1. Consider the function

$$f(x) = e^{1+x^2} - 20 \qquad \text{for } x \in \mathbb{R}.$$

a) Determine the linearization of $f(x)$ in $x_0 = 1$. 3 points

b) Determine the range of $f(x)$. 2 points

c) Let $F(x)$ be the solution of the initial value problem

$$\begin{cases} F'(x) = f(x) \\ F(0) = 2. \end{cases}$$

Is $F(1)$ bigger or smaller than 2? You do **not** have to compute $F(x)$.
Do not forget to justify your solution. 3 points

2. **a)** Determine the general solution of the differential equation

$$y'' - 2y' + 5y = 8e^{-x} \ .$$

4 points

b) Solve the initial value problem

$$2yy' = y^2 + 3, \quad y(0) = 2 \ .$$

4 points

3. Consider the matrix

$$A = \begin{pmatrix} 1 & 2 & 2 & 1 \\ 2 & 6 & 6 & 6 \\ 0 & 0 & 1 & 2 \end{pmatrix}.$$

a) Is the system

$$A\vec{x} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

solvable? If yes, determine a special solution.
If no, explain why no solution exists. 3 points

b) Determine a basis of the solution set of the matrix equation $A\vec{x} = \vec{0}$. 3 points

c) What is the dimension of the space of vectors $\vec{v}$, for which the matrix equation $A\vec{x} = \vec{v}$ is solvable? 2 points

4. Consider the function

$$f(x,y) = x^2 + 2y^3 - 3y^2 + 1 \qquad \text{for } (x,y) \in \mathbb{R}^2.$$

a) Determine and classify the critical points of f (as local maximum, local minimum or saddle point). 4 points

b) We consider the composition $f(x(t), y(t))$ of $f(x,y)$ with the following parametrization of the unit circle:

$$\begin{cases} x(t) = \cos t \\ y(t) = \sin t \end{cases} \qquad \text{for } t \in [0, 2\pi].$$

Determine the derivative of this composition for $t = \pi$. 3 points

c) Let $\vec{F} = \text{grad}(f)$. Determine the line integral of the vector field $\vec{F}$ along the *quarter circle* parametrized by

$$\begin{cases} x(t) = \cos t \\ y(t) = \sin t \end{cases} \qquad \text{for } t \in [0, \tfrac{\pi}{2}].$$

3 points

5. The plane with the equation $y = 2z$ intersects the solid straight circular cylinder $\{(x,y,z) \in \mathbb{R}^3 \mid x^2 + y^2 \leq 4\}$ in a surface S.

a) Parametrize S using cylindrical coordinates. 2 points

b) Determine the area of S. 4 points

c) Using Stokes Theorem, determine the work $\oint_C \vec{F} \cdot d\vec{r}$ done by the vector field

$$\vec{F}(x,y,z) = \begin{pmatrix} yz \\ 2xz \\ x \end{pmatrix} \qquad \text{for } (x,y,z) \in \mathbb{R}^3,$$

when going once around S along the boundary curve C of S in positive direction, when observed from above. 4 points

6. Consider the following vector field

$$\vec{F}(x,y) = \begin{pmatrix} y - \frac{y}{x^2+y^2} \\ x + \frac{x}{x^2+y^2} \end{pmatrix},$$

which is defined for $(x, y) \in \mathbb{R}^2 \setminus \{(0, 0)\}$.

a) Is $\vec{F}$ a gradient field on the first quadrant 3 points

$$Q := \left\{(x, y) \in \mathbb{R}^2 \,\middle|\, x, y > 0\right\}?$$

Yes: ☐ No: ☐

Justification:

