Eidgenössische Technische Hochschule Zürich
Swiss Federal Institute of Technology Zurich

Family name:	**Department:**
First name:	**ETH ID No.:**

For the grading:

	1K	2K	Points	Comments:
1				
2				
3				
4				
5				
6				
7-26				
Total				

MATHEMATICS I AND II EXAM

for students of Agricultural Science, Earth Sciences, Environmental Sciences, and Food Science

Important:

- Please fill the header on the cover page and lay your ETH-card visible on the table.
- Please write neatly with a non erasable blue or black pen, in particular not with a pencil. Beware that something that is too hard to read could be ignored.
- Please leave some empty space on the margins for the correction.
- This exam has 26 questions and lasts for 180 minutes.

For questions 1-6:

- Please write down all intermediate steps of your calculations and solutions.
- Write your name and ETH ID / Legi-Nr. on each additional sheet.
- The maximal score of each exercise part is given in the right margin.

For questions 7-26:

- Mark your answers clearly.
- There is always only one correct answer and 2 points per question.

Permitted aids:

- Written notes up to 40 A4-Pages, one English dictionary,
- **no** calculator, **no** mobile phone, **no** laptop.
- Please switch off your mobile phone and stow it away.

Good Luck!

1. Consider the function

$$f(x) = \frac{e^{3x}}{x} \text{ for } x \textbf{ positive}.$$

a) Determine and classify the local extrema of $f(x)$. 4 points

b) Determine the range of $f(x)$. 3 points

c) Let $F(x),\ x > 0$ be a function with

$$\begin{cases} F'(x) = f(x) \\ F(1) = 0 \end{cases}$$

and let $G(x)$ be the inverse function of $F(x)$. Then we have that $G(0) = 1$. Determine $G'(0)$. You **do not** have to determine $F(x)$. 3 points

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2. Determine the general solution of each of the following differential equations:

a) $y'' + 2\sqrt{2}y' + 2 = 0$ 5 points

b) $y' - 2xy - x = 0$ für $x > 0$. 5 points

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3. Consider the matrix

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 4 & 1 \end{pmatrix}.$$

a) Determine the eigenvalues of A. 4 points

b) Is A diagonalizable? 2 points

c) For which vectors $\vec{x}_0$ is

$$\vec{x}(t) = e^t \vec{x}_0$$

a solution of

$$\dot{\vec{x}} = A\vec{x} \quad ?$$

4 points

4. Consider the vector field

$$f(x,y) = \ln(x - y^2) \text{ for } x > y^2.$$

and its gradient $\vec{F} = \mathrm{grad}(f)$.

a) Determine the vector field $\vec{F}$. 2 points

b) In which direction does f increase most rapidly at the point $(x,y) = (2,0)$? 2 points

c) Does the equation

$$f(x,y) = 0$$

define a differentiable function of the form $x = x(y)$ or of the form $y = y(x)$ in a neighborhood of the point $(x,y) = (1,0)$? 3 points

d) Determine the line integral of $\vec{F}$ along the straight line C from the point $(1,0)$ to the point $(3,1)$. 3 points

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5. Consider the vector field

$$\vec{F}(x,y,z) = \begin{pmatrix} -y \\ x \\ z^2 \end{pmatrix}$$

and the sphere

$$A : x^2 + y^2 + z^2 = 5.$$

a) Determine $\mathrm{div}(\vec{F})$. 2 points

b) Determine the flux of $\vec{F}$ through A outwards. 3 points

c) Parametrize the intersection curve of A with the plane $z = 1$ (in an arbitrary direction). 3 points

d) Determine the circulation of $\vec{F}$ along the curve from part c) in positive direction when looking from above. 4 points

6. Consider problems of the form

$$\begin{cases} u_t = u_{xx} \\ u_x(0,t) = u_x(1,t) = 0 \\ u(x,0) = f(x) \end{cases}$$

for an unknown function $u(x, y)$ and $0 \leq x \leq 1$, $t > 0$.

a) Determine the solution $u(x, y)$ of the problem with

$$f(x) = 33 - 2\cos(5\pi x).$$

You may use relevant eigenfunctions **without** deriving them. 4 points

b) With the ansatz $u(x, y) = X(x)T(t)$ the PDE can be separated into a system of ODEs

$$X'' - kX = 0 \text{ and } T' - kT = 0.$$

Which boundary conditions does the function $X(x)$ have to fulfill? Determine all functions $X(x)$ that fulfill the boundary condition. 4 points

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For exercises 7-26: Each question gives 2 points. Wrong or multiple answers give 0 points. Mark your answers on this exam.

