1. Consider the function

$$f : \mathbb{R} \to \mathbb{R}, \qquad f(x) = x^2 e^{-x}.$$

a) Determine the monotonicity properties of f, i.e., determine its critical points and the intervals in which f is strictly monotone increasing, resp. decreasing.

5 Punkte

b) Determine the limits $\lim_{x \to -\infty} f(x)$ and $\lim_{x \to +\infty} f(x)$. 2 Punkte

c) What is the range of $f(x)$? 3 Punkte

2. **a)** Consider the differential equation

$$y'' + 4y = e^x.$$

2 Punkte

Which of the following statements are true?

> Mark your answers directly on this sheet by ticking the correct circle.
> Each statement gives one point and no justification is needed.
> Wrong or multiple answers give zero points.

true	false	
◯	◯	Each solution of the corresponding homogeneous differential equation is a bounded function.
◯	◯	There exists a particular solution of the given inhomogeneous differential equation that is of the form Ae^{-2x} with $A \in \mathbb{R}$.

b) Solve the following initial value problem for $x > 0$

6 Punkte

$$y' + \frac{2y}{x} = \frac{\cos x}{x^2}, \quad y(\pi) = 1 \ .$$

3. **a)** For which values of the parameter $c \in \mathbb{R}$ does the following system have a solution? Why?

$$\begin{pmatrix} 1 & 0 & 0 & 3 \\ 2 & 2 & \boldsymbol{c} & 6 \\ 1 & 4 & 1 & 3 \end{pmatrix} \vec{x} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

4 Punkte

b) We consider the above coefficient matrix with $c = -1$, that is

$$A = \begin{pmatrix} 1 & 0 & 0 & 3 \\ 2 & 2 & -1 & 6 \\ 1 & 4 & 1 & 3 \end{pmatrix}.$$

3 Punkte

Which of the following statements are true?

> Mark your answers directly on this sheet by ticking the correct circle.
> Each statement gives one point and no justification is needed.
> Wrong or multiple answers give zero points.

true	false	
◯	◯	Rank$(A) = 3$.
◯	◯	The system $A\vec{x} = \vec{0}$ has a unique solution.
◯	◯	There exist solutions to the system $A\vec{x} = \vec{b}$ for all $\vec{b} \in \mathbb{R}^3$.

4. Consider the function

$$f(x,y) = x^3 - y^2 - 3xy + 1 \qquad \text{for } (x,y) \in \mathbb{R}^2.$$

a) Which of the following statements are true? 4 Punkte

Mark your answers directly on this sheet by ticking the correct circle.
Each statement gives one point and no justification is needed.
Wrong or multiple answers give zero points.

true	false	
◯	◯	The gradient of f is $\nabla f(x,y) = \begin{pmatrix} 3x^2 - 3y \\ -3x - 2y \end{pmatrix}$.
◯	◯	At the point $(-1,1)$ the function f has the biggest decrease in direction $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$.
◯	◯	The function f has three critical points.
◯	◯	The function f has a local minimum at the point $(-\frac{3}{2}, \frac{9}{4})$.

b) Determine the slope of the level set of f at the point $(-1,0)$.

Hint: Use the tangent to that level set. 4 Punkte

5. **a)** Consider the surface S given by

$$z = x^2 + y^2, \quad 0 \leq z \leq 1.$$

Use Stokes' Theorem in order to compute the downward flux of the rotation of the vectorfield

$$\overrightarrow{F}(x,y,z) := \begin{pmatrix} y+1 \\ z-x \\ x-2 \end{pmatrix}$$

through S. That is, compute

$$\iint_S \text{rot}\,(\overrightarrow{F}) \cdot \overrightarrow{n}\; dA$$

where $\overrightarrow{n}$ points into the negative z-direction. 7 Punkte

b) Let $\overrightarrow{G}$ be a divergence-free vector field on the whole of $\mathbb{R}^3$. 3 Punkte

Which of the following statements are true in general?

> Mark your answers directly on this sheet by ticking the correct circle.
> Each statement gives one point and no justification is needed.
> Wrong or multiple answers give zero points.

true	false	
◯	◯	For the three components of $\overrightarrow{G}$ we have everywhere $$\frac{\partial G_1}{\partial x} + \frac{\partial G_2}{\partial y} + \frac{\partial G_3}{\partial z} = 0.$$
◯	◯	The line integral $\oint_C \overrightarrow{G} \cdot \overrightarrow{dr}$ along a closed curve C is always equal to 0.
◯	◯	The flux $\iint_A \overrightarrow{G} \cdot \overrightarrow{n}\; dA$ out of a closed surface A is always equal to 0.

