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Symplectic Origami

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An origami manifold is a manifold equipped with a closed 2-form which is symplectic except on a hypersurface, where it is like the pullback of a symplectic form by a folding map and its kernel fibrates with oriented circle fibers over a compact base. We can move back and forth between origami and symplectic manifolds using cutting (unfolding) and radial blow-up (folding), modulo compatibility conditions. We prove an origami convexity theorem for Hamiltonian torus actions, classify toric origami manifolds by polyhedral objects resembling paper origami and discuss examples. We also prove a cobordism result and compute the cohomology of a special class of origami manifolds.

1 Introduction

This is the third in a series of papers on *folded* symplectic manifolds. The first of these papers [8] contains a description of the basic local and semi-global features of these manifolds and of the *folding* and *unfolding* operations; in the second [7] it is shown that a manifold is folded symplectic if and only if it is stable complex and, in particular,

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that every oriented 4-manifold is folded symplectic. (Other recent papers on the topology of folded symplectic manifolds are [2, 3].)

In this third paper, we take up the theme of Hamiltonian group actions on folded symplectic manifolds. We focus on a special class of folded symplectic manifolds which we call *origami manifolds*. (Jean-Claude Hausmann pointed out to us that the term "origami" had once been proposed for another class of spaces: what are now known as orbifolds.) For the purposes of this introduction, let us say that a folded symplectic manifold is a triple (M, Z, ω) where M is an oriented $2n$-dimensional manifold, ω a closed 2-form and $Z \overset{i}{\hookrightarrow} M$ a hypersurface. "Folded symplectic" requires that ω be symplectic on $M \setminus Z$ and that the restriction of ω to Z be odd-symplectic, that is,

$$(i^*\omega)^{n-1} \neq 0.$$

From this one gets on Z a *null foliation* by lines and (M, Z, ω) is "origami" if this foliation is fibrating with compact connected oriented fibers. In this case one can *unfold M* by taking the closures of the connected components of $M \setminus Z$ and identifying boundary points on the same leaf of the null foliation. We will prove that this unfolding defines a cobordism between (a compact) M and a disjoint union of (compact) symplectic manifolds M_i:

$$M \sim \bigsqcup_i M_i. \tag{1}$$

Moreover, if M is a Hamiltonian G-manifold we will prove that the M_i's are as well. The *origami* results of this paper involve reconstructing the moment data of M (and in the toric case M itself) from the moment data of the M_i's.

Precise definitions of "folded symplectic" and "origami" are given in Section 2.1. In Section 2.2 we describe in detail the unfolding operation (1) and in Section 2.3 how one can refold the terms on the right to reconstruct M via a radial blow-up operation. Then in Sections 2.4 and 2.5 we prove that folding and unfolding are inverse operations: unfolding followed by folding gives one the manifold one started with and vice versa.

We turn in Section 3 to the main theme of this paper: torus actions on origami manifolds. In Section 3.1, we define for such actions an origami version of the notion of moment polytope, which turns out to be a collection of convex polytopes with compatibility conditions, or *folding instructions* on facets. We then concentrate in Section 3.2 on the toric case and prove in Section 3.3 an origami version of the Delzant theorem. More explicitly, we show that toric origami manifolds are classified by *origami templates*:

pairs $(\mathcal{P}, \mathcal{F})$, where $\mathcal{P}$ is a finite collection of oriented n-dimensional Delzant polytopes and $\mathcal{F}$ a collection of pairs of facets of these polytopes satisfying:

(a) for each pair of facets $\{F_1, F_2\} \in \mathcal{F}$ the corresponding polytopes in $\mathcal{P}$ have opposite orientations and are identical in a neighborhood of these facets;

(b) if a facet occurs in a pair, then neither itself nor any of its neighboring facets occur in any other pair;

(c) the topological space constructed from the disjoint union of all the $\Delta_i \in \mathcal{P}$ by identifying facet pairs in $\mathcal{F}$ is connected.

Without the assumption that M be origami, that is, that the null foliation be fibrating, it is *not* possible to classify Hamiltonian torus actions on folded symplectic manifolds by a finite set of combinatorial data; why not is illustrated by Example 3.11. Nonetheless, Chris Lee has shown that a (more intricate) classification of these objects by moment data *is* possible at least in dimension four [15]. We found this result of Lee very helpful in putting our own results into perspective.

Throughout this introduction, we have been assuming that our origami manifolds are oriented. However, all the definitions and results extend to the case of nonorientable origami manifolds, and that is how they will be presented in this paper. In particular, the notion of origami template explained above becomes that of Definition 3.12, which drops the orientations of the polytopes in $\mathcal{P}$ and allows for sets of single facets in $\mathcal{F}$. Moreover, as we show in Section 3, some of the most curious examples of origami manifolds (such as $\mathbb{RP}^{2n}$ and the Klein bottle) are nonorientable.

The final two sections of this paper contain results that hold only for oriented origami manifolds.

In Section 4, we prove that (1) is a cobordism and, in fact, an equivariant cobordism in the presence of group actions. We show that this cobordism is a *symplectic cobordism*, that is, there exists a closed two form on the cobording manifold the restriction of which to M is the folded symplectic form on M and on the symplectic cut pieces is the symplectic form on those manifolds. Moreover, in the presence of a (Hamiltonian) compact group action, this cobordism is a (Hamiltonian) equivariant cobordism. Using these results and keeping track of stable almost complex structures, one can give in the spirit of [11] a proof that the equivariant spin-$\mathbb{C}$ quantization of M is, as a virtual vector space (and in the presence of group actions as a virtual representation), equal to the spin-$\mathbb{C}$ quantizations of its symplectic cut pieces. However, we will not do so here. We refer the reader instead to the proof of this result in [8, Section 8], which is essentially a cobordism proof of this type.

Section 5 is devoted to the origami version of a theorem in the standard theory of Hamiltonian actions: In it we compute the cohomology groups of an oriented toric origami manifold, under the assumption that the folding hypersurface be connected.

Origami manifolds and higher codimension analogs arise naturally when converting Hamiltonian torus actions on symplectic manifolds into free actions by generalizations of *radial blow-up* along orbit-type strata. We intend to pursue this direction to obtain free Hamiltonian torus actions on compact presymplectic manifolds, complementing recent work by Karshon and Lerman on noncompact symplectic toric manifolds [14].

2 Origami Manifolds

2.1 Folded symplectic and origami forms

Definition 2.1. A *folded symplectic form* on a $2n$-dimensional manifold M is a closed 2-form ω the top power ω^n of which vanishes transversally on a submanifold Z, called the *folding hypersurface* or *fold*, and the restriction to that submanifold of which has maximal rank. The pair (M, ω) is then called a *folded symplectic manifold*. □

By transversality, the folding hypersurface Z of a folded symplectic manifold is indeed of codimension 1 and embedded. An analog of Darboux's theorem for folded symplectic forms [8, 17] says that near any point $p \in Z$ there is a coordinate chart centered at p where the form ω is

$$x_1 \, dx_1 \wedge dy_1 + dx_2 \wedge dy_2 + \cdots + dx_n \wedge dy_n.$$

Let (M, ω) be a $2n$-dimensional folded symplectic manifold. Let $i : Z \hookrightarrow M$ be the inclusion of the folding hypersurface Z. Away from Z, the form ω is nondegenerate, so $\omega^n|_{M \setminus Z} \neq 0$. The induced restriction $i^*\omega$ has a one-dimensional kernel at each point: the line field V on Z, called the *null foliation*. Note that $V = TZ \cap E \subset i^*TM$ where E is the rank 2 bundle over Z the fiber at each point of which is the kernel of ω.

Remark 2.2. When a folded symplectic manifold (M, ω) is an oriented manifold, the complement $M \setminus Z$ decomposes into open subsets M^+ where $\omega^n > 0$ and M^- where $\omega^n < 0$. This induces a coorientation on Z and hence an orientation on Z. From the form $(i^*\omega)^{n-1}$ we obtain an orientation of the quotient bundle $(i^*TM)/E$ and hence an orientation of E. From the orientations of TZ and of E, we obtain an orientation of their intersection, the null foliation V. □

We concentrate on the case of fibrating null foliation.

Definition 2.3. An *origami manifold* is a folded symplectic manifold (M, ω) the null foliation of which is fibrating with oriented circle fibers, π, over a compact *base*, B. (It would be natural to extend this definition admitting *Seifert fibrations* and *orbifold bases*.)

$$\begin{tikzcd} Z \arrow[d, "\pi"] \\ B. \end{tikzcd}$$

The form ω is called an *origami form* and the null foliation, that is, the vertical bundle of π, is called the *null fibration*. ☐

Remark 2.4. When an origami manifold is oriented we assume that any chosen orientation of the null fibration or any principal S^1-action matches the induced orientation of the null foliation V. By definition, a nonorientable origami manifold still has an orientable null foliation. ☐

Note that, on an origami manifold, the base B is naturally symplectic: as in symplectic reduction, there is a unique symplectic form ω_B on B satisfying

$$i^*\omega = \pi^*\omega_B.$$

Note also that the fold Z is necessarily compact since it is the total space of a circle fibration with a compact base. We can choose different principal S^1-actions on Z by choosing nonvanishing (positive) vertical vector fields with periods 2π.

Example 2.5. Consider the unit sphere S^{2n} in euclidean space $\mathbb{R}^{2n+1} \simeq \mathbb{C}^n \times \mathbb{R}$ with coordinates $x_1, y_1, \ldots, x_n, y_n, h$. Let ω_0 be the restriction to S^{2n} of $\mathrm{d}x_1 \wedge \mathrm{d}y_1 + \cdots + \mathrm{d}x_n \wedge \mathrm{d}y_n = r_1 \mathrm{d}r_1 \wedge \mathrm{d}\theta_1 + \cdots + r_n \mathrm{d}r_n \wedge \mathrm{d}\theta_n$. Then ω_0 is a folded symplectic form. The folding hypersurface is the equator sphere given by the intersection with the plane $h = 0$. The null foliation is the Hopf foliation since

$$\imath_{\frac{\partial}{\partial\theta_1} + \cdots + \frac{\partial}{\partial\theta_n}} \omega_0 = -r_1\,\mathrm{d}r_1 - \cdots - r_n\,\mathrm{d}r_n$$

vanishes on Z, hence a null fibration is $S^1 \hookrightarrow S^{2n-1} \twoheadrightarrow \mathbb{CP}^{n-1}$. Thus, (S^{2n}, ω_0) is an orientable origami manifold. ☐

Example 2.6. The standard folded symplectic form ω_0 on $\mathbb{RP}^{2n} = S^{2n}/\mathbb{Z}_2$ is induced by the restriction to S^{2n} of the $\mathbb{Z}_2$-invariant form $dx_1 \wedge \mathrm{d}x_2 + \cdots + \mathrm{d}x_{2n-1} \wedge \mathrm{d}x_{2n}$ in $\mathbb{R}^{2n+1}$ [8]. The folding hypersurface is $\mathbb{RP}^{2n-1} \simeq \{[x_1, \ldots, x_{2n}, 0]\}$, a null fibration is the $\mathbb{Z}_2$-quotient of the Hopf fibration $S^1 \hookrightarrow \mathbb{RP}^{2n-1} \twoheadrightarrow \mathbb{CP}^{n-1}$, and $(\mathbb{RP}^{2n}, \omega_0)$ is a nonorientable origami manifold. □

The following definition regards *symplectomorphism* in the sense of *presymplectomorphism*.

Definition 2.7. Two (oriented) origami manifolds (M, ω) and (M', ω') are *symplectomorphic* if there is a (orientation-preserving) diffeomorphism $\rho : M \to \widetilde{M}$ such that $\rho^* \widetilde{\omega} = \omega$. □

This notion of equivalence between origami manifolds stresses the importance of the null foliation being fibrating, and not a particular choice of principal circle fibration. We might sometimes identify symplectomorphic origami manifolds.

2.2 Cutting

The folding hypersurface Z plays the role of an *exceptional divisor* as it can be *blown-down* to obtain honest symplectic pieces. (Origami manifolds may hence be interpreted as *birationally symplectic manifolds*. However, in algebraic geometry the designation *birational symplectic manifolds* was used by Huybrechts [13] in a different context, that of birational equivalence for complex manifolds equipped with a holomorphic nondegenerate 2-form.) This process, called *cutting* (or *blowing-down* or *unfolding*), is essentially symplectic cutting and was described in [8, Theorem 7] in the orientable case.

Example 2.8. Cutting the origami manifold (S^{2n}, ω_0) from Example 2.5 produces $\mathbb{CP}^n$ and $\overline{\mathbb{CP}^n}$ each equipped with the same multiple of the Fubini-study form with total volume equal to that of an original hemisphere, $n!(2\pi)^n$. □

Example 2.9. Cutting the origami manifold $(\mathbb{RP}^{2n}, \omega_0)$ from Example 2.6 produces a single copy of $\mathbb{CP}^n$. □

Proposition 2.10 ([8]). Let (M^{2n}, ω) be an oriented origami manifold.

Then the unions $M^+ \sqcup B$ and $M^- \sqcup B$, each admits a structure of $2n$-dimensional symplectic manifold, denoted (M_0^+, ω_0^+) and (M_0^-, ω_0^-) respectively, with ω_0^+ and ω_0^- restricting to ω on M^+ and M^- and with a natural embedding of (B, ω_B) as a symplectic submanifold with radially projectivized normal bundle isomorphic to the null fibration $Z \xrightarrow{\pi} B$.

The orientation induced from the original orientation on M matches the symplectic orientation on M_0^+ and is opposite to the symplectic orientation on M_0^-. □

The proof relies on origami versions of Moser's trick and of Lerman's cutting. Lerman's cutting applies to a Hamiltonian circle action, defined as in the symplectic case:

Definition 2.11. The action of a Lie group G on an origami manifold (M, ω) is *Hamiltonian* if it admits a *moment map*, $\mu : M \to \mathfrak{g}^*$, satisfying the conditions:

- μ collects Hamiltonian functions, that is, $d\langle \mu, X\rangle = \imath_{X^\#}\omega$, $\forall X \in \mathfrak{g} := \mathrm{Lie}(G)$, where $X^\#$ is the vector field generated by X;
- μ is equivariant with respect to the given action of G on M and the coadjoint action of G on the dual vector space $\mathfrak{g}^*$.

