QUANTIZATION OF SYMPLECTIC ORBIFOLDS AND GROUP ACTIONS

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ABSTRACT. To a compact symplectic orbifold with a hamiltonian action of a group G (and with other compatible data), we can associate a quantization space V which is a representation of G. This paper reports on the decomposition of V into irreducible representations of G in terms of data at the fixed points, and on the semiclassical limit of this decomposition, for the case where G is a compact connected semisimple Lie group.

1. INTRODUCTION

Let (M, ω) be a compact symplectic orbifold such that $[\omega]$ is the Chern class of a complex line bundle L over M. Choose an almost complex structure, J, on M compatible with ω (i.e. such that $(u, v) \mapsto \omega(u, Jv)$ is a positive metric), and choose a hermitian connection on L with curvature given by ω. To such data we can associate a first order elliptic differential operator, called the Dolbeault-Dirac operator [8, 6]:

$$\bar{\partial} : C^\infty(M; \wedge^{0,\text{even}} T^*M \otimes L) \longrightarrow C^\infty(M; \wedge^{0,\text{odd}} T^*M \otimes L) \ .$$

Inspired by Bott [2], we define **quantization** of (M, ω), to be the (K-theoretic) index of $\bar{\partial}$, i.e.

$$Q(M, \omega) := \text{kernel} \, \bar{\partial} - \text{cokernel} \, \bar{\partial} \ .$$

This index depends only on (M, ω) and not on the auxiliary data.

If a group G acts on M, the action lifts to L and preserves all data (in particular, the action must be hamiltonian [12]), then the quantization space becomes a linear representation of G. We will first address the following question: What is the decomposition of $Q(M, \omega)$ into irreducible representations of G?

As explained in this paper, not only does the answer depend exclusively on data at the fixed points, but also this answer can be given by a very explicit formula useful for computations. The prototype for

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Date: November 1, 1998.

this formula is a formula of Kostant [15] for multiplicities of group representations. The prototype for the proof is an observation of Cartier that we will now review.

From now on, let G be a compact connected semisimple Lie group with a maximal torus T. Let $\rho : G \to \mathrm{GL}(V)$ be an irreducible representation with maximal weight λ. As a T-representation, V breaks up into weight spaces

$$V = \bigoplus_{\text{weights } \mu} V_\mu \ ,$$

where $\mathrm{mult}(\mu) := \dim V_\mu$. Kostant [15] gave a formula for the multiplicity of a weight μ in ρ:

$$(1.1) \qquad \mathrm{mult}(\mu) = \sum_{\omega \in W} (-1)^\omega \mathcal{P}(\omega(\delta + \lambda) - (\delta + \mu)) \ ,$$

where W is the Weyl group, $\delta = \frac{1}{2} \sum_{\alpha \in \Delta^+} \alpha$, Δ^+ is a chosen set of positive roots, and $\mathcal{P}(\nu)$ is the number of ways that ν may be partitioned into a sum of positive roots:

$$\mathcal{P}(\nu) := \{(n_\alpha \in \mathbb{Z}_0^+) \mid \nu = \sum_{\alpha \in \Delta^+} n_\alpha \alpha\} \ .$$

Cartier [7] showed that the Kostant formula is equivalent to the Weyl character formula, which states that

$$(1.2) \qquad \chi(\exp \xi) = \frac{\sum_{\omega \in W}(-1)^\omega e^{(\delta + \lambda)(\omega \xi)}}{e^{\delta(\xi)} \prod_{\alpha \in \Delta^+}(1 - e^{-\alpha(\xi)})} \ ,$$

χ being the character of ρ and $\xi \in \mathrm{Lie}(T)$. Cartier's simple argument starts by expanding the Weyl denominator in a power series, to obtain

$$\frac{1}{\prod_{\alpha \in \Delta^+}(1 - e^{-\alpha})} = \sum_{\text{weights } \nu} \mathcal{P}(\nu) e^{-\nu} \ ,$$

where $\mathcal{P}$ is the Kostant partition function. Now the Weyl formula can be written

$$\sum_\mu \mathrm{mult}(\mu) e^{\mu(\xi)} = \sum_{\omega \in W} \sum_\nu (-1)^\omega \mathcal{P}(\nu) e^{(\delta + \lambda)(\omega \xi)} e^{-(\delta + \nu)(\xi)} \ .$$

Formula (1.1) follows from equating the coefficients of $e^{\mu(\xi)}$.

The upshot of this argument is that formulas (1.1) and (1.2) are dual of each other by a Fourier expansion. This is less trivial than it sounds: There are several ways of expanding the Weyl denominator into a Fourier series, and for some of these expansions the coefficient of e^ν will be given by a divergent infinite sum.

In §4 we obtain a formula (Theorem 1) resembling Kostant's for the multiplicity of a weight μ in $Q(M, \omega)$ for an action of a torus with the fixed points not necessarily isolated. We start from the equivariant index formula [1] modified for orbifolds [19, 8, 17], and apply a suitable Fourier expansion. Our proof uses the language of associated orbifolds explained in §A.3; Appendix A also reviews other relevant orbifold definitions. To ensure that each coefficient in the Fourier series is given by a convergent series, in §3 we define a polarization [10]. §2 describes how to handle an action of a compact connected semisimple Lie group by a combination of tricks.