6. Solve the following wave equation

$$\begin{cases} u_{tt} &= u_{xx} \ , \\ u_x(0,t) &= 0 \ , \\ u_x(1,t) &= 0 \ , \\ u(x,0) &= 2\cos^2(3\pi x) - \cos(7\pi x) \ , \\ u_t(x,0) &= \cos(4\pi x). \end{cases}$$

You may use here the following basis solutions without proof

$$1, \quad \cos(n\pi t)\cdot\cos(n\pi x), \quad \sin(n\pi t)\cdot\cos(n\pi x), \quad n = 1, 2, \ldots.$$

Hint: $\cos^2\theta = \frac{1}{2} + \frac{\cos(2\theta)}{2}$. 7 Punkte

For exercises 7-31: Each question gives two points. Wrong or multiple answers give zero points. **Only** answers **on the answer sheet** count.

7. In which point x has the tangential to the graph of $f(x) = 2x^4$ slope 1?

(a) $x = \dfrac{1}{\sqrt{8}}$ (b) $x = \dfrac{1}{2}$ (c) $x = -\dfrac{1}{2}$ (d) $x = -\dfrac{1}{\sqrt{8}}$

8. Which of the following statements about the function $f(x) = \dfrac{3x^2}{x^2 - 1}$ is **false**?

(a) $\lim\limits_{x \to 1} f(x) = +\infty.$

(b) $\lim\limits_{x \to +\infty} f(x) = 3.$

(c) $\lim\limits_{x \to -\infty} f(x) = 3.$

(d) $\lim\limits_{x \to 0} f(x) = 0.$

9. Which of the following statements about the extremal values of the function

$$f\colon [1,2] \to \mathbb{R},\ f(x) = x \ln x$$

is true?

(a) The minimum is 0 and the maximum is $\ln 4$.

(b) The minimum is 1 and the maximum is $\ln 4$.

(c) The minimum is 0 and the maximum is $\ln(2e)$.

(d) The minimum is 1 and the maximum is $\ln(2e)$.

10. We consider a point (x, y) on the parabola $3y = x^2 - 2$. Let ℓ be the distance of the point (x, y) to the origin $(0, 0)$. Write ℓ as a function of x.

(a) $\ell(x) = |x|$

(b) $\ell(x) = \sqrt{x^4 - 2x^2 + 6}$

(c) $\ell(x) = \frac{1}{3}|x^2 - 2|$

(d) $\ell(x) = \frac{1}{3}\sqrt{x^4 + 5x^2 + 4}$

11. Using separation of variables and partial fraction decomposition one sees that the differential equation

$$y' = e^x(y^2 + y)$$

is equivalent to

(a) $\int \left(\frac{1}{y+1} - \frac{1}{y}\right) dy = e^x + c$ where $c \in \mathbb{R}$.

(b) $\int \left(\frac{2}{y+1} - \frac{1}{y}\right) dy = e^x + c$ where $c \in \mathbb{R}$.

(c) $\int \left(\frac{2}{y} - \frac{1}{y+1}\right) dy = e^x + c$ where $c \in \mathbb{R}$.

(d) $\int \left(\frac{1}{y} - \frac{1}{y+1}\right) dy = e^x + c$ where $c \in \mathbb{R}$.

12. Which statement about the stability of the equilibrium points of the equation

$$y' = y^3 - 4y^2$$

is correct?

(a) $y = 0$ and $y = 4$ are both stable equilibrium points.

(b) $y = 0$ and $y = 4$ are both unstable equilibrium points.

(c) $y = 0$ is a stable equilibrium point and $y = 4$ is unstable.

(d) $y = 0$ is an unstable equilibrium point and $y = 4$ is stable.

