

EXISTENCE AND UNIQUENESS RESULTS FOR THE CONTINUITY EQUATION AND APPLICATIONS TO THE CHROMATOGRAPHY SYSTEM

LUIGI AMBROSIO*, GIANLUCA CRIPPA †, ALESSIO FIGALLI ‡, AND
LAURA V. SPINOLO §

Abstract. We discuss some new well-posedness results for the continuity equation in arbitrary space dimension and we then illustrate applications to a system of conservation laws in one space dimension known as the chromatography system. In the last section, we discuss some related open problems.

Key words. Continuity equation, chromatography system, BV functions.

AMS(MOS) subject classifications. 35F10, 35L65.

1. Introduction. We are concerned with the Cauchy problem obtained by coupling the continuity equation

$$(1.1) \quad \partial_t u + \operatorname{div}_x (bu) = 0 \quad (t, x) \in [0, +\infty[\times \mathbb{R}^d, \quad b(t, x) \in \mathbb{R}^d, \quad u \in \mathbb{R}$$

with the initial datum

$$(1.2) \quad u(0, x) = \bar{u}(x).$$

In particular, we focus on the case when b has low regularity with respect to the space variable. In this context, DiPerna and Lions [23] established existence and uniqueness results under the hypothesis that b has Sobolev regularity with respect to the space variable. More recently, Ambrosio [2] proved well-posedness results in the BV (bounded total variation) case: for a discussion concerning these topics, we refer for example to the lecture notes by Ambrosio [3] and Ambrosio and Crippa [5], and to Crippa [17].

In both [2, 23], the analysis required some mild regularity assumptions on the divergence $\operatorname{div}_x b$. These hypotheses, however, are not completely natural in view of the applications, provided by Ambrosio, Bouchut and DeLellis [4, 7], to the Keyfitz and Kranzer system [25] in several space variables. This was a motivation for introducing the notion of *nearly incompressible vector field* (see Definition 2.1 in Section 2.1): the related theory is discussed in the lecture notes by De Lellis [21] and can be more

*Scuola Normale Superiore, Piazza dei Cavalieri 7, 56126 Pisa, Italy. Email: l.ambrosio@sns.it

†Dipartimento di Matematica, Università degli Studi di Parma, viale G.P. Usberti 53/A (Campus), 43100 Parma, Italy. Email: gianluca.crippa@unipr.it

‡Department of Mathematics, The University of Texas at Austin, 1 University Station, C1200, Austin, Texas 78712-1082, USA. Email: figalli@math.utexas.edu

§Centro De Giorgi, Collegio Puteano, Scuola Normale Superiore, Piazza dei Cavalieri 3, 56126 Pisa, Italy. Email: laura.spinolo@sns.it

directly applied to the analysis of the Keyfitz and Kranzer system. As a side remark, we point out that a counter-example due to Crippa and De Lellis [18] rules out the possibility of extending to general systems of conservation laws in several space dimensions the approach used in [4] to tackle the Keyfitz and Kranzer system.

The exposition in here is organized as follows. In Section 2 we describe new well-posedness results for nearly incompressible vector fields, handling in particular the case when the BV norm of $b(t, \cdot)$ blows up as $t \rightarrow 0^+$. In Section 3 we illustrate applications to the chromatography system and in Section 4 we discuss some related open problems.

2. Well-posedness results for the continuity equation.

2.1. Nearly incompressible vector fields. In this section we recall the definition of near incompressibility and a related well-posedness result concerning the Cauchy problem (1.1), (1.2). As mentioned in the introduction, a detailed discussion on these topics can be found in the lecture notes by De Lellis [21].

DEFINITION 2.1. *The bounded vector field b is nearly incompressible if there exists a measurable function ρ and a constant $C > 0$ such that*

$$(2.1) \quad \partial_t \rho + \operatorname{div}_x (b\rho) = 0$$

and

$$(2.2) \quad 0 < \frac{1}{C} \leq \rho(t, x) \leq C \quad \text{for } \mathcal{L}^{d+1} - \text{a.e. } (t, x) \in]0, +\infty[\times \mathbb{R}^d.$$

By combining Corollary 3.14 and Lemma 5.10 in De Lellis [21] one obtains

THEOREM 2.1. *Let $b \in L_{\text{loc}}^\infty \cap BV_{\text{loc}}([0, +\infty[\times \mathbb{R}^d; \mathbb{R}^d)$ be a nearly incompressible vector field and assume that the function ρ in equation (2.1), besides satisfying the bound (2.2), enjoys the following regularity:*

$$\rho \in BV_{\text{loc}}([0, +\infty[\times \mathbb{R}^d), \quad \text{and} \quad \rho(0, \cdot) \in BV_{\text{loc}}(\mathbb{R}^d).$$

Then, for any $\bar{u} \in L^\infty(\mathbb{R}^d)$, there exists a unique bounded, distributional solution of the Cauchy problem (1.1), (1.2).

