2, \dots, |z_d|^2\big) + (\lambda_1, \dots, \lambda_d),$$

where the constant is chosen for later convenience. The subtorus N acts on $\mathbb{C}^d$ in a Hamiltonian way with moment map $i^* \circ \phi : \mathbb{C}^d \to \mathfrak{n}^*$. Let $Z = (i^* \circ \phi)^{-1}(0)$.

In order to show that Z (a closed set) is compact it suffices (by the Heine–Borel theorem) to show that Z is bounded. Let Δ' be the image of Δ by π^*. First we show that $\phi(Z) = \Delta'$. A value $y \in (\mathbb{R}^d)^*$ is in the image of Z by ϕ if and only if

(a) y is in the image of ϕ and (b) $i^* y = 0$

if and only if (using the expression for ϕ and the third exact sequence)

(a) $\langle y, e_i \rangle \leqslant \lambda_i$ for $i = 1, \dots, d$ and

(b) $y = \pi^*(x)$ for some $x \in (\mathbb{R}^n)^*$.

Suppose that $y = \pi^*(x)$. Then

$$\begin{aligned} \langle y, e_i \rangle \leqslant \lambda_i, \quad \forall i \quad &\Longleftrightarrow \quad \big\langle x, \pi(e_i) \big\rangle \leqslant \lambda_i, \quad \forall i \\ &\Longleftrightarrow \quad \langle x, v_i \rangle \leqslant \lambda_i, \quad \forall i \quad \Longleftrightarrow \quad x \in \Delta. \end{aligned}$$

Thus, $y \in \phi(Z) \Leftrightarrow y \in \pi^*(\Delta) = \Delta'$. Since Δ' is compact, ϕ is proper and $\phi(Z) = \Delta'$, we conclude that Z must be bounded, and hence compact.

In order to show that N acts freely on Z, pick a vertex p of Δ, and let $I = \{i_1, \dots, i_n\}$ be the set of indices for the n facets meeting at p. Pick $z \in Z$ such that $\phi(z) = \pi^*(p)$. Then p is characterized by n equations $\langle p, v_i \rangle = \lambda_i$ where $i \in I$:

$$\begin{aligned} \langle p, v_i \rangle = \lambda_i \quad &\Longleftrightarrow \quad \big\langle p, \pi(e_i) \big\rangle = \lambda_i \\ &\Longleftrightarrow \quad \big\langle \pi^*(p), e_i \big\rangle = \lambda_i \\ &\Longleftrightarrow \quad \big\langle \phi(z), e_i \big\rangle = \lambda_i \\ &\Longleftrightarrow \quad i\text{th coordinate of } \phi(z) \text{ is equal to } \lambda_i \\ &\Longleftrightarrow \quad -\frac{1}{2}|z_i|^2 + \lambda_i = \lambda_i \\ &\Longleftrightarrow \quad z_i = 0. \end{aligned}$$

Hence, those z's are points whose coordinates in the set I are zero, and whose other coordinates are nonzero. Without loss of generality, we may assume that $I = \{1, \dots, n\}$. The stabilizer of z is

$$\big(\mathbb{T}^d\big)_z = \big\{(t_1, \dots, t_n, 1, \dots, 1) \in \mathbb{T}^d\big\}.$$

As the restriction $\pi : (\mathbb{R}^d)_z \to \mathbb{R}^n$ maps the vectors $e_1, \ldots, e_n$ to a $\mathbb{Z}$-basis $v_1, \ldots, v_n$ of $\mathbb{Z}^n$ (respectively), at the level of groups $\pi : (\mathbb{T}^d)_z \to \mathbb{T}^n$ must be bijective. Since $N = \ker(\pi : \mathbb{T}^d \to \mathbb{T}^n)$, we conclude that $N \cap (\mathbb{T}^d)_z = \{e\}$, i.e., $N_z = \{e\}$. Hence all N-stabilizers at points mapping to vertices are trivial. But this was the worst case, since other stabilizers $N_{z'}$ ($z' \in Z$) are contained in stabilizers for points z that map to vertices. We conclude that N acts freely on Z.