13. Which of the following integral diverges?

(a) $\int_0^1 \frac{1}{x^2+1}\, dx$

(b) $\int_0^1 \frac{1}{x^2-1}\, dx$

(c) $\int_0^1 \frac{1}{\sqrt{x}}\, dx$

(d) $\int_0^1 \frac{1}{\sqrt{1-x}}\, dx$

14. Let $z \in \mathbb{C}\backslash\{0\}$. Which of the following complex numbers is always purely imaginary?

(a) $e^{i\,\mathrm{Im}\,z}$ (b) $\dfrac{1}{i|z|}$ (c) $e^{i|z|}$ (d) $\dfrac{z}{\bar{z}}$

15. Which of the following statements is in general **false** for real invertible $n \times n$-matrices A und B?

(a) $\det(A+B) = \det A + \det B$

(b) $\det(-A) = (-1)^n \det A$

(c) $\det(AB) = \det(BA)$

(d) $\det(A^{-1}) = (\det A)^{-1}$

16. The matrix

$$\begin{pmatrix} 4 & 0 & -2 & 3 \\ -7 & 1 & 4 & 8 \\ 4 & 0 & -2 & 6 \\ 0 & 0 & 0 & 5 \end{pmatrix}$$

has eigenvalues 0, 1, 5 and ...

(a) -2. (b) 3. (c) -3. (d) 2.

17. Consider a differential equation of the form

$$y'' + by' + c = 0,$$

whose zeros of the characteristic equation form a pair of complex conjugate numbers, $\alpha \pm \beta i$. Here $\alpha, \beta, b, c \in \mathbb{R}$ and $\beta \neq 0$. When exactly do all solutions to the differential equation stay bounded for all $t \geq 0$?

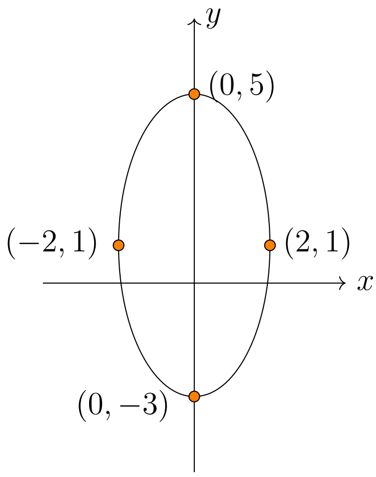
(a) When $\alpha < 1$ and $\beta < 0$.

(b) When $\alpha \leq 0$ and β arbitrary.

(c) When $\alpha \geq 0$ and $\beta < 0$.

(d) When $\alpha > 1$ and β is arbitrary.

18. Which is a parametrisation of the following ellipse, with $t \in [0, 2\pi]$?



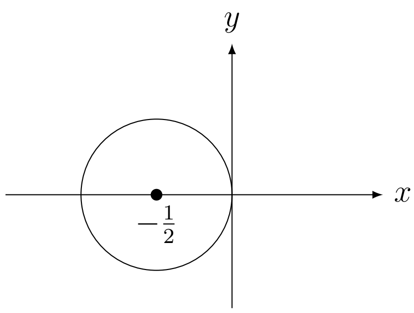
(a) $\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} 2\sin t \\ 1 + 4\cos t \end{pmatrix}$.

(b) $\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} 2\sin t \\ \cos t - 1 \end{pmatrix}$.

(c) $\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} \frac{\sin t}{2} \\ \frac{\cos t - 1}{4} \end{pmatrix}$.

(d) $\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} \frac{1}{2}\sin t \\ \frac{1}{4} + \cos t \end{pmatrix}$.

19. One of the following equations describes the circle depicted below in polar coordinates. Which one is it?



(a) $r = \theta^2$

(b) $r = \sin(2\theta)$

(c) $r = -\cos(\theta)$

(d) $r = -\tan(\theta)$

20. The intersection curves of planes of the form $y =$ Const. with the graph of the function $f(x, y) = 2xy - 3y^2 + y$ are

(a) straight lines.

(b) ellipses

(c) parabola

(d) hyperbolas

21. Let

$$T(x, y) = 1 - 2x + \frac{1}{e}(x^2 + y^2)$$

be the quadratic Taylor polynomial at the point $(x, y) = (0, 0)$ of a function of the form

$$f(x, y) = \ln(x^2 + y^2 + e) + 2kx.$$

Then

(a) $k = -1$

(b) $k = -2$

(c) $k = 1$

(d) $k = 2$

22. Which point $P = (x, y)$ on the branch $x > 0$ of the hyperbola $x^2 - y^2 = 4$ has minimal distance to the point $(0, 2)$?

(a) $P = (3, \sqrt{5})$

(b) $P = (\sqrt{8}, 2)$

(c) $P = (\sqrt{5}, 1)$

(d) $P = (2, \sqrt{2})$

23. For which b is the double integral

$$\int_0^b \int_0^\pi y \sin x \, dx \, dy$$

equal to 2?

(a) $b = \frac{\sqrt{2}}{2}$.

(b) $b = \sqrt{2}$.