In the statement of the previous theorem uniqueness can be interpreted as follows. Classical results (see e.g. Dafermos [20, Lemma 1.3.3]) ensure that any bounded, distributional solution of the continuity equation (1.1) admits a representative such that the map $t \mapsto u(t, \cdot)$ is continuous from \mathbb{R} to $L^\infty(\mathbb{R}^d)$ provided that $L^\infty(\mathbb{R}^d)$ is endowed with the w^* -topology. Hence the statement of Theorem 2.1 can be reformulated by saying that the Cauchy problem (1.1), (1.2) admits a unique solution in the space of bounded functions u such that the map $t \mapsto u(t, \cdot)$ is w^* -continuous. In the following, we will always identify a bounded, distributional solution of (1.1) with its w^* -continuous representative.

2.2. A well-posedness result in the class of strongly continuous solutions. Before introducing our first well-posedness result (Theorem 2.2), we make some further preliminary considerations.

An example due to Depauw [22] ensures that if in the statement of Theorem 2.1 one removes the hypothesis that b has BV regularity then the uniqueness result fails. More precisely, Depauw exhibited a time dependent, bounded vector field b taking values in \mathbb{R}^2 and satisfying the following conditions. The vector field $b(t, \cdot)$ has locally bounded total variation for every $t > 0$, but, for any compact set $K \subseteq \mathbb{R}^2$, the norm $\|b(t, \cdot)\|_{BV(K)}$ blows up as $t \rightarrow 0^+$ and the function $t \mapsto \|b(t, \cdot)\|_{BV(K)}$ is not integrable up to $t = 0$. Also, the vector field b is divergence free and hence nearly incompressible (one can take $\rho(t, x) \equiv 1$ in Definition 2.1). In this case the uniqueness part in the statement of Theorem 2.1 fails because Depauw exhibits a non trivial solution of (1.1) satisfying the Cauchy datum $u(0, x) \equiv 0$, namely

$$(2.3) \quad \lim_{t \rightarrow 0^+} u(t, \cdot) = 0 \quad \text{weakly}^* \text{ in } L^\infty(\mathbb{R}^d).$$

Loosely speaking, our first new well-posedness result for the Cauchy problem (1.1)-(1.2) states that, for vector fields b like the one in Depauw's example, uniqueness can be restored provided one imposes additional regularity requirements on the solution u . The point is the following: the space $L^\infty(\mathbb{R}^d)$ can be endowed with the strong topology of $L^1_{\text{loc}}(\mathbb{R}^d)$. As a matter of fact, one can verify that the non trivial solution u in Depauw's example is highly oscillating as $t \rightarrow 0^+$ and hence the limit (2.3) is *not* satisfied if, instead of the w^* topology, we consider the strong topology of $L^1_{\text{loc}}(\mathbb{R}^2)$. Hence, there is hope of obtaining a uniqueness result for the Cauchy problem (1.1)-(1.2) in the class of bounded, distributional solutions u such that the map $t \mapsto u(t, \cdot)$ is continuous with respect to the strong topology in $L^1_{\text{loc}}(\mathbb{R}^d)$.

Before stating our precise well-posedness result, we have to introduce some notations: we denote by B_R the ball of radius R and center at $\vec{0}$ in \mathbb{R}^d . Also, assume that the function $\rho(t, x)$ has, for a given t , locally bounded total variation with respect to the x variable, then the distributional derivative $|D\rho(t, \cdot)|$ is a Radon measure and we denote by $|D\rho(t, \cdot)|(B_R)$ the value assumed on the set B_R .