(M, ω, G, μ) denotes an origami manifold equipped with a Hamiltonian action of a Lie group G having moment map μ. □

Moser's trick needs to be adapted as in [8, Theorem 1]. We start from a tubular neighborhood model defined as follows.

Definition 2.12. A *Moser model* for an oriented origami manifold (M, ω) with null fibration $Z \xrightarrow{\pi} B$ is a diffeomorphism

$$\varphi : Z \times (-\varepsilon, \varepsilon) \to \mathcal{U},$$

where $\varepsilon > 0$ and $\mathcal{U}$ is a tubular neighborhood of Z such that $\varphi(x, 0) = x$ for all $x \in Z$ and

$$\varphi^*\omega = p^* i^*\omega + d(t^2 p^*\alpha),$$

with $p : Z \times (-\varepsilon, \varepsilon) \to Z$ the projection onto the first factor, $i : Z \hookrightarrow M$ the inclusion, t the real coordinate on the interval $(-\varepsilon, \varepsilon)$ and α an S^1-connection form for a chosen principal S^1-action along the null fibration. □

A choice of a principal S^1-action along the null fibration, $S^1 \hookrightarrow Z \xrightarrow{\pi} B$, corresponds to a vector field v on Z generating the principal S^1-bundle. Following [8, Theorem 1], a Moser model can then be found after choices of a connection form α, a small enough positive real number ε and a vector field w over a tubular neighborhood of

Z such that, at each $x \in Z$, the pair (w_x, v_x) is an oriented basis of the kernel of ω_x. The orientation on this kernel is determined by the given orientation of TM and the symplectic orientation of TM modulo the kernel. Conversely, a Moser model for an oriented origami manifold gives a connection 1-form α by contracting $\varphi^*\omega$ with the vector field $\frac{1}{2t}\frac{\partial}{\partial t}$ (and hence gives a vertical vector field v such that $\imath_v \alpha = 1$ which generates an S^1-action), an ε from the width of the symmetric real interval and a vector field $w = \varphi_*(\frac{\partial}{\partial t})$.

Lemma 2.13. Any two Moser models $\varphi_i : Z \times (-\varepsilon_i, \varepsilon_i) \to \mathcal{U}_i$, with $i = 0, 1$, admit isotopic restrictions to $Z \times (-\varepsilon, \varepsilon)$ for ε small enough, that is, those restrictions can be smoothly connected by a family of Moser models. □

Proof. Let v_0 and v_1 be the vector fields generating the S^1-actions for models φ_0 and φ_1, and let w_0 and w_1 be the vector fields $w_i = (\varphi_i)_*(\frac{\partial}{\partial t})$. The vector fields $v_t = (1-t)v_0 + tv_1$ on $\mathcal{U}_0 \cap \mathcal{U}_1$ correspond to a connecting family of S^1-actions all with the same orbits, orientation and periods 2π. Connect the vector fields w_0 and w_1 on $\mathcal{U}_0 \cap \mathcal{U}_1$ by a smooth family of vector fields w_t forming oriented bases (w_t, v_t) of $\ker \omega$ over Z. Note that the v_t are all positively proportional and the set of all possible vector fields w_t is contractible. By compactness of Z, we can even take the convex combination $w_t = (1-t)w_0 + tw_1$, for ε small enough. Pick a smooth family of connections α_t: for instance, using a metric pick 1-forms β_t such that $\beta_t(v_t) = 1$ and then average each β_t by the S^1-action generated by v_t. For the claimed isotopy, use a corresponding family of Moser models $\varphi_t : Z \times (-\varepsilon, \varepsilon) \to \mathcal{U}_t$ with ε sufficiently small so that integral curves of all w_t starting at points of Z are defined for $t \in (-\varepsilon, \varepsilon)$. ■

With these preliminaries out of the way, we recall and expand the proof from [8] for Proposition 2.10.

Proof. Choose a principal S^1-action along the null fibration, $S^1 \hookrightarrow Z \xrightarrow{\pi} B$. Let $\varphi : Z \times (-\varepsilon, \varepsilon) \to \mathcal{U}$ be a Moser model, and let $\mathcal{U}^+$ denote $M^+ \cap \mathcal{U} = \varphi(Z \times (0, \varepsilon))$. The diffeomorphism

$$\psi : Z \times (0, \varepsilon^2) \to \mathcal{U}^+, \quad \psi(x, s) = \varphi(x, \sqrt{s})$$

induces a symplectic form

$$\psi^*\omega = p^* i^* \omega + d(sp^*\alpha) =: \nu$$

on $Z \times (0, \varepsilon^2)$ that extends by the same formula to $Z \times (-\varepsilon^2, \varepsilon^2)$.

As in standard symplectic cutting [16], form the product $(Z \times (-\varepsilon^2, \varepsilon^2), \nu) \times (\mathbb{C}, -\omega_0)$ where $\omega_0 = \frac{i}{2}\, dz \wedge d\bar{z}$. The product action of S^1 on $Z \times (-\varepsilon^2, \varepsilon^2) \times \mathbb{C}$ by

$$\mathrm{e}^{\mathrm{i}\theta} \cdot (x, s, z) = (\mathrm{e}^{\mathrm{i}\theta} \cdot x, s, \mathrm{e}^{-\mathrm{i}\theta} z)$$

is Hamiltonian and $\mu(x, s, z) = s - \frac{|z|^2}{2}$ is a moment map. Zero is a regular value of μ and the corresponding level is a codimension-1 submanifold which decomposes into

$$\mu^{-1}(0) = Z \times \{0\} \times \{0\} \sqcup \{(x, s, z) \mid s > 0, |z|^2 = 2s\}.$$

Since S^1 acts freely on $\mu^{-1}(0)$, the quotient $\mu^{-1}(0)/S^1$ is a manifold and the point-orbit map is a principal S^1-bundle. Moreover, we can view it as

$$\mu^{-1}(0)/S^1 \simeq B \sqcup \mathcal{U}^+.$$

Indeed, B embeds as a codimension-2 submanifold via

$$\begin{aligned} j : B &\longrightarrow \mu^{-1}(0)/S^1, \\ \pi(x) &\longmapsto [x, 0, 0] \quad \text{for } x \in Z \end{aligned}$$

and $\mathcal{U}^+$ embeds as an open dense submanifold via

$$\begin{aligned} j^+ : \mathcal{U}^+ &\longrightarrow \mu^{-1}(0)/S^1 \\ \psi(x, s) &\longmapsto [x, s, \sqrt{2s}]. \end{aligned}$$

The symplectic form Ω_{red} on $\mu^{-1}(0)/S^1$ obtained by symplectic reduction is such that the above embeddings of (B, ω_B) and of $(\mathcal{U}^+, \omega|_{\mathcal{U}^+})$ are symplectic.

The normal bundle to $j(B)$ in $\mu^{-1}(0)/S^1$ is the quotient over S^1-orbits (upstairs and downstairs) of the normal bundle to $Z \times \{0\} \times \{0\}$ in $\mu^{-1}(0)$. This latter bundle is the product bundle $Z \times \{0\} \times \{0\} \times \mathbb{C}$ where the S^1-action is

$$\mathrm{e}^{\mathrm{i}\theta} \cdot (x, 0, 0, z) = (\mathrm{e}^{\mathrm{i}\theta} \cdot x, 0, 0, \mathrm{e}^{-\mathrm{i}\theta} z).$$

Performing $\mathbb{R}^+$-projectivization and taking the S^1-quotient we get the bundle $Z \to B$ with the isomorphism

$$\begin{tikzcd}
(Z \times \{0\} \times \{0\} \times \mathbb{C}^*)/S^1 \ni [x, 0, 0, r\,\mathrm{e}^{\mathrm{i}\theta}] \arrow[r, mapsto] \arrow[d] & \mathrm{e}^{\mathrm{i}\theta} x \in Z \arrow[d] \\
(Z \times \{0\} \times \{0\})/S^1 \ni [x, 0, 0] \arrow[r, mapsto] & \pi(x) \in B.
\end{tikzcd}$$

By gluing the rest of M^+ along $\mathcal{U}^+$, we produce a $2n$-dimensional symplectic manifold (M_0^+, ω_0^+) with a symplectomorphism $\overline{j^+} : M^+ \to M_0^+ \setminus j(B)$ extending j^+.

For the other side, the map $\psi_- : Z \times (0, \varepsilon^2) \to \mathcal{U}^- := M^- \cap \mathcal{U}$, $(x, s) \mapsto \varphi(x, -\sqrt{s})$ reverses orientation and $(\psi_-)^*\omega = \nu$. The base B embeds as a symplectic submanifold of $\mu^{-1}(0)/S^1$ by the previous formula. The embedding

$$\begin{aligned} j^- : \mathcal{U}^- &\longrightarrow \mu^{-1}(0)/S^1, \\ \psi_-(x, s) &\longmapsto [x, s, -\sqrt{2s}] \end{aligned}$$

is an orientation-reversing symplectomorphism. By gluing the rest of M^- along $\mathcal{U}^-$, we produce (M_0^-, ω_0^-) with a symplectomorphism $\overline{j^-} : M^- \to M_0^- \setminus j(B)$ extending j^-. ■

Remark 2.14. The cutting construction in the previous proof produces a symplectomorphism γ between tubular neighborhoods $\mu^{-1}(0)/S^1$ of the embeddings of B in M_0^+ and M_0^-, comprising $\mathcal{U}^+ \to \mathcal{U}^-$, $\varphi(x, t) \mapsto \varphi(x, -t)$, and the identity map on B:

$$\begin{aligned} \gamma : \mu^{-1}(0)/S^1 &\longrightarrow \mu^{-1}(0)/S^1, \\ [x, s, \sqrt{2s}] &\longmapsto [x, s, -\sqrt{2s}]. \end{aligned}$$

□

Definition 2.15. Symplectic manifolds (M_0^+, ω_0^+) and (M_0^-, ω_0^-) obtained by cutting are called *symplectic cut pieces* of the oriented origami manifold (M, ω) and the embedded copies of B are called *centers*. □

The next proposition states that symplectic cut pieces of an origami manifold are unique up to symplectomorphism.

Proposition 2.16. Different choices of a Moser model for a tubular neighborhood of the fold in an origami manifold yield symplectomorphic symplectic cut pieces. □

Proof. Let φ_0 and φ_1 be two Moser models for a tubular neighborhood $\mathcal{U}$ of the fold Z in an origami manifold (M, ω). Let $(M_0^\pm, \omega_0^\pm)$ and $(M_1^\pm, \omega_1^\pm)$ be the corresponding symplectic manifolds obtained by the above cutting. Let $\varphi_t : Z \times (-\varepsilon, \varepsilon) \to \mathcal{U}_t$ be an isotopy between (restrictions of) φ_0 and φ_1. By suitably rescaling t, we may assume that φ_t is a technical isotopy in the sense of [6, p.89], that is, $\varphi_t = \varphi_0$ for t near 0 and $\varphi_t = \varphi_1$ for t near 1. Let

$$j_t^\pm : \mathcal{U}_t^\pm \longrightarrow \mu^{-1}(0)/S^1$$

be the corresponding isotopies of symplectic embeddings, where $\mu^{-1}(0)/S^1$ is equipped with $(\Omega_{\mathrm{red}})_t$, and let $(M_t^\pm, \omega_t^\pm)$ be the corresponding families of symplectic manifolds obtained from gluing: for instance, (M_t^+, ω_t^+) is the quotient of the disjoint union

$$M^+ \sqcup \mu^{-1}(0)/S^1$$

by the equivalence relation which sets each point in $\mathcal{U}^+$ equivalent to its image by the symplectomorphism j_t^+.

Let $\mathcal{U}_c := \overline{\mathcal{U}_{\mathrm{out}}} \setminus \mathcal{U}_{\mathrm{in}}$ be a compact subset of each $\mathcal{U}_t$ where $\mathcal{U}_{\mathrm{in}}$ and $\mathcal{U}_{\mathrm{out}}$ are tubular neighborhoods of Z with $\overline{\mathcal{U}_{\mathrm{in}}} \subset \mathcal{U}_{\mathrm{out}}$. Let $\mathcal{U}_*^+ = \mathcal{U}_* \cap M^+$ where $*$ stands for the subscripts c, *out* or *in*. Let C be a compact neighborhood of $j^+([0,1] \times \mathcal{U}_c^+) := \cup_{t\in[0,1]} j_t^+(\mathcal{U}_c^+)$ in $\mu^{-1}(0)/S^1$, such that $B \cap C = \emptyset$.

By Theorem 10.9 in [6], there is a smooth family of diffeomorphisms

$$H_t : \mu^{-1}(0)/S^1 \longrightarrow \mu^{-1}(0)/S^1, \quad t \in [0,1],$$

which hold fixed all points outside C (in particular, the H_t fix a neighborhood of B), with H_0 the identity map and such that

$$j_t^+|_{\mathcal{U}_c^+} = H_t \circ j_0^+|_{\mathcal{U}_c^+}.$$

The diffeomorphism H_t restricted to $D := \overline{j_0^+(\mathcal{U}_{\mathrm{out}}^+)}$ and the identity diffeomorphism on $M^+ \setminus \mathcal{U}_{\mathrm{in}}^+$ together define a diffeomorphism

$$\phi_t : M_0^+ \longrightarrow M_t^+.$$

All forms in the family $\phi_t^* \omega_t^+$ on M_0^+ are symplectic, have the same restriction to B, and are equal to ω_0^+ away from the set D which retracts to B. Hence, all $\phi_t^* \omega_t^+$ are in the same cohomology class and, moreover,

$$\frac{\mathrm{d}}{\mathrm{d}t} \phi_t^* \omega_t^+ = \mathrm{d}\beta_t$$

for some smooth family of 1-forms β_t supported in the compact set D [18, p. 95].