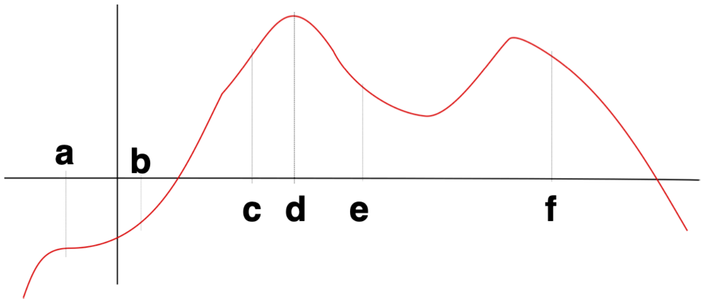
b) Is $\vec{F}$ a gradient field on $\mathbb{R}^2 \setminus \{(0, 0)\}$? 3 points

Yes: ☐ No: ☐

Justification:

For exercises 7-31: Each question gives two points. Wrong or multiple answers give zero points. **Only** answers **on the answer sheet** count.

7. Consider the twice differentiable function f with the following graph. Choose the correct statement:



(a) f' has at least 5 zeros.

(b) f' has at least 2 saddle points.

(c) It holds: $f'(e) \cdot f''(e) \leq 0$.

(d) It holds: $f''(d) \geq f''(b)$.

8. What is the limit

$$\lim_{x \to 0^+} x^{\frac{4}{x}} \quad ?$$

(a) 0 (b) 1 (c) 2 (d) $+\infty$

9. Let $g(y)$ be the inverse of the function

$$y = f(x) = x^3 + e^{(x+1)^5 - 1}.$$

What is the value of the derivative of g at the point $y = f(0) = 1$?

(a) $g'(1) = -5$

(b) $g'(1) = \frac{1}{5}$

(c) $g'(1) = -\frac{1}{5}$

(d) $g'(1) = 5$

10. Determine the number of zeros of the function $f(x) = 2e^x - x - 2$ for $x \in \mathbb{R}$.

(a) 0 (b) 1 (c) 2 (d) 3

11. What is the value of the derivative of the function

$$f(x) = \int_0^{2x} \ln(t^4 + e^4)\, dt$$

in the origin?

(a) $f'(0) = 0$

(b) $f'(0) = 2$

(c) $f'(0) = 4$

(d) $f'(0) = 8$

12. Which of the following statements about the function

$$f(x) = x^4 + x^3$$

on the interval $[-1, 0]$ is true?

(a) f attains on $[-1, 0]$ its global minimum in the point $x = -\frac{1}{3}$.

(b) f attains on $[-1, 0]$ its global minimum in the point $x = -\frac{3}{4}$.

(c) f attains on $[-1, 0]$ its global maximum in the point $x = -\frac{3}{4}$.

(d) f attains on $[-1, 0]$ its global maximum in the point $x = -\frac{1}{3}$.

13. The zeros of the polynomial $p(\lambda) = \lambda^4 + 81$ are:

(a) $-3, 3, -3i, 3i$

(b) $3e^{i\frac{\pi}{4}}, 3e^{i\frac{3\pi}{4}}, 3e^{i\frac{5\pi}{4}}, 3e^{i\frac{7\pi}{4}}$

(c) $-9, 9, -9i, 9i$

(d) $9e^{i\frac{\pi}{4}}, 9e^{i\frac{3\pi}{4}}, 9e^{i\frac{5\pi}{4}}, 9e^{i\frac{7\pi}{4}}$

14. The determinant of the matrix $\begin{pmatrix} 2 & 2 & 0 & 0 \\ 1 & 4 & 1 & 0 \\ 2 & 2 & 3 & 1 \\ 1 & 3 & 0 & 0 \end{pmatrix}$ is

(a) $-4.$ (b) $-1.$ (c) $1.$ (d) $4.$

15. Which equation characterizes the plane in $\mathbb{R}^3$, which goes through the point $(1, 0, -1)$ and is orthogonal to the vector $(6, 5, 4)$?

(a) $x + 2y + 3z = 0$

(b) $x + 2y + 3z = -2$

(c) $6x + 5y + 4z = 0$

(d) $6x + 5y + 4z = 2$

16. For which value of the parameter $c \in \mathbb{R}$ does the matrix

$$A = \begin{pmatrix} c & 1 & 0 \\ -1 & 0 & 0 \\ \pi & 2 & 3 \end{pmatrix}$$

have at least one eigenvalue, which is *not real*?

