4-Manifolds with a Symplectic Bias

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Abstract. This text reviews some state of the art and open questions on (smooth) 4-manifolds from the point of view of symplectic geometry.

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INTRODUCTION

This text evolved from the first two lectures of a short course presented at the *International Fall Workshop on Geometry and Physics 2007* which took place in Lisbon, 5-8/September/2007.

Section 1 corresponds to the first lecture and focuses on **4-manifolds**. Whereas (closed simply connected) topological 4-manifolds are completely classified, the panorama for smooth 4-manifolds is quite wild: we see how the existence of a smooth structure imposes strong topological constraints, yet for the same topology there can be infinite different smooth structures.

Section 2 – the second lecture – discusses **symplectic 4-manifolds**, in particular, existence and uniqueness of symplectic forms on a given 4-manifold. These questions are particularly relevant to 4-dimensional topology and to mathematical physics, where symplectic manifolds occur as building blocks or as key examples.

Both of these sections describe examples/constructions and invariants/classification with an effort to keep prerequisites to a minimum, essentially to basic differential geometry and topology.

The original course included a third lecture explaining the existence on any orientable 4-manifold of a *folded symplectic form* [9], that is, a closed 2-form which is symplectic except on a separating hypersurface where the form singularities are like the pullback of a symplectic form by a folding map.

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1. 4-MANIFOLDS

1.1. Intersection Form

Very little was known about 4-dimensional manifolds[1] until 1981, when Freedman [24] provided a complete classification of closed[2] simply connected *topological* 4-manifolds, and soon thereafter Donaldson [12] showed that the panorama for *smooth* 4-manifolds was much wilder. Key to this understanding was the *intersection form*.

The **intersection form** of an oriented topological closed 4-manifold M is the symmetric bilinear pairing

$$Q_M : H^2(M;\mathbb{Z}) \times H^2(M;\mathbb{Z}) \to \mathbb{Z} , \qquad Q_M(\alpha,\beta) := \langle \alpha \cup \beta, [M] \rangle ,$$

where $\alpha \cup \beta$ is the *cup product* and $[M]$ is the *fundamental class*. For smooth simply connected manifolds and smooth differential forms representing (non-torsion) cohomology classes, this pairing is simply $Q_M([f],[g]) = \int_M f \wedge g$ on 2-forms $[f]$ and $[g]$.[3]

Since the intersection form Q_M always vanishes on torsion elements, it descends to the quotient group $H^2(M;\mathbb{Z})/\text{torsion}$ where it is represented by a matrix with integer entries after choosing a basis of this free abelian group. The quotient Q_M is a unimodular pairing: the determinant of a matrix representing Q_M is ± 1 by Poincaré duality. Hence, the corresponding (symmetric) matrix is diagonalizable over $\mathbb{R}$ with eigenvalues ± 1. We denote by b_2^+ (respectively, b_2^-) the number of positive (resp. negative) eigenvalues of Q_M counted with multiplicities, i.e., the dimension of a maximal subspace where Q_M is positive-definite (resp. negative-definite).

The **signature** of M is the difference $\sigma := b_2^+ - b_2^-$, whereas the second Betti number is the sum $b_2 = b_2^+ + b_2^-$, i.e., the **rank** of Q_M (the dimension of the domain). The **type** of an intersection form is **definite** if it is positive or negative definite (i.e., $|\sigma| = b_2$) and **indefinite** otherwise. We say that the **parity** of an intersection form Q_M is **even** when $Q_M(\alpha,\alpha)$ is always even, and is **odd** otherwise.

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[1] An n-dimensional **topological manifold** is a second countable Hausdorff space in which every point has a neighborhood homeomorphic to an open euclidean n-dimensional ball. An n-dimensional **smooth manifold** is an an n-dimensional topological manifold admitting homeomorphisms on overlapping neighborhoods which are diffeomorphisms, so that we may define a set of differentiable functions on the whole manifold as functions which are differentiable in each neighborhood. Other kinds of manifolds may be considered with additional structure, the structure on each map being consistent with the overlapping maps.

[2] A **closed manifold** is a compact manifold (without boundary).

[3] For smooth closed oriented 4-manifolds, every element of $H_2(M;\mathbb{Z})$ can be represented by an embedded surface: elements of $H^2(M;\mathbb{Z})$ are in one-to-one correspondence with complex line bundles over M via the Euler class; the zero set of a generic section of a bundle with Euler class α is a smooth surface representing the Poincaré dual of α. If Σ_α and Σ_β are generic surface representatives of the Poincaré duals of $\alpha, \beta \in H^2(M;\mathbb{Z})$, so that their intersections are transverse, then $Q_M(\alpha,\beta)$ is the number of intersection points in $\Sigma_\alpha \cap \Sigma_\beta$ counted with signs depending on the matching of orientations – called the **intersection number**, $\Sigma_\alpha \cdot \Sigma_\beta$. Also, for topological closed simply connected manifolds, since $\pi_2(M) \simeq H_2(M;\mathbb{Z})$, each element of $H_2(M;\mathbb{Z})$ can be represented by an immersed sphere for which double points can be surgically eliminated or perturbed at the cost of increasing the genus, thus yielding (topologically) embedded surfaces as representatives.

1.2. Up to the 80's

We restrict to closed simply connected[4] topological 4-manifolds.

Before the work of Freedman, it had been proved by Rokhlin [64] in 1952 that if such a smooth manifold M has even intersection form Q_M (this amounts to the second Stiefel-Whitney class $w_2(M)$ vanishing), then the signature of Q_M must be a multiple of 16. Milnor [54] showed that, as a consequence of a theorem of Whitehead [85], two such topological manifolds are homotopy equivalent if and only if they have the same intersection form.

Example. Consider the even positive-definite form given by the matrix

$$E_8 := \begin{bmatrix} 2 & 1 & & & & & & \\ 1 & 2 & 1 & & & & & \\ & 1 & 2 & 1 & & & & \\ & & 1 & 2 & 1 & & & \\ & & & 1 & 2 & 1 & & 1 \\ & & & & 1 & 2 & 1 & \\ & & & & & 1 & 2 & \\ & & & & 1 & & & 2 \end{bmatrix}$$

related to the lattice of the Lie algebra with the same name. Since E_8 has signature 8, by Rokhlin's result it cannot occur as the intersection form of a smooth 4-manifold. Indeed there is a topological manifold, called the E_8 *manifold*, with this intersection form, built by plumbing based on the E_8 Dynkin diagram. ♢

Freedman [24] showed that, modulo homeomorphism, such topological manifolds are essentially classified by their intersection forms:

- for an *even* intersection form there is exactly one class, whereas
- for an *odd* intersection form there are exactly two classes distinguished by the *Kirby-Siebenmann invariant* [40] in $\mathbb{Z}/2$, at most one of which admits smooth representatives (smoothness requires vanishing invariant).

Example. For instance, whereas the standard complex projective plane $\mathbb{CP}^2$ has odd intersection form

$$Q_{\mathbb{CP}^2} = [1] \ ,$$

there is a topological manifold, called the *fake projective plane*, with the same intersection form (hence the same homotopy type) yet not homeomorphic to $\mathbb{CP}^2$ and admitting no smooth structure. ♢

The 4-dimensional topological Poincaré conjecture is a corollary of Freedman's theorem: when $H_2(M) = 0$, the manifold M must be homeomorphic to the sphere S^4.

By the way, Freedman's work can extend to a few other simple enough fundamental groups. Very little is known when the fundamental group is large. Yet, any finitely

[4] A manifold is **simply connected** if it is path-connected and every path between two points can be continuously transformed into every other.

presented group occurs as the fundamental group of a closed (smooth) 4-manifold, and such groups are not classifiable.

Freedman's work reduced the classification of closed simply connected topological 4-manifolds to the algebraic problem of classifying unimodular symmetric bilinear forms. Milnor and Husemoller [56] showed that indefinite forms are classified by rank, signature and parity. Up to isomorphism, intersection forms can be:

- if odd indefinite, then $n[1] \oplus m[-1]$;
- if even indefinite, then $\pm 2nE_8 \oplus kH$;
- if definite, there are too many possibilities. For each rank there is a finite number which grows very fast. For instance, there are more than 10^{50} different definite intersection forms with rank 40 [56], so this classification is hopeless in practice.

On the other hand, Donaldson [12] showed that for a smooth manifold an intersection form which is definite must be a diagonal either of 1s or of -1s which we represent by $n[1]$ and $m[-1]$. In particular, it cannot be an even form (unless it is empty, i.e., $H_2(M) = \{0\}$).