In §5 we compute the **semiclassical limit** of the multiplicity formula (4.1): We replace the line bundle, L, by its nth tensor power, to get a T-invariant elliptic first order differential operator $\bar{\partial}^{(n)}$. Setting $\text{mult}^{(n)}(\mu)$ to be the multiplicity with which μ occurs as a weight of the representation of T on

$$\text{kernel } \bar{\partial}^{(n)} - \text{cokernel } \bar{\partial}^{(n)} \ ,$$

we show that, as the integer n tends to infinity, the function of μ given by

$$n^{-d}\text{mult}^{(n)}(n\mu) \ ,$$

where $d = \frac{1}{2}\dim M - \dim T$, tends to the Radon-Nikodym derivative for the Duistermaat-Heckman measure relative to the Lebesgue measure on the dual of the Lie algebra of T: Theorem 2 of §5 and Theorem 3 of §6. The proof of Theorem 2 relies on complex analysis and on elementary results of invariant theory for finite groups; the proof of Theorem 3 relies on Fourier calculus and is analogous to the manifold proof in [6].

The case of torus actions on symplectic manifolds with only isolated fixed points was solved by the second author with Lerman and Sternberg [10]; their multiplicity formula was derived from the Atiyah-Bott-Lefschetz formula by an argument along the lines of Cartier's. Later he and Prato [11] tackled the case of nonabelian actions (the argument in this paper is different from theirs and is an expansion of the remarks in [6]). The present paper is a continuation of [6] which contained the corresponding results for the case of manifolds. The extension to orbifolds is important since, not only are the generic reduced spaces of symplectic manifolds orbifolds, but also the generic reduced spaces of symplectic orbifolds are still orbifolds. Number-theoretic applications of multiplicity formulas for orbifolds can be found in [5].

2. Quantization of Nonabelian Group Actions

Let $\rho : G \to \mathrm{GL}(V)$ be a representation of a compact connected semisimple Lie group G with a maximal torus T. The dimension of the space, V^G, of G-invariants is given by

$$
\begin{aligned}
\dim V^G \;&=\; \int_G \chi(g)\,dg \\[2mm]
&=\; \frac{1}{|W|} \int_T \underbrace{\prod_{\text{roots }\alpha}(1 - e^{\alpha(\xi)})\,\chi(t)}_{\sum_\mu c_\mu e^{\mu(\xi)}}\; dt \qquad \text{where } \exp\xi = t \\[2mm]
&=\; \frac{1}{|W|} \sum_\mu c_\mu \dim V_\mu \;.
\end{aligned}
$$

The middle equality follows from Weyl's integral formula (see, for instance, [3]), the others follow from orthogonality relations for characters, and the coefficients c_μ are constants. Hence, the dimension of the space of G-invariants is a linear combination of the multiplicities for the maximal torus with coefficients independent of the G-representation.[1]

Now let $V = Q(M,\omega)$ be the quantization space of a compact symplectic orbifold as described in §1, and let $\mathcal{O}$ be a coadjoint orbit of G equipped with its canonical Lie-Poisson symplectic form, $\omega^{\mathcal{O}}$. The coadjoint action of G is hamiltonian with moment map given by inclusion. By the Borel-Weil theorem, any irreducible complex G-representation, $\rho : G \to \mathrm{GL}(V^\rho)$ is the quantization space of a unique (integral) coadjoint orbit $\mathcal{O}$,

$$
Q(\mathcal{O}, \omega^{\mathcal{O}}) \simeq V^\rho \;.
$$

The "shifting trick" [12] is the observation that

$$
\dim\big(Q(M \times \mathcal{O}, \omega - \omega^{\mathcal{O}})\big)^G = \text{ multiplicity of } \rho \text{ in } Q(M,\omega) \;.
$$

We conclude that multiplicities for a nonabelian group action are linear combinations of multiplicities for the action of a maximal torus. From now on we restrict attention to actions of tori.

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[1]We thank Michèle Vergne for showing us this trick.

3. POLARIZATION

Let (M, ω) be a compact symplectic orbifold such that $[\omega]$ is the Chern class of a complex line bundle L over M. Suppose that a torus T acts on (M, ω), and that this action is effective, lifts to L and has a moment map Φ. Choose an almost complex structure, J, on M compatible with ω and choose a hermitian connection on L with curvature given by ω, both invariant by T.

The normal bundle, NF, to a connected component, F, of the fixed point set splits into a direct sum of vector subbundles

$$(3.1) \qquad E_{F,1} \oplus \ldots \oplus E_{F,m} \ ,$$

m depending on F, such that the isotropy representation of T on $E_{F,j}$ has a fixed rational weight, $\alpha_{F,j} \in \operatorname{Lie}(T)^*$, where $\alpha_{F,j} \neq \alpha_{F,k}$ for $j \neq k$. Since T is compact, each F is non-degenerate, in the sense that all $\alpha_{F,j} \neq 0$. Hence, we can polarize these weights as in [10] by choosing an element, v, of $\operatorname{Lie}(T)$ such that $\alpha_{F,j}(v) \neq 0$ for all i, j, and setting

$$\alpha_{F,j}^+ = \varepsilon_{F,j} \alpha_{F,j} \qquad \text{where} \qquad \varepsilon_{F,j} = \operatorname{sign} \alpha_{F,j}(v) \ .$$

The polarized weights all lie in the half-space $\xi(v) > 0$.