By solving Moser's equation

$$\imath_{w_t} \omega_t^+ + \beta_t = 0$$

we find a time-dependent vector field w_t compactly supported on D. The isotopy $\rho_t : M_0^+ \to M_0^+$, $t \in \mathbb{R}$, corresponding to this vector field satisfies $\rho_t \equiv \mathrm{id}$ away from D and

$$\rho_t^*(\phi_t^*\omega_t^+) = \omega_0^+ \quad \text{for all } t.$$

The map $\phi_1 \circ \rho_1$ is a symplectomorphism between (M_0^+, ω_0^+) and (M_1^+, ω_1^+). Similarly for (M_0^-, ω_0^-) and (M_1^-, ω_1^-). ■

Cutting may be performed for any *nonorientable* origami manifold (M, ω) by working with its orientable double cover. The double cover involution yields a symplectomorphism from one symplectic cut piece to the other. Hence, we regard these pieces as a trivial double cover (of one of them) and call their $\mathbb{Z}_2$-quotient the *symplectic cut space* of (M, ω). In the case where $M \setminus Z$ is connected, the symplectic cut space is also connected; see Example 2.9.

Definition 2.17. The *symplectic cut space* of an origami manifold (M, ω) is the natural $\mathbb{Z}_2$-quotient of symplectic cut pieces of its orientable double cover. □

Note that, when the original origami manifold is compact, the symplectic cut space is also compact.

2.3 Radial blow-up

We can reverse the cutting procedure using an origami (and simpler) analog of Gompf's gluing construction [10]. *Radial blow-up* is a local operation on a symplectic tubular neighborhood of a codimension-2 symplectic submanifold modeled by the following example.

Example 2.18. Consider the standard symplectic $(\mathbb{R}^{2n}, \omega_0)$ with its standard euclidean metric. Let B be the symplectic submanifold defined by $x_1 = y_1 = 0$ with unit normal bundle N identified with the hypersurface $x_1^2 + y_1^2 = 1$. The map $\beta : N \times \mathbb{R} \to \mathbb{R}^{2n}$ defined by

$$\beta((p, \mathrm{e}^{\mathrm{i}\theta}), r) = p + (r\cos\theta, r\sin\theta, 0, \ldots, 0) \quad \text{for } p \in B$$

induces by pullback an origami form on the cylinder $N \times \mathbb{R} \simeq S^1 \times \mathbb{R}^{2n-1}$, namely

$$\beta^*\omega_0 = r\,\mathrm{d}r \wedge \mathrm{d}\theta + \mathrm{d}x_2 \wedge \mathrm{d}y_2 + \cdots + \mathrm{d}x_n \wedge \mathrm{d}y_n.$$ □

Let (M, ω) be a symplectic manifold with a codimension-2 symplectic submanifold B. Let $i: B \hookrightarrow M$ be the inclusion map. Consider the radially projectivized normal bundle over B

$$\mathcal{N} := \mathbb{P}^{+}(i^{*}TM/TB) = \{x \in (i^{*}TM)/TB,\, x \neq 0\}/\sim$$

where $\lambda x \sim x$ for $\lambda \in \mathbb{R}^{+}$. We choose an S^1 action making $\mathcal{N}$ a principal circle bundle over B. Let $\varepsilon > 0$.

Definition 2.19. A *blow-up model* for a tubular neighborhood $\mathcal{U}$ of B in (M, ω) is a map

$$\beta : \mathcal{N} \times (-\varepsilon, \varepsilon) \longrightarrow \mathcal{U}$$

which factors as

$$\beta : \mathcal{N} \times (-\varepsilon, \varepsilon) \xrightarrow{\beta_0} \mathcal{N} \times_{S^1} \mathbb{C} \xrightarrow{\eta} \mathcal{U}$$

$$(x, t) \longmapsto [x, t]$$

where $e^{i\theta} \cdot (x, t) = (e^{i\theta} \cdot x,\, t\, e^{-i\theta})$ for $(x, t) \in \mathcal{N} \times \mathbb{C}$ and $\eta : \beta_0(\mathcal{N} \times (-\varepsilon, \varepsilon)) \to \mathcal{U}$ is a tubular bundle diffeomorphism. By *tubular bundle diffeomorphism* we mean a bundle diffeomorphism covering the identity $B \to B$ and isotopic to a diffeomorphism given by a geodesic flow for some choice of metric on $\mathcal{U}$. □

In practice, a blow-up model may be obtained by choosing a Riemannian metric to identify $\mathcal{N}$ with the unit bundle inside the geometric normal bundle $TB^{\perp}$, and then by using the exponential map: $\beta(x, t) = \exp_p(tx)$ where p is the projection onto B of $x \in \mathcal{N}$.

Remark 2.20. From the properties of β_0, it follows that:

(i) the restriction of β to $\mathcal{N} \times (0, \varepsilon)$ is an orientation-preserving diffeomorphism onto $\mathcal{U} \setminus B$;
(ii) $\beta(-x, -t) = \beta(x, t)$;
(iii) the restriction of β to $\mathcal{N} \times \{0\}$ is the bundle projection $\mathcal{N} \to B$;
(iv) for the vector fields ν generating the vertical bundle of $\mathcal{N} \to B$ and $\frac{\partial}{\partial t}$ tangent to $(-\varepsilon, \varepsilon)$ we have that $D\beta(\nu)$ intersects zero transversally and $D\beta(\frac{\partial}{\partial t})$ is never zero. □

Lemma 2.21. If $\beta : \mathcal{N} \times (-\varepsilon, \varepsilon) \to \mathcal{U}$ is a blow-up model for the neighborhood $\mathcal{U}$ of B in (M, ω), then the pull-back form $\beta^*\omega$ is an origami form the null foliation of which is the circle fibration $\pi : \mathcal{N} \times \{0\} \to B$. □

Proof. By properties (i) and (ii) in Remark 2.20, the form $\beta^*\omega$ is symplectic away from $\mathcal{N} \times \{0\}$. By property (iii), on $\mathcal{N} \times \{0\}$ the kernel of $\beta^*\omega$ has dimension 2 and is fibrating. By property (iv) the top power of $\beta^*\omega$ intersects zero transversally. ■

All blow-up models share the same germ up to diffeomorphism. More precisely, if $\beta_1 : \mathcal{N} \times (-\varepsilon, \varepsilon) \to \mathcal{U}_1$ and $\beta_2 : \mathcal{N} \times (-\varepsilon, \varepsilon) \to \mathcal{U}_2$ are two blow-up models for neighborhoods $\mathcal{U}_1$ and $\mathcal{U}_2$ of B in (M, ω), then there are possibly narrower tubular neighborhoods of B, $\mathcal{V}_i \subseteq \mathcal{U}_i$ and a diffeomorphism $\gamma : \mathcal{V}_1 \to \mathcal{V}_2$ such that $\beta_2 = \gamma \circ \beta_1$. Moreover we have the following.

Lemma 2.22. Any two blow-up models $\beta_i : \mathcal{N} \times (-\varepsilon_i, \varepsilon_i) \to \mathcal{U}_i$, $i = 1, 2$, are isotopic, that is, can be smoothly connected by a family of blow-up models. □

Proof. By definition, the blow-up models factor as

$$\beta_i = \eta_i \circ \beta_0, \quad i = 1, 2,$$

for some tubular neighborhood diffeomorphisms, η_1 and η_2, which are isotopic since the set of Riemannian metrics on $\mathcal{U}$ is convex and different geodesic flows are isotopic. ■

Let (M, ω) be a symplectic manifold with a codimension-2 symplectic submanifold B.

Definition 2.23. A *model involution* of a tubular neighborhood $\mathcal{U}$ of B is a symplectic involution $\gamma : \mathcal{U} \to \mathcal{U}$ preserving B such that on the connected components $\mathcal{U}_i$ of $\mathcal{U}$ where $\gamma(\mathcal{U}_i) = \mathcal{U}_i$ we have $\gamma|_{\mathcal{U}_i} = \mathrm{id}_{\mathcal{U}_i}$. □

A model involution γ induces a bundle involution $\Gamma : \mathcal{N} \to \mathcal{N}$ covering $\gamma|_B$ by the formula

$$\Gamma[v] = [d\gamma_p(v)] \quad \text{for } v \in T_pM, \ p \in B.$$

This is well-defined because $\gamma(B) = B$. We denote by $-\Gamma : \mathcal{N} \to \mathcal{N}$ the involution $[v] \mapsto [-d\gamma_p(v)]$.

Remark 2.24. When B is the disjoint union of B_1 and B_2, and correspondingly $\mathcal{U} = \mathcal{U}_1 \sqcup \mathcal{U}_2$, if $\gamma(B_1) = B_2$ then

$$\gamma_1 := \gamma|_{\mathcal{U}_1} : \mathcal{U}_1 \to \mathcal{U}_2 \quad \text{and} \quad \gamma|_{\mathcal{U}_2} = \gamma_1^{-1} : \mathcal{U}_2 \to \mathcal{U}_1.$$

In this case, $B/\gamma \simeq B_1$ and $\mathcal{N}/-\Gamma \simeq \mathcal{N}_1$ is the radially projectivized normal bundle to B_1. □

Proposition 2.25. Let (M, ω) be a (compact) symplectic manifold, B a compact codimension-2 symplectic submanifold and $\mathcal{N}$ its radially projectivized normal bundle. Let $\gamma : \mathcal{U} \to \mathcal{U}$ be a model involution of a tubular neighborhood $\mathcal{U}$ of B and $\Gamma : \mathcal{N} \to \mathcal{N}$ the induced bundle map.

Then there is a (compact) origami manifold $(\widetilde{M}, \widetilde{\omega})$ with symplectic part $\widetilde{M} \setminus Z$ symplectomorphic to $M \setminus B$, folding hypersurface diffeomorphic to $\mathcal{N}/-\Gamma$ and null fibration isomorphic to $\mathcal{N}/-\Gamma \to B/\gamma$. □

Proof. Choose $\beta : \mathcal{N} \times (-\varepsilon, \varepsilon) \to \mathcal{U}$ a blow-up model for the neighborhood $\mathcal{U}$ such that $\gamma \circ \beta = \beta \circ \Gamma$. This is always possible: For components $\mathcal{U}_i$ of $\mathcal{U}$ where $\gamma(\mathcal{U}_i) = \mathcal{U}_i$ this condition is trivial; for disjoint neighborhood components $\mathcal{U}_i$ and $\mathcal{U}_j$ such that $\gamma(\mathcal{U}_i) = \mathcal{U}_j$ (as in Remark 2.24), this condition amounts to choosing any blow-up model on one of these components and transporting it to the other by γ.

Then $\beta^*\omega$ is a folded symplectic form on $\mathcal{N} \times (-\varepsilon, \varepsilon)$ with folding hypersurface $\mathcal{N} \times \{0\}$ and null foliation integrating to the circle fibration $S^1 \hookrightarrow \mathcal{N} \xrightarrow{\pi} B$. We define

$$\widetilde{M} = (M \setminus B \sqcup \mathcal{N} \times (-\varepsilon, \varepsilon))/\sim$$

where we quotient by

$$(x, t) \sim \beta(x, t) \text{ for } t > 0 \quad \text{and} \quad (x, t) \sim (-\Gamma(x), -t).$$

The forms ω on $M \setminus B$ and $\beta^*\omega$ on $\mathcal{N} \times (-\varepsilon, \varepsilon)$ induce on $\widetilde{M}$ an origami form $\widetilde{\omega}$ with folding hypersurface $\mathcal{N}/-\Gamma$. Indeed β is a symplectomorphism for $t > 0$, and $(-\Gamma, -\mathrm{id})$ on $\mathcal{N} \times (-\varepsilon, \varepsilon)$ is a symplectomorphism away from $t = 0$ (since β and γ are) and at points where $t = 0$ it is a local diffeomorphism. ■

Definition 2.26. An origami manifold $(\widetilde{M}, \widetilde{\omega})$ as just constructed is called a *radial blow-up of* (M, ω) *through* (γ, B). □

The next proposition states that radial blow-ups of (M, ω) through (γ, B) are unique up to symplectomorphism.

Proposition 2.27. Let (M, ω) be a symplectic manifold, B a compact codimension-2 symplectic submanifold and $\gamma : \mathcal{U} \to \mathcal{U}$ a model involution of a tubular neighborhood $\mathcal{U}$ of B. Then different choices of a blow-up model for $\mathcal{U}$ yield symplectomorphic radial blow-ups through (γ, B). □

Proof. Let β_0 and β_1 be two blow-up models for $\mathcal{U}$ such that $\gamma \circ \beta_i = \beta_i \circ \Gamma$. We restrict them to the same domain with ε small enough and still denote

$$\beta_i : \mathcal{N} \times (-\varepsilon, \varepsilon) \longrightarrow \mathcal{U}_i, \quad i = 0, 1.$$

Let $\beta_t : \mathcal{N} \times (-\varepsilon, \varepsilon) \to \mathcal{U}_t$ be a technical isotopy between β_0 and β_1 in the sense of [6, p. 89]. Each of the corresponding origami manifolds $(\widetilde{M}_t, \widetilde{\omega}_t)$ is defined as a quotient of the disjoint union of $(M \setminus B, \omega)$ with $(\mathcal{N} \times (-\varepsilon, \varepsilon), \beta_t^* \omega)$ by the equivalence relation $\sim_t$ given by β_t and Γ as in the proof of Proposition 2.25. Let C be a compact neighborhood of $\beta([0, 1] \times \mathcal{N} \times [\delta, \varepsilon - \delta]) := \cup_{t \in [0,1]} \beta_t(\mathcal{N} \times [\delta, \varepsilon - \delta])$ in $M \setminus B$, where $\delta < \frac{\varepsilon}{2}$.