Consequently, *the homeomorphism class of a connected simply connected closed oriented smooth 4-manifold is determined by the two integers* (b_2, σ) *– the second Betti number and the signature – and the parity of the intersection form.*

Whereas the existence of a smooth structure imposes strong constraints on the topological type of a manifold, Donaldson also showed that for the same topological manifold there can be infinite different smooth structures. In other words, by far not all intersection forms can occur for smooth 4-manifolds and the same intersection form may correspond to nondiffeomorphic manifolds.

Donaldson's tool was a set of gauge-theoretic invariants, defined by counting with signs the equivalence classes (modulo gauge equivalence) of connections on $\mathrm{SU}(2)$- (or $\mathrm{SO}(3)$-) bundles over M whose curvature has vanishing self-dual part. For a dozen years there was hard work on the invariants discovered by Donaldson but limited advancement on the understanding of smooth 4-manifolds.

1.3. Topological Coordinates

As a consequence of the work of Freedman and Donaldson in the 80's, the numbers (b_2, σ) – the second Betti number and the signature – can be treated as **topological coordinates** determining, together with the parity, the homeomorphism class of a connected simply connected closed oriented *smooth* 4-manifold. Yet, for each pair (b_2, σ) there could well be infinite different (i.e., nondiffeomorphic) smooth manifolds.

Traditionally, the numbers used are $(c_1^2, c_2) := (3\sigma + 2\chi, \chi) = (3\sigma + 4 + 2b_2, 2 + b_2)$, and frequently just the **slope** c_1^2/c_2 is considered. If M admits an almost complex structure J, then (TM, J) is a complex vector bundle, hence has Chern classes [11] $c_1 = c_1(M, J)$ and $c_2 = c_2(M, J)$. Both $c_1^2 := c_1 \cup c_1$ and c_2 may be regarded as numbers since $H^4(M; \mathbb{Z}) \simeq \mathbb{Z}$. They satisfy

- $c_1^2 = 3\sigma + 2\chi$ (by Hirzebruch's signature formula) [87] and

- $c_2 = \chi$ (because the top Chern class is always the Euler class),

justifying the notation for the topological coordinates in this case.

Examples.

- The manifold $\mathbb{CP}^2$ has $(b_2,\sigma) = (1,1)$, i.e., $(c_1^2,c_2) = (9,3)$. We have that $H_2(\mathbb{CP}^2) \simeq \mathbb{Z}$ is generated by the class of a complex projective line inside $\mathbb{CP}^2$. The corresponding intersection form is represented by the matrix $[1]$, translating the fact that two lines meet at one point.
  Reversing the orientation, $\overline{\mathbb{CP}^2}$ has $(b_2,\sigma) = (1,-1)$, i.e., $(c_1^2,c_2) = (3,3)$. The intersection form is now represented by $[-1]$.
- The connected sum[5] $\mathbb{CP}^2 \# \overline{\mathbb{CP}^2}$ has $(b_2,\sigma) = (2,0)$, i.e., $(c_1^2,c_2) = (8,0)$. The corresponding intersection form is represented by

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} .$$

- The product $S^2 \times S^2$ also has $(b_2,\sigma) = (2,0)$ i.e., $(c_1^2,c_2) = (8,4)$. But $\mathbb{CP}^2 \# \overline{\mathbb{CP}^2}$ has an *odd* intersection form whereas $S^2 \times S^2$ has an *even* intersection form represented by

$$H := \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} .$$

  The standard generators of $H_2(S^2 \times S^2)$ are the classes of each factor times a point in the other factor.
- The quartic hypersurface in $\mathbb{CP}^3$

$$K3 = \{[z_0,z_1,z_2,z_3] \in \mathbb{CP}^3 \mid z_0^4 + z_1^4 + z_2^4 + z_3^4 = 0\}$$

  (named in honor of Kummer, Kähler and Kodaira or/and after the famous K2 mountain in the Himalayas) has intersection form represented by

$$-2E_8 \oplus 3H .$$

  This can be seen from studying $K3$ as a singular fibration $E(2)$.

$\diamondsuit$

Geography problems are problems on the existence of simply connected closed oriented 4-dimensional manifolds with some additional structure (such as, a symplectic form or a complex structure) for each pair of topological coordinates; see Section 2.3.

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[5] A **connected sum** $M\#N$ of two 4-manifolds M and N is a manifold formed by cutting out a 4-ball inside each of M and N and identifying the resulting boundary 3-spheres. The intersection form of a connected sum $M\#N$ is (isomorphic to) the direct sum of the intersection forms of the manifold summands: $Q_{M\#N} \simeq Q_M \oplus Q_N$. *Topologically*, the converse is also true as a consequence of Freedman's theorem: if for a simply connected manifold the intersection form splits as a direct sum of two forms, then that manifold is the connected sum of two topological manifolds with those forms.

1.4. Smooth Representatives

Donaldson's work together with work of Furuta [26] in the 90's showed that, for the intersection form Q_M of a smooth manifold M,

- if Q_M is odd, then $Q_M \simeq n[1] \oplus m[-1]$,
- if Q_M is even, then $Q_M \simeq \pm 2nE_8 \oplus kH$ with $k > 2n$ or Q_M is trivial (i.e., $H_2(M) = \{0\}$).

The first set is realized by connected sums

$$M = \left(\#_n \mathbb{CP}^2\right) \# \left(\#_m \overline{\mathbb{CP}^2}\right) .$$

For the second set, notice that with $k > 2n$ and $n \neq 0$ we have

$$\frac{b_2}{|\sigma|} = \frac{16n+2k}{16n} > \frac{16n+4n}{16n} = \frac{5}{4} .$$

When $k \geq 3n$, the forms $\pm 2nE_8 \oplus kH$ are represented by

$$\left(\#_n \overline{K3}\right) \# \left(\#_{k-3n} S^2 \times S^2\right) \qquad \text{and} \qquad \left(\#_n K3\right) \# \left(\#_{k-3n} S^2 \times S^2\right) .$$

Indeed, recall that

$$Q_{K3} = -2E_8 \oplus 3H \qquad \text{and} \qquad Q_{S^2 \times S^2} = H$$

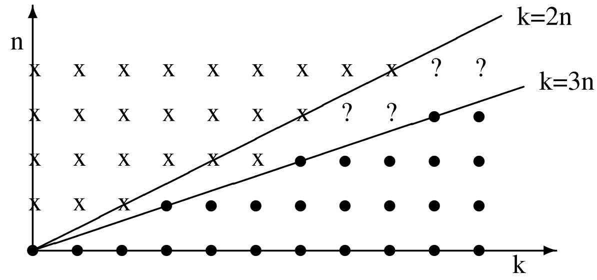
and notice that $H \simeq -H$ by flipping the sign of one of the generators, and $E_8 \oplus (-E_8) \simeq 8H$. In the case $k \geq 3n$, and with $n \neq 0$, we have that

$$\frac{b_2}{|\sigma|} \geq \frac{16n+6n}{16n} = \frac{11}{8} .$$

The question of whether the forms $\pm 2nE_8 \oplus kH$ are realized as the intersection forms Q_M and $Q_{\overline{M}}$ for a smooth manifold M has thus been answered affirmatively for $k \geq 3n$ (represented by dots in the following diagram) and negatively for $k \leq 2n$ (represented by crosses).

The $\frac{11}{8}$ *conjecture* [45] claims that the answer is also no for all points between the two lines. The case corresponding to $n = 2$ and $k = 5$ has been confirmed by Furuta, Kametani and Matsue [27], yet all others (starting with $n = 3$ and $k = 7$ for which the rank is 62), represented by question marks in the following diagram, are still open. If this conjecture holds, then any smooth 4-manifold is *homeomorphic* to either

$$\left(\#_n \mathbb{CP}^2\right) \# \left(\#_m \overline{\mathbb{CP}^2}\right) \text{ or } \left(\#_n K3\right) \# \left(\#_{k-3n} S^2 \times S^2\right) \text{ or } \left(\#_n \overline{K3}\right) \# \left(\#_{k-3n} S^2 \times S^2\right) .$$



1.5. Exotic Manifolds

In dimensions up to 3, each topological manifold has exactly one smooth structure, and in dimensions 5 and higher each topological manifold has at most finitely many smooth structures. Yet there are no known finiteness results for the smooth types of a given topological 4-manifold. Using riemannian geometry, Cheeger [10] showed that there are at most *countably many* different smooth types for closed 4-manifolds.

For open manifolds, the contrast of behavior for dimensions 4 and other is at least as striking. Whereas each topological $\mathbb{R}^n$, $n \neq 4$, admits a unique smooth structure, Taubes [75] showed that the topological $\mathbb{R}^4$ admits uncountably many smooth structures.

A manifold homeomorphic but not diffeomorphic to a smooth manifold M is called an **exotic** M. Finding exotic smooth structures on closed simply connected manifolds with small b_2, dubbed *small 4-manifolds*, has long been an interesting problem, especially in view of the smooth Poincaré conjecture for 4-manifolds: if M is a closed smooth 4-manifold homotopy equivalent to the sphere S^4, is M necessarily diffeomorphic to S^4?