Let $n_{F,j}$ be half of the real rank of the vector bundle $E_{F,j}$, and let

$$\alpha_F^\# = \sum_{\varepsilon_{F,j}=-1} n_{F,j} \alpha_{F,j} \qquad \text{and} \qquad n_F = \sum_{\varepsilon_{F,j}=-1} n_{F,j} \ .$$

For every m-tuple of non-negative integers, $k = (k_1, \ldots, k_m)$, let $E_F(k)$ be the tensor product

$$\left(\bigotimes_{j=1}^m \mathcal{S}^{k_j}(E_{F,j}^+) \right)^* \otimes \left(\bigotimes_{\varepsilon_{F,j}=-1} \wedge^{n_{F,j}}(E_{F,j}^+) \right)^*$$

where $\mathcal{S}^{k_j}(E_{F,j}^+)$ is the k_j-th symmetric power of $E_{F,j}^+$ and

$$E_{F,j}^+ = \begin{cases} E_{F,j} & \text{if } \varepsilon_{F,j} = 1 \\ E_{F,j}^* & \text{if } \varepsilon_{F,j} = -1 \ . \end{cases}$$

4. MULTIPLICITY FORMULA FOR ORBIFOLDS

Given a weight μ of T, let $\Delta_F(\mu)$ be the convex polytope in $\mathbb{R}^m$ consisting of all m-tuples, $(s_1, \ldots, s_m)$, $s_i \geq 0$, for which

$$\Phi_F - \sum_j s_j \alpha_{F,j}^+ = \mu$$

where Φ_F is the value of the moment map Φ on the connected component F of the fixed point set. In this expression $\alpha_{F,j}^+$ and Φ_F are rational, whereas μ is integral. The fact that the $\alpha_{F,j}^+$'s are polarized (see §4) implies that $\Delta_F(\mu)$ is compact.

Theorem 1. *The multiplicity with which μ occurs as a weight of the representation of T on $Q(M, \omega)$ is equal to the sum*

$$\text{mult}(\mu) = \sum_F (-1)^{n_F} \mathcal{N}_F(\mu)$$

where

$$(4.1) \qquad \mathcal{N}_F(\mu) = \sum_{k \in \Delta_F(\mu - \alpha_F^\#)} \text{KRR}(F, E_F(k) \otimes L_F)$$

$\text{KRR}(F, E_F(k) \otimes L)$ *being the Kawasaki-Riemann-Roch number* [14] *of the orbifold vector bundle $E_F(k) \otimes L_F$ over F, with respect to the almost complex structure induced on F by J.*

Proof.   The equivariant index theorem for orbifolds [19, 8] states that, for a generic ξ in $\text{Lie}(T)$ (generic in the sense that $\exp \xi$ generates a dense subgroup of T), the trace of $\exp \xi$ on the (virtual) vector space $Q(M, \omega)$ is equal to the sum over the fixed point components, F, of "local traces", $\chi_F(\xi)$, whose definition we will now undertake.

Let $\widehat{F}$ be the orbifold associated to F, as described in §A.3; $\widehat{F}$ will in general have various connected components of possibly different dimensions, depending on the canonical stratification of F (see §A.2). Let $L_{\widehat{F}}$, $E_{\widehat{F},j}$ and $E_{\widehat{F},j}^+$ be the pull-backs of $L|_F$, $E_{F,j}$ and $E_{F,j}^+$ by the natural map $\mu : \widehat{F} \to F$; according to (3.1), the pull-back of the normal bundle NF splits into $E_{\widehat{F},1} \oplus \ldots \oplus E_{\widehat{F},m}$. These bundles over $\widehat{F}$ have canonical automorphisms as explained in §A.4, which we denote

$$A_F := A(L_{\widehat{F}}), \qquad A_{F,j} := A(E_{\widehat{F},j}), \qquad A_{F,j}^+ := A(E_{\widehat{F},j}^+).$$

Finally, let $-2\pi i \Omega_{F,j}$ be the curvature form associated with a hermitian connection on $E_{F,j}$, and we define $\Omega_{F,j}^+$, $\Omega_{\widehat{F},j}$ and $\Omega_{\widehat{F},j}^+$ similarly for the bundles $E_{F,j}^+$, $E_{\widehat{F},j}$ and $E_{\widehat{F},j}^+$, respectively.

The local traces are then

$$\chi_F(\exp \xi) =$$
$$(4.2) \quad e^{\Phi_F(\xi)} \int_{\widehat{F}} \frac{1}{m_{\widehat{F}}} \cdot \frac{A_F \exp(\omega_{\widehat{F}}) \text{Todd}(\widehat{F})}{D_{\widehat{F}} \prod_j \det\left(I - A_{F,j}^{-1} \exp(-\alpha_{F,j}(\xi)I - \Omega_{\widehat{F},j})\right)},$$

where $\omega_{\widehat{F}}$ is the pull-back of the symplectic form ω to $\widehat{F}$, $m_{\widehat{F}}$ is the order function of $\widehat{F}$ (see §A.2) and $D_{\widehat{F}}$ is the associated characteristic form defined in §A.4 – the innocent looking $D_{\widehat{F}}$ is actually an elaborate characteristic form involving the normal bundles of the orbifold strata inside F.