We now apply reduction. Since i^* is surjective, $0 \in \mathfrak{n}^*$ is a regular value of $i^* \circ \phi$. Hence, Z is a compact submanifold of $\mathbb{C}^d$ of (real) dimension $2d - (d - n) = d + n$. The orbit space $M_\Delta = Z/N$ is a compact manifold of (real) dimension $\dim Z - \dim N = (d + n) - (d - n) = 2n$. The point-orbit map $p : Z \to M_\Delta$ is a principal N-bundle over M_Δ. Consider the diagram

$$\begin{array}{ccc} Z & \overset{j}{\hookrightarrow} & \mathbb{C}^d \\ p \downarrow & & \\ M_\Delta & & \end{array}$$

where $j : Z \hookrightarrow \mathbb{C}^d$ is inclusion. The Marsden–Weinstein–Meyer theorem (Theorem 6.1) guarantees the existence of a symplectic form ω_Δ on M_Δ satisfying

$$p^*\omega_\Delta = j^*\omega_0.$$

Since Z is connected, the symplectic manifold $(M_\Delta, \omega_\Delta)$ is also connected.

It remains to show that $(M_\Delta, \omega_\Delta)$ is a Hamiltonian $\mathbb{T}^n$-space with a moment map μ_Δ having image $\mu_\Delta(M_\Delta) = \Delta$. Let z be such that $\phi(z) = \pi^*(p)$ where p is a vertex of Δ. Let $\sigma : \mathbb{T}^n \to (\mathbb{T}^d)_z$ be the inverse for the earlier bijection $\pi : (\mathbb{T}^d)_z \to \mathbb{T}^n$. This is a *section*, i.e., a right inverse for π, in the sequence

$$\begin{array}{ccccccccc} 1 & \longrightarrow & N & \overset{i}{\longrightarrow} & \mathbb{T}^d & \overset{\pi}{\longrightarrow} & \mathbb{T}^n & \longrightarrow & 1, \\ & & & & & \underset{\sigma}{\longleftarrow} & & & \end{array}$$

so it *splits*, i.e., becomes like a sequence for a product, as we obtain an isomorphism $(i, \sigma) : N \times \mathbb{T}^n \overset{\simeq}{\longrightarrow} \mathbb{T}^d$. The action of the $\mathbb{T}^n$ factor (or, more rigorously, $\sigma(\mathbb{T}^n) \subset \mathbb{T}^d$) descends to the quotient $M_\Delta = Z/N$. Consider the diagram

$$\begin{array}{l} Z \;\; \overset{j}{\hookrightarrow} \mathbb{C}^d \overset{\phi}{\longrightarrow} \left(\mathbb{R}^d\right)^* \simeq \eta^* \oplus (\mathbb{R}^n)^* \overset{\sigma^*}{\longrightarrow} (\mathbb{R}^n)^* \\ p \downarrow \\ M_\Delta \end{array}$$

where the last horizontal map is projection onto the second factor. Since the composition of the horizontal maps is constant along N-orbits, it descends to a map

$$\mu_\Delta : M_\Delta \longrightarrow (\mathbb{R}^n)^*$$

which satisfies $\mu_\Delta \circ p = \sigma^* \circ \phi \circ j$. By reduction for product groups (Section 6.2), this is a moment map for the action of $\mathbb{T}^n$ on $(M_\Delta, \omega_\Delta)$. The image of μ_Δ is

$$\mu_\Delta(M_\Delta) = (\mu_\Delta \circ p)(Z) = (\sigma^* \circ \phi \circ j)(Z) = (\sigma^* \circ \pi^*)(\Delta) = \Delta,$$

because $\phi(Z) = \pi^*(\Delta)$ and $\pi \circ \sigma = \mathrm{id}$. We conclude that $(M_\Delta, \omega_\Delta, \mathbb{T}^n, \mu_\Delta)$ is the required toric manifold corresponding to Δ. This construction via reduction also shows that symplectic toric manifolds are in fact Kähler.