By [6, Theorem 10.9], there is a smooth family of diffeomorphisms

$$H_t : M \longrightarrow M, \quad t \in [0, 1],$$

which hold fixed all points outside C (in particular, the H_t fix B), with H_0 the identity map and such that

$$\beta_t|_{\mathcal{N} \times [\delta, \varepsilon - \delta]} = H_t \circ \beta_0|_{\mathcal{N} \times [\delta, \varepsilon - \delta]}.$$

By property (ii) in Remark 2.20, the same holds on $\mathcal{N} \times [-\varepsilon + \delta, -\delta]$.

The diffeomorphism H_t restricted to $D := M \setminus \beta_0(\mathcal{N} \times (-\delta, \delta))$ and the identity diffeomorphism on $\mathcal{N} \times (-\varepsilon + \delta, \varepsilon - \delta)$ together define a diffeomorphism

$$\phi_t : \widetilde{M}_0 \longrightarrow \widetilde{M}_t$$

which fixes the fold $\mathcal{N} / - \Gamma$.

All forms in the family $\phi_t^* \widetilde{\omega}_t$ on $\widetilde{M}_0$ are origami with the same fold, are equal at points of that fold, and are all equal to $\widetilde{\omega}_0$ away from

$$C \cup \beta_0(\mathcal{N} \times [-\delta, \delta])$$

which is a compact neighborhood of the fold in $\widetilde{M}_0$ retracting to the fold. Hence, by a folded version of Moser's trick (see the proof of [8, Theorem 1]) there is an isotopy

$\rho_t : \widetilde{M}_0 \to \widetilde{M}_0$, $t \in \mathbb{R}$, fixing the fold such that

$$\rho_t^*(\phi_t^* \widetilde{\omega}_t) = \widetilde{\omega_0} \qquad \text{for all } t.$$

The map $\phi_1 \circ \rho_1$ is a symplectomorphism between $(\widetilde{M}_0, \widetilde{\omega}_0)$ and $(\widetilde{M}_1, \widetilde{\omega}_1)$. ■

Different model involutions $\gamma_i : \mathcal{U}_i \to \mathcal{U}_i$ of tubular neighborhoods of B give rise to symplectomorphic origami manifolds as long as the induced bundle maps $\Gamma_i : \mathcal{N} \to \mathcal{N}$ are isotopic. Examples 2.29 and 2.30 illustrate the dependence on the model involution.

Example 2.28. Let M be a 2-sphere, B one point on it, and γ the identity map on a neighborhood of that point. Then a radial blow-up $\widetilde{M}$ is $\mathbb{RP}^2$ and $\widetilde{\omega}$ a form which folds along a circle. □

Example 2.29. Let M be a 2-sphere, B the union of two (distinct) points on it, and γ the identity map on a neighborhood of those points. Then a radial blow-up $\widetilde{M}$ is a Klein bottle and $\widetilde{\omega}$ a form which folds along two circles. □

Example 2.30. Again, let M be a 2-sphere, B the union of two (distinct) points on it, and now γ defined by a symplectomorphism from a Darboux neighborhood of one point to a Darboux neighborhood of the other. Then a radial blow-up $\widetilde{M}$ is a Klein bottle and $\widetilde{\omega}$ a form which folds along a circle. □

Remark 2.31. The quotient $\mathcal{N} \times (-\varepsilon, \varepsilon)/(-\Gamma, -\mathrm{id})$ provides a collar neighborhood of the fold in $(\widetilde{M}, \widetilde{\omega})$.

When B splits into two disjoint components interchanged by γ as in Remark 2.24, this collar is orientable so the fold is coorientable. Example 2.30 illustrates a case where, even though the fold is coorientable, the radial blow-up $(\widetilde{M}, \widetilde{\omega})$ is not orientable.

When γ is the identity map, as in Example 2.28, the collar is nonorientable and the fold is not coorientable. In the latter case, the collar is a bundle of Möbius bands $S^1 \times (-\varepsilon, \varepsilon)/(-\mathrm{id}, -\mathrm{id})$ over B.

In general, γ will be the identity over some connected components of B and will interchange other components, so some components of the fold will be coorientable and others will not. □

Remark 2.32. For the radial blow-up $(\widetilde{M}, \widetilde{\omega})$ to be orientable, the starting manifold (M, ω) must be the disjoint union of symplectic manifolds (M_1, ω_1) and (M_2, ω_2) with $B = B_1 \cup B_2$, $B_i \subset M_i$, such that $\gamma(B_1) = B_2$ as in Remark 2.24. In this case $(\widetilde{M}, \widetilde{\omega})$ may be equipped with an orientation such that

$$\widetilde{M}^+ \simeq M_1 \setminus B_1 \quad \text{and} \quad \widetilde{M}^- \simeq M_2 \setminus B_2.$$

The folding hypersurface is diffeomorphic to $\mathcal{N}_1$ (or $\mathcal{N}_2$) and we have

$$\widetilde{\omega} \simeq \begin{cases} \omega_1 & \text{on } M_1 \setminus B_1, \\ \omega_2 & \text{on } M_2 \setminus B_2, \\ \beta^*\omega_1 & \text{on } \mathcal{N} \times (-\varepsilon, \varepsilon). \end{cases}$$

We then say that $(\widetilde{M}, \widetilde{\omega})$ *is the blow-up of* (M_1, ω_1) *and* (M_2, ω_2) *through* (γ_1, B_1) where γ_1 is the restriction of γ to a tubular neighborhood of B_1. □

Remark 2.33. Radial blow-up may be performed on an origami manifold at a symplectic submanifold B (away from the fold). When we start with two folded surfaces and radially blow them up at one point (away from the folding curves), topologically the resulting manifold is a connected sum at a point, $M_1\#\overline{M_2}$, with all the previous folding curves plus a new closed curve. Since all $\mathbb{RP}^{2n}$ are folded symplectic manifolds, the standard real blow-up of a folded symplectic manifold at a point away from its folding hypersurface still admits a folded symplectic form, obtained by viewing this operation as a connected sum. □

2.4 Cutting a radial blow-up

Proposition 2.34. Let (M, ω) be a radial blow-up of the symplectic manifolds (M_1, ω_1) and (M_2, ω_2) through (γ_1, B_1) where γ_1 is a symplectomorphism of tubular neighborhoods of codimension-2 symplectic submanifolds B_1 and B_2 of M_1 and M_2, respectively, taking B_1 to B_2.

Then cutting (M, ω) yields manifolds symplectomorphic to (M_1, ω_1) and (M_2, ω_2) where the symplectomorphisms carry B to B_1 and B_2. □

Proof. We first exhibit a symplectomorphism ρ_1 between a cut space (M_0^+, ω_0^+) of (M, ω) and the original manifold (M_1, ω_1).

Let $\mathcal{N}$ be the radially projectivized normal bundle to B_1 in M_1 and let $\beta : \mathcal{N} \times (-\varepsilon, \varepsilon) \to \mathcal{U}_1$ be a blow-up model. The cut space M_0^+ is obtained gluing the reduced space

$$\mu^{-1}(0)/S^1 = \left\{(x, s, z) \in Z \times [0, \varepsilon^2) \times \mathbb{C} | s = \tfrac{|z|^2}{2}\right\} / S^1$$

with the manifold

$$M_1 \setminus B_1$$

via the diffeomorphisms

$$\begin{aligned} \mathcal{N} \times (0,\varepsilon) &\longrightarrow \mu^{-1}(0)/S^1 \quad \text{and} \quad \mathcal{N} \times (0,\varepsilon) \longrightarrow \mathcal{U}_1 \setminus B_1, \\ (x,t) &\longmapsto [x, t^2, t\sqrt{2}], \quad (x,t) \longmapsto \beta(x,t) \end{aligned}$$

that is, the gluing is by the identification $[x,t^2,t\sqrt{2}] \sim \beta(x,t)$ for $t>0$ over $\mathcal{U}_1 \setminus B_1$. The symplectic form ω_0^+ on M_0^+ is equal to the reduced symplectic form on $\mu^{-1}(0)/S^1$ and equal to ω_1 on $M_1 \setminus B_1$ (the gluing diffeomorphism $[x,t^2,t\sqrt{2}] \mapsto \beta(x,t)$ is a symplectomorphism).

We want to define a map $\rho_1 : M_1 \to M_0^+$ which is the identity on $M_1 \setminus B_1$ and on $\mathcal{U}_1$ is the composed diffeomorphism

$$\begin{aligned} \delta_1 : \mathcal{U}_1 \longrightarrow (\mathcal{N} \times \mathbb{C})/S^1 &\longrightarrow \mu^{-1}(0)/S^1, \\ [x,z] &\longmapsto [x, |z|^2, z\sqrt{2}], \end{aligned}$$

where the first arrow is the inverse of the bundle isomorphism given by the blow-up model. In order to show that ρ_1 is well defined we need to verify that $u_1 \in \mathcal{U}_1 \setminus B_1$ is equivalent to its image $\delta_1(u_1) \in \mu^{-1}(0)/S^1 \setminus B$. Indeed u_1 must correspond to $[x,z] \in (\mathcal{N} \times \mathbb{C})/S^1$ with $z \neq 0$. We write z as $z = t\,\mathrm{e}^{\mathrm{i}\theta}$ with $t>0$. Since $[x,z] = [\mathrm{e}^{\mathrm{i}\theta}x, t]$, we have $u_1 = \beta(\mathrm{e}^{\mathrm{i}\theta}x, t)$ and $\delta_1(u_1) = [\mathrm{e}^{\mathrm{i}\theta}x, |t|^2, t\sqrt{2}]$. These two are equivalent under $\beta(x,t) \sim [x,t^2,t\sqrt{2}]$, so ρ_1 is well defined.

Furthermore, M_1 and M_0^+ are symplectic manifolds equipped with a diffeomorphism which is a symplectomorphism on the common dense subset $M_1 \setminus B_1$. We conclude that M_1 and M_0^+ must be globally symplectomorphic.

Now we tackle (M_2, ω_2) and (M_0^-, ω_0^-). The cut space M_0^- is obtained gluing the same reduced space $\mu^{-1}(0)/S^1$ with the manifold $M_2 \setminus B_2$ via the diffeomorphisms

$$\begin{aligned} \mathcal{N} \times (-\varepsilon, 0) &\longrightarrow \overline{\mu^{-1}(0)/S^1}, \\ (x,t) &\longmapsto [x, t^2, t\sqrt{2}] \end{aligned}$$

and

$$\begin{aligned} \mathcal{N} \times (-\varepsilon, 0) &\longrightarrow \overline{\mathcal{U}_1 \setminus B_1} \xrightarrow{\gamma} \overline{\mathcal{U}_2 \setminus B_2}, \\ (x,t) &\longmapsto \beta(x,t) \longmapsto \gamma(\beta(x,t)), \end{aligned}$$

that is, the gluing is by the identification $[x, t^2, t\sqrt{2}] \sim \gamma(\beta(x, t))$ for $t < 0$ over $\mathcal{U}_2 \setminus B_2$. The symplectic form ω_0^- on M_0^- restricts to the reduced form on $\mu^{-1}(0)/S^1$ and ω_2 on $M_2 \setminus B_2$.

We want to define a map $\rho_2 : M_2 \to M_0^-$ as being the identity on $M_2 \setminus B_2$ and on $\mathcal{U}_2$ being the composed diffeomorphism

$$\delta_2 : \mathcal{U}_2 \xrightarrow{\gamma^{-1}} \mathcal{U}_1 \longrightarrow (\mathcal{N} \times \mathbb{C})/S^1 \longrightarrow \mu^{-1}(0)/S^1,$$

$$[x, z] \longmapsto [x, |z|^2, z\sqrt{2}],$$

where the second arrow is the inverse of the bundle isomorphism given by the blow-up model. In order to show that ρ_2 is well defined we need to verify that $u_2 = \gamma(u_1) \in \mathcal{U}_2 \setminus B_2$ is equivalent to its image $\delta_2(u_2) \in \mu^{-1}(0)/S^1 \setminus B$. Indeed u_1 must correspond to $[x, z] \in (\mathcal{N} \times \mathbb{C})/S^1$ with $z \neq 0$. We write z as $z = -t\,\mathrm{e}^{\mathrm{i}\theta}$ with $t < 0$. From $[x, z] = [-\mathrm{e}^{\mathrm{i}\theta} x, t]$, we conclude that $u_2 = \gamma(\beta(-\mathrm{e}^{\mathrm{i}\theta} x, t))$ and $\delta_2(u_2) = [-\mathrm{e}^{\mathrm{i}\theta} x, |t|^2, t\sqrt{2}]$. These two are equivalent under $\gamma(\beta(x, t)) \sim [x, t^2, t\sqrt{2}]$, so ρ_2 is well defined.

As before, we conclude that M_2 and M_0^- must be globally symplectomorphic. ■

Lemma 2.35. Let (M, ω) be a radial blow-up of the symplectic manifold (M_s, ω_s) through (γ, B). We write $B = B_0 \sqcup B_1 \sqcup B_2$ and the domain of γ as $\mathcal{U} = \mathcal{U}_0 \sqcup \mathcal{U}_1 \sqcup \mathcal{U}_2$ where γ is the identity map on $\mathcal{U}_0$ and exchanges $\mathcal{U}_1$ and $\mathcal{U}_2$.

Let $(\overline{M_s}, \overline{\omega_s})$ be the trivial double cover of (M_s, ω_s) with $\bar{B} = B^\uparrow \sqcup B^\downarrow$, $\bar{\mathcal{U}} = \mathcal{U}^\uparrow \sqcup \mathcal{U}^\downarrow$ the double covers of B and $\mathcal{U}$. Let $\bar{\gamma} : \bar{\mathcal{U}} \to \bar{\mathcal{U}}$ be the lift of γ satisfying

$$\bar{\gamma}(\mathcal{U}_0^\uparrow) = \mathcal{U}_0^\downarrow, \quad \bar{\gamma}(\mathcal{U}_1^\uparrow) = \mathcal{U}_2^\downarrow \quad \text{and} \quad \bar{\gamma}(\mathcal{U}_2^\uparrow) = \mathcal{U}_1^\downarrow$$

and let $(\bar{M}, \bar{\omega})$ be a radial blow-up of $(\overline{M_s}, \overline{\omega_s})$ through $(\bar{\gamma}, \bar{B})$.