If $\varepsilon_{F,j} = -1$, we rewrite the jth term in the denominator in polarized form as

(4.3)
$$(-1)^{n_{F,j}} \det A_{F,j}^{-1} \, e^{-n_{F,j}\alpha_{F,j}(\xi)} \det \exp\left(-\Omega_{\widehat{F},j}\right) \cdot$$
$$\det\left(I - A_{F,j}\exp(\alpha_{F,j}(\xi)I + \Omega_{\widehat{F},j})\right) \ .$$

Let $E_{\widehat{F}}^{\#}$ be the line bundle

$$\left(\bigotimes_{\varepsilon_{F,j}=-1} \wedge^{n_{F,j}}\left(E_{\widehat{F},j}^{+}\right)\right)^{*}$$

having curvature form $-2\pi i\Omega_{\widehat{F}}^{\#}$ and canonical automorphism $A_{F}^{\#} = (\prod_{\varepsilon_{F,j}=-1} \det A_{F,j}^{+})^{-1}$. Since $\exp(\operatorname{trace} B) = \det(\exp B)$ for an endomorphism B, and the curvature form satisfies $\operatorname{trace}\Omega(E) = \Omega(\wedge^{n}E)$, where n is the rank of the vector bundle E, if we substitute (4.3) into (4.2), we can rewrite (4.2) in polarized form:

(4.4)
$$\chi_{F}(\exp\xi) =$$
$$(-1)^{n_{F}} e^{(\Phi_{F}-\alpha_{F}^{\#})(\xi)} \int_{\widehat{F}} \frac{1}{m_{\widehat{F}}} \cdot \frac{A_{F} A_{F}^{\#} \exp\left(\omega_{\widehat{F}} + \Omega_{\widehat{F}}^{\#}\right) \operatorname{Todd}(\widehat{F})}{D_{\widehat{F}} \prod_{j} \det\left(I - (A_{F,j}^{+})^{-1}\exp(-\alpha_{F,j}^{+}(\xi)I - \Omega_{\widehat{F},j}^{+})\right)} \ .$$

By Lemma (B.1) of [6], we can expand (4.4) into an infinite trigonometric series

(4.5)
$$\chi_{F}(\exp\xi) = (-1)^{n_{F}} \sum_{k} c_{k}\, e^{(\Phi_{F}+\alpha_{F}^{\#}-k_{1}\alpha_{F,1}^{+}-\dots-k_{m}\alpha_{F,m}^{+})(\xi)}$$

summed over all non-negative integer m-tuples, k, where c_{k} equals

(4.6)
$$\int_{\widehat{F}} \frac{1}{m_{\widehat{F}}} \cdot \frac{\left(\prod_{j} \operatorname{trace} \mathcal{S}^{k_{F,j}}((A_{F,j}^{+})^{-1}\exp(-\Omega_{\widehat{F},j}^{+}))\right) A_{F} A_{F}^{\#} \exp(\omega_{\widehat{F}} + \Omega_{\widehat{F}}^{\#})\operatorname{Todd}(\widehat{F})}{D_{\widehat{F}}} \ .$$

By definition of associated Chern character (see §A.4), formula (4.6) is simply

$$\int_{\widehat{F}} \frac{1}{m_{\widehat{F}}} \cdot \frac{\widehat{\mathrm{Ch}}(E_{\widehat{F}}(k) \otimes L_{\widehat{F}})\mathrm{Todd}(\widehat{F})}{D_{\widehat{F}}}$$

which is the Riemann-Roch number of the orbifold line bundle $E_F(k) \otimes L_F$ over F as proved by Kawasaki [14].

On the other hand, for $\xi \in \mathrm{Lie}(T)$ the trace of $\exp \xi$ on the (virtual) vector space $Q(M, \omega)$ is equal to

$$(4.7) \qquad \sum_{\text{weights } \mu} \mathrm{mult}(\mu) e^{\mu(\xi)} \ ,$$

and by comparing (4.5) with (4.7) one obtains the identity (4.1). $\quad \square$

Remark. In the manifold case, we recover the result in [6] where the Kawasaki-Riemann-Roch number is the usual Riemann-Roch number:

$$\mathrm{RR}(F, E_F(k) \otimes L_F) = \int_F \mathrm{Ch}(E_F(k) \otimes L_F)\mathrm{Todd}(F) \ .$$

5. Semiclassical Limit

Let n be a positive integer. Replacing the line bundle, L, by its nth tensor power, one gets, as in §1, a T-invariant elliptic first order differential operator $\bar{\partial}^{(n)}$. Denote by $\mathrm{mult}^{(n)}(\mu)$ the multiplicity with which μ occurs as a weight of the representation of T on kernel $\bar{\partial}^{(n)} -$ cokernel $\bar{\partial}^{(n)}$, and let $d = \frac{1}{2}\dim M - \dim T$.