(a) $c = 1$ (b) $c = 2$ (c) $c = 3$ (d) $c = 4$

17. We consider the initial value problem

$$\dot{\vec{y}}(t) = A\vec{y}(t), \qquad \vec{y}(0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix},$$

where the coefficient matrix A is real and $3i - 4$ is an eigenvalue of A. Which of the following statements about the solution $\vec{y}(t)$ of this IVP is true?

(a) $|\vec{y}(t)|$ stays for sufficiently large t always bigger than 10.

(b) $|\vec{y}(t)|$ stays for sufficiently large t always smaller than $\frac{1}{10}$.

(c) $|\vec{y}(t)|$ stays always constant equal to $\sqrt{2}$.

(d) $|\vec{y}(t)|$ oscillates between values bigger than 10 and smaller than $\frac{1}{10}$.

18. What is the value of the solution of the following initial value problem at time $t = 1$?

$$\dot{\vec{r}}(t) = \begin{pmatrix} -4(e^{-4t} + t) \\ 6t^2 - 1 \end{pmatrix}, \quad \vec{r}(0) = \begin{pmatrix} 1 \\ 5 \end{pmatrix}.$$

(a) $\vec{r}(1) = \begin{pmatrix} -e^4 - 2 \\ 1 \end{pmatrix}$

(b) $\vec{r}(1) = \begin{pmatrix} -e^4 - 2 \\ 6 \end{pmatrix}$

(c) $\vec{r}(1) = \begin{pmatrix} \frac{1}{e^4} - 2 \\ 1 \end{pmatrix}$

(d) $\vec{r}(1) = \begin{pmatrix} \frac{1}{e^4} - 2 \\ 6 \end{pmatrix}$

19. Which integral computes the arc length of the helix with polar equation

$$r = \theta^2 , \quad 0 \le \theta \le 3 \ ?$$

(a) $\displaystyle\int_0^3 \theta\sqrt{\theta^2 + 4}\, d\theta$

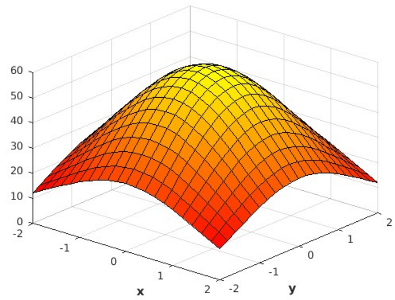
(b) $\displaystyle\int_0^3 \theta^2\sqrt{\theta^2 + 4}\, d\theta$

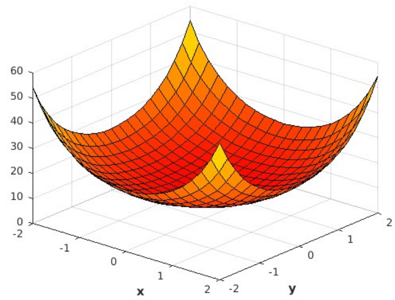
(c) $\displaystyle\int_0^9 \theta\sqrt{9 - \theta}\, d\theta$

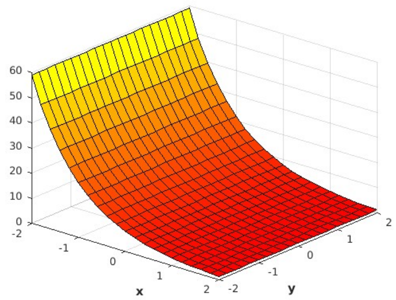
(d) $\displaystyle\int_0^9 \theta^2\sqrt{9 - \theta}\, d\theta$

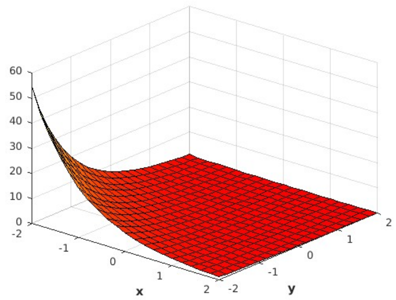
20. Which picture shows the graph of the function

$$f(x, y) = e^{-x-y} \ ?$$

(a)


(b)


(c)


(d)


21. For which value of the parameter b does the equation

$$2x + by + 3z = 11$$

describe the tangent plane of the graph of the function

$$f(x, y) = 4 - \sqrt{1 + x^2 + y^2}$$

at the point $(x_0, y_0) = (2, -2)$?