EXAMPLE. Here are the details of Delzant's construction for the case of a segment $\Delta = [0, a] \subset \mathbb{R}^*$ $(n = 1, d = 2)$. Let $v(= 1)$ be the standard basis vector in $\mathbb{R}$. Then Δ is described by $\langle x, -v\rangle \leqslant 0$ and $\langle x, v\rangle \leqslant a$, where $v_1 = -v$, $v_2 = v$, $\lambda_1 = 0$ and $\lambda_2 = a$. The projection $\mathbb{R}^2 \xrightarrow{\pi} \mathbb{R}$, $e_1 \mapsto -v$, $e_2 \mapsto v$, has kernel equal to the span of $(e_1 + e_2)$, so that N is the diagonal subgroup of $\mathbb{T}^2 = S^1 \times S^1$. The exact sequences become

$$\begin{array}{ccccccc}
1 \longrightarrow N & \xrightarrow{i} & \mathbb{T}^2 & \xrightarrow{\pi} & S^1 & \longrightarrow 1, \\
t & \longmapsto & (t, t), & & & \\
 & & (t_1, t_2) & \longmapsto & t_1^{-1} t_2, &
\end{array}$$

$$\begin{array}{ccccccc}
0 \longrightarrow \mathfrak{n} & \xrightarrow{i} & \mathbb{R}^2 & \xrightarrow{\pi} & \mathbb{R} & \longrightarrow 0, \\
x & \longmapsto & (x, x), & & & \\
 & & (x_1, x_2) & \longmapsto & x_2 - x_1, &
\end{array}$$

$$\begin{array}{ccccccc}
0 \longrightarrow \mathbb{R}^* & \xrightarrow{\pi^*} & (\mathbb{R}^2)^* & \xrightarrow{i^*} & \mathfrak{n}^* & \longrightarrow 0, \\
x & \longmapsto & (-x, x), & & & \\
 & & (x_1, x_2) & \longmapsto & x_1 + x_2. &
\end{array}$$

The action of the diagonal subgroup $N = \{(e^{it}, e^{it}) \in S^1 \times S^1\}$ on $\mathbb{C}^2$ by

$$\left(e^{it}, e^{it}\right) \cdot (z_1, z_2) = \left(e^{it} z_1, e^{it} z_2\right)$$

has moment map $(i^* \circ \phi)(z_1, z_2) = -\frac{1}{2}(|z_1|^2 + |z_2|^2) + a$, with zero-level set

$$(i^* \circ \phi)^{-1}(0) = \left\{(z_1, z_2) \in \mathbb{C}^2\colon\ |z_1|^2 + |z_2|^2 = 2a\right\}.$$

Hence, the reduced space is a projective space, $(i^* \circ \phi)^{-1}(0)/N = \mathbb{CP}^1$.

6.6. *Duistermaat–Heckman theorems*

Throughout this subsection, let (M, ω, G, μ) be a Hamiltonian G-space, where G is an n-torus[79] and the moment map μ is proper.

[79] The discussion in this subsection may be extended to Hamiltonian actions of other compact Lie groups, not necessarily tori; see [66, Exercises 2.1–2.10].

If G acts freely on $\mu^{-1}(0)$, it also acts freely on nearby levels $\mu^{-1}(t)$, $t \in \mathfrak{g}^*$ and $t \approx 0$. (Otherwise, assume only that 0 is a regular value of μ and work with orbifolds.) We study the variation of the reduced spaces by relating

$$\left(M_{\mathrm{red}} = \mu^{-1}(0)/G, \omega_{\mathrm{red}}\right) \quad \text{and} \quad \left(M_t = \mu^{-1}(t)/G, \omega_t\right).$$