Examples.

- The first exotic smooth structures on a rational surface $\mathbb{CP}^2\#_n\overline{\mathbb{CP}^2}$ were found in the late 80's when Donaldson [13] proved that the Dolgachev surface $E(1)_{2,3}$ is homeomorphic but not diffeomorphic to $\mathbb{CP}^2\#_9\overline{\mathbb{CP}^2}$ by using his invariant based on $\mathrm{SU}(2)$ gauge theory. Shortly thereafter, Friedman and Morgan [25] and Okonek and Van de Ven [59] produced an infinite family of manifolds homeomorphic but not diffeomorphic to $\mathbb{CP}^2\#_9\overline{\mathbb{CP}^2}$. Later Kotschick [42] and Okonek and Van de Ven [60] applied $\mathrm{SO}(3)$ gauge theory to prove that the Barlow surface is homeomorphic but not diffeomorphic to $\mathbb{CP}^2\#_8\overline{\mathbb{CP}^2}$.
- There was no progress until work of Jongil Park [62] in 2004 constructing a symplectic exotic $\mathbb{CP}^2\#_7\overline{\mathbb{CP}^2}$ and using this to exhibit a third distinct smooth structure on $\mathbb{CP}^2\#_8\overline{\mathbb{CP}^2}$, thus illustrating how the existence of symplectic forms links to the existence of different smooth structures. This stimulated research by Fintushel, J. Park, Stern, Stipsicz and Szabó [72, 21, 63], which shows that there are infinitely many exotic smooth structures on $\mathbb{CP}^2\#_n\overline{\mathbb{CP}^2}$ for $n = 5, 6, 7, 8$.

- Last year Fintushel, Doug Park and Stern [20] announced an infinite family of distinct smooth structures on $\mathbb{CP}^2 \#_3 \overline{\mathbb{CP}^2}$, following work by Akhmedov and D. Park [1] and Balbridge and Kirk [7] providing one such exotic structure.

$\diamondsuit$

Still, up to date there is no classification of smooth structures on any given *smoothable* topological 4-manifold. It could well be that any such manifold has infinite smooth structures. There are not even standing structural conjectures. It was speculated that perhaps any simply connected closed smooth 4-manifold other than S^4 is diffeomorphic to a connected sum of symplectic manifolds, where any orientation is allowed on each summand – the so-called *minimal conjecture* for smooth 4-manifolds. Szabó [73, 74] provided counterexamples in a family of irreducible[6] simply connected closed non-symplectic smooth 4-manifolds.

2. SYMPLECTIC 4-MANIFOLDS

2.1. Kähler Structures & Co.

A **symplectic 4-manifold** (M,ω) is a smooth oriented 4-manifold M equipped with a closed 2-form ω such that $\omega \wedge \omega$ is a volume form. In other dimensions, necessarily even, a symplectic manifold is a smooth manifold equipped with a closed nondegenerate 2-form. The form ω is then called a **symplectic form**. Hence, both an algebraic condition – nondegeneracy – and an analytic condition – closedness – come into *symplectiness*. Just as any n-dimensional manifold is locally diffeomorphic to $\mathbb{R}^n$, the *Darboux theorem* states that any symplectic manifold (M^{2n},ω) is locally *symplectomorphic* to $(\mathbb{R}^{2n},\omega_0)$ where $\omega_0 = \sum_{i=1}^n dx_i \wedge dy_i$ in terms of linear coordinates $(x_1,\dots,x_n,y_1,\dots,y_n)$ in $\mathbb{R}^{2n}$. A **symplectomorphism** is a diffeomorphism from one symplectic manifold to another taking one symplectic form to the other.

A **complex manifold** (M,ω) is a smooth manifold M equipped with an atlas of complex coordinate charts for which the transition maps are biholomorphic. On such a manifold, multiplication by i induces a field of linear maps on the tangent spaces $J_p : T_pM \to T_pM$ with $J_p^2 = -\mathrm{Id}$ for each $p \in M$, called an **almost complex structure** J. More concretely, if z_1, z_2 are local complex coordinates on a complex surface (real 4-manifold) with $z_k = x_k + iy_k$, then $\frac{\partial}{\partial x_1}, \frac{\partial}{\partial y_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial y_2}$ span the tangent space at each point and we have that

$$J_p\left(\left.\frac{\partial}{\partial x_k}\right|_p\right) = \left.\frac{\partial}{\partial y_k}\right|_p \qquad \text{and} \qquad J_p\left(\left.\frac{\partial}{\partial y_k}\right|_p\right) = -\left.\frac{\partial}{\partial x_k}\right|_p .$$

This is globally well-defined thanks to the Cauchy-Riemann equations.

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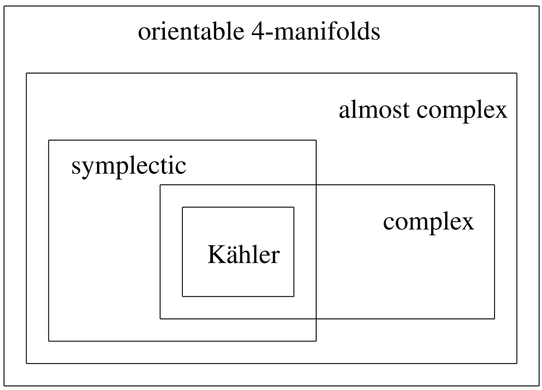
[6] A (smooth) manifold is **irreducible** when it is not a connected sum of other (smooth) manifolds except if one of the summands is a *homotopy sphere*. A **homotopy sphere** is a closed n-manifold which is homotopy equivalent to the n-sphere.

A **Kähler manifold** is a symplectic manifold (M, ω) which is also a complex manifold and where the map that assigns to each point $p \in M$ the bilinear pairing $g_p : T_pM \times T_pM \to \mathbb{R}$, $g_p(u,v) := \omega_p(u, J_pv)$ is a riemannian metric, the map J being the almost complex structure induced by the complex coordinates. This **compatibility condition** comprises the *positivity* $\omega_p(v, J_pv) > 0$ for all $v \neq 0$ and the *symmetry* which translates into $\omega_p(J_pu, J_pv) = \omega_p(u,v)$ for all u, v. The symplectic form ω is then called a **Kähler form**.

A linear algebra argument known as the *polar decomposition* shows that any symplectic vector space, i.e., a vector space equipped with a nondegenerate skew-symmetric bilinear pairing Ω, admits a *compatible* linear complex structure, that is, a linear complex structure J such that $\Omega(\cdot, J\cdot)$ is an inner product. It is enough to start with a choice of an arbitrary inner product G, take the matrix A that satisfies the relation $\Omega(u,v) = G(Au, v)$ for all vectors u, v and consider $J := (\sqrt{AA^t})^{-1}A$, where the square root is well-defined for a positive symmetric matrix.

Being canonical after the choice of G, the above argument may be performed *smoothly* on a symplectic manifold with some riemannian metric. This shows that any symplectic manifold admits compatible almost complex structures.

The following diagram faithfully represents the relations among these structures for *closed* 4-manifolds, where each region admits representatives presented in Section 2.2.



Not all 4-dimensional manifolds are almost complex. A result of Wu [87] gives a necessary and sufficient condition in terms of the signature σ and the Euler characteristic χ of a 4-dimensional closed manifold M for the existence of an almost complex structure: $3\sigma + 2\chi = h^2$ *for some* $h \in H^2(M;\mathbb{Z})$ *congruent with the second Stiefel-Whitney class* $w_2(M)$ *modulo 2*.

Example. The sphere S^4 and $(S^2 \times S^2)\#(S^2 \times S^2)$ are not almost complex. $\diamondsuit$

When an almost complex structure exists, the first Chern class of the tangent bundle (regarded as a complex vector bundle) satisfies the condition for h. The sufficiency of

Wu's condition is the remarkable part.[7]

The Newlander-Nirenberg theorem [58] gives a necessary and sufficient condition for an almost complex manifold (M,J) to actually be complex, i.e., for a J to be actually induced by an underlying complex atlas. That condition can be phrased in terms of a Dolbeault operator or in terms of the vanishing of the *Nijenhuis tensor*:

$$\mathscr{N}(X,Y) := [JX,JY] - J[JX,Y] - J[X,JY] - [X,Y] \ ,$$

for vector fields X and Y on M, $[\cdot,\cdot]$ being the usual bracket.[8]

According to Kodaira's classification [41], a closed complex surface admits a Kähler structure if and only if its first Betti number b_1 is even. The necessity of this condition is a Hodge relation on the Betti numbers: for a compact Kähler manifold, the Hodge theorems [38] imply that the Betti numbers must be the sum of *Hodge numbers* $b^k = \sum_{\ell+m=k} h^{\ell,m}$ where $h^{\ell,m} = h^{m,\ell}$ are integers, hence the odd Betti numbers must be even.