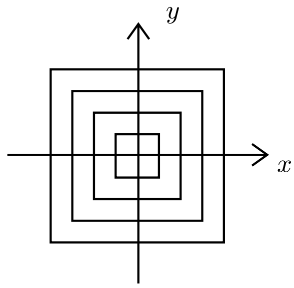
(a) $b = -4$

(b) $b = -2$

(c) $b = 3$

(d) $b = 5$

22. The following picture shows the level sets of a function $f : \mathbb{R}^2 \to \mathbb{R}$.



Which of the following functions has level sets as in the picture?

(a) $f(x, y) = |x| + |y|$

(b) $f(x, y) = |x| - |y|$

(c) $f(x, y) = |x + y| + |x - y|$

(d) $f(x, y) = |x + y| - |x - y|$

23. The lemiscate $y^2(1 - y^2) = x^2$ can be described...

(a) near the point $(1, 0)$ as a graph of a function in x.

(b) near the point $(1, 0)$ as a graph of a function in y.

(c) near the point $(0, 1)$ as a graph of a function in x.

(d) near the point $(0, 1)$ as a graph of a function in y.

24. What is the coefficent c in the quadratic Taylor polynom

$$1 + x - y + cxy + \frac{x^2}{2} + \frac{y^2}{2}$$

of the function $f(x, y) = e^{x-y}$ at the point $(0, 0)$?

(a) $c = -1$ (b) $c = -\frac{1}{2}$ (c) $c = \frac{1}{4}$ (d) $c = 2$

25. For which of the following differential equations is the function

$$f(t,x) = \cos(x-3t) + e^{x+3t}$$

a solution?

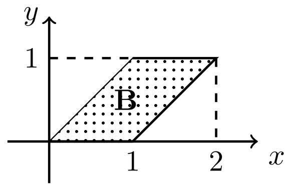
(a) $f_{tt} - 3f_x = 0$

(b) $f_{tt} + 3f_x = 0$

(c) $f_{tt} - 9f_{xx} = 0$

(d) $f_{tt} + 9f_{xx} = 0$

26. Which expression computes the integral of an arbitrary integrable function $f(x,y)$ over the domain B showed in the picture?



(a) $\int_0^1 \int_{1-x}^2 f(x,y)\,dy\,dx$

(b) $\int_0^1 \int_y^{1+y} f(x,y)\,dx\,dy$

(c) $\int_0^2 \int_x^{2-x} f(x,y)\,dy\,dx$

(d) $\int_0^2 \int_1^{y-1} f(x,y)\,dx\,dy$

27. Which integral is in general equal to

$$\int_0^{\sqrt{2}} \int_0^x f(x,y)\,dy\,dx\ ?$$

(a) $\int_0^{\frac{\pi}{4}} \int_0^{\frac{\sqrt{2}}{\cos\theta}} r f(r\cos\theta, r\sin\theta)\,dr\,d\theta.$

(b) $\int_0^{\frac{\pi}{4}} \int_{\frac{\sqrt{2}}{\sin\theta}}^{2} r f(r\cos\theta, r\sin\theta)\,dr\,d\theta.$

(c) $\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \int_0^{\frac{\sqrt{2}}{\sin\theta}} r f(r\cos\theta, r\sin\theta)\,dr\,d\theta.$

(d) $\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \int_{\frac{\sqrt{2}}{\cos\theta}}^{2} r f(r\cos\theta, r\sin\theta)\,dr\,d\theta.$

28. Which of the following equations is satisfied by the surface parametrized by

$$\vec{r}(u,t) = \begin{pmatrix} 1 + u\cos t \\ -1 + u\sin t \\ \ln(u^2+1) \end{pmatrix}, \quad \text{for } u \geq 0, 0 \leq t \leq 2\pi \text{ ?}$$

(a) $(x+1)^2 + (y-1)^2 = e^{z-1}$.