Theorem 2. *As n tends to infinity, the quantity $n^{-d}\mathrm{mult}^{(n)}(n\mu)$ tends to*

$$(5.1) \qquad \sum_{F \subseteq M^T} (-1)^{\sigma_F} \int_{\Delta_F(\mu)} \mathrm{Res}_F(s) ds \ ,$$

where $\mathrm{Res}_F(s)$ is the sum of residues of

$$(5.2) \qquad e^{\Sigma s_j z_j} \frac{1}{|\Gamma_F|} \int_F \frac{\exp \omega_F}{c_{F,1}(z_1) \dots c_{F,m}(z_m)} \ ,$$

Γ_F being the structure group of F and $c_{F,j}(z)$ the Chern polynomial of $E_{F,j}^+$.

Proof. By Theorem 1, $\text{mult}^{(n)}(n\mu)$ is equal to the sum

$$\sum_F (-1)^{n_F} \mathcal{N}_F^{(n)}(n\mu)$$

where

(5.3) $\qquad \mathcal{N}_F^{(n)}(n\mu) = \sum \int_{\widehat{F}} \frac{1}{m_{\widehat{F}}} \cdot \frac{\widehat{\text{Ch}}(E_{\widehat{F}}(k) \otimes L_{\widehat{F}}^n)\text{Todd}(\widehat{F})}{D_{\widehat{F}}}$

summed over all non-negative integral solutions, k, of the equation

$$n\Phi_F - k_1\alpha_{F,1}^+ - \ldots - k_m\alpha_{F,m}^+ + \alpha_F^{\#} = n\mu \ .$$

(If we replace L by L^n, we must replace ω by $n\omega$, Φ by $n\Phi$ and A_F by A_F^n.) We set $\dim F = 2p$, and $\dim \Delta_F(\mu) = q$. By (5.3), we have

(5.4)

$$n^{-d}\mathcal{N}^{(n)}(n\mu) = n^{-(d-p)} \sum_k \int_{\widehat{F}} n^{-p} \frac{1}{m_{\widehat{F}}} \cdot \frac{\widehat{\text{Ch}}(E_{\widehat{F}}(k) \otimes L_{\widehat{F}}^n)\text{Todd}(\widehat{F})}{D_{\widehat{F}}} \ .$$

Lemma 5.5. *Up to an error of order* $O\left(\frac{1}{n}\right)$, *the sum in* (5.4) *equals*

(5.6)

$$\sum_k \frac{1}{|\Gamma_F|} \sum_{g \in \Gamma_F} \rho_{F,0}^n(g)\rho_F^{\#}(g) \int_F \exp \omega_F \prod_j \text{trace } \mathcal{S}^{k_j}\left(\rho_{F,j}^+(g) \exp\left(-\frac{\Omega_{F,j}^+}{n}\right)\right) \ ,$$

where $\rho_{F,0}$, $\rho_F^{\#}$, $\rho_{F,j}^+$ *are the representations of the structure group* Γ_F *of* F *on the orbifold charts of* L, $E^{\#}$ *and* E_j^+ *over* F.

Proof. Let $\widehat{F}_l$ be a connected component of $\widehat{F}$ and let $2p_l = \dim \widehat{F}_l$. With ω and A_F replaced by $n\omega$ and A_F^n in (4.6), the integrand in this expression can be expanded into a sum of terms of the form

$$\pm \frac{1}{m_{\widehat{F}_l}} \cdot \frac{A_F^n A_F^{\#} \mathcal{A} \, (n\omega_{\widehat{F}})^r \wedge \Omega_{i_1} \wedge \ldots \wedge \Omega_{i_s} \wedge (\Omega_{\widehat{F}}^{\#})^{\nu} \wedge T^{\mu}}{D_{\widehat{F}_l}} \ ,$$

where $\mathcal{A}$ is a factor involving the A_j^+'s, Ω_{i_a} is a coefficient of the curvature form $\Omega_{\widehat{F},i_a}^+$, and T^{μ} is the component of degree 2μ of $\text{Todd}(\widehat{F}_l)$. However, this term can only contribute to the integral if $r+s+\nu+\mu = p_l$, in which case it can be rewritten as

$$\pm \frac{1}{m_{\widehat{F}_l}} \cdot \frac{A_F^n A_F^{\#} \mathcal{A} \, n^{p_l} \, \omega_{\widehat{F}}^r \wedge (\Omega_{i_1}/n) \wedge \ldots \wedge (\Omega_{i_s}/n) \wedge (\Omega_{\widehat{F}}^{\#}/n)^{\nu} \wedge T^{\mu}/n^{\mu}}{D_{\widehat{F}_l}} \ .$$