Then $(\bar{M}, \bar{\omega})$ is an orientable double cover of (M, ω). □

Proof. Since it is the double cover of an oriented manifold, we write

$$\overline{M_s} = M_s^\uparrow \sqcup M_s^\downarrow$$

with each component diffeomorphic to M_s. By Remark 2.32, the blow-up $(\bar{M}, \bar{\omega})$ is orientable and has

$$\bar{M}^+ \simeq M_s^\uparrow \setminus B^\uparrow \quad \text{and} \quad \bar{M}^- \simeq M_s^\downarrow \setminus B^\downarrow$$

with fold $\mathcal{N}^\uparrow$ fibering over $B^\uparrow$. There is a natural two-to-one smooth projection $\bar{M} \to M$ taking $M_s^\uparrow \setminus B^\uparrow$ and $M_s^\downarrow \setminus B^\downarrow$ each diffeomorphically to $M \setminus Z$ where Z is the fold of (M, ω), and taking the fold $\mathcal{N}^{\uparrow} \simeq \mathcal{N}$ of $(\bar{M}, \bar{\omega})$ to $Z \simeq \mathcal{N}/-\Gamma$ with $\Gamma : \mathcal{N} \to \mathcal{N}$ the bundle map induced by γ (the map $-\Gamma$ having no fixed points). ■

Corollary 2.36. Let (M, ω) be a radial blow-up of the symplectic manifold (M_s, ω_s) through (γ, B). Then cutting (M, ω) yields a manifold symplectomorphic to (M_s, ω_s) where the symplectomorphism carries the base to B. □

Proof. Let $(M_{\text{cut}}, \omega_{\text{cut}})$ be a symplectic cut space of (M, ω). Let $(\overline{M_s}, \overline{\omega_s})$ and $(\overline{M_{\text{cut}}}, \overline{\omega_{\text{cut}}})$ be the trivial double covers of (M_s, ω_s) and $(M_{\text{cut}}, \omega_{\text{cut}})$. By Lemma 2.35, the radial blow-up $(\bar{M}, \bar{\omega})$ of $(\overline{M_s}, \overline{\omega_s})$ through $(\bar{\gamma}, \bar{B})$ is an orientable double cover of (M, ω). As a consequence of Definition 2.17, $(\overline{M_{\text{cut}}}, \overline{\omega_{\text{cut}}})$ is the symplectic cut space of $(\bar{M}, \bar{\omega})$. By Proposition 2.34, $(\overline{M_s}, \overline{\omega_s})$ and $(\overline{M_{\text{cut}}}, \overline{\omega_{\text{cut}}})$ are symplectomorphic relative to the centers. It follows that (M_s, ω_s) and $(M_{\text{cut}}, \omega_{\text{cut}})$ are symplectomorphic relative to the centers. ■

2.5 Radially blowing-up cut pieces

Proposition 2.37. Let (M, ω) be an oriented origami manifold with null fibration $Z \xrightarrow{\pi} B$.

Let (M_1, ω_1) and (M_2, ω_2) be its symplectic cut pieces, B_1 and B_2 the natural symplectic embedded images of B in each and $\gamma_1 : \mathcal{U}_1 \to \mathcal{U}_2$ the symplectomorphism of tubular neighborhoods of B_1 and B_2 as in Remark 2.14.

Let $(\widetilde{M}, \widetilde{\omega})$ be a radial blow-up of (M_1, ω_1) and (M_2, ω_2) through (γ_1, B_1).

Then (M, ω) and $(\widetilde{M}, \widetilde{\omega})$ are equivalent origami manifolds. □

Proof. Choose a principal S^1 action $S^1 \hookrightarrow Z \xrightarrow{\pi} B$ and let $\varphi : Z \times (-\varepsilon, \varepsilon) \to \mathcal{U}$ be a *Moser model* for a tubular neighborhood $\mathcal{U}$ of Z in M as in the proof of Proposition 2.10. Let $\mathcal{N}$ be the radial projectivized normal bundle to B_1 in M_1. By Proposition 2.10, the natural embedding of B in M_1 with image B_1 lifts to a bundle isomorphism from $\mathcal{N} \to B_1$ to $Z \to B$. Under this isomorphism, we pick the following blow-up model for the neighborhood $\mu^{-1}(0)/S^1$ of B_1 in (M_1, ω_1):

$$\beta : Z \times (-\varepsilon, \varepsilon) \longrightarrow \mu^{-1}(0)/S^1,$$

$$(x, t) \longmapsto [x, t^2, t\sqrt{2}].$$

By recalling the construction of the reduced form ω_1 on $\mu^{-1}(0)/S^1$ (see proof of Proposition 2.10) we find that $\beta^*\omega_1 = \varphi^*\omega$. Hence in this case the origami manifold $(\widetilde{M}, \widetilde{\omega})$ has

$$\widetilde{M} = M_1 \setminus B_1 \cup \overline{M_2 \setminus B_2} \cup Z \times (-\varepsilon, \varepsilon))/\sim$$

where we quotient identifying

$$Z \times (0, \varepsilon) \overset{\beta}{\simeq} \mu^{-1}(0)/S^1 \subset \mathcal{U}_1 \setminus B_1$$

and

$$Z \times (-\varepsilon, 0) \overset{\beta}{\simeq} \overline{\mu^{-1}(0)/S^1} \overset{\gamma}{\simeq} \overline{\mu^{-1}(0)/S^1} \subset \overline{\mathcal{U}_2 \setminus B_2}$$

and we have

$$\widetilde{\omega} := \begin{cases} \omega_1 & \text{on } M_1 \setminus B_1, \\ \omega_2 & \text{on } M_2 \setminus B_2, \\ \beta^*\omega_1 & \text{on } Z \times (-\varepsilon, \varepsilon). \end{cases}$$

The symplectomorphisms (from the proof of Proposition 2.10) $\overline{j^+} : M^+ \to M_1 \setminus B_1$ and $\overline{j^-} : M^- \to M_2 \setminus B_2$ extending $\varphi(x, t) \mapsto [x, t^2, t\sqrt{2}]$ make the following diagrams (one for $t > 0$, the other for $t < 0$) commute:

$$\begin{tikzcd} M^+ \supset \mathcal{U}^+ \arrow[rr, "j^+"] & & \mu^{-1}(0)/S^1 \subset M_1 \setminus B_1 \\ & Z \times (0, \varepsilon) \arrow[ul, "\varphi"] \arrow[ur, "\beta"'] & \end{tikzcd}$$

and

$$\begin{tikzcd} M^- \supset \mathcal{U}^- \arrow[rr, "j^-"] & & \overline{\mu^{-1}(0)/S^1} \subset \overline{M_2 \setminus B_2} \\ & Z \times (-\varepsilon, 0) \arrow[ul, "\varphi"] \arrow[ur, "\beta"'] & \end{tikzcd}$$

Therefore, the maps $\overline{j^+}$, $\overline{j^-}$, and φ^{-1} together define a diffeomorphism from M to $\widetilde{M}$ pulling back $\widetilde{\omega}$ to ω. ■

Corollary 2.38. Let (M, ω) be an origami manifold with null fibration $Z \xrightarrow{\pi} B$.

Let $(M_{\text{cut}}, \omega_{\text{cut}})$ be its symplectic cut space, B_{cut} the natural symplectic embedded image of B in M_{cut}, and $\gamma : \mathcal{U} \to \mathcal{U}$ a symplectomorphism of a tubular neighborhood $\mathcal{U}$ of B_{cut} as in Remark 2.14.

Let $(\widetilde{M}, \widetilde{\omega})$ be a radial blow-up of $(M_{\text{cut}}, \omega_{\text{cut}})$ through (γ, B_{cut}).

Then (M, ω) and $(\widetilde{M}, \widetilde{\omega})$ are symplectomorphic origami manifolds. □

Proof. We pass to the orientable double covers. By Proposition 2.37, the orientable double cover of (M, ω) is symplectomorphic to the blow-up of its cut space. By definition, the cut space of the double cover of (M, ω) is the double cover of $(M_{\mathrm{cut}}, \omega_{\mathrm{cut}})$. By Lemma 2.35, the blow-up of the latter double cover is the double cover of $(\widetilde{M}, \widetilde{\omega})$. ■

3 Origami Polytopes

3.1 Origami convexity

Definition 3.1. If F_i is a face of a polytope Δ_i, $i=1,2$, in $\mathbb{R}^n$, we say that Δ_1 near F_1 *agrees* with Δ_2 near F_2 when $F_1=F_2$ and there is an open subset $\mathcal{U}$ of $\mathbb{R}^n$ containing F_1 such that $\mathcal{U} \cap \Delta_1 = \mathcal{U} \cap \Delta_2$. □

The following is an origami analog of the Atiyah–Guillemin–Sternberg convexity theorem.

Theorem 3.2. Let (M, ω, G, μ) be a connected compact origami manifold with null fibration $Z \overset{\pi}{\to} B$ and a Hamiltonian action of an m-dimensional torus G with moment map $\mu : M \to \mathfrak{g}^*$. Then:

(a) The image $\mu(M)$ of the moment map is the union of a finite number of convex polytopes Δ_i, $i=1,\dots,N$, each of which is the image of the moment map restricted to the closure of a connected component of $M \setminus Z$.

(b) Over each connected component Z' of Z, the null fibration is given by a subgroup of G if and only if $\mu(Z')$ is a facet of each of the one or two polytopes corresponding to the neighboring component(s) of $M \setminus Z$, and when those are two polytopes, they agree near the facet $\mu(Z')$. □

We call such images $\mu(M)$ *origami polytopes*.

Remark 3.3. When M is oriented, the facets from part (b) are always shared by *two* polytopes. In general, a component Z' is coorientable if and only if $\mu(Z')$ is a facet of two polytopes. □

Proof. (a) Since the G-action preserves ω, it also preserves each connected component of the folding hypersurface Z and its null foliation V. Choose an oriented trivializing section w of V. Average w so that it is G-invariant, that is, replace it with

$$\frac{1}{|G|}\int_G g_*(w_{g^{-1}(p)})\,\mathrm{d}g.$$

Next, scale it uniformly over each orbit so that its integral curves all have period 2π, producing a vector field v which generates an action of S^1 on Z that commutes with the G-action. This S^1-action also preserves the moment map μ: for any $X \in \mathfrak{g}$ with corresponding vector field $X^{\#}$ on M, we have over Z

$$\mathcal{L}_v \langle \mu, X \rangle = \imath_v d\langle \mu, X \rangle = -\imath_v \imath_{X^{\#}} \omega = \omega(v, X^{\#}) = 0.$$

Using this v, the cutting construction from Section 2 has a Hamiltonian version. Let (M_i, ω_i), $i = 1, \dots, N$, be the resulting compact connected components of the symplectic cut space. Let B_i be the union of the components of the base B which naturally embed in M_i. Each $M_i \setminus B_i$ is symplectomorphic to a connected component $\mathcal{W}_i \subset M \setminus Z$ and M_i is the closure of $M_i \setminus B_i$. Each (M_i, ω_i) inherits a Hamiltonian action of G with moment map μ_i which matches $\mu|_{\mathcal{W}_i}$ over $M_i \setminus B_i$ and is the well defined S^1-quotient of $\mu|_Z$ over B_i.

By the Atiyah–Guillemin–Sternberg convexity theorem [1, 13], each $\mu_i(M_i)$ is a convex polytope Δ_i. Since $\mu(M)$ is the union of the $\mu_i(M_i)$, we conclude that

$$\mu(M) = \bigcup_{i=1}^{N} \Delta_i.$$

(b) Assume first that M is orientable.

Let Z' be a connected component of Z with null fibration $Z' \to B'$. Let $\mathcal{W}_1$ and $\mathcal{W}_2$ be the two neighboring components of $M \setminus Z$ on each side of Z', $(M_1, \omega_1, G, \mu_1)$ and $(M_2, \omega_2, G, \mu_2)$ the corresponding cut spaces with moment polytopes Δ_1 and Δ_2.

Let $\mathcal{U}$ be a G-invariant tubular neighborhood of Z' with a G-equivariant diffeomorphism $\varphi : Z' \times (-\varepsilon, \varepsilon) \to \mathcal{U}$ such that

$$\varphi^* \omega = p^* i^* \omega + d(t^2 p^* \alpha),$$

where G acts trivially on $(-\varepsilon, \varepsilon)$, $p : Z' \times (-\varepsilon, \varepsilon) \to Z'$ is the projection onto the first factor, t is the real coordinate on the interval $(-\varepsilon, \varepsilon)$ and α is a G-invariant S^1-connection on Z' for a chosen principal S^1 action, $S^1 \hookrightarrow Z \overset{\pi}{\to} B$. The existence of such φ follows from an equivariant Moser trick, analogous to that in the proof of Proposition 2.10.

Without loss of generality, $Z' \times (0, \varepsilon)$ and $Z' \times (-\varepsilon, 0)$ correspond via φ to the two sides $\mathcal{U}_1 := \mathcal{U} \cap \mathcal{W}_1$ and $\mathcal{U}_2 := \mathcal{U} \cap \mathcal{W}_2$, respectively. The involution

$\tau : \mathcal{U} \to \mathcal{U}$ translating $t \mapsto -t$ in $Z' \times (-\varepsilon, \varepsilon)$ is a G-equivariant (orientation-reversing) diffeomorphism preserving Z', switching $\mathcal{U}_1$ and $\mathcal{U}_2$ but preserving ω. Hence the moment map satisfies $\mu \circ \tau = \mu$ and $\mu(\mathcal{U}_1) = \mu(\mathcal{U}_2)$.

When the null fibration is given by a subgroup of G, we cut the G-space $\mathcal{U}$ at the level Z'. The image $\mu(Z')$ is the intersection of $\mu(\mathcal{U})$ with a hyperplane and thus a facet of both Δ_1 and Δ_2.