We could go through the previous discussion restricting to closed 4-dimensional examples *with a specific fundamental group*. For simply connected manifolds, it is a consequence of Wu's result [87] that such a manifold admits an almost complex structure if and only if b_2^+ is odd. By Kodaira's classification [41], a simply connected complex surface always admits a compatible symplectic form (since $b^1 = 0$ is even), i.e., it is always Kähler. Hence, the previous picture collapses in this class where being complex implies being Kähler.

2.2. Examples

- The complex projective plane $\mathbb{CP}^2$ with the Fubini-Study form[9] might be called the simplest example of a closed Kähler 4-manifold. All of $\mathbb{CP}^2\#_m\overline{\mathbb{CP}^2}$ are also simply connected Kähler manifolds because they are *pointwise blow-ups* of $\mathbb{CP}^2$. An **ε-blow-up** of a symplectic 4-manifold (M,ω) at a point p is modeled on the ε-blow-up of $\mathbb{C}^2$ at the origin.[10] The resulting symplectic 4-manifold is diffeomorphic

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[7] Moreover, such solutions h are in one-to-one correspondence with *isomorphism* classes of almost complex structures.

[8] The **bracket** of vector fields X and Y is the vector field $[X,Y]$ characterized by the property that $\mathscr{L}_{[X,Y]}f := \mathscr{L}_X(\mathscr{L}_Y f) - \mathscr{L}_Y(\mathscr{L}_X f)$, for $f \in C^\infty(M)$, where $\mathscr{L}_X f = df(X)$.

[9] The 2-form

$$\omega_{\text{FS}} = \tfrac{i}{2}\partial\bar{\partial}\log(|z|^2+1)$$

is a Kähler form on $\mathbb{C}^n$, called the **Fubini-Study form** on $\mathbb{C}^n$. Since ω_{FS} is preserved by the transition maps of the usual complex atlas on $\mathbb{CP}^n$, it induces forms on each chart which glue well together to form the **Fubini-Study form** on $\mathbb{CP}^n$.

[10] *Symplectic blow-up* extends the blow-up operation in algebraic geometry and is due to Gromov according to the first printed exposition of this operation in [46]. The standard *blow-up* of $\mathbb{C}^n$ at the origin replaces it by the total space $\widetilde{\mathbb{C}}^n = \{([p],z) \mid [p] \in \mathbb{CP}^{n-1},\ z = \lambda p \text{ for some } \lambda \in \mathbb{C}\}$ of the tautological line bundle over $\mathbb{CP}^{n-1}$. The fibers of $\pi : \widetilde{\mathbb{C}}^n \to \mathbb{CP}^{n-1}$, $([p],z) \mapsto [p]$, are the complex lines in $\mathbb{C}^n$ represented by each point $[p]$. Under the holomorphic map $\beta : \widetilde{\mathbb{C}}^n \to \mathbb{C}^n$, $([p],z) \mapsto z$, the zero section – called the **exceptional divisor** E and diffeomorphic to $\mathbb{CP}^{n-1}$ – is mapped to the origin, whereas

to the connected sum $M\#\overline{\mathbb{CP}^2}$.

- The **Kodaira-Thurston example** [80] first demonstrated that a manifold that admits both a symplectic and a complex structure does not have to admit any Kähler structure. Take $\mathbb{R}^4$ with $dx_1 \wedge dy_1 + dx_2 \wedge dy_2$, and Γ the discrete group generated by the four symplectomorphisms:

$$\begin{array}{rcl}(x_1,x_2,y_1,y_2) & \longmapsto & (x_1+1,x_2,y_1,y_2)\\ (x_1,x_2,y_1,y_2) & \longmapsto & (x_1,x_2+1,y_1,y_2)\\ (x_1,x_2,y_1,y_2) & \longmapsto & (x_1,x_2+y_2,y_1+1,y_2)\\ (x_1,x_2,y_1,y_2) & \longmapsto & (x_1,x_2,y_1,y_2+1)\end{array}$$

  Then $M = \mathbb{R}^4/\Gamma$ is a symplectic manifold that is a 2-torus bundle over a 2-torus. Kodaira's classification [41] shows that M has a complex structure. However, $\pi_1(M) = \Gamma$, hence $H_1(\mathbb{R}^4/\Gamma;\mathbb{Z}) = \Gamma/[\Gamma,\Gamma]$ has rank 3, so $b_1 = 3$ is *odd*.
- Fernández-Gotay-Gray [19] first exhibited (non simply connected) symplectic manifolds that do not admit any complex structure at all. Their examples are circle bundles over circle bundles (i.e., a *tower* of circle bundles) over a 2-torus.
- There is a family of simply connected manifolds obtained from $\mathbb{CP}^2\#_9\overline{\mathbb{CP}^2} =: E(1)$ by a *knot surgery* that were shown by Fintushel and Stern [22] to be symplectic and confirmed by Jongil Park [61] not to admit a complex structure.[11]
- The **Hopf surface** is the (non simply connected) complex surface diffeomorphic to $S^1 \times S^3$ obtained as the quotient $\mathbb{C}^2\backslash\{0\}/\Gamma$ where $\Gamma = \{2^n\mathrm{Id} \mid n \in \mathbb{Z}\}$ is a group of complex transformations, i.e., we factor $\mathbb{C}^2\backslash\{0\}$ by the equivalence relation $(z_1,z_2) \sim (2z_1,2z_2)$. The Hopf surface is not symplectic because its second cohomology group vanishes (and it is compact).
- The manifold $\mathbb{CP}^2\#\mathbb{CP}^2\#\mathbb{CP}^2$ is almost complex but is neither complex (since it does not fit Kodaira's classification [41]), nor symplectic as shown by Taubes [76, 77] using Seiberg-Witten invariants. Taubes showed that, *when a compact symplectic 4-manifold is of the form $M = M_1\#M_2$, one of the M_i's must have negative definite intersection form.*
- The connected sum $\#_m\mathbb{CP}^2\#_n\overline{\mathbb{CP}^2}$ (of m copies of $\mathbb{CP}^2$ with n copies of $\overline{\mathbb{CP}^2}$) has an almost complex structure if and only if m is odd.

$\widetilde{\mathbb{C}}^n \setminus E$ is diffeomorphic to $\mathbb{C}^n \setminus \{0\}$. The map β is $\mathrm{U}(n)$-equivariant for the action of the unitary group on $\widetilde{\mathbb{C}}^n$ induced by the standard linear action on $\mathbb{C}^n$. Guillemin and Sternberg [37] showed that a $\mathrm{U}(n)$-invariant symplectic form ω on $\widetilde{\mathbb{C}}^n$ for which the difference $\omega - \beta^*\omega_0$ is compactly supported (where $\omega_0 = \frac{i}{2}\sum_{k=1}^n dz_k \wedge d\bar{z}_k$ is the standard symplectic form on $\mathbb{C}^n$) is determined up to $\mathrm{U}(n)$-equivariant diffeomorphism by its restriction to E. Hence, a **symplectic ε-blow-up** of $(\mathbb{C}^n,\omega_0)$ at the origin is defined to be a symplectic manifold $(\widetilde{\mathbb{C}}^n,\omega)$, where ω is $\mathrm{U}(n)$-invariant, $\omega - \beta^*\omega_0$ is compactly supported and the restriction of ω to E is the multiple $\varepsilon\omega_{\mathrm{FS}}$ of the Fubini-Study form. Moreover, we can define a *blow-up of a symplectic manifold (M,ω) along a symplectic submanifold.*

[11] The first example of a closed simply connected symplectic manifold that cannot be Kähler, was a 10-dimensional manifold obtained by McDuff [46] by blowing-up $(\mathbb{CP}^5,\omega_{FS})$ along the image of a symplectically embedded [36, 81] Kodaira-Thurston example $\mathbb{R}^4/\Gamma$.

2.3. Geography and Botany

Symplectic geography [33, 71] addresses the following existence question (cf. Section 1.3):

– What is the set of pairs of integers $(m,n) \in \mathbb{Z} \times \mathbb{Z}$ for which there exists a connected simply connected closed *symplectic* 4-manifold M having second Betti number $b_2(M) = m$ and signature $\sigma(M) = n$?

This problem includes the usual geography of simply connected complex surfaces, since all such surfaces are Kähler according to Kodaira's classification [41]. Often, instead of the numbers (b_2, σ), the question is equivalently phrased in terms of the Chern numbers $(c_1^2, c_2) = (3\sigma + 2\chi, \chi)$ for a compatible almost complex structure, where $\chi = b_2 + 2$ is the *Euler number*; cf. Section 1.3. Usually only *minimal*[12] or *irreducible* (Section 1.5) manifolds are considered to avoid trivial examples. These questions could be posed for other fundamental groups.