THEOREM 2.2. *Let $b \in L^\infty \cap BV_{\text{loc}}([0, +\infty[\times \mathbb{R}^d; \mathbb{R}^d)$ be a nearly incompressible vector field and assume that the function ρ in equation (2.1), besides fulfilling the bound (2.2), satisfies $\rho \in BV_{\text{loc}}([0, +\infty[\times \mathbb{R}^d)$ and enjoys the following regularity properties: the map $t \mapsto \rho(t, \cdot)$ is continuous from $[0, +\infty[$ in the space $L^1_{\text{loc}}(\mathbb{R}^d)$ endowed with the strong topology. Also, the map $t \mapsto |D\rho(t, \cdot)|(B_R)$ belongs to $L^\infty_{\text{loc}}([0, +\infty[)$ for every $R > 0$.*

Then, for any $\bar{u} \in L^\infty(\mathbb{R}^d)$, there exists a unique bounded, distributional solution of the Cauchy problem (1.1)-(1.2) such that the map $t \mapsto u(t, \cdot)$ is continuous with respect to the strong topology in $L^1_{\text{loc}}(\mathbb{R}^d)$.

The proof of Theorem 2.2 is provided in [6]: the main issue is actually establishing the existence of a strongly continuous solutions, which is

achieved by approximating b by a sequence of smooth, nearly incompressible vector fields.

2.3. A new well-posedness result for weakly continuous solutions. In this section we discuss Theorem 2.3, another well-posedness result for the Cauchy problem (1.1)-(1.2) which, as Theorem 2.2, can be regarded as an extension of Theorem 2.1. More precisely, in the statement of Theorem 2.3 we keep the same regularity assumptions on b and ρ as in Theorem 2.1, but we relax the assumption that the function ρ in Definition 2.1 is bounded away from 0 by handling the case when ρ attains the value zero.

THEOREM 2.3. *Let $b \in BV_{\text{loc}}([0, +\infty[\times \mathbb{R}^d; \mathbb{R}^d)$ be a bounded vector field and assume that there exists a nonnegative, locally bounded function $\rho \in BV_{\text{loc}}([0, +\infty[\times \mathbb{R}^d)$ satisfying the following assumptions: first, the map $t \mapsto |D\rho(t, \cdot)|(B_R)$ belongs to $L_{\text{loc}}^\infty([0, +\infty[)$ for all $R > 0$. Also, ρ is a distributional solution of*

$$\partial_t \rho + \text{div}_x (b\rho) = 0.$$

If, for some positive constant $K > 0$, the initial datum \bar{u} satisfies $|\bar{u}(x)| \leq K\rho(0, x)$ for a.e. $x \in \mathbb{R}^d$, then there exists a unique bounded, distributional solution of the Cauchy problem (1.1), (1.2) such that

$$(2.4) \quad |u(t, x)| \leq K\rho(t, x) \quad \text{for every } t \geq 0 \text{ and for a.e. } x \in \mathbb{R}^d.$$

Note that, as mentioned before, we are identifying any solution of the continuity equation (1.1) with its w^* -continuous representative and hence the inequality (2.4) is well-defined for every $t \geq 0$.

3. Applications to the chromatography system. In this section we discuss how Theorems 2.2 and 2.3 can be applied to the analysis of the so-called chromatography system, namely

$$(3.1) \quad \begin{cases} \partial_t u_1 + \partial_x \left(\frac{u_1}{1 + u_1 + u_2} \right) = 0 \\ \partial_t u_2 + \partial_x \left(\frac{u_2}{1 + u_1 + u_2} \right) = 0. \end{cases}$$

In the previous expression, $(t, x) \in [0, +\infty[\times \mathbb{R}$ and u_1 and u_2 are both real valued functions. The analysis of (3.1) is motivated by the study of the two-component chromatography (see e.g. Bressan [12, page 102]) and hence one is usually interested in finding nonnegative solutions, $u_1 \geq 0$ and $u_2 \geq 0$. As a matter of fact, both Theorems 3.1 and 3.2 can be easily extended to the case of k components,

$$\partial_t u_i + \partial_x \left(\frac{u_i}{1 + \sum_{j=1}^k u_j} \right) = 0, \quad u_i \geq 0 \quad \text{for } i = 1, \dots, k.$$

However, to simplify the exposition in the following we focus on the case (3.1). Also, we are concerned with the Cauchy problem obtained by coupling (3.1) with the initial data

$$(3.2) \quad u_1(0, x) = \bar{u}_1(x) \quad \text{and} \quad u_2(0, x) = \bar{u}_2(x).$$