7. Let

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \\ 0 & 1 & 3 \\ 0 & 1 & 2 \end{pmatrix}.$$

What are the rank and the dimension of the kernel of A?

(a) $\text{rank}(A) = 2$ und $\dim(\ker(A)) = 0$.

(b) $\text{rank}(A) = 2$ und $\dim(\ker(A)) = 1$.

(c) $\text{rank}(A) = 3$ und $\dim(\ker(A)) = 0$.

(d) $\text{rank}(A) = 3$ und $\dim(\ker(A)) = 1$.

8. Let

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 3 & 2 & 0 \\ -4 & 5 & -1 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 2 & 5 & 0 \\ 0 & 3 & -4 \\ 0 & -1 & 2 \end{pmatrix}.$$

Which of the following claims is **wrong**?

(a) $\det(2B^{-1}A^{-1}) = -1.$

(b) $\det(-B^{-1}A^2) = 1.$

(c) $\det(2AB^{-1}) = -4.$

(d) $\det(-2A^{-1}) = 4.$

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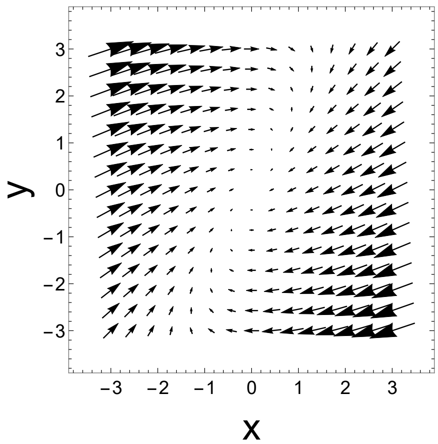
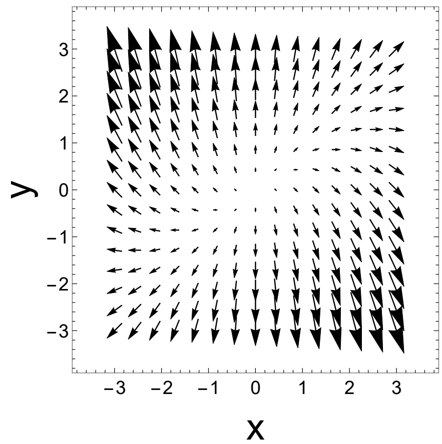
9. Which picture shows the phase portrait of the system

$$\frac{d\vec{x}}{dt} = \begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix} \vec{x} \quad ?$$

(a)



(c)



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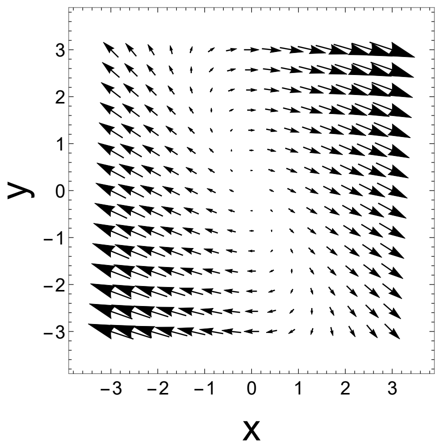

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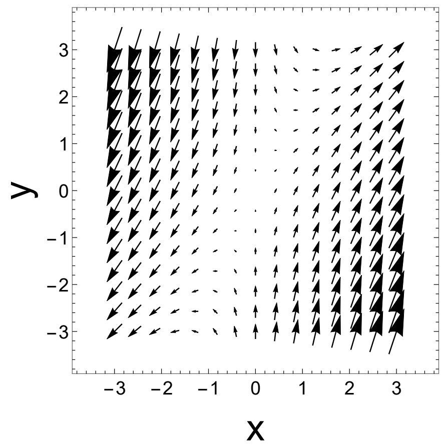

10. A relationship between two measurements x and y has the following form in double logarithmic representation (i.e. instead of x and y, the quantities $a = \log_{10}(x)$ and $b = \log_{10}(y)$ are plotted against the axes):

$$b = 3a - 1.$$

Which function $y = f(x)$ represents this relationship?

(a) $y = 10^{1-3x}$. (b) $y = 10^{3x-1}$. (c) $y = \frac{1}{10}x^3$. (d) $y = 10x^3$.

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11. Which of the following limits exist?

(I) $\lim_{x\to 0} \frac{e^{2x}-2e^x+1}{x^2}$

(II) $\lim_{x\to +\infty} \frac{\cos(x)}{\ln(x)}$

(a) Both limits exist.

(b) Limit (I) exists, but limit (II) does not exist.