A naïve attempt to produce new symplectic manifolds from old is to use connected sums. Yet, in dimensions other than 2 and 6, a connected sum $M_0 \# M_1$ of closed symplectic manifolds (M_0, ω_0) and (M_1, ω_1) does not admit a symplectic form isotopic to ω_i on each M_i minus a ball, $i = 0,1$. The reason is that such a symplectic form on $M_0 \# M_1$ would allow to construct an almost complex structure on the sphere formed by the union of the two removed balls [3], which is known not to exist except on S^2 and S^6.

For connected sums to work in the symplectic category, in particular for 4-manifolds, they should be done along codimension-2 symplectic submanifolds. The following construction, already mentioned in [36], was dramatically explored by Gompf [30]. Let (M_0, ω_0) and (M_1, ω_1) be two $2n$-dimensional symplectic manifolds. Suppose that a compact symplectic manifold (X, α) of dimension $2n-2$ admits symplectic embeddings to both $i_0 : X \hookrightarrow M_0$, $i_1 : X \hookrightarrow M_1$. For simplicity, assume that the corresponding normal bundles are trivial (in general, they need to have symmetric Euler classes). By the *symplectic neighborhood theorem*,[13] there exist symplectic embeddings $j_0 : X \times B_\varepsilon \to M_0$ and $j_1 : X \times B_\varepsilon \to M_1$ (called **framings**) where B_ε is a ball of radius ε and centered at the origin in $\mathbb{R}^2$ such that $j_k^* \omega_k = \alpha + dx \wedge dy$ and $j_k(p,0) = i_k(p)\ \forall p \in X$, $k = 0,1$. Chose an area- and orientation-preserving diffeomorphism ϕ of the annulus $B_\varepsilon \setminus B_\delta$ for $0 < \delta < \varepsilon$ that interchanges the two boundary components. Let $\mathscr{U}_k = j_k(X \times B_\delta) \subset M_k$,

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[12] Following algebraic geometry, a $2n$-dimensional symplectic manifold (M, ω) is **minimal** if it has no symplectically embedded $(\mathbb{CP}^{n-1}, \omega_{FS})$ with normal bundle isomorphic to the tautological bundle, so that (M, ω) is not the blow-up at a point of another symplectic manifold. In dimension 4, a manifold is minimal if it does not contain any embedded sphere S^2 with self-intersection -1. Indeed, by the work of Taubes [76, 78], if such a sphere S exists, then either the homology class $[S]$ or its symmetric $-[S]$ can be represented by a *symplectically* embedded sphere with self-intersection -1.

[13] The **symplectic neighborhood theorem** of Weinstein's [83] says that, *if a compact manifold X embeds as a symplectic submanifold into two symplectic manifolds (M_0, ω_0) and (M_1, ω_1), $i_0 : X \hookrightarrow M_0$, $i_1 : X \hookrightarrow M_1$, with an isomorphism $\widetilde{\phi} : NX_0 \to NX_1$ of the corresponding symplectic normal bundles covering a symplectomorphism $\phi : (X, i_0^* \omega_0) \to (X, i_1^* \omega_1)$, then there exist neighborhoods $\mathscr{U}_0 \subset M_0$, $\mathscr{U}_1 \subset M_1$ of $X_0 := i_0(X)$, $X_1 := i_1(X)$, and a symplectomorphism $\varphi : \mathscr{U}_0 \to \mathscr{U}_1$ such that the restriction of $d\varphi$ to the normal bundle NX_0 is $\widetilde{\phi}$.*

$k = 0,1$. A **symplectic sum** of M_0 and M_1 along X is defined to be

$$M_0 \#_X M_1 := (M_0 \setminus \mathscr{U}_0) \cup_\phi (M_1 \setminus \mathscr{U}_1)$$

where the symbol $\cup_\phi$ means that we identify $j_1(p,q)$ with $j_0(p,\phi(q))$ for all $p \in X$ and $\delta < |q| < \varepsilon$. As ω_0 and ω_1 agree on the regions under identification, they induce a symplectic form on $M_0 \#_X M_1$. The result depends on j_0, j_1, δ and ϕ.

Gompf [30] used symplectic sums to prove that *every finitely-presented group occurs as the fundamental group $\pi_1(M)$ of a compact symplectic 4-manifold (M,ω)*. He also showed that his surgery construction can be adapted to produce *non*-Kähler examples. Since finitely-presented groups are not classifiable, this shows that compact symplectic 4-manifolds are not classifiable.

Symplectic botany [23] addresses the following uniqueness question (cf. Section 1.3):

– Given a pair of integers $(m,n) \in \mathbb{Z} \times \mathbb{Z}$, what are all the connected simply connected closed *symplectic* 4-manifolds M having second Betti number $b_2(M) = m$ and signature $\sigma(M) = n$ (up to diffeomorphism)?

The answer here is still less clear. There has been significant research on classes of surgery operations that can be used to produce examples, such as fiber sums, surgery on tori, blow-up and rational blow-downs. In particular, if a symplectic 4-manifold has nontrivial Seiberg-Witten invariants and contains a symplectically embedded minimal genus torus with self-intersection zero and with simply connected complement, then by *knot surgery* one can show that it also admits infinitely many distinct smooth symplectic structures (as well as infinitely many distinct smooth nonsymplectic structures) [23].

Instead of *smoothly*, the uniqueness question can be studied *symplectically*, where different identifications compete. Let (M,ω_0) and (M,ω_1) be two symplectic manifolds (with the same underlying manifold M).

- (M,ω_0) and (M,ω_1) are **symplectomorphic** if there is a diffeomorphism $\varphi : M \to M$ such that $\varphi^*\omega_1 = \omega_0$.
- (M,ω_0) and (M,ω_1) are **strongly isotopic** if there is an isotopy $\rho_t : M \to M$ such that $\rho_1^*\omega_1 = \omega_0$.
- (M,ω_0) and (M,ω_1) are **deformation-equivalent** if there is a smooth family ω_t of symplectic forms joining ω_0 to ω_1.
- (M,ω_0) and (M,ω_1) are **isotopic** if they are deformation-equivalent and the de Rham cohomology class $[\omega_t]$ is independent of t.
- (M,ω_0) and (M,ω_1) are **equivalent** if they are related by a combination of deformation-equivalences and symplectomorphisms.

Hence, *equivalence* is the relation generated by deformations and diffeomorphisms. The corresponding equivalence classes can be viewed as the connected components of the moduli space of symplectic forms up to diffeomorphism. *Equivalence* deserves this simple designation because this notion allows the cleanest statements about uniqueness when focusing on topological properties.

Examples.

- The complex projective plane $\mathbb{CP}^2$ has a unique symplectic structure up to symplectomorphism and scaling. This was shown by Taubes [77] relating Seiberg-Witten invariants to pseudoholomorphic curves (Section 2.4) to prove the existence of a pseudoholomorphic sphere. Previous work of Gromov [35] and McDuff [48] showed that the existence of a pseudoholomorphic sphere implies that the symplectic form is standard.
- Lalonde and McDuff [43] concluded similar classifications for symplectic ruled surfaces and for symplectic rational surfaces.[14] The symplectic form on a symplectic ruled surface is unique up to symplectomorphism in its cohomology class, and is isotopic to a standard Kähler form. In particular, any symplectic form on $S^2 \times S^2$ is symplectomorphic to $a\pi_1^*\sigma + b\pi_2^*\sigma$ for some $a, b > 0$ where σ is the standard area form on S^2.
- Li and Liu [44] showed that the symplectic structure on $\mathbb{CP}^2 \#_n \overline{\mathbb{CP}^2}$ for $2 \leq n \leq 9$ is unique up to equivalence.
- McMullen and Taubes [53] first exhibited simply connected closed 4-manifolds admitting inequivalent symplectic structures. Their examples were constructed using 3-dimensional topology, and distinguished by analyzing the structure of Seiberg-Witten invariants to show that the first Chern classes of the two symplectic structures lie in disjoint orbits of the diffeomorphism group. In higher dimensions there were previously examples of manifolds with inequivalent symplectic forms; see for instance [65].
- With symplectic techniques and avoiding gauge theory, Smith [69] showed that, for each $n \geq 2$, there is a simply connected closed 4-manifold that admits at least n inequivalent symplectic forms, also distinguished via the first Chern classes. It is not yet known whether there exist inequivalent symplectic forms on a 4-manifold with the same first Chern class.

$\diamondsuit$

2.4. Pseudoholomorphic Curves

Whereas an almost complex manifold (M, J) tends to have no J-holomorphic functions $M \to \mathbb{C}$ at all,[15] it has plenty of *pseudoholomorphic curves* $\mathbb{C} \to M$. In the mid 80's, Gromov first realized that these curves provide a powerful tool in symplectic topology in an extremely influential paper [35].