(c) $b = 2\sqrt{2}$.

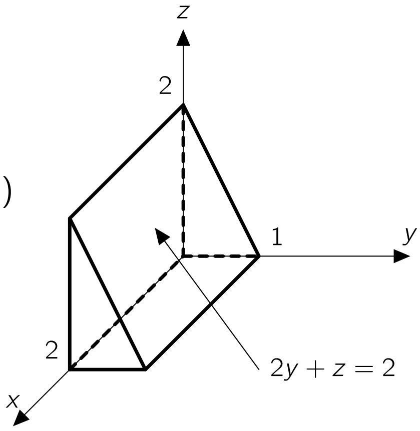
(d) $b = \frac{1}{2}$.

24. Consider the integral

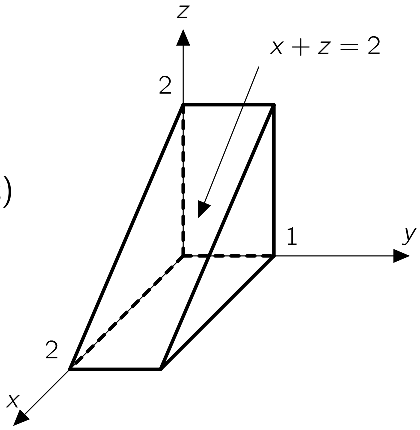
$$\int_0^1 \int_0^2 \int_0^{2-2y} f(x, y, z) \, dz \, dx \, dy.$$

What is the corresponding region of integration?

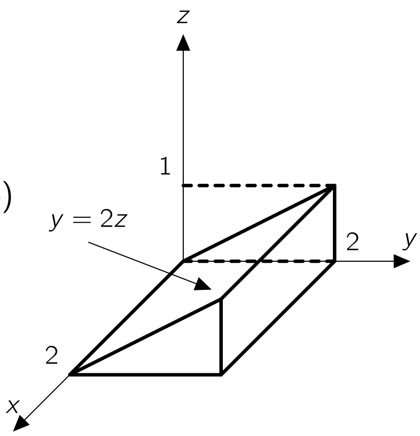
(a)



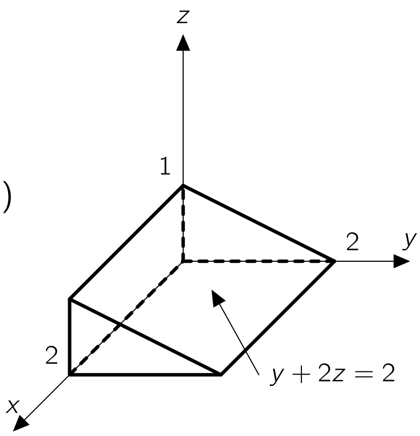
(c)



(b)



(d)



25. The integral

$$\iint_B \sqrt{4 - x^2 - y^2}\, dx\, dy$$

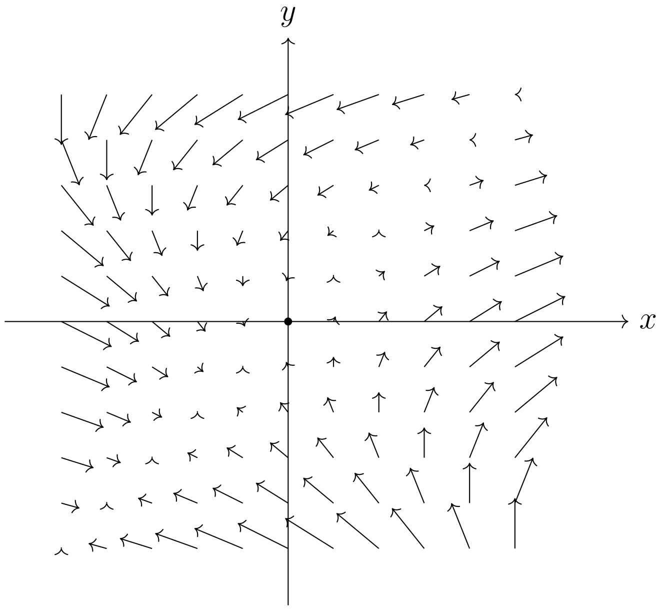
over the region

$$B = \Big\{(x, y) \,|\, x, y \leq 0, 1 \leq x^2 + y^2 \leq 4\Big\}$$

is equal to

(a) $\dfrac{\pi\sqrt{3}}{4}$. (b) $\dfrac{3\pi}{2}$. (c) $\dfrac{3\pi}{4}$. (d) $\dfrac{\pi\sqrt{3}}{2}$.