When multiplied by n^{-p} (as in (5.4)), the terms in this sum for which ν or μ is positive or $p > p_l$ can be discarded since they contribute errors

of order $O(\frac{1}{n})$. We are left with the components of $\widehat{F}$ whose dimension is $2p$. Such components are indexed by the conjugacy classes of Γ_F (see §A.3). Hence the sum in (5.4) is equal to

$$\sum_k \sum_{\gamma \in \mathrm{Conj}(\Gamma_F)} \frac{1}{m_{\widehat{F}_\gamma}} \int_{\widehat{F}_\gamma} \frac{A_F^n A_F^\# \exp \omega_{F_\gamma} \prod_j \mathrm{trace}\, \mathcal{S}^{k_j} \left(\left(A_{F,j}^+ \right)^{-1} \exp \left(-\frac{\Omega_{\widehat{F},j}^+}{n} \right) \right)}{D_{\widehat{F}_\gamma}} .$$

On the component $\widehat{F}_\gamma$, corresponding to $\gamma \in \mathrm{Conj}(\Gamma_F)$, we have

$$A_F = \rho_{F,0}(\gamma) , \quad A^\# = \rho_F^\#(\gamma) , \quad \text{and} \quad A_j^+ \text{ is conjugate to } \rho_j^+(\gamma)$$

(we use the same symbol for a class function on a group and its induced function on conjugacy classes of that group). We also have $D_{\widehat{F}_\gamma} = 1$ since $N_{\widehat{F}_\gamma}$ is trivial, $m_{\widehat{F}_\gamma}$ is the common order of the centralizers of elements in the conjugacy class γ, and any class function, χ, satisfies

$$\sum_{\gamma \in \mathrm{Conj}(\Gamma_F)} \frac{\chi(\gamma)}{m_{\widehat{F}_\gamma}} = \frac{1}{|\Gamma_F|} \sum_{g \in \Gamma_F} \chi(g) .$$

Finally, as the natural immersions $\mu : \widehat{F}_\gamma \to F$ are bijective, the integral over $\widehat{F}_\gamma$ coincides with the integral over F. $\qquad \square$

Since the action of T is effective, we have $q = m - \dim T$, and hence $d - p = q + \sum_{j=1}^m (n_j - 1)$. By Lemma 5.5 and Lemma B.1, (5.4) is, up to an error of order $O(\frac{1}{n})$, equal to

$$(5.7) \qquad n^{-q} \sum_k \mathcal{W}_{F,k} \mathrm{Res} \left[e^{\frac{k_1}{n} z_1 + \ldots + \frac{k_m}{n} z_m} \int_F \frac{\exp \omega_F}{\prod_j \det(z_j I + \Omega_{F,j}^+)} \right]$$

where the operator Res takes the sum of residues and the number $\mathcal{W}_{F,k}$ equals

$$\frac{1}{|\Gamma_F|} \sum_{g \in \Sigma_F} \rho_{F,0}^n(g) \rho_F^\#(g) \prod_j (\rho_{F,j}^+(g))^{-k_j} ,$$

Σ_F being the abelian subgroup of Γ_F consisting of the elements, g, for which each $\rho_{F,j}^+(g)$ is a scalar multiple of the identity. By the orthogonality relations for characters we know that

$$\sum_{g \in \Sigma_F} \left(\rho_{F,0}^n \rho_F^\# \prod_j (\rho_{F,j}^+)^{-k_j} \right) (g) = 0$$

unless $\left(\rho_{F,0}^n \rho_F^\# \prod_j (\rho_{F,j}^+)^{-k_j} \right)$ is the trivial character of Σ_F, in which case this sum is $|\Sigma_F|$. Hence, (5.7) is equal to

$$(5.8) \qquad n^{-q} \sum_k \frac{|\Sigma_F|}{|\Gamma_F|} \mathrm{Res} \left[e^{\sum_j \frac{k_j}{n} z_j} \int_F \frac{\exp \omega_F}{\prod_j \det(z_j I + \Omega_{F,j}^+)} \right]$$

summed over all k satisfying both

$$\Phi_F - \frac{k_1}{n} \alpha_{F,1}^+ - \ldots - \frac{k_m}{n} \alpha_{F,m}^+ + \frac{\alpha_F^\#}{n} = \mu$$

and

$$(5.9) \qquad \rho_{F,0}^n \rho_F^\# \prod_j (\rho_{F,j}^+)^{-k_j} = \mathrm{id}$$

in the character group of Σ_F. By representation theory for finite abelian groups, condition (5.9) is picking out a sublattice of order $|\Sigma_F|$ inside $\mathbb{Z}^m$. Therefore, as n tends to infinity (5.8) tends to the integral

$$(5.10) \qquad \int_{\Delta_F(\mu)} \mathrm{Res} \left[e^{sz} \int_F \frac{1}{|\Gamma_F|} \cdot \frac{\exp \omega_F}{c_1(z_1) \ldots c_m(z_m)} \right] ds \;,$$

where $c_j(z_j) := \det \left(z_j I + \Omega_{F,j}^+ \right)$ is the Chern polynomial of $E_{F,j}^+$. $\qquad \square$

Remark. When the dimension of $\Delta_F(\mu)$ is $q > 0$ expression (5.8) converges pointwise to (5.10) for all rational μ. However, when $q = 0$ it converges to (5.10) in a weak sense: for μ rational and any continuous test function f, the distribution (5.8) evaluated at f tends to the evaluation of the distribution (5.10) at f, as n tends to infinity.