3.1. The chromatography system. The chromatography system (3.1) belongs to the so-called Temple class [31], and hence well-posedness results are available even for initial data that violate the classical hypothesis of small total variation (see e.g. the book by Bressan [12] for an overview of the theory of one-dimensional systems of conservation laws with Cauchy data having small total variation). Concerning Temple systems, Serre [30] established existence results under the hypothesis that the total variation of the initial data is bounded but arbitrarily large, while later on Bianchini [10] obtained existence and uniqueness results for initial data that are merely bounded. See also Baiti and Bressan [9] and Bressan and Goatin [14]. The well-posedness results in both Serre [30] and Bianchini [10] apply to general Temple systems, while here we restrict to the chromatography system. However, in [10, 30] the analysis requires a standard hypothesis of so-called *strict hyperbolicity* (see e.g. Dafermos [20, Chapter 7] for the exact definition), which is violated by the chromatography system if the sum $u_1 + u_2$ attains the value 0. For general systems of conservation laws, when strict hyperbolicity is violated even the existence of bounded, distributional solutions may fail (see again Dafermos [20, Section 9.6] for a related discussion). However, in the case of the chromatography system well-posedness results can be extended to the case the sum $u_1 + u_2$ attains the value 0, see Theorem 3.2.

The main difference between the well-posedness results in Serre [30] and Bianchini [10] and Theorems 3.1 and 3.2 lies in their approach: indeed, the analysis in both [10, 30] is based on the introduction of suitable approximation schemes. Conversely, Theorems 3.1 and 3.2 are obtained by introducing a change of variables which transforms (3.1) in the coupling between a scalar conservation law and a continuity equation with a weakly differentiable vector field (see Section 3.2 in here for more details). This is similar to the approach in Bressan and Shen [15] and Panov [28, Remark 4, page 140]. See also the analysis by Ambrosio, Bouchut and De Lellis [4, 7] concerning the so-called Keyfitz and Kranzer system [25], which was inspired by considerations in Bressan [13].

3.2. Well-posedness results for the chromatography system.

In order to apply Theorems 2.2 and 2.3 to the analysis of the chromatography system (3.1) we first introduce the change of variables

$$(3.3) \quad v := u_1 + u_2 \quad w := u_1 - u_2,$$

which transforms (3.1) in the coupling between the scalar conservation law

$$(3.4) \quad \partial_t v + \partial_x \left(\frac{v}{1+v} \right) = 0$$

and the continuity equation

$$(3.5) \quad \partial_t w + \partial_x \left(\frac{w}{1+w} \right) = 0.$$

By applying Kruřkov's theory [26], we get that, for any bounded initial datum, the Cauchy problem for (3.4) admits a unique bounded, entropy admissible, distributional solution. For the definition of the entropy admissibility criterion for conservation laws see e.g. Dafermos [20, Section 3.2].

We can then plug the solution v into (3.5), thus obtaining a continuity equation in the form (1.1) with $b(t, x) := 1/[1+v(t, x)]$. However, it is well known (see again Dafermos [20, Section 4.3]) that, even if the initial datum is extremely regular, in general classical solutions of scalar conservation laws break down in finite time. Hence, when handling (3.5) we are forced to take into account the possibility that the vector field $1/(1+v)$ has low regularity. Nevertheless, note that Kruřkov's theory [26] ensures that equation (3.4) preserves the BV regularity, namely $v(t, \cdot) \in BV(\mathbb{R})$ for every $t > 0$ if the initial datum $v(0, \cdot) \in BV(\mathbb{R})$. Also, in the case of equation (3.4) the flux $v \mapsto 1/(1+v)$ is strictly concave, hence we can apply Oleřnik's estimate [27] and get that, even if the initial datum is merely bounded, the entropy admissible solution of (3.4) satisfies $v(t, \cdot) \in BV(\mathbb{R})$ for every $t > 0$. Note, however, that in this case the BV norm of $v(t, \cdot)$ blows up as $t \rightarrow 0^+$.

Our first well-posedness result for the Cauchy problem (3.1)-(3.2) is the following.