26. Which vector field corresponds to this drawing?



(a) $\overrightarrow{F} = \begin{pmatrix} x+y \\ x^2 \end{pmatrix}$

(b) $\overrightarrow{F} = \begin{pmatrix} x^2-y^2 \\ x-y \end{pmatrix}$

(c) $\overrightarrow{F} = \begin{pmatrix} -x \\ 2y \end{pmatrix}$

(d) $\overrightarrow{F} = \begin{pmatrix} y \\ x^2+y^2 \end{pmatrix}$

27. Which of the following vector fields $\overrightarrow{F}$ has a potential on $\mathbb{R}^2$?

(a) $\overrightarrow{F}(x,y) = \begin{pmatrix} y-x \\ x-y \end{pmatrix}$.

(b) $\overrightarrow{F}(x,y) = \begin{pmatrix} x-y \\ x+y \end{pmatrix}$.

(c) $\overrightarrow{F}(x,y) = \begin{pmatrix} x^2-y \\ 2xy \end{pmatrix}$.

(d) $\overrightarrow{F}(x,y) = \begin{pmatrix} y-x^2 \\ x^2+y \end{pmatrix}$.

28. On which of the following regions is the vector field

$$\overrightarrow{G}(x,y) = \frac{1}{(x-2)^2+(y-3)^2}\begin{pmatrix} -y+3 \\ x-2 \end{pmatrix}$$

a gradient field?

Hint: The vorticity of $\overrightarrow{G}$ is zero for all $(x,y) \neq (2,3)$.

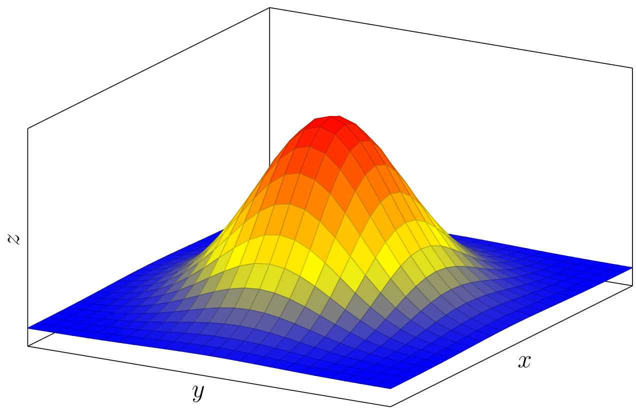
(a) In the first quadrant without $(2,3)$.

(b) In the halfplane $y \leq 0$.

(c) In the annulus $1 \leq (x-2)^2 + (y-3)^2 \leq 2$.

(d) In the plane $\mathbb{R}^2$ without the line segment from the origin to $(2,3)$.

29. Which of the following parametrisations (where (u,v) varies on a square) describes the depicted surface?



(a) $\vec{r}(u,v) = \begin{pmatrix} u \\ v \\ e^{-u^2-v^2} \end{pmatrix}$

(c) $\vec{r}(u,v) = \begin{pmatrix} u \\ v \\ \sin(u^2+v^2) \end{pmatrix}$

(b) $\vec{r}(u,v) = \begin{pmatrix} u \\ v \\ \sqrt{u^2+v^2} \end{pmatrix}$

(d) $\vec{r}(u,v) = \begin{pmatrix} u \\ v \\ 1-u^2-v^2 \end{pmatrix}$

30. What is the outward flux of the vector field

$$\vec{F}(x,y) = \begin{pmatrix} 2x - x\cos(xy) \\ x + y\cos(xy) \end{pmatrix}$$

through the circle $x^2 + y^2 = 1$?

(a) π (b) -2π (c) 2π (d) $-\pi$

31. We consider the partial differential equation

$$u_{xx} = u_{tt} - 3u.$$

With the Ansatz

$$u(x,t) = X(x)T(t)$$

this PDE decomposes into a system of ODE's for $X(x)$ and $T(t)$ in dependence of a parameter $k \in \mathbb{R}$. Which system?

(a) $X'' - kX = 0$ and $T'' + (k-3)T = 0$.

(b) $X'' - kX = 0$ and $T'' - kT = 0$.

(c) $X'' - kX = 0$ and $T'' + kT = 0$.

(d) $X'' - kX = 0$ and $T'' - (k+3)T = 0$.