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[14] A **symplectic rational surface** is a symplectic 4-manifold (M, ω) that can be obtained from the standard $(\mathbb{CP}^2, \omega_{FS})$ by blowing up and blowing down.

[15] Recently, the study of *asymptotically J-holomorphic functions* has been developed for symplectic manifolds [14, 16, 6] leading in particular to a topological description of symplectic 4-manifolds; see Section 2.5.

Fix a closed Riemann surface (Σ, j), that is, a closed complex 1-dimensional manifold Σ equipped with the canonical almost complex structure j. A parametrized **pseudoholomorphic curve** (or *J-holomorphic curve*) in (M,J) is a (smooth) map $u:\Sigma\to M$ whose differential intertwines j and J, that is, $du_p\circ j_p = J_p\circ du_p$, $\forall p\in\Sigma$. The last condition, requiring that du_p be complex-linear, amounts to the **Cauchy-Riemann equation**: $du + J\circ du\circ j = 0$, a well-behaved (elliptic) system of first order partial differential equations.

When J is a compatible almost complex structure on a symplectic manifold (M,ω), pseudoholomorphic curves are related to parametrized 2-dimensional symplectic submanifolds.[16] If a pseudoholomorphic curve $u:\Sigma\to M$ is an embedding, then its image $S:=u(\Sigma)$ is a 2-dimensional almost complex submanifold, hence a symplectic submanifold. Conversely, the inclusion $i:S\hookrightarrow M$ of a 2-dimensional symplectic submanifold can be seen as a pseudoholomorphic curve. An appropriate compatible almost complex structure J on (M,ω) can be constructed starting from S, such that TS is J-invariant. The restriction j of J to TS is necessarily integrable because S is 2-dimensional.

The group G of complex diffeomorphisms of (Σ, j) acts on (parametrized) pseudoholomorphic curves by reparametrization: $u\mapsto u\circ\gamma$, for $\gamma\in G$. This usually means that each curve u has a noncompact orbit under G. The orbit space $\mathscr{M}_g(A,J)$, called the **moduli space of unparametrized pseudoholomorphic curves** of genus g representing the class A, is the set of unparametrized pseudoholomorphic curves in (M,J) whose domain Σ has genus g and whose image $u(\Sigma)$ has homology class $A\in H_2(M;\mathbb{Z})$. For generic J, Fredholm theory shows that pseudoholomorphic curves occur in finite-dimensional smooth families, so that the moduli spaces $\mathscr{M}_g(A,J)$ can be manifolds, after avoiding singularities given by *multiple coverings*.[17]

Example. Often Σ is the Riemann sphere $\mathbb{CP}^1$ whose complex diffeomorphisms are those given by *fractional linear transformations* (or *Möbius transformations*). So the 6-dimensional noncompact group of projective linear transformations $\mathrm{PSL}(2;\mathbb{C})$ acts on *pseudoholomorphic spheres* by reparametrization $u\mapsto u\circ\gamma_A$, where $A=\begin{bmatrix} a & b\\ c & d\end{bmatrix}\in \mathrm{PSL}(2;\mathbb{C})$ acts by $\gamma_A:\mathbb{CP}^1\to\mathbb{CP}^1$, $\gamma_A[z,1]=[\frac{az+b}{cz+d},1]$. ♢

When J is an almost complex structure *compatible* with a symplectic form ω, the area of the image of a pseudoholomorphic curve u (with respect to the metric $g_J(\cdot,\cdot)=\omega(\cdot,J\cdot)$) is determined by the class A that it represents. The number

$$E(u):=[\omega](A)=\int_\Sigma u^*\omega = \text{ area of the image of } u \text{ with respect to } g_J$$

is called the **energy** of the curve u and is a topological invariant: it only depends on the cohomology class $[\omega]$ and on the homotopy class of u. Gromov proved that the constant energy of all the pseudoholomorphic curves representing a homology class A ensured that the space $\mathscr{M}_g(A,J)$, though not necessarily compact, had natural *compactifications*

[16] A **symplectic submanifold** of a symplectic manifold (M,ω) is a submanifold X of M where, at each $p\in X$, the restriction of ω_p to the subspace T_pX is nondegenerate.

[17] A curve $u:\Sigma\to M$ is a **multiple covering** if u factors as $u=u'\circ\sigma$ where $\sigma:\Sigma\to\Sigma'$ is a holomorphic map of degree greater than 1.

$\overline{\mathscr{M}}_g(A,J)$ by including what he called *cusp-curves*. **Gromov's compactness theorem** states that, *if (M,ω) is a compact manifold equipped with a generic compatible almost complex structure J, and if u_j is a sequence of pseudoholomorphic curves in $\mathscr{M}_g(A,J)$, then there is a subsequence that weakly converges to a cusp-curve in $\overline{\mathscr{M}}_g(A,J)$.*

Hence the cobordism class of the compactified moduli space $\overline{\mathscr{M}}_g(A,J)$ is a nice symplectic invariant of (M,ω), as long as it is not empty or null-cobordant. Actually, a nontrivial regularity criterion for J ensures the existence of pseudoholomorphic curves. And even when $\overline{\mathscr{M}}_g(A,J)$ is null-cobordant, we can define an invariant to be the (signed) number of pseudoholomorphic curves of genus g in class A that intersect a specified set of representatives of homology classes in M [66, 77, 86]. For more on pseudoholomorphic curves, see for instance [51] (for a comprehensive discussion of the genus 0 case) or [4] (for higher genus). Here is a selection of applications of (developments from) pseudoholomorphic curves:

- Proof of the **nonsqueezing theorem** [35]: for $R > r$ there is no symplectic embedding of a ball B_R^{2n} of radius R into a cylinder $B_r^2 \times \mathbb{R}^{2n-2}$ of radius r, both in $(\mathbb{R}^{2n},\omega_0)$.
- Proof that there are *no lagrangian spheres*[18] in $(\mathbb{C}^n,\omega_0)$, except for the circle in $\mathbb{C}^2$, and more generally *no compact exact lagrangian submanifolds*, in the sense that the tautological 1-form α restricts to an exact form [35].
- Proof that if (M,ω) is a connected symplectic 4-manifold symplectomorphic to $(\mathbb{R}^4,\omega_0)$ outside a compact set and containing no symplectic S^2's, then (M,ω) symplectomorphic to $(\mathbb{R}^4,\omega_0)$ [35].
- Study questions of **symplectic packing** [8, 50, 82] such as: for a given $2n$-dimensional symplectic manifold (M,ω), what is the maximal radius R for which there is a symplectic embedding of N disjoint balls B_R^{2n} into (M,ω)?
- Study **groups of symplectomorphisms** of 4-manifolds (for a review see [49]). Gromov [35] showed that the groups of symplectomorphisms of $(\mathbb{CP}^2,\omega_{\text{FS}})$ and of $(S^2 \times S^2, \mathrm{pr}_1^*\sigma \oplus \mathrm{pr}_2^*\sigma)$ deformation retract onto the corresponding groups of standard isometries.
- Development of **Gromov-Witten invariants** allowing to prove, for instance, the nonexistence of symplectic forms on $\mathbb{CP}^2\#\mathbb{CP}^2\#\mathbb{CP}^2$ or the classification of symplectic structures on *ruled surfaces*.[19]
- Development of **Floer homology** to prove the Arnold conjecture [2, Appendix 9] on the fixed points of symplectomorphisms of compact symplectic manifolds, or on the intersection of lagrangian submanifolds (see, for instance, [17, 67]).

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[18] A submanifold X of a symplectic manifold (M,ω) is **lagrangian** if, at each $p \in X$, the restriction of ω_p to the subspace T_pX is trivial and $\dim X = \frac{1}{2}\dim M$.

[19] A (rational) **ruled surface** is a complex (Kähler) surface that is the total space of a holomorphic fibration over a Riemann surface with fiber $\mathbb{CP}^1$. When the base is also a sphere, these are the **Hirzebruch surfaces** $\mathbb{P}(L \oplus \mathbb{C})$ where L is a holomorphic line bundle over $\mathbb{CP}^1$. A **symplectic ruled surface** is a symplectic 4-manifold (M,ω) that is the total space of an S^2-fibration where ω is nondegenerate on the fibers.

- Development of **symplectic field theory** introduced by Eliashberg, Givental and Hofer [18] extending Gromov-Witten theory, exhibiting a rich algebraic structure and also with applications to *contact geometry* [29].