For simplicity, assume G to be the circle S^1. Let $Z = \mu^{-1}(0)$ and let $i : Z \hookrightarrow M$ be the inclusion map. Fix a connection form $\alpha \in \Omega^1(Z)$ for the principal bundle

$$\begin{array}{ccc} S^1 & \hookrightarrow & Z \\ & & \downarrow \pi \\ & & M_{\mathrm{red}} \end{array}$$

that is, $\mathcal{L}_{X^\#}\alpha = 0$ and $\imath_{X^\#}\alpha = 1$, where $X^\#$ is the infinitesimal generator for the S^1-action. Construct a 2-form on the product manifold $Z \times (-\varepsilon, \varepsilon)$ by the recipe

$$\sigma = \pi^*\omega_{\mathrm{red}} - d(x\alpha),$$

where x is a linear coordinate on the interval $(-\varepsilon, \varepsilon) \subset \mathbb{R} \simeq \mathfrak{g}^*$. (By abuse of notation, we shorten the symbols for forms on $Z \times (-\varepsilon, \varepsilon)$ that arise by pullback via projection onto each factor.)

LEMMA 6.12. *The 2-form σ is symplectic for ε small enough.*

PROOF. At points where $x = 0$, the form $\sigma|_{x=0} = \pi^*\omega_{\mathrm{red}} + \alpha \wedge dx$ satisfies $\sigma|_{x=0}(X^\#, \frac{\partial}{\partial x}) = 1$, so σ is nondegenerate along $Z \times \{0\}$. Since nondegeneracy is an open condition, we conclude that σ is nondegenerate for x in a sufficiently small neighborhood of 0. Closedness is clear. $\square$

Notice that σ is invariant with respect to the S^1-action on the first factor of $Z \times (-\varepsilon, \varepsilon)$. This action is Hamiltonian with moment map $x : Z \times (-\varepsilon, \varepsilon) \to (-\varepsilon, \varepsilon)$ given by projection onto the second factor (since $\mathcal{L}_{X^\#}\alpha = 0$ and $\imath_{X^\#}\alpha = 1$):

$$\imath_{X^\#}\sigma = -\imath_{X^\#}d(x\alpha) = -\mathcal{L}_{X^\#}(x\alpha) + d\imath_{X^\#}(x\alpha) = dx.$$

LEMMA 6.13. *There exists an equivariant symplectomorphism between a neighborhood of Z in M and a neighborhood of $Z \times \{0\}$ in $Z \times (-\varepsilon, \varepsilon)$, intertwining the two moment maps, for ε small enough.*

PROOF. The inclusion $i_0 : Z \hookrightarrow Z \times (-\varepsilon, \varepsilon)$ as $Z \times \{0\}$ and the natural inclusion $i : Z \hookrightarrow M$ are S^1-equivariant coisotropic embeddings. Moreover, they satisfy $i_0^*\sigma = i^*\omega$ since both sides are equal to $\pi^*\omega_{\mathrm{red}}$, and the moment maps coincide on Z because $i_0^*x = 0 = i^*\mu$. Replacing ε by a smaller positive number if necessary, the result follows from the equivariant version of the coisotropic embedding theorem (Theorem 2.9).[80] $\square$

[80]**Equivariant coisotropic embedding theorem:** *Let (M_0, ω_0), (M_1, ω_1) be symplectic manifolds of dimension $2n$, G a compact Lie group acting on (M_i, ω_i), $i = 0, 1$, in a Hamiltonian way with moment maps μ_0*

Therefore, in order to compare the reduced spaces $M_t = \mu^{-1}(t)/S^1$ for $t \approx 0$, we can work in $Z \times (-\varepsilon, \varepsilon)$ and compare instead the reduced spaces $x^{-1}(t)/S^1$.

PROPOSITION 6.14. *The space (M_t, ω_t) is symplectomorphic to $(M_{\mathrm{red}}, \omega_{\mathrm{red}} - t\beta)$ where β is the curvature form of the connection α.*