(c) Limit (I) does not exist, but limit (II) exists.

(d) Both limits do not exist.

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12. The expression $\frac{(2-4i)^2}{i-3}$ can be transformed into

(a) $-2-6i$. (b) $-2+6i$. (c) $2-6i$. (d) $2+6i$.

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13. Let $z = \frac{1}{2} - \frac{\sqrt{3}}{2}i$. What is the real part of z^9?

(a) $-\frac{\sqrt{3}}{2}$ (b) -1 (c) 0 (d) $\frac{1}{2}$

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14. What is the derivative of the function

$$f(x) = \int_{e^{-x}}^{0} \cos(t^2)\, dt$$

at the point $x = 0$?

(a) -1. (b) $-\cos(1)$. (c) $\cos(1)$. (d) 1.

15. Which is the equation in polar coordinates of the following parabola branch?

$$y = x^2, \quad x > 0$$

(a) $r = \frac{\sin(\theta)}{\cos^2(\theta)},\ \frac{\pi}{4} < \theta < \frac{\pi}{3}.$

(b) $r = \frac{\sin(\theta)}{\cos^2(\theta)},\ 0 < \theta < \frac{\pi}{2}.$

(c) $r = \cos^2(\theta) - \sin(\theta),\ \frac{\pi}{4} \leq \theta < \frac{\pi}{2}.$

(d) $r = \cos^2(\theta) - \sin(\theta),\ 0 < \theta \leq \frac{\pi}{3}.$

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16. The trajectory of a particle satisfies the following initial value problem:

$$\begin{cases} \dfrac{d\vec{r}}{dt} = \begin{pmatrix} -2\sin(t) \\ \cos(t) \end{pmatrix} \\ \vec{r}(0) = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \end{cases}$$

Which of the following points does **not** lie on the trajectory of the particle?

(a) $\begin{pmatrix} -2 \\ 1 \end{pmatrix}$

(b) $\begin{pmatrix} 0 \\ 2 \end{pmatrix}$

(c) $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$

(d) $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$

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17. The domain D of the function

$$f(x, y) = \sqrt{x^2 + y^2 - 3}$$

is

(a) open and bounded.

(b) open and unbounded.

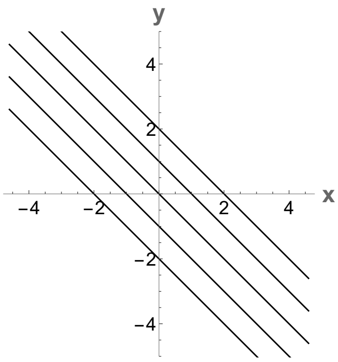
(c) closed and bounded.

(d) closed and unbounded.

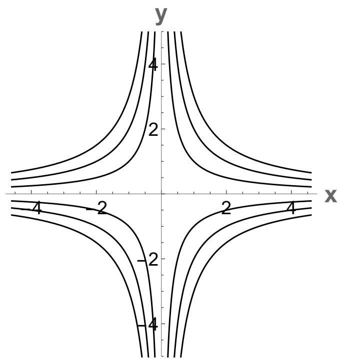
18. Which of the following pictures shows the level curves of the function

$$f(x, y) = e^{x+y+4} \quad ?$$

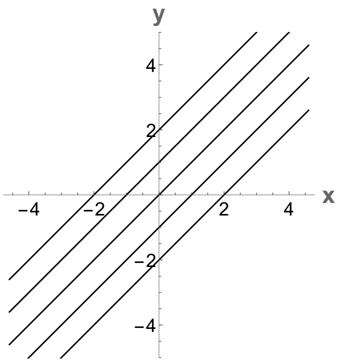
(a)



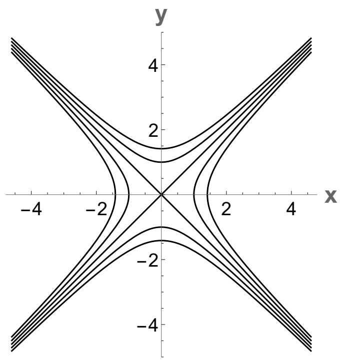
(c)



(b)



(d)



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19. Consider the composition of the function

$$f(x, y) = x^2 y$$

with differentiable functions $x(u)$ and $y(u)$. Calculate the derivative

$$\frac{d}{du} f(x(u), y(u)).$$

(a) $2x(u)x'(u)y'(u)$.

(b) $2x(u)x'(u) + y'(u)$.

(c) $2\,(x(u))^2\, x'(u)y(u)y'(u)$.