2.5. Lefschetz Pencils

Lefschetz pencils in symplectic geometry imitate linear systems in complex geometry. Whereas holomorphic functions on a projective surface must be constant, there are interesting functions on the complement of a finite set, and generic such functions have only quadratic singularities. A Lefschetz pencil can be viewed as a complex Morse function [55] or as a very singular fibration, in the sense that, not only some fibers are singular (have ordinary double points) but all *fibers* go through some points.

A **Lefschetz pencil** on an oriented 4-manifold M is a map $f : M \setminus \{b_1, \ldots, b_n\} \to \mathbb{CP}^1$ defined on the complement of a finite set in M, called the **base locus**, that is a submersion away from a finite set $\{p_1, \ldots, p_{n+1}\}$, and obeying local models $(z_1, z_2) \mapsto z_1/z_2$ near the b_j's and $(z_1, z_2) \mapsto z_1 z_2$ near the p_j's, where (z_1, z_2) are oriented local complex coordinates.

Usually it is also required that each fiber contains at most one singular point. By blowing-up M at the b_j's, we obtain a map to $\mathbb{CP}^1$ on the whole manifold, called a **Lefschetz fibration**. Lefschetz pencils and Lefschetz fibrations can be defined on higher dimensional manifolds where the b_j's are replaced by codimension-4 submanifolds. By working on the Lefschetz fibration, Gompf [31, 32] proved that a structure of Lefschetz pencil (with a nontrivial base locus) gives rise to a symplectic form, canonical up to isotopy, such that the fibers are symplectic.

Using asymptotically holomorphic techniques [5, 14], Donaldson [16] proved that symplectic 4-manifolds admit Lefschetz pencils. More precisely, *if J is a compatible almost complex structure on a compact symplectic 4-manifold (M, ω) where the class $[\omega]$ is integral, i.e., lies in $H^2(M;\mathbb{Z})$, then J can be deformed through almost complex structures to an almost complex structure J' such that M admits a Lefschetz pencil with J'-holomorphic fibers.*

The closure of a smooth fiber of the Lefschetz pencil is a symplectic submanifold Poincaré dual to $k[\omega]$. The starting point is actually a theorem of Donaldson's [14] on the existence of such manifolds: *if (M, ω) is a compact symplectic manifold with $[\omega]$ integral, then, for every sufficiently large integer k, there exists a connected codimension-2 symplectic submanifold representing the Poincaré dual of the integral cohomology class $k[\omega]$.*

Other perspectives on Lefschetz pencils have been explored, including in terms of representations of the free group $\pi_1(\mathbb{CP}^1 \setminus \{p_1, \ldots, p_{n+1}\})$ in the mapping class group Γ_g of the generic fiber surface [70].

Similar techniques were used by Auroux [6] to realize symplectic 4-manifolds as *branched covers* of $\mathbb{CP}^2$, and thus reduce the classification of symplectic 4-manifolds to a (hard) algebraic question about factorization in the braid group. Let M and N be compact oriented 4-manifolds, and let ν be a symplectic form on N. A map $f : M \to N$ is a **symplectic branched cover** if for any $p \in M$ there are complex charts centered at p and $f(p)$ such that ν is positive on each complex line and where f is given by: a local diffeomorphism $(x,y) \to (x,y)$, or a simple branching $(x,y) \to (x^2,y)$, or an ordinary cusp $(x,y) \to (x^3 - xy, y)$. Auroux proved that, *if* (M,ω) *is a compact symplectic 4-manifold with* $[\omega]$ *integral and k is a sufficiently large integer, then there is a symplectic branched cover* $f_k : (M, k\omega) \to \mathbb{CP}^2$*, that is canonical up to isotopy for k large enough. Conversely, given a symplectic branched cover* $f : M \to N$*, the domain M inherits a symplectic form canonical up to isotopy in the class* $f^*[\nu]$.

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REFERENCES

1. Akhmedov, A., Park, B. D., Exotic smooth structures on small 4-manifolds, preprint (2007) math.GT/0701664.
2. Arnold, V., *Mathematical Methods of Classical Mechanics*, Graduate Texts in Math. **60**, Springer-Verlag, New York, 1978.
3. Audin, M., Exemples de variétés presque complexes, *Enseign. Math.* **37** (1991), 175-190.
4. Audin, M., Lafontaine, J., Eds., *Holomorphic Curves in Symplectic Geometry*, Progress in Mathematics **117**, Birkhäuser Verlag, Basel, 1994.
5. Auroux, D., Asymptotically holomorphic families of symplectic submanifolds, *Geom. Funct. Anal.* **7** (1997), 971-995.
6. Auroux, D., Symplectic 4-manifolds as branched coverings of $\mathbb{CP}^2$, *Invent. Math.* **139** (2000), 551-602.
7. Baldridge, S., Kirk, P., A symplectic manifold homeomorphic but not diffeomorphic to $\mathbb{CP}^2 \#_3 \overline{\mathbb{CP}^2}$, preprint (2007) math.GT/0702211.
8. Biran, P., A stability property of symplectic packing, *Invent. Math.* **136** (1999), 123-155.
9. Cannas da Silva, A., Fold-forms for four-folds, preprint (2002).
10. Cheeger, J., Finiteness theorems for Riemannian manifolds, *Amer. J. Math.* **92** (1970), 61-74.
11. Chern, S.S., *Complex Manifolds Without Potential Theory*, with an appendix on the geometry of characteristic classes, second edition, Universitext, Springer-Verlag, New York-Heidelberg, 1979.
12. Donaldson, S., An application of gauge theory to four-dimensional topology, *J. Differential Geom.* **18** (1983), 279-315.
13. Donaldson, S., Irrationality and the h-cobordism conjecture, *J. Differential Geom.* **26** (1987), 141-168.
14. Donaldson, S., Symplectic submanifolds and almost-complex geometry, *J. Differential Geom.* **44** (1996), 666-705.
15. Donaldson, S., Lefschetz fibrations in symplectic geometry, *Proceedings of the I.C.M.*, vol. II (Berlin, 1998), *Doc. Math.* **1998**, extra vol. II, 309-314.
16. Donaldson, S., Lefschetz pencils on symplectic manifolds, *J. Differential Geom.* **53** (1999), 205-236.
17. Donaldson, S., *Floer Homology Groups in Yang-Mills Theory*, with the assistance of M. Furuta and D. Kotschick, Cambridge Tracts in Mathematics **147**, Cambridge University Press, Cambridge, 2002.
18. Eliashberg, Y., Givental, A., Hofer, H., Introduction to symplectic field theory, GAFA 2000 (Tel Aviv, 1999), *Geom. Funct. Anal.* 2000, special volume, part II, 560-673.
19. Fernández, M., Gotay, M., Gray, A., Compact parallelizable four-dimensional symplectic and complex manifolds, *Proc. Amer. Math. Soc.* **103** (1988), 1209-1212.
20. Fintushel, R., Park, B. D., Stern, R., Reverse engineering small 4-manifolds, preprint (2007) math.GT/0701846.
21. Fintushel, R., Stern, R., Double node neighborhoods and families of simply connected 4-manifolds with $b^+ = 1$, *J. Amer. Math. Soc.* **19** (2006), 171-180.