Each $\mathcal{U}_i \cup B'$ is equivariantly symplectomorphic to a neighborhood $\mathcal{V}_i$ of B' in $(M_i, \omega_i, G, \mu_i)$ with $\mu_i(\mathcal{V}_i) = \mu(\mathcal{U}_i) \cup \mu(Z')$, $i = 1, 2$. As a map to its image, the moment map is open [4]. Since $\mu_1(\mathcal{V}_1) = \mu_2(\mathcal{V}_2)$, we conclude that Δ_1 and Δ_2 agree near the facet $\mu(Z')$.

For a general null fibration, we cut the $G \times S^1$-space $\mathcal{U}$ with moment map (μ, t^2) at Z', the S^1-level $t^2 = 0$. The image of Z' by the $G \times S^1$-moment map is the intersection of the image of the full $\mathcal{U}$ with a hyperplane. We conclude that the image $\mu(Z')$ is the first factor projection $\pi : \mathfrak{g}^* \times \mathbb{R} \to \mathfrak{g}^*$ of a facet of a polytope $\widetilde{\Delta}$ in $\mathfrak{g}^* \times \mathbb{R}$, so it can be of codimension 0 or one; see Example 3.5.

If $\pi|_{\widetilde{\Delta}} : \widetilde{\Delta} \to \Delta_1$ is one-to-one, then facets of $\widetilde{\Delta}$ map to facets of Δ_1 and $\widetilde{\Delta}$ is contained in a hyperplane surjecting onto $\mathfrak{g}^*$. The normal to that hyperplane corresponds to a circle subgroup of $G \times S^1$ acting trivially on $\mathcal{U}$ and surjecting onto the S^1-factor. This allows us to express the S^1-action in terms of a subgroup of G.

If $\pi|_{\widetilde{\Delta}} : \widetilde{\Delta} \to \Delta_1$ is not one-to-one, it cannot map the facet $\widetilde{F}_{Z'}$ of $\widetilde{\Delta}$ corresponding to Z' to a facet of Δ_1: otherwise, $\widetilde{F}_{Z'}$ would contain nontrivial *vertical* vectors $(0, x) \in \mathfrak{g}^* \times \mathbb{R}$ which would contradict the fact that the S^1 direction is that of the null fibration on Z'. Hence, the normal to $\widetilde{F}_{Z'}$ in $\widetilde{\Delta}$ must be transverse to $\mathfrak{g}^*$, and the corresponding null fibration circle subgroup is not a subgroup of G.

When M is not necessarily orientable, we consider its orientable double cover and lift the Hamiltonian torus action. The lifted moment map is the composition of the two-to-one projection with the original double map, and the result follows. ■

Example 3.4. Consider $(S^4, \omega_0, \mathbb{T}^2, \mu)$ where (S^4, ω_0) is a sphere as in Example 2.5 with $\mathbb{T}^2$ acting by

$$(e^{i\theta_1}, e^{i\theta_2}) \cdot \underbrace{(z_1, z_2, h)}_{\in \mathbb{C}^2 \times \mathbb{R} \simeq \mathbb{R}^5} = (e^{i\theta_1} z_1, e^{i\theta_2} z_2, h)$$

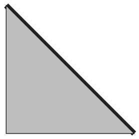

Fig. 1. An origami polytope for a 4-sphere.

and the moment map defined by

$$\mu(z_1, z_2, h) = \left(\underbrace{\frac{|z_1|^2}{2}}_{x_1}, \underbrace{\frac{|z_2|^2}{2}}_{x_2} \right)$$

the image of which is the triangle $x_1 \geq 0$, $x_2 \geq 0$, $x_1 + x_2 \leq \frac{1}{2}$. The image $\mu(Z)$ of the folding hypersurface (the equator) is the hypotenuse.

The null foliation is the Hopf fibration given by the diagonal circle subgroup of $\mathbb{T}^2$. In this case, Theorem 3.2 says that the triangle is the union of two identical triangles, each of which is the moment polytope of one of the $\mathbb{CP}^2$'s obtained by cutting; see Example 2.8. Likewise, if $(S^4, \omega_0, \mathbb{T}^2, \mu)$ was blown-up at a pole, the triangle in Figure 1 would be the superposition of the same triangle with a trapezoid. ☐

Example 3.5. Consider $(S^2 \times S^2, \omega_s \oplus \omega_f, S^1, \mu)$, where (S^2, ω_s) is a standard symplectic sphere, (S^2, ω_f) is a folded symplectic sphere with folding hypersurface given by a parallel, and S^1 acts as the diagonal of the standard rotation action of $S^1 \times S^1$ on the product manifold. Then the moment map image is a line segment and the image of the folding hypersurface is a nontrivial subsegment. Indeed, the image of μ is a 45° projection of the image of the moment map for the full $S^1 \times S^1$ action, that is, a rectangle in which the folding hypersurface surjects to one of the sides; see Figure 2.

By considering the first or second factors of $S^1 \times S^1$ alone, we get the two extreme cases in which the image of the folding hypersurface is either the full line segment or simply one of the boundary points.

The analogous six-dimensional examples $(S^2 \times S^2 \times S^2, \omega_s \oplus \omega_s \oplus \omega_f, \mathbb{T}^2, \mu)$ produce moment images which are rational projections of a cube, with the folding hypersurface mapped to rhombi; see Figure 3. ☐

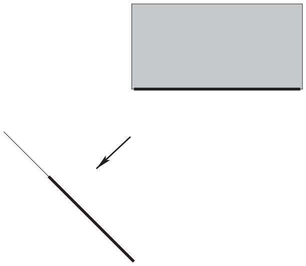

Fig. 2. Origami polytopes for a product of two 2-spheres.

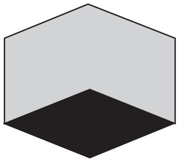

Fig. 3. An origami polytope for a product of three 2-spheres.

3.2 Toric case

Definition 3.6. A *toric origami manifold* (M, ω, G, μ) is a compact connected origami manifold (M, ω) equipped with an effective Hamiltonian action of a torus G with $\dim G = \frac{1}{2} \dim M$ and with a choice of a corresponding moment map μ. □

For a toric origami manifold (M, ω, G, μ), orbits with trivial isotropy—the *principal orbits*—form a dense open subset of M [5, p.179]. Any *coorientable* connected component Z' of Z has a G-invariant tubular neighborhood modeled on $Z' \times (-\varepsilon, \varepsilon)$ with a $G \times S^1$ Hamiltonian action having moment map (μ, t^2). As the orbits are isotropic submanifolds, the principal orbits of the $G \times S^1$-action must still have dimension $\dim G$. Their stabilizer must be a one-dimensional compact connected subgroup surjecting onto S^1. Hence, over those connected components of Z, the null fibration is given by a subgroup of G. A similar argument holds for *noncoorientable* connected components of Z, using orientable double covers. We have thus proved the following corollary of Theorem 3.2.

Corollary 3.7. When (M, ω, G, μ) is a toric origami manifold, the moment map image of each connected component Z' of Z is a facet of each of the one or two polytopes corresponding to the neighboring component(s) of $M \setminus Z$, and when those are two polytopes, they agree near the facet $\mu(Z')$. □

Delzant spaces, also known as *symplectic toric manifolds*, are closed symplectic $2n$-dimensional manifolds equipped with an effective Hamiltonian action of an n-dimensional torus and with a corresponding moment map. Delzant's theorem [9] says that the image of the moment map (a polytope in $\mathbb{R}^n$) determines the Delzant space (up to an equivariant symplectomorphism intertwining the moment maps). The *Delzant conditions* on polytopes are conditions characterizing exactly those polytopes that occur as moment polytopes for Delzant spaces. A polytope in $\mathbb{R}^n$ is *Delzant* if:

- there are n edges meeting at each vertex;
- each edge meeting at vertex p is of the form $p + tu_i$, $t \geq 0$, where $u_i \in \mathbb{Z}^n$;
- for each vertex, the corresponding $u_1, \ldots, u_n$ can be chosen to be a $\mathbb{Z}$-basis of $\mathbb{Z}^n$.

Corollary 3.7 says that for a toric origami manifold (M, ω, G, μ) the image $\mu(M)$ is the superimposition of Delzant polytopes with certain compatibility conditions. Section 3.3 will show how all such (compatible) superimpositions occur and, in fact, classify toric origami manifolds.

For a Delzant space, G-equivariant symplectic neighborhoods of connected components of the orbit-type strata are simple to infer just by looking at the polytope.

Lemma 3.8. Let $G = \mathbb{T}^n$ be an n-dimensional torus and $(M_i^{2n}, \omega_i, \mu_i)$, $i = 1, 2$, two symplectic toric manifolds. If the moment polytopes $\Delta_i := \mu_i(M_i)$ agree near facets $F_1 \subset \mu_1(M_1)$ and $F_2 \subset \mu_2(M_2)$, then there are G-invariant neighborhoods $\mathcal{U}_i$ of $B_i = \mu_i^{-1}(F_i)$, $i = 1, 2$, with a G-equivariant symplectomorphism $\gamma : \mathcal{U}_1 \to \mathcal{U}_2$ extending a symplectomorphism $B_1 \to B_2$ and such that $\gamma^* \mu_2 = \mu_1$. □

Proof. Let $\mathcal{U}$ be an open set containing $F_1 = F_2$ such that $\mathcal{U} \cap \Delta_1 = \mathcal{U} \cap \Delta_2$.

Perform symplectic cutting [16] on M_1 and M_2 by slicing Δ_i along a hyperplane parallel to F_i such that:

- the moment polytope $\widetilde{\Delta}_i$ containing F_i is in the open set $\mathcal{U}$;
- the hyperplane is close enough to F_i to guarantee that $\widetilde{\Delta}_i$ satisfies the Delzant conditions. (For generic rational hyperplanes, the third of the Delzant conditions fails inasmuch as we only get a $\mathbb{Q}^n$-basis, thus we need to consider orbifolds.)

Then $\widetilde{\Delta}_1 = \widetilde{\Delta}_2$. By Delzant's theorem, the corresponding cut spaces $\widetilde{M}_1$ and $\widetilde{M}_2$ are G-equivariantly symplectomorphic, the symplectomorphism pulling back one moment

Fig. 4. Polytopes agreeing near the left vertical edges.

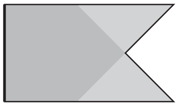

Fig. 5. Origami polytope for a radial blow-up of two Hirzebruch surfaces.

map to the other. Restricting the previous symplectomorphism gives us a G-equivariant symplectomorphism between G-equivariant neighborhoods $\mathcal{U}_i$ of B_i in M_i pulling back one moment map to the other. ■

Example 3.9. The polytopes in Figure 4 represent four different symplectic toric 4-manifolds: twice the topologically nontrivial S^2-bundle over S^2 (these are Hirzebruch surfaces), an $S^2 \times S^2$ blown-up at one point and an $S^2 \times S^2$ blown-up at two points. If any two of these polytopes are translated so that their left vertical edges exactly superimpose, we get examples to which Lemma 3.8 applies, the relevant facets being the vertical facets on the left. □

Example 3.10. Let $(M_1, \omega_1, \mathbb{T}^2, \mu_1)$ and $(M_2, \omega_2, \mathbb{T}^2, \mu_2)$ be the first two symplectic toric manifolds from Example 3.9 (Hirzebruch surfaces). Let $(B, \omega_B, \mathbb{T}^2, \mu_B)$ be a symplectic S^2 with a Hamiltonian (noneffective) $\mathbb{T}^2$-action and Hamiltonian embeddings j_i into $(M_i, \omega_i, \mathbb{T}^2, \mu_i)$ as preimages of the vertical facets. By Lemma 3.8, there exists a $\mathbb{T}^2$-equivariant symplectomorphism $\gamma : \mathcal{U}_1 \to \mathcal{U}_2$ between invariant tubular neighborhoods $\mathcal{U}_i$ of $j_i(B)$ extending a symplectomorphism $j_1(B) \to j_2(B)$ such that $\gamma^*\mu_2 = \mu_1$. The corresponding radial blow-up has the origami polytope in Figure 5.

Different shades of gray distinguish regions where each point represents one orbit (lighter) or two orbits (darker), which results from the superimposition of two Hirzebruch polytopes.

This example may be considerably generalized; see Section 3.3. □

Example 3.11. Dropping the origami hypothesis gives us much less rigid moment map images. For instance, take any symplectic toric manifold (M', ω', G, μ'), for example,

Fig. 6. Folded (non-origami) moment map images.

$S^2 \times S^2$, and use a regular closed curve inside the moment image to scoop out a G-invariant open subset corresponding to the region inside the curve. Let f be a defining function for the curve such that f is positive on the exterior. Consider the manifold

$$M = \{(p, x) \in M' \times \mathbb{R} \mid x^2 = f(p)\}.$$

This is naturally a *toric* folded symplectic manifold. However, the null foliation on Z is not fibrating: at points where the slope of the curve is irrational, the corresponding leaf is not compact.

For instance, when $M' = S^2 \times S^2$ and we take some closed curve, the moment map image is as on the left of Figure 6. If instead we discard the region corresponding to the outside of the curve (by choosing a function f positive on the interior of the curve), the moment map image is as on the right. □

3.3 Classification of toric origami manifolds

Let (M, ω, G, μ) be a toric origami manifold. By Theorem 3.2, the image $\mu(M)$ is the superimposition of the Delzant polytopes corresponding to the connected components of its symplectic cut space. Moreover, μ maps the folding hypersurface to certain facets possibly shared by two polytopes which agree near those facets.

Conversely, we will see that, given a *template* of an allowable superimposition of Delzant polytopes, we can construct a toric origami manifold the moment image of which is that superimposition. Moreover, such templates classify toric origami manifolds.