(d) $2x(u)x'(u)y(u) + (x(u))^2\, y'(u)$.

20. Which point is a saddle point of

$$f(x,y) = 3x^2 - 3y^2 + 2y^3 + 6xy \quad ?$$

(a) $(-2,2)$. (b) $(-1,-1)$. (c) $(0,-1)$. (d) $(0,0)$.

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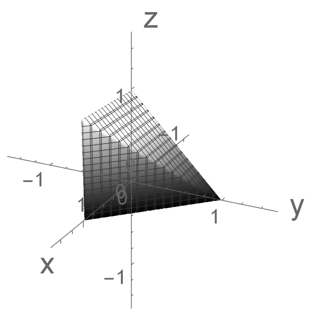
21. Consider an integral of the form

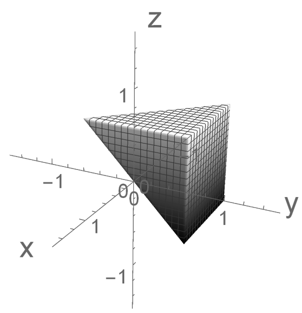
$$\int_0^1 \int_0^{1-z} \int_0^{1-y-z} f(x,y,z)\, dx\, dy\, dz.$$

What is the corresponding domain of integration?

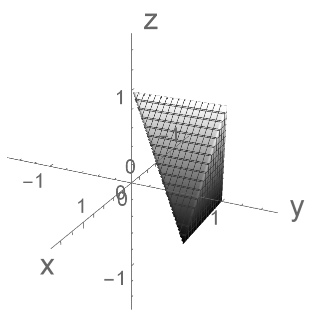
(a)



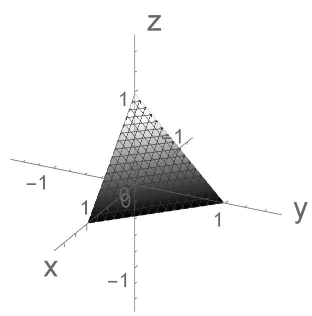
(c)



(b)



(d)



22. Which of the following inequalities describes the solid $V \subseteq \mathbb{R}^3$ given by

$$z \leq \sqrt{3(x^2 + y^2)}$$

in spherical coordinates?

(a) $0 \leq \varphi \leq \frac{\pi}{6}$.

(b) $0 \leq \varphi \leq \frac{\pi}{3}$.

(c) $\frac{\pi}{6} \leq \varphi \leq \pi$.

(d) $\frac{\pi}{3} \leq \varphi \leq \pi$.

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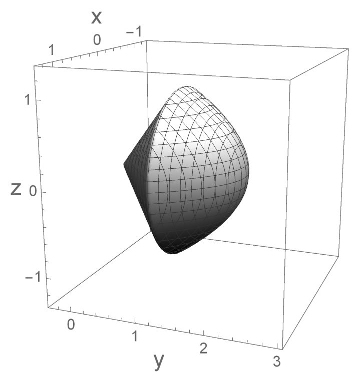
23. The volume of the bounded solid defined by

$$\sqrt{x^2 + z^2} \leq y \leq 2 - x^2 - z^2$$

is



(a) $\frac{5}{12}\pi$.

(b) $\frac{7}{12}\pi$.

(c) $\frac{5}{6}\pi$.

(d) $\frac{7}{6}\pi$.

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24. What is the outwards flux of the vector field

$$\vec{F} = \begin{pmatrix} x - x^2 + 3xy^2 \\ 2x - y^3 - y \end{pmatrix}$$

through the boundary curve of the rectangle

$$0 \leq x \leq 1, \quad -1 \leq y \leq 1 \quad ?$$

(a) -2.

(b) -1.

(c) 1.

(d) 2.

25. Let C be a closed curve in $\mathbb{R}^3$ and let $\vec{F}(x, y, z)$ be a vector field, whose work along C is 2π. Which of the following properties can $\vec{F}$ **not** have?

(a) $\vec{F}$ is divergence-free.

(b) $\vec{F}$ is curl-free.

(c) $\vec{F}$ is a gradient.

(d) $\vec{F}$ is a curl.

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26. Which of the following surfaces in $\mathbb{R}^3$ is **not** simply connected?

(a) The ellipsoid $x^2 + 2y^2 + 3z^2 = 1.$

(b) The cylinder segment $x^2 + y^2 = 1,\ -1 \leq z \leq 1.$

(c) The hemisphere $x^2 + y^2 + z^2 = 1,\ z \leq 0.$

(d) The cone segment $z^2 = x^2 + y^2,\ -1 \leq z \leq 1.$