(b) $(x-1)^2 + (y+1)^2 = e^z - 1$.

(c) $\sqrt{(x-1)^2 + (y+1)^2} = e^z - 1$.

(d) $\sqrt{(x+1)^2 + (y-1)^2} = e^{z-1}$.

29. The three-dimensional domain V is described in cartesian coordinates by the following inequalities:

$$2 \leq x^2 + y^2 + z^2 \leq 4, \quad z^2 \leq x^2 + y^2.$$

Which of the following inequalities describes V in spherical coordinates?

(a) $2 \leq \rho \leq 4, \quad \frac{\pi}{4} \leq \varphi \leq \frac{\pi}{2}, \quad 0 \leq \theta \leq 2\pi.$

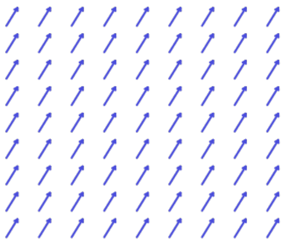
(b) $2 \leq \rho \leq 4, \quad \frac{\pi}{4} \leq \varphi \leq \frac{3\pi}{4}, \quad 0 \leq \theta \leq 2\pi.$

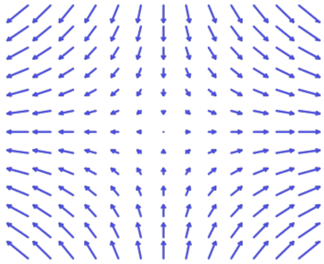
(c) $\sqrt{2} \leq \rho \leq 2, \quad \frac{\pi}{4} \leq \varphi \leq \frac{\pi}{2}, \quad 0 \leq \theta \leq 2\pi.$

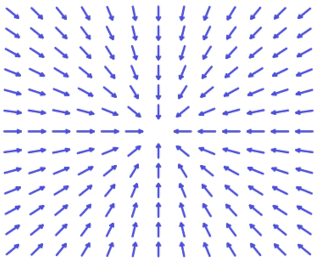
(d) $\sqrt{2} \leq \rho \leq 2, \quad \frac{\pi}{4} \leq \varphi \leq \frac{3\pi}{4}, \quad 0 \leq \theta \leq 2\pi.$

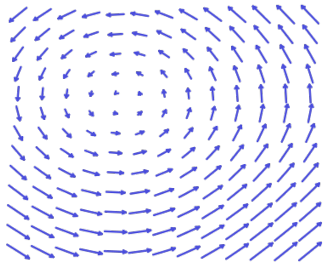
30. Which picture shows the vector field

$$\vec{F}(x,y) = \begin{pmatrix} x \\ -y \end{pmatrix} \; ?$$

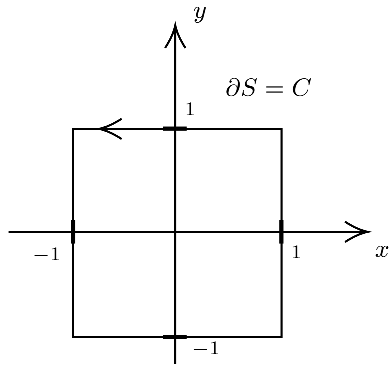
(a)


(c)


(b)


(d)


31. Let S be the square with boundary curve C as shown in the picture.



Let $\vec{F}_a$ be the a-dependent vector field

$$\vec{F}_a(x,y) = \begin{pmatrix} x - 4xy - 2y \\ 2a(3x - y) \end{pmatrix}.$$

For which a is the work of $\vec{F}_a$ along the curve C in counter-clockwise direction equal to 4? That is, for which a is $\oint_C \vec{F}_a \cdot d\vec{r} = 4$?

(a) $a = -\frac{1}{2}$

(b) $a = -\frac{1}{6}$

(c) $a = \frac{1}{6}$

(d) $a = \frac{1}{2}$