Definition 3.12. An n-dimensional *origami template* is a pair $(\mathcal{P}, \mathcal{F})$, where $\mathcal{P}$ is a (nonempty) finite collection of n-dimensional Delzant polytopes and $\mathcal{F}$ is a collection of facets and pairs of facets of polytopes in $\mathcal{P}$ satisfying the following properties:

(a) for each pair $\{F_1, F_2\} \in \mathcal{F}$, the corresponding polytopes in $\mathcal{P}$ agree near those facets;

(b) if a facet F occurs in $\mathcal{F}$, either by itself or as a member of a pair, then neither F nor any of its neighboring facets occur elsewhere in $\mathcal{F}$;

(c) the topological space constructed from the disjoint union $\sqcup \Delta_i$, $\Delta_i \in \mathcal{P}$, by identifying facet pairs in $\mathcal{F}$ is connected. □

Theorem 3.13. Toric origami manifolds are classified by origami templates up to equivariant symplectomorphism preserving the moment maps. More specifically, at the level of symplectomorphism classes (on the left hand side), there is a one-to-one correspondence

$$\{2n\text{-diml toric origami manifolds}\} \longrightarrow \{n\text{-diml origami templates}\}$$

$$(M^{2n}, \omega, \mathbb{T}^n, \mu) \longmapsto \mu(M). \qquad \square$$

Proof. To build a toric origami manifold from a template $(\mathcal{P}, \mathcal{F})$, take the Delzant spaces corresponding to the Delzant polytopes in $\mathcal{P}$ and radially blow up the inverse images of the facets occurring in sets in $\mathcal{F}$: for pairs $\{F_1, F_2\} \in \mathcal{F}$ the model involution γ uses the symplectomorphism from Lemma 3.8; for single faces $F \in \mathcal{F}$, the map γ must be the identity. The uniqueness part follows from an equivariant version of Corollary 2.38. ■

Remark 3.14. There is also a one-to-one correspondence between *oriented* origami toric manifolds (up to equivariant symplectomorphism) and *oriented* origami templates. We say that an origami template is *oriented* if the polytopes in $\mathcal{P}$ come with an orientation and $\mathcal{F}$ consists solely of pairs of facets which belong to polytopes with opposite orientations; see Section 1. Indeed, for $\{F_1, F_2\} \in \mathcal{F}$, with $F_1 \in \Delta_1$ and $F_2 \in \Delta_2$, the opposite orientations on the polytopes Δ_1 and Δ_2 induce opposite orientations on the corresponding components of $M \setminus Z$ which piece together to a global orientation of M, and vice versa. □

Example 3.15. Unlike ordinary toric manifolds, toric origami manifolds may come from nonsimply connected templates. Let M be the manifold $S^2 \times S^2$ blown up at two points, with one S^2 factor having three times the area of the other: the associated polytope Δ is a rectangle with two corners removed. We can construct an origami template $(\mathcal{P}, \mathcal{F})$ where $\mathcal{P}$ consists of four copies of Δ arranged in a square and $\mathcal{F}$ is four pairs of edges coming from the blowups. The result is shown in Figure 7. Note that the associated origami manifold is also not simply connected. □

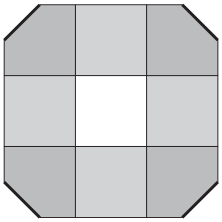

Fig. 7. Nonsimply connected toric origami template.

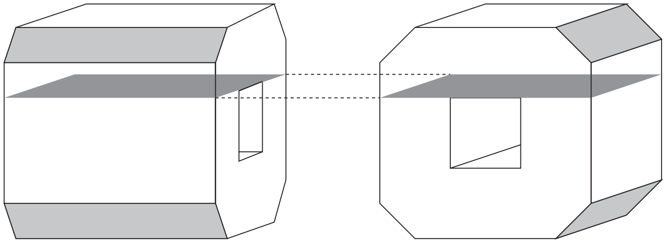

Fig. 8. Non-2-connected toric origami template.

Example 3.16. We can form higher-dimensional analogs of the previous example which fail to be k-connected for $k \geq 2$. In the case $k = 2$, for instance, let Δ' be the polytope associated to $M \times S^2$, and construct an origami template $(\mathcal{P}', \mathcal{F}')$ just as before: this gives the three-dimensional figures on the left and right of Figure 8. We now superimpose these two solids along the dark shaded facets (the bottom facets of the top copies of Δ'), giving us a ninth pair of facets and the desired non-2-connected template.

Note that even though the moment map image has interesting π_2, the template (thought of as the polytopes glued along facets) has trivial H_2 and π_2. This is indeed a general feature of origami templates, see Remark 3.20. □

Example 3.17. Although the facets of $\mathcal{F}$ are necessarily paired if the origami manifold is oriented, the converse fails. As shown in Figure 9, one can form a template of three polytopes, each corresponding to an $S^2 \times S^2$ blown up at two points, and three paired facets. Since each fold flips orientation, the resulting topological space is nonorientable. □

Example 3.18. Recall that the polytope associated to $\mathbb{CP}^2$ is a triangle (shown on the right of Figure 10). The sphere S^4 (shown left) is the orientable toric origami manifold the template of which is two copies of this triangle glued along one edge. Similarly, $\mathbb{RP}^4$ (shown center) is the nonorientable manifold the template of which is a single copy of

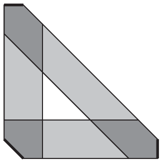

Fig. 9. Nonorientable toric origami template with co-orientable folds.

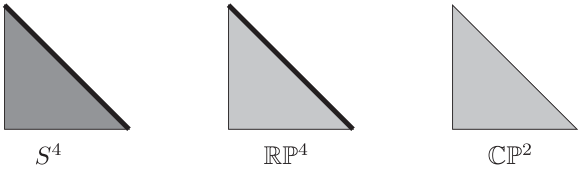

Fig. 10. Three origami templates with the same origami polytope.

the triangle with a single folded edge. This exhibits S^4 as a double cover of $\mathbb{RP}^4$ at the level of templates. □

Example 3.19. We can classify all two-dimensional toric origami manifolds by classifying one-dimensional templates (Figures 11–14). These are disjoint unions of n segments (one-dimensional Delzant polytopes) connected at vertices (the facets of those polytopes) with zero angle: internal vertices and endpoints marked with bullets represent folds. Each segment (resp. marking) gives a component of $M \setminus Z$ (resp. Z), while each unmarked endpoint corresponds to a fixed point. There are four families (Instead of drawing segments superimposed, we open up angles slightly to show the number of components. All pictures ignore segment lengths which account for continuous parameters of symplectic area in components of $M \setminus Z$.):

- Templates with two unmarked endpoints give manifolds diffeomorphic to S^2: they have two fixed points and $n-1$ components of Z (Figure 11).
- Templates with one marked and one unmarked endpoints give manifolds diffeomorphic to $\mathbb{RP}^2$: they have one fixed point and n components of Z (Figure 12).
- Templates with two marked endpoints give manifolds diffeomorphic to the Klein bottle: they have no fixed points and $n+1$ components of Z (Figure 13).
- Templates with no endpoints give manifolds diffeomorphic to $\mathbb{T}^2$: they have no fixed points and an even number n of components of Z (Figure 14). □

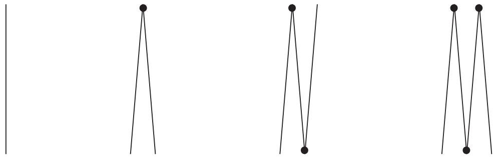

Fig. 11. Toric origami 2-spheres.

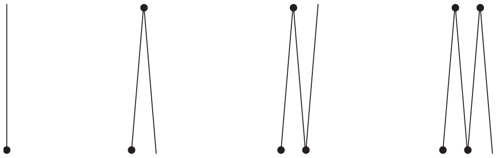

Fig. 12. Toric origami real projective planes.

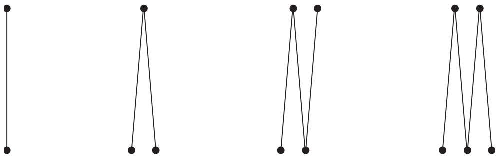

Fig. 13. Toric origami Klein bottles.

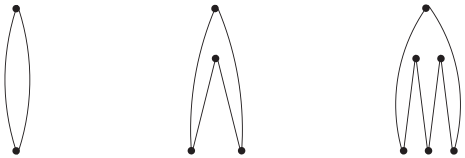

Fig. 14. Toric origami 2-tori.

Remark 3.20. As illustrated by Example 3.11, it is not possible to classify toric folded symplectic manifolds by combinatorial moment data as is the case for toric origami manifolds. Such a classification must be more intricate for the general folded case, and Lee [15] gives a partial result that sheds some light on the type of classification that might be possible.

A *toric folded symplectic manifold* $(M, \omega, \mathbb{T}, \mu)$ is a compact connected folded symplectic manifold (M^{2n}, ω) endowed with an effective Hamiltonian action of a half-dimensional torus $\mathbb{T}^n$ and a corresponding moment map μ. The *orbital moment map* is the map on the orbit space $M/\mathbb{T}$ induced by the moment map. Two toric folded symplectic 4-manifolds $(M, \omega, \mathbb{T}, \mu)$ and $(M', \omega', \mathbb{T}, \mu')$ are symplectomorphic (as folded symplectic manifolds) if $H^2(M/\mathbb{T}, \mathbb{Z}) = 0$ and there exists a diffeomorphism between orbit spaces preserving orbital moment maps.

When $(M, \omega, \mathbb{T}, \mu)$ is a toric origami manifold, $M/\mathbb{T}$ can be realized as the topological space obtained by identifying the polytopes of its origami template along the common facets (see point (c) in Definition 3.12). This space has the same homotopy type as the graph obtained by replacing each polytope by a point and each "glued" double facet by an edge between the points corresponding to the polytopes that the facet belongs to. Therefore, $H^2(M/\mathbb{T}, \mathbb{Z}) = 0$. The existence of a diffeomorphism between orbit spaces implies that $(M', \omega', \mathbb{T}, \mu')$ is an origami manifold as well, and that its origami template is the same as that of $(M, \omega, \mathbb{T}, \mu)$, which makes the manifolds symplectomorphic by Theorem 3.13. □

4 Cobordism

We will now prove the following conjecture of Yael Karshon's stating that an oriented origami manifold is *symplectically* cobordant to its symplectic cut space. By *symplectic* cobordism we mean, following [11], a cobordism manifold endowed with a closed 2-form which restricts to the origami and symplectic forms on its boundary.

Theorem 4.1. Let (M, ω) be an oriented origami manifold and let $(M_0^{\pm}, \omega_0^{\pm})$ be its symplectic cut pieces.

Then there is a manifold W equipped with a closed 2-form Ω such that the boundary of W equipped with the restriction of Ω is symplectomorphic to

$$(M, \omega) \sqcup (M_0^+, \omega_0^+) \sqcup (M_0^-, \omega_0^-).$$

Moreover, in the presence of a (Hamiltonian) compact group action, this cobordism can be made equivariant (or Hamiltonian). □

Proof. Choose an S^1-action making the null fibration into a principal fibration, $S^1 \hookrightarrow Z \xrightarrow{\pi} B$. Let $\mathbb{L} \xrightarrow{\pi_{\mathbb{L}}} B$ be the associated hermitian line bundle $Z \times_{S^1} \mathbb{C}$ for the standard multiplication action of S^1 on $\mathbb{C}$. Let $r : \mathbb{L} \to \mathbb{R}$ given by $r(\ell) = \sqrt{\langle \ell, \ell \rangle_{\pi_{\mathbb{L}}(\ell)}}$ be the hermitian length and let $i_Z : Z \hookrightarrow \mathbb{L}$, $i_Z(x) = [x, 1]$.

For ε small enough, let $\varphi : Z \times (-\varepsilon, \varepsilon) \to \mathcal{U}$ be a Moser model for M (see Definition 2.12). Take a small enough δ and choose a nondecreasing smooth function $g : \mathbb{R}_0^+ \to \mathbb{R}$ such that $g(s) = s$ for $s < \delta^2$ and $g(s) = 1$ for $s > 4\delta^2$.

The set

$$\left\{(r,t) \in \mathbb{R}_0^+ \times (-\varepsilon,\varepsilon) \mid g(t^2) - \frac{\delta^2}{4} \leq r^2 \leq 1\right\}$$

is depicted in Figure 15, where $R := \sqrt{1 - \frac{\delta^2}{4}}$.

Then the set

$$W_\varepsilon := \left\{(\ell,t) \in \mathbb{L} \times (-\varepsilon,\varepsilon) \mid g(t^2) - \frac{\delta^2}{4} \leq r(\ell)^2 \leq 1\right\}$$

is the manifold with boundary sketched in Figure 16 with B represented by a point.

The boundary of W_ε is made up of the following three pieces:

$$C := \{(\ell,t) \in \mathbb{L} \times (-\varepsilon,\varepsilon) \mid r(\ell) = 1\},$$

$$C^+ := \left\{(\ell,t) \in \mathbb{L} \times \left[\frac{\delta}{2},\varepsilon\right) \mid g(t^2) = r(\ell)^2 + \frac{\delta^2}{4}\right\},$$

$$C^- := \left\{(\ell,t) \in \mathbb{L} \times \left(-\varepsilon,-\frac{\delta}{2}\right] \mid g(t^2) = r(\ell)^2 + \frac{\delta^2}{4}\right\}.$$

The set C is the image of $\mathcal{U}$ under the diffeomorphism

$$\mathcal{U} \xrightarrow{\varphi^{-1}} Z \times (-\varepsilon,\varepsilon) \xrightarrow{i_Z \times \mathrm{id}} \mathbb{L} \times (-\varepsilon,\varepsilon). \tag{2}$$

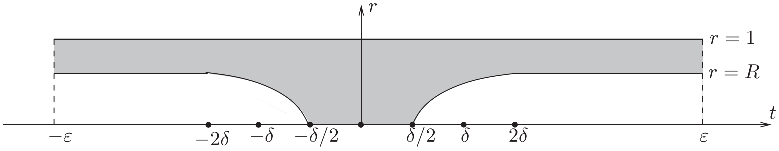

Fig. 15. The shaded area is the set $\{(r,t) \in \mathbb{R}_0^+ \times (-\varepsilon,\varepsilon) \mid g(t^2) - \frac{\delta^2}{4} \leq r^2 \leq 1\}$.