PROOF. By Lemma 6.13, (M_t, ω_t) is symplectomorphic to the reduced space at level t for the Hamiltonian space $(Z \times (-\varepsilon, \varepsilon), \sigma, S^1, x)$. Since $x^{-1}(t) = Z \times \{t\}$, where S^1 acts on the first factor, all the manifolds $x^{-1}(t)/S^1$ are diffeomorphic to $Z/S^1 = M_{\mathrm{red}}$. As for the symplectic forms, let $\iota_t : Z \times \{t\} \hookrightarrow Z \times (-\varepsilon, \varepsilon)$ be the inclusion map. The restriction of σ to $Z \times \{t\}$ is

$$\iota_t^* \sigma = \pi^* \omega_{\mathrm{red}} - t\, d\alpha.$$

By definition of curvature, $d\alpha = \pi^* \beta$. Hence, the reduced symplectic form on $x^{-1}(t)/S^1$ is $\omega_{\mathrm{red}} - t\beta$. □

In loose terms, Proposition 6.14 says that the reduced forms ω_t vary linearly in t, for t close enough to 0. However, the identification of M_t with M_{red} as abstract manifolds is not natural. Nonetheless, any two such identifications are isotopic. By the homotopy invariance of de Rham classes, we obtain:

THEOREM 6.15 (Duistermaat–Heckman [38]). *Under the hypotheses and notation before, the cohomology class of the reduced symplectic form $[\omega_t]$ varies linearly in t. More specifically, if $c = [-\beta] \in H^2_{\mathrm{deRham}}(M_{\mathrm{red}})$ is the first Chern class*[81] *of the S^1-bundle $Z \to M_{\mathrm{red}}$, we have*

$$[\omega_t] = [\omega_{\mathrm{red}}] + tc.$$

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and μ_1, respectively, Z a manifold of dimension $k \geqslant n$ with a G-action, and $\iota_i : Z \hookrightarrow M_i$, $i = 0, 1$, G-equivariant coisotropic embeddings. Suppose that $\iota_0^* \omega_0 = \iota_1^* \omega_1$ and $\iota_0^* \mu_0 = \iota_1^* \mu_1$. Then there exist G-invariant neighborhoods $\mathcal{U}_0$ and $\mathcal{U}_1$ of $\iota_0(Z)$ and $\iota_1(Z)$ in M_0 and M_1, respectively, and a G-equivariant symplectomorphism $\varphi : \mathcal{U}_0 \to \mathcal{U}_1$ such that $\varphi \circ \iota_0 = \iota_1$ and $\mu_0 = \varphi^* \mu_1$.

[81] Often the Lie algebra of S^1 is identified with $2\pi i \mathbb{R}$ under the exponential map $\exp : \mathfrak{g} \simeq 2\pi i \mathbb{R} \to S^1$, $\xi \mapsto e^{\xi}$. Given a principal S^1-bundle, by this identification the infinitesimal action maps the generator $2\pi i$ of $2\pi i \mathbb{R}$ to the generating vector field $X^{\#}$. A connection form A is then an imaginary-valued 1-form on the total space satisfying $\mathcal{L}_{X^{\#}} A = 0$ and $\iota_{X^{\#}} A = 2\pi i$. Its curvature form B is an imaginary-valued 2-form on the base satisfying $\pi^* B = dA$. By the Chern–Weil isomorphism, the *first Chern class* of the principal S^1-bundle is $c = [\frac{i}{2\pi} B]$.

Here we identify the Lie algebra of S^1 with $\mathbb{R}$ and implicitly use the exponential map $\exp : \mathfrak{g} \simeq \mathbb{R} \to S^1$, $t \mapsto e^{2\pi i t}$. Hence, given a principal S^1-bundle, the infinitesimal action maps the generator 1 of $\mathbb{R}$ to $X^{\#}$, and here a connection form α is an ordinary 1-form on the total space satisfying $\mathcal{L}_{X^{\#}} \alpha = 0$ and $\iota_{X^{\#}} \alpha = 1$. The curvature form β is an ordinary 2-form on the base satisfying $\pi^* \beta = d\alpha$. Consequently, we have $A = 2\pi i \alpha$, $B = 2\pi i \beta$ and the first Chern class is given by $c = [-\beta]$.