22. Fintushel, R., Stern, R., Knots, links, and 4-manifolds, *Invent. Math.* **134** (1998), 363-400.
23. Fintushel, R., Stern, R., Six lectures on four 4-manifolds, preprint (2007) `math.GT/0610700`.
24. Freedman, M., The topology of four-dimensional manifolds, *J. Differential Geom.* **17** (1982), 357-453.
25. Friedman, R., Morgan, J., On the diffeomorphism types of certain algebraic surfaces, I, *J. Differential Geom.* **27** (1988), 297-369.
26. Furuta, M., Monopole equation and the $\frac{11}{8}$-conjecture, *Math. Res. Lett.* **8** (2001), 279-291.
27. Furuta, M., Kametani, Y., Matsue, H., Spin 4-manifolds with signature $= -32$, *Math. Res. Lett.* **8** (2001), 293-301.
28. Gay, D., Kirby, R., Constructing symplectic forms on 4-manifolds which vanish on circles, *Geom. Topol.* **8** (2004), 743-777.
29. Geiges, H., Contact geometry, *Handbook of Differential Geometry*, vol. II, 315-382, Elsevier/North-Holland, Amsterdam, 2006.
30. Gompf, R., A new construction of symplectic manifolds, *Ann. of Math.* **142** (1995), 527-595.
31. Gompf, R., Toward a topological characterization of symplectic manifolds, *J. Symp. Geom.* **2** (2004), 177-206.
32. Gompf, R., Symplectic structures from Lefschetz pencils in high dimensions, *Geometry and Topology Monographs* **7** (2004), 267-290.
33. Gompf, R., Stipsicz, A., *4-Manifolds and Kirby Calculus*, *Graduate Studies in Mathematics* **20**, Amer. Math. Soc., Providence, 1999.
34. Griffiths, P., Harris, J., *Principles of Algebraic Geometry*, reprint of the 1978 original, Wiley Classics Library, John Wiley & Sons, Inc., New York, 1994.
35. Gromov, M., Pseudoholomorphic curves in symplectic manifolds, *Invent. Math.* **82** (1985), 307-347.
36. Gromov, M., *Partial Differential Relations*, Ergebnisse der Mathematik und ihrer Grenzgebiete **9**, Springer-Verlag, Berlin-New York, 1986.
37. Guillemin, V., Sternberg, S., Birational equivalence in the symplectic category, *Invent. Math.* **97** (1989), 485-522.
38. Hodge, W., *The Theory and Applications of Harmonic Integrals*, 2nd edition, Cambridge University Press, Cambridge, 1952.
39. Honda, K., Transversality theorems for harmonic forms, *Rocky Mountain J. Math.* **34** (2004), 629-664.
40. Kirby, R., Siebenmann, L., *Foundational Essays on Topological Manifolds, Smoothings, and Triangulations*, with notes by J. Milnor and M. Atiyah, Annals of Mathematics Studies **88**, Princeton University Press, Princeton, 1977.
41. Kodaira, K., On the structure of compact complex analytic surfaces, I, *Amer. J. Math.* **86** (1964), 751-798.
42. Kotschick, D., On manifolds homeomorphic to $\mathbb{CP}^2\#8\overline{\mathbb{CP}^2}$, *Invent. Math.* **95** (1989), 591-600.
43. Lalonde, F., McDuff, D., J-curves and the classification of rational and ruled symplectic 4-manifolds, *Contact and Symplectic Geometry (Cambridge, 1994)*, 3-42, *Publ. Newton Inst.* **8**, Cambridge Univ. Press, Cambridge, 1996.
44. Li, T., Liu, A., Symplectic structure on ruled surfaces and a generalized adjunction formula, *Math. Res. Lett.* **2** (1995), 453-471.
45. Matsumoto, Y., On the bounding genus of homology 3-spheres, *J. Fac. Sci. Univ. Tokyo Sect. IA Math.* **29** (1982), 287-318.
46. McDuff, D., Examples of simply-connected symplectic non-Kählerian manifolds, *J. Differential Geom.* **20** (1984), 267-277.
47. McDuff, D., Rational and ruled symplectic 4-manifolds, *Geometry of low-dimensional manifolds* 2 (Durham, 1989), 7-14, *London Math. Soc. Lecture Note Ser.* **151**, Cambridge Univ. Press, Cambridge, 1990.
48. McDuff, D., The structure of rational and ruled symplectic 4-manifolds, *J. Amer. Math. Soc.* **3** (1990), 679-712.
49. McDuff, D., Lectures on groups of symplectomorphisms, *Rend. Circ. Mat. Palermo* (2) **72** (2004), 43-78.
50. McDuff, D., Polterovich, L., Symplectic packings and algebraic geometry, with an appendix by Yael Karshon, *Invent. Math.* **115** (1994), 405-434.

51. McDuff, D., Salamon, D., *J-holomorphic Curves and Symplectic Topology*, Amer. Math. Soc. Colloquium Publications **52**, Amer. Math. Soc., Providence, 2004.
52. McDuff, D., Salamon, D., *Introduction to Symplectic Topology*, Oxford Mathematical Monographs, Oxford University Press, New York, 1995.
53. McMullen, C., Taubes, C., 4-manifolds with inequivalent symplectic forms and 3-manifolds with inequivalent fibrations, *Math. Res. Lett.* **6** (1999), 681-696.
54. Milnor, J., On simply connected 4-manifolds, *1958 International Symposium on Algebraic Topology* 122-128, Universidad Nacional Autónoma de México and UNESCO, Mexico City.
55. Milnor, J., *Morse Theory*, based on lecture notes by M. Spivak and R. Wells, Annals of Mathematics Studies **51**, Princeton University Press, Princeton, 1963.
56. Milnor, J., Husemoller, D., *Symmetric Bilinear Forms*, Springer-Verlag, New York-Heidelberg, 1973.
57. Moser, J., On the volume elements on a manifold, *Trans. Amer. Math. Soc.* **120** (1965), 286-294.
58. Newlander, A., Nirenberg, L., Complex analytic coordinates in almost complex manifolds, *Ann. of Math.* **65** (1957), 391-404.
59. Okonek, C., Van de Ven, A., Stable bundles and differentiable structures on certain elliptic surfaces, *Invent. Math.* **86** (1986), 357-370.
60. Okonek, C., Van de Ven, A., Γ-type-invariants associated to $PU(2)$-bundles and the differentiable structure of Barlow's surface, *Invent. Math.* **95** (1989), 601-614.
61. Park, J., Non-complex symplectic 4-manifolds with $b_2^+ = 1$, *Bull. London Math. Soc.* **36** (2004), 231-240.
62. Park, J., Simply connected symplectic 4-manifolds with $b_2^+ = 1$ and $c_1^2 = 2$, *Invent. Math.* **159** (2005), 657-667.
63. Park, J., Stipsicz, A., Szabó, Z., Exotic smooth structures on $\mathbb{CP}^2\#5\overline{\mathbb{CP}^2}$, *Math. Res. Lett.* **12** (2005), 701-712.
64. Rohlin, V. A., New results in the theory of four-dimensional manifolds, *Doklady Akad. Nauk SSSR (N.S.)* **84** (1952), 221-224.
65. Ruan, Y., Symplectic topology on algebraic 3-folds, *J. Differential Geom.* **39** (1994), 215-227.
66. Ruan, Y., Topological sigma model and Donaldson-type invariants in Gromov theory, *Duke Math. J.* **83** (1996), 461-500.
67. Salamon, D., Lectures on Floer homology, *Symplectic Geometry and Topology* (Eliashberg, Y., Traynor, L., eds.), 143-229, IAS/Park City Math. Ser. **7**, Amer. Math. Soc., Providence, 1999.
68. Scorpan, A., *The Wild World of 4-Manifolds*, Amer. Math. Soc., Providence, 2005.
69. Smith, I., On moduli spaces of symplectic forms, *Math. Res. Lett.* **7** (2000), 779-788.
70. Smith, I., Geometric monodromy and the hyperbolic disc, *Q. J. Math.* **52** (2001), 217-228.
71. Stipsicz, A., The geography problem of 4-manifolds with various structures, *Acta Math. Hungar.* **7** (2000), 267-278.
72. Stipsicz, A., Szabó, Z., An exotic smooth structure on $\mathbb{CP}^2\#6\overline{\mathbb{CP}^2}$, *Geom. Topol.* **9** (2005), 813-832.
73. Szabó, Z., Exotic 4-manifolds with $b_2^+ = 1$, *Math. Res. Lett.* **3** (1996), 731-741.
74. Szabó, Z., Simply-connected irreducible 4-manifolds with no symplectic structures, *Invent. Math.* **132** (1998), 457-466.
75. Taubes, C., Gauge theory on asymptotically periodic 4-manifolds, *J. Differential Geom.* **25** (1987), 363-430.
76. Taubes, C., The Seiberg-Witten invariants and symplectic forms, *Math. Res. Lett.* **1** (1994), 809-822.
77. Taubes, C., The Seiberg-Witten and Gromov invariants, *Math. Res. Lett.* **2** (1995), 221-238.
78. Taubes, C., SW $\Rightarrow$ Gr: from the Seiberg-Witten equations to pseudo-holomorphic curves, *J. Amer. Math. Soc.* **9** (1996), 845-918.
79. Taubes, C., Seiberg-Witten invariants and pseudo-holomorphic subvarieties for self-dual, harmonic 2-forms, *Geom. Topol.* **3** (1999), 167-210.
80. Thurston, W., Some simple examples of symplectic manifolds, *Proc. Amer. Math. Soc.* **55** (1976), 467-468.
81. Tischler, D., Closed 2-forms and an embedding theorem for symplectic manifolds, *J. Differential Geometry* **12** (1977), 229-235.
82. Traynor, L., Symplectic packing constructions, *J. Differential Geom.* **42** (1995), 411-429.
83. Weinstein, A., Symplectic manifolds and their lagrangian submanifolds, *Advances in Math.* **6** (1971), 329-346.

84. Weinstein, A., *Lectures on Symplectic Manifolds*, Regional Conference Series in Mathematics **29**, Amer. Math. Soc., Providence, 1977.
85. Whitehead, J. H. C., On simply connected, 4-dimensional polyhedra, *Comment. Math. Helv.* **22** (1949), 48-92.
86. Witten, E., Topological sigma models, *Comm. Math. Phys.* **118** (1988), 411-449.
87. Wu, W.-T., Sur les classes caractéristiques des structures fibrées sphériques, *Actualités Sci. Ind.* **1183** (1952).