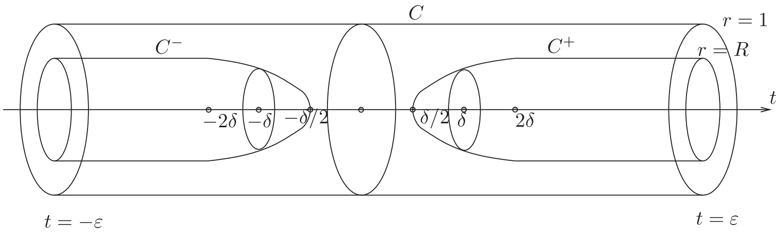

Fig. 16. The key portion W_ε of the cobordism manifold W.

By the tubular neighborhood theorem, the set C^+ is diffeomorphic to a neighborhood of B in the symplectic cut space M_0^+ (and similarly for C^- and M_0^-); indeed the normal bundle of B in M_0^+ is also $\mathbb{L}$.

We can now extend the cobordism W_ε between C and $C^+ \sqcup C^-$ to a global cobordism between M and $M_0^+ \sqcup M_0^-$ (modulo diffeomorphisms). Let $M_{2\delta} := \varphi^{-1}(Z \times (-2\delta, 2\delta))$ be a narrower tubular neighborhood of Z in M. We form the global cobordism W by gluing W_ε to $(M \setminus M_{2\delta}) \times [R, 1]$ using the restriction to $\mathcal{U} \setminus M_{2\delta}$ of the diffeomorphism (2) and using the identity map on $[R, 1]$. Note that any subset of $\mathcal{U}$ not containing Z, for example $\mathcal{U} \setminus M_{2\delta}$, can be viewed also as a subset of $M_0^+ \sqcup M_0^-$ via $j^+ \sqcup j^-$ (for j^+ and j^- see proof of Proposition 2.10).

We will next exhibit a closed 2-form on W restricting to the given origami and symplectic forms on the boundary.

Let α be the S^1-connection form on Z from the Moser model. Denote by B_0 the zero-section in $\mathbb{L}$. Since $\mathbb{L} \setminus B_0 = Z \times_{S^1} \mathbb{C}^* \approx Z \times \mathbb{R}^+$, we can extend α to $\mathbb{L} \setminus B_0$ by pullback via the projection $Z \times \mathbb{R}^+ \to Z$. Although α is not defined over B_0, the product $r^2\alpha$ is a smooth 1-form on $\mathbb{L}$: On open sets where the fibration $Z \xrightarrow{\pi} B$ is trivial, we have $\alpha = d\theta + \pi^*\xi$ for some $\xi \in \Omega^1(B)$, and $r^2\alpha = r^2\, d\theta + r^2\pi^*\xi$ is well defined on the whole $\mathbb{L}$ because $r^2\, d\theta$ is no longer singular.

Now let $h(r, t)$ be a smooth function defined on the shaded region in Figure 15 which is even in t (i.e., $h(r, t) = h(r, -t)$) and in particular equal to t^2 on the region $|t| > 2\delta$ and on the line $r = 1$, and the restriction $\widetilde{h}$ to the curve $g(t^2) = r^2 + \frac{\delta^2}{4}$ of which is strictly increasing as a function of t and equal to r^2 for $|t| < \delta$. An exercise in two-dimensional interpolation shows such a function exists; note that, by definition of g, we have that $r^2 = t^2 - \frac{\delta^2}{4}$ on the latter curve when $|t| < \delta$.

We define the following closed 2-form on W_ε:

$$\Omega = \pi^*\omega_B + d(h\alpha).$$

Because the restriction $\widetilde{h}$ is unique up to pre-composition with a diffeomorphism, the restrictions of Ω to C^+ and C^- are unique up to symplectomorphism. Furthermore, by the Weinstein tubular neighborhood theorem [19], $\Omega|_{C^+}$ and $\Omega|_{C^-}$ are, up to symplectomorphism, the symplectic forms ω_0^+ and ω_0^- on corresponding open subsets of M_0^+ and M_0^-. On the other hand, restricting to C, we have $\Omega = \pi^*\omega_B + d(t^2\alpha)$ which under the map in (2) is the origami form on $\mathcal{U}$.

In particular, on the region $|t| > 2\delta$ in Figure 16, the folded symplectic form on C, the symplectic forms on C^+ and C^-, and the cobording form Ω on W_ε are all identical and equal to $\pi^*\omega_B + \mathrm{d}t^2\alpha$. Moreover, the "up to symplectomorphism" part of these statements involves symplectomorphisms which are the identity on the region $|t| > 2\delta$, and hence Ω can be extended to a cobording 2-form on all of W by letting it be the constant (constant on r) cobording 2-form on $|t| > \varepsilon$.

Suppose now that M is equipped with an action of a compact Lie group G which preserves ω. One then gets a G-action on B, Z, and $\mathbb{L}$, and by averaging we can choose the α in the Moser model to be G-invariant. Thus all the data involved in the definition of the cobording 2-form Ω above are G-invariant and hence Ω itself is G-invariant. Moreover, if ω is G-Hamiltonian, the form ω_B is as well, and since α is G-invariant the 2-form $d(h\alpha)$ is G-Hamiltonian with moment map

$$v \in \mathfrak{g} \longmapsto \phi^v = \imath_{v^\#}(h\alpha),$$

where $v^\#$ is the vector field on $\mathbb{L} \times (-\varepsilon, \varepsilon)$ associated with the action of G on this space. Thus, W_ε with the form Ω is a Hamiltonian cobordism, and it extends to a Hamiltonian cobordism W. ■

Remark 4.2. If M and its cut pieces are pre-quantizable one has an isomorphism of virtual vector spaces (and in the presence of group actions, virtual representation spaces)

$$\mathcal{Q}(M) = \mathcal{Q}(M_0^+) - \mathcal{Q}(M_0^-),$$

where $\mathcal{Q}$ is the spin-$\mathbb{C}$ quantization functor. For a proof of this, see [8]. An alternative proof of this result is based on the "quantization commutes with cobordism" theorem of [11]. □

Remark 4.3. By the "cobordism commutes with reduction" theorem of [11], the symplectic reduction of M at a regular level (for an Hamiltonian abelian Lie group action) is cobordant to the symplectic reduction of $M_0^+ \sqcup M_0^-$ at that level. □

Remark 4.4. In the nonorientable case, we would obtain an orbifold cobordism. However, this is not interesting, since any manifold M bounds an orbifold, $(M \times [-1, 1])/\mathbb{Z}_2$. □

5 Cohomology of Toric Origami

In this section, we assume connectedness of the folding hypersurface Z. This assumption is essential for the argument below: the more general case of a nonconnected folding hypersurface remains open.

Let $(M, \omega, \mathbb{T}, \mu)$ be a $2n$-dimensional oriented toric origami manifold with null fibration $Z \overset{\pi}{\twoheadrightarrow} B$ and connected folding hypersurface Z. Let $S^1 \subset \mathbb{T}$ be the circle group generating the null fibration and $f : M \to \mathbb{R}$ a corresponding moment map with $f = 0$ on Z and $f > 0$ on $M \setminus Z$. Note that, on a tubular neighborhood of Z given by a Moser diffeomorphism $\varphi : Z \times (-\varepsilon, \varepsilon) \to \mathcal{U}$, the origami form is $\varphi^*\omega = p^*i^*\omega + d(t^2 p^*\alpha)$ and hence the moment map is $\varphi^* f(x, t) = \frac{t^2}{2}$.

Near the folding hypersurface Z, the function f is essentially $\frac{t^2}{2}$, and away from it, f restricts to a moment map on an honest symplectic manifold $M \setminus Z$, and is thus Morse–Bott. Furthermore, Z is a nondegenerate critical manifold of codimension 1.

Define $g : M \to \mathbb{R}$ as

$$g = \begin{cases} \sqrt{f} & \text{on } M^+, \\ 0 & \text{on } Z, \\ -\sqrt{f} & \text{on } M^-. \end{cases}$$

We claim that g is Morse–Bott and its critical manifolds are those of f, excluding Z. But this follows easily from the fact that, on $\mathcal{U}$, we have $\varphi^* g = \frac{t}{\sqrt{2}}$, the derivative of which never vanishes, while on $M \setminus Z$, $\mathrm{d}g$ vanishes if and only if $\mathrm{d}f$ vanishes:

$$\mathrm{d}g = \begin{cases} \mathrm{d}f/(2\sqrt{f}) & \text{on } M^+, \\ -\mathrm{d}f/(2\sqrt{f}) & \text{on } M^-. \end{cases}$$

Morse(–Bott) theory then gives us the cohomology groups $H^k_{\mathbb{T}}(M)$ in terms of the cohomology groups of the critical manifolds of g, $X \subset M \setminus Z$:

$$H^k_{\mathbb{T}}(M) = \begin{cases} 0 & \text{if } k \text{ odd}, \\ \sum_X H^{k-r_X}_{\mathbb{T}}(X) & \text{if } k \text{ even}, \end{cases}$$

where $r_X = \mathrm{Ind}(X, g)$ is the index of the critical manifold X with respect to the function g, and is $\mathrm{Ind}(X, f)$ if $X \subset M^+$, and $2d - \mathrm{Ind}(X, f)$ if $X \subset M^-$.

Similarly to the symplectic toric manifold case, the cohomology groups are easily read from the template.

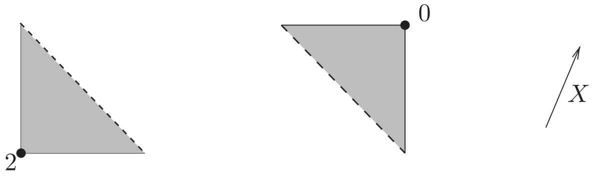

Fig. 17. The *unfolded* set $\mathcal{P}^u$ for a toric 4-sphere.

Let $(\mathcal{P}, \mathcal{F})$ be the template of a $2n$-dimensional oriented toric origami manifold $(M, \omega, \mathbb{T}, \mu)$ with connected folding hypersurface Z. Since the manifold is oriented, the set $\mathcal{P}$ is a finite collection of oriented n-dimensional Delzant polytopes and the set $\mathcal{F}$ a collection of pairs of facets of these polytopes (see Section 1). We say that a polytope is *positively oriented* if its orientation matches that of $\mathfrak{g}^*$ and *negatively oriented* otherwise. Let $S^1 \subset \mathbb{T}$ be the circle group generating the null fibration and let $\mathbb{H} \subset \mathbb{T}$ be a complementary $(n-1)$-dimensional subtorus. Let $\mathfrak{s}^* \oplus \mathfrak{h}^*$ be the induced decomposition of the dual of the Lie algebra of $\mathbb{T}$, and let $\mathrm{refl}(a \oplus b) = -a \oplus b$ be the corresponding reflection along $\mathfrak{s}^*$. Let $\mathcal{P}^u$ be the collection of all positively oriented polytopes in $\mathcal{P}$ and of the images by refl of all negatively oriented polytopes in $\mathcal{P}$. The set $\mathcal{P}^u$ can be thought of in terms of unfolding of the moment polytope. Let $\mathcal{F}^u$ be the set of pairs of facets of polytopes in $\mathcal{P}^u$ corresponding to the pairs in $\mathcal{F}$. We will call $(\mathcal{P}^u, \mathcal{F}^u)$ the *unfolded template* of $(M, \omega, \mathbb{T}, \mu)$.

Corollary 5.1. Let $(\mathcal{P}^u, \mathcal{F}^u)$ be the unfolded template of a $2n$-dimensional oriented toric origami manifold $(M, \omega, \mathbb{T}, \mu)$ with connected folding hypersurface Z, as defined above. Let $X \in \mathfrak{g}$ generate an irrational flow.

For k even, the degree-k cohomology group of M has a dimension equal to the number of vertices v of polytopes in $\mathcal{P}^u$ such that:

(i) at v there are exactly $\frac{k}{2}$ primitive inward-pointing edge vectors which point up relative to the projection along X, and
(ii) v does not belong to any facet in $\mathcal{F}^u$.

All odd-degree cohomology groups of M are zero. □

Example 5.2. For the toric 4-sphere $(S^4, \omega_0, \mathbb{T}^2, \mu)$ from Example 3.4, the set $\mathcal{P}^u$ contains two triangles, one being a mirror image of the other, as in Figure 17, where the dashed hypotenuses form the unique pair of facets in the corresponding $\mathcal{F}^u$.

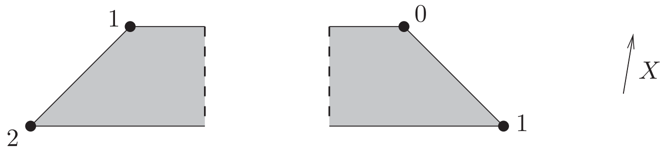

Fig. 18. The *unfolded* set $\mathcal{P}^u$ for a toric manifold with a flag-like moment polytope.

For the chosen direction X, the numbers next to the relevant vertices count the edge vectors which point up relative to X. Indeed, $\dim H^4(S^4) = \dim H^0(S^4) = 1$, all other groups being trivial. □

Example 5.3. For the toric 4-manifold $(M, \omega, \mathbb{T}^2, \mu)$ from Example 3.10, the set $\mathcal{P}^u$ contains two trapezoids, as in Figure 18, where the dashed vertical sides form the unique pair of facets in its $\mathcal{F}^u$.

For the chosen direction X, the numbers next to the relevant vertices count the edge vectors which point up relative to X. The conclusion is that

$$H^k(M; \mathbb{Z}) = \begin{cases} 0 & \text{if } k \text{ odd}, \\ \mathbb{Z} & \text{if } k = 0 \text{ or } k = 4, \\ \mathbb{Z}^2 & \text{if } k = 2 \end{cases}$$

which happen to coincide with the groups for an ordinary Hirzebruch surface. □

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