DEFINITION 6.16. The *Duistermaat–Heckman measure*, m_{DH}, on $\mathfrak{g}^*$ is the push-forward of the Liouville measure[82] by $\mu: M \to \mathfrak{g}^*$, that is, for any Borel subset $\mathcal{U}$ of $\mathfrak{g}^*$, we have

$$m_{\mathrm{DH}}(\mathcal{U}) = \int_{\mu^{-1}(\mathcal{U})} \frac{\omega^n}{n!}.$$

The integral with respect to the Duistermaat–Heckman measure of a compactly-supported function $h \in C^\infty(\mathfrak{g}^*)$ is

$$\int_{\mathfrak{g}^*} h\, dm_{\mathrm{DH}} := \int_M (h \circ \mu) \frac{\omega^n}{n!}.$$

On $\mathfrak{g}^*$ regarded as a vector space, say $\mathbb{R}^n$, there is also the Lebesgue (or Euclidean) measure, m_0. The relation between m_{DH} and m_0 is governed by the *Radon–Nikodym derivative*, denoted by $\frac{dm_{\mathrm{DH}}}{dm_0}$, which is a *generalized function* satisfying

$$\int_{\mathfrak{g}^*} h\, dm_{\mathrm{DH}} = \int_{\mathfrak{g}^*} h \frac{dm_{\mathrm{DH}}}{dm_0}\, dm_0.$$

THEOREM 6.17 (Duistermaat–Heckman [38]). *Under the hypotheses and notation before, the Duistermaat–Heckman measure is a piecewise polynomial multiple of Lebesgue measure on $\mathfrak{g}^* \simeq \mathbb{R}^n$, that is, the Radon–Nikodym derivative $f = \frac{dm_{\mathrm{DH}}}{dm_0}$ is piecewise polynomial. More specifically, for any Borel subset $\mathcal{U}$ of $\mathfrak{g}^*$, we have $m_{\mathrm{DH}}(\mathcal{U}) = \int_{\mathcal{U}} f(x)\, dx$, where $dx = dm_0$ is the Lebesgue volume form on $\mathcal{U}$ and $f: \mathfrak{g}^* \simeq \mathbb{R}^n \to \mathbb{R}$ is polynomial on any region consisting of regular values of μ.*

This Radon–Nikodym derivative f is called the *Duistermaat–Heckman polynomial.* In the case of a toric manifold, the Duistermaat–Heckman polynomial is a universal constant equal to $(2\pi)^n$ when Δ is n-dimensional. Thus the symplectic volume of $(M_\Delta, \omega_\Delta)$ is $(2\pi)^n$ times the Euclidean volume of Δ.

EXAMPLE. For the standard spinning of a sphere $(S^2, \omega = d\theta \wedge dh, S^1, \mu = h)$, the image of μ is the interval $[-1, 1]$. The Lebesgue measure of $[a, b] \subseteq [-1, 1]$ is $m_0([a, b]) = b - a$. The Duistermaat–Heckman measure of $[a, b]$ is

$$m_{\mathrm{DH}}\big([a, b]\big) = \int_{\{(\theta, h) \in S^2 \mid a \leqslant h \leqslant b\}} d\theta\, dh = 2\pi(b - a),$$

[82]On an arbitrary symplectic manifold (M^{2n}, ω), with symplectic volume $\frac{\omega^n}{n!}$, the *Liouville measure* (or *symplectic measure*) of a Borel subset $\mathcal{U}$ of M is

$$m_\omega(\mathcal{U}) = \int_{\mathcal{U}} \frac{\omega^n}{n!}.$$

The set $\mathcal{B}$ of *Borel subsets* is the σ*-ring* generated by the set of compact subsets, i.e., if $A, B \in \mathcal{B}$, then $A \setminus B \in \mathcal{B}$, and if $A_i \in \mathcal{B}$, $i = 1, 2, \ldots$, then $\bigcup_{i=1}^\infty A_i \in \mathcal{B}$.

i.e., $m_{\mathrm{DH}} = 2\pi m_0$. Consequently, *the area of the spherical region between two parallel planes depends only on the distance between the planes*, a result that was known to Archimedes around 230 BC.

PROOF. We sketch the proof of Theorem 6.17 for the case $G = S^1$. The proof for the general case, which follows along similar lines, can be found in, for instance, [66], besides the original articles.