6. Duistermaat-Heckman Measure

In [6] it was proved that, in the case of manifolds, the function of μ defined by (5.1) is the Radon-Nikodym derivative

$$(6.1) \qquad \frac{d\mu_{\mathrm{DH}}}{d\mu_{\mathrm{Lebesgue}}}$$

where μ_{DH} is the Duistermaat-Heckman measure and μ_{Lebesgue} is the standard Lebesgue measure on $\mathrm{Lie}(T)^*$ (suitably normalized). The same is true for orbifolds:

Theorem 3. *The piecewise polynomial function defined by (5.1) is the Radon-Nikodym derivative (6.1).*

The proof is analogous to that in [6]. Instead of the usual abelian localization formula, the orbifold version should be used [17], where the contributions of the fixed point components are affected by a factor equal to the inverse of the order of the orbifold structure group of that component [4].

Appendix A. Orbifolds

A.1.   An n-dimensional **orbifold**, M, is a Hausdorff topological space, $|M|$, equipped with a complete atlas of orbifold charts, that is, a collection of triples $(\widetilde{\mathcal{U}}, \Gamma, \phi)$ where $\widetilde{\mathcal{U}}$ is a connected open subset of $\mathbb{R}^n$, Γ is a finite group acting on $\widetilde{\mathcal{U}}$ by linear transformations,[2] and $\phi : \widetilde{\mathcal{U}} \to |M|$ is a continuous Γ-invariant map inducing a homeomorphism from $\widetilde{\mathcal{U}}/\Gamma$ to $\mathcal{U} := \phi(\widetilde{\mathcal{U}})$; satisfying appropriate compatibility conditions and such that the $\mathcal{U}$'s cover $|M|$ and this collection of charts is maximal; more details and a review of orbifold differential geometry can be found in [4, 8, 13, 16, 17, 18]. For instance, a smooth function on an orbifold is a collection of smooth invariant functions on each orbifold chart which agree on overlaps, and similarly for many other concepts.

A.2.   Let M and N be two orbifolds with a continuous inclusion of the underlying topological spaces. Suppose that M has an orbifold atlas where on each chart $(\widetilde{\mathcal{U}}, \Gamma, \phi)$ the pre-image of N, if not empty, is given by the intersection of $\widetilde{\mathcal{U}}$ with a linear subspace V of $\mathbb{R}^n$. Let Γ_V be the subgroup of those elements in Γ whose action preserves V. Then N is a **suborbifold** of M if the collection $\{(\widetilde{\mathcal{U}} \cap V, \Gamma_V, \phi|_{\widetilde{\mathcal{U}} \cap V})\}$, together with the induced injections, forms an atlas of orbifold charts for N.

If $(\widetilde{\mathcal{U}}, \Gamma, \phi)$ is an orbifold chart for M, the **structure group**, Γ_p, of a point $p \in \mathcal{U}$ is the isotropy group of a pre-image of p under ϕ; Γ_p is defined up to isomorphism. The type of the structure group induces a **canonical stratification** of M into suborbifolds. On each connected component of M, there is an open dense set of points for which the order of the structure group is minimal, called the principal stratum of M. When M is connected, the **structure group**, Γ_M, of M is the (abstract) isotropy group of its principal stratum; the **order** of M is the order of Γ_M. When M is not connected the orders of its connected components define the locally constant **order function** $m_M : M \to \mathbb{N}$.

---

[2]We assume that the set of all fixed points of Γ has codimension at least two. We do *not* assume the action of Γ to be effective. The result of the action of $g \in \Gamma$ on $u \in \mathcal{U}$ is denoted by $g \cdot u$.

A.3. Given an orbifold M, the **associated orbifold** [14, 9] $\widehat{M}$ has charts $(\widetilde{\mathcal{V}}, \Gamma, \psi)$ built as follows: For each chart, $(\widetilde{\mathcal{U}}, \Gamma, \phi)$, of M, let

$$\widetilde{\mathcal{V}} = \{(u, g) \in \widetilde{\mathcal{U}} \times \Gamma \mid g \cdot u = u\} .$$

Γ acts on $\widetilde{\mathcal{V}}$ via $h \cdot (u, g) = (h \cdot u, hgh^{-1})$. Let $\mathcal{V} = \widetilde{\mathcal{V}}/\Gamma$ be the space of orbits with projection $\psi : \widetilde{\mathcal{V}} \to \mathcal{V}$. The orbifold charts $(\widetilde{\mathcal{V}}, \Gamma, \psi)$ inherit the compatibility conditions from the $(\widetilde{\mathcal{U}}, \Gamma, \phi)$. As a set,

$$\widehat{M} = \bigsqcup_{p \in M} \mathrm{Conj}(\Gamma_p) ,$$

where $\mathrm{Conj}(\Gamma_p)$ is the set of conjugacy classes in Γ_p. For example, when M is a teardrop orbifold (i.e. M is homeomorphic to a sphere and has just one singular point, p, with structure group $\mathbb{Z}/n$ for some n), then $\widehat{M}$ has n components: one component diffeomorphic to M plus $n - 1$ points each with structure group $\mathbb{Z}/n$, corresponding to p paired with the nonzero elements of $\mathbb{Z}/n$. Associated orbifolds may be viewed as a bookkeeping device which allows for concise orbifold index formulas.