Let (M, ω, S^1, μ) be a Hamiltonian S^1-space of dimension $2n$ and let (M_x, ω_x) be its reduced space at level x. Proposition 6.14 or Theorem 6.15 imply that, for x in a sufficiently narrow neighborhood of 0, the symplectic volume of M_x,

$$\mathrm{vol}(M_x) = \int_{M_x} \frac{\omega_x^{n-1}}{(n-1)!} = \int_{M_{\mathrm{red}}} \frac{(\omega_{\mathrm{red}} - x\beta)^{n-1}}{(n-1)!},$$

is a polynomial in x of degree $n - 1$. This volume can be also expressed as

$$\mathrm{vol}(M_x) = \int_Z \frac{\pi^*(\omega_{\mathrm{red}} - x\beta)^{n-1}}{(n-1)!} \wedge \alpha,$$

where α is a connection form for the S^1-bundle $Z \to M_{\mathrm{red}}$ and β is its curvature form. Now we go back to the computation of the Duistermaat–Heckman measure. For a Borel subset $\mathcal{U}$ of $(-\varepsilon, \varepsilon)$, the Duistermaat–Heckman measure is, by definition,

$$m_{\mathrm{DH}}(\mathcal{U}) = \int_{\mu^{-1}(\mathcal{U})} \frac{\omega^n}{n!}.$$

Using the fact that $(\mu^{-1}(-\varepsilon, \varepsilon), \omega)$ is symplectomorphic to $(Z \times (-\varepsilon, \varepsilon), \sigma)$ and, moreover, they are isomorphic as Hamiltonian S^1-spaces, we obtain

$$m_{\mathrm{DH}}(\mathcal{U}) = \int_{Z \times \mathcal{U}} \frac{\sigma^n}{n!}.$$

Since $\sigma = \pi^*\omega_{\mathrm{red}} - d(x\alpha)$, its power is $\sigma^n = n(\pi^*\omega_{\mathrm{red}} - x\, d\alpha)^{n-1} \wedge \alpha \wedge dx$. By the Fubini theorem, we then have

$$m_{\mathrm{DH}}(\mathcal{U}) = \int_{\mathcal{U}} \left[\int_Z \frac{\pi^*(\omega_{\mathrm{red}} - x\beta)^{n-1}}{(n-1)!} \wedge \alpha \right] \wedge dx.$$

Therefore, the Radon–Nikodym derivative of m_{DH} with respect to the Lebesgue measure, dx, is

$$f(x) = \int_Z \frac{\pi^*(\omega_{\mathrm{red}} - x\beta)^{n-1}}{(n-1)!} \wedge \alpha = \mathrm{vol}(M_x).$$

The previous discussion proves that, for $x \approx 0$, $f(x)$ is a polynomial in x. The same holds for a neighborhood of any other regular value of μ, because we may change the moment map μ by an arbitrary additive constant. $\square$

Duistermaat and Heckman [38] also applied these results when M is compact to provide a formula for the oscillatory integral $\int_M e^{i\mu^X} \frac{\omega^n}{n!}$ for $X \in \mathfrak{g}$ as a sum of contributions of the fixed points of the action of the one-parameter subgroup generated by X. They hence showed that the *stationary phase approximation*[83] is exact in the case of the moment map. When G is a maximal torus of a compact connected simple Lie group acting on a coadjoint orbit, the Duistermaat–Heckman formula reduces to the Harish–Chandra formula. It was observed by Berline and Vergne [14] and by Atiyah and Bott [5] that the Duistermaat–Heckman formula can be derived by *localization in equivariant cohomology*. This is an instance of *Abelian localization*, i.e., a formula for an integral (in equivariant cohomology) in terms of data at the fixed points of the action, and typically is used for the case of Abelian groups (or of maximal tori). Later *non-Abelian localization* formulas were found, where integrals (in equivariant cohomology) are expressed in terms of data at the zeros of the moment map, normally used for the case of non-Abelian groups. Both localizations gave rise to computations of the cohomology ring structure of reduced spaces [80].

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