A.4. A rank k **orbifold vector bundle**, $\pi : E \to M$, over an orbifold M is a collection of Γ-equivariant rank k vector bundles $\pi_{\widetilde{\mathcal{U}}} : \widetilde{E}_{\mathcal{U}} \to \widetilde{\mathcal{U}}$ on each chart $(\widetilde{\mathcal{U}}, \Gamma, \phi)$, together with suitable compatibility conditions. The fibers $\pi^{-1}(p)$ are in general diffeomorphic to some quotient of a k-dimensional vector space by an the action of the structure group Γ_p. Additional structures on an orbifold bundle are defined as collections of Γ-invariant objects on each $\pi_{\widetilde{\mathcal{U}}} : \widetilde{E}_{\mathcal{U}} \to \widetilde{\mathcal{U}}$ agreeing on overlaps.

An orbifold vector bundle, $E \to \widehat{M}$, over an associated orbifold has a **canonical automorphism**, $A(E)$, given on each chart $(\widetilde{\mathcal{V}}, \Gamma, \psi)$ by the natural action of $g \in \Gamma$ on the fiber of $\widetilde{E}_{\mathcal{V}}$ above $(u, g) \in \widetilde{\mathcal{V}}$ [17].

Suppose that M has an almost complex structure. Then the normal bundle, $N_{\widehat{M}}$, of the natural immersion $\mu : \widehat{M} \to M$ can be endowed with a hermitian connection; let $-2\pi i \Omega(N_{\widehat{M}})$ be its curvature. The **associated characteristic form** (appearing in the index formulas) is

$$D_{\widehat{M}} = \det \left(I - A^{-1}(N_{\widehat{M}}) \exp(-\Omega(N_{\widehat{M}})) \right) .$$

The **associated Chern character** of a complex vector bundle $E \to \widehat{M}$ with curvature $-2\pi i \Omega$ with respect to a hermitian connection is

$$\widehat{\mathrm{Ch}}(E) = \mathrm{trace} \left(A(E) \exp \Omega \right) .$$

APPENDIX B. INTEGRAL ESTIMATE

Lemma B.1. *Let V be a d-dimensional complex vector space, and let A, B be commuting elements of $\mathrm{GL}(V)$, with A diagonalizable. Then*

(B.2)

$$
n^{-(d-1)}\mathrm{trace}\,\mathcal{S}^k(A\exp\tfrac{B}{n}) =
\begin{cases}
\lambda^k\mathrm{Res}\left[\dfrac{e^{\frac{d+k-1}{n}z}}{\det(z-B)}\right]\left(1+O\left(\tfrac{1}{n}\right)\right) & \text{if } A = \lambda I \\[2em]
O\left(\tfrac{1}{n}\right) & \text{otherwise}
\end{cases}
$$

where the $O(\tfrac{1}{n})$'s are uniform in k, and the operator Res takes the integral along a contour about the origin containing the zeroes of $\det(z-B)$ and divides it by $2\pi i$.

Proof.   Without loss of generality, we can assume that A and B are simultaneously diagonalizable with eigenvalues $\lambda_1, \dots, \lambda_d$ (of A) and $\mu_1, \dots, \mu_d$ (of B), and that $e^{\mu_1}, \dots, e^{\mu_d}$ are distinct. Let C be a contour about the origin containing $\lambda_1, \dots, \lambda_d$. For n large enough, the left hand side of (B.2) is equal to the contour integral

$$
n^{-(d-1)}\frac{1}{2\pi i}\int_C \frac{z^{d+k-1}}{\prod_j\left(z - \lambda_j e^{\mu_j/n}\right)}dz \ .
$$

The pole at $\lambda_j e^{\mu_j/n}$ has residue

(B.3) $$
n^{-(d-1)}\frac{\left(\lambda_j e^{\mu_j/n}\right)^{d+k-1}}{\prod_{i\neq j}\left(\lambda_j e^{\mu_j/n} - \lambda_i e^{\mu_i/n}\right)}
$$

where

$$
\lambda_j e^{\mu_j/n} - \lambda_i e^{\mu_i/n} =
\begin{cases}
\lambda_j\frac{\mu_j-\mu_i}{n}\left(1+O(\tfrac{1}{n})\right) & \text{if } \lambda_j = \lambda_i \\
(\lambda_j - \lambda_i)\left(1+O(\tfrac{1}{n})\right) & \text{if } \lambda_j \neq \lambda_i \ .
\end{cases}
$$

There are $(d-1)$ factors in the denominator of (B.3). Therefore, if A has at least two different eigenvalues, all residues (B.3) are of order $O(\tfrac{1}{n})$. If $A = \lambda I$, then we express $n^{-(d-1)}\mathrm{trace}\,\mathcal{S}^k(\exp\tfrac{B}{n})$ as in Theorem (B2) of [6]. □

By analyticity, (B.2) is valid for any endomorphism $B : V \to V$, not necessarily in $\mathrm{GL}(V)$. When k is of order $O(n)$ (i.e. $k \to \infty$ as $n \to \infty$, but $\frac{k}{n} = O(1)$), (B.2) remains true if all eigenvalues of A have absolute value at most 1.

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