THEOREM 3.1. *Assume that the initial data (3.2) satisfy the following conditions:*

- $\bar{u}_1, \bar{u}_2 \in L_{\text{loc}}^\infty(\mathbb{R})$;
- $\bar{u}_1(x) \geq 0$ and $\bar{u}_2(x) \geq 0$ for a.e. $x \in \mathbb{R}$;
- for every $R > 0$ there exists $\delta_R > 0$ such that $\bar{u}_1 + \bar{u}_2 \geq \delta_R$ for a.e. $x \in]-R, R[$.

Then there exists a unique vector valued function (u_1, u_2) satisfying

1. (u_1, u_2) is an entropy admissible, distributional solution of the Cauchy problem (3.1)-(3.2);
2. the maps $t \mapsto u_1(t, \cdot)$ and $t \mapsto u_2(t, \cdot)$ are both continuous with respect to the strong topology of $L_{\text{loc}}^1(\mathbb{R})$.

Also, the unique vector valued function (u_1, u_2) satisfying 1. and 2. above enjoys the additional property that for every $t \geq 0$ and for a.e. $x \in \mathbb{R}$ $u_1(t, x) \geq 0$ and $u_2(t, x) \geq 0$.

For the proof of Theorem 3.1, we refer to [6], however the main ingredients are the changes of variables (3.3), Kruřkov's theory [26], Theorem 2.2,

Oleřnik's estimate [27], a regularity result due to Chen and Rascle [16] and a result which classify the entropies of (3.1) and ensures that the entropy admissible solutions of (3.1) are exactly those such that the sum $v = u_1 + u_2$ is an entropy admissible solution of (3.4) and the difference $w = u_1 - u_2$ is a distributional solution of (3.5) (see Lemma 4.1 and Proposition 4.2 in [6] for a more precise statement).

In our second well-posedness theorem we remove the assumption that the sum $\bar{u}_1 + \bar{u}_2$ is locally bounded away from 0, but the price we have to pay is that now we impose that $\bar{u}_1 + \bar{u}_2$ has BV regularity.

THEOREM 3.2. *Assume that the initial data (3.2) satisfy the following conditions:*

- $\bar{u}_1, \bar{u}_2 \in L_{\text{loc}}^\infty(\mathbb{R})$;
- $\bar{u}_1(x) \geq 0$ and $\bar{u}_2(x) \geq 0$ for a.e. $x \in \mathbb{R}$;
- $(\bar{u}_1 + \bar{u}_2) \in BV_{\text{loc}}(\mathbb{R})$.

Then the Cauchy problem (3.1), (3.2) admits a unique entropy admissible, distributional solution (u_1, u_2) such that

1. *the maps $t \mapsto u_1(t, \cdot)$ and $t \mapsto u_2(t, \cdot)$ are both continuous with respect to the w^* topology of $L_{\text{loc}}^\infty(\mathbb{R})$;*
2. *for every $t > 0$, $u_1(t, \cdot), u_2(t, \cdot) \in L_{\text{loc}}^\infty(\mathbb{R})$ and $u_1(t, x), u_2(t, x) \geq 0$ for a.e. $x \in \mathbb{R}$.*

Also, this unique solution satisfies the additional regularity property that the maps $t \mapsto u_1(t, \cdot)$ and $t \mapsto u_2(t, \cdot)$ are both continuous with respect to the strong topology of $L_{\text{loc}}^1(\mathbb{R})$.

The proof of Theorem 3.2 is given in [6] and relies on Theorem 2.3 and, again, on Kruřkov's theory [26], on the regularity result by Chen and Rascle [16] and on the classification of the entropies of (3.1).

Note that by applying Theorems 2.2 and 2.3 one can also get well-posedness results for the so-called Keyfitz and Kranzer system [25], in the same spirit as those in Ambrosio, Bouchut and De Lellis [4] (multi-dimensional case with BV data), Freistühler [24] and Panov [28] (one-dimensional case with bounded initial data). See [6, Section 5] for the details.

REMARK 3.1. *Theorems 2.2 and 2.3 are valid in any space dimension, but the applications in Theorems 3.1 and 3.2 are concerned with the one-dimensional case only. In the one-dimensional case, well-posedness results for the continuity equation (1.1) are available under much weaker assumption than those required to deal with higher dimensions: see for example Panov [29]. By combining the classification of the entropies provided by Lemma 4.1 in [6] with the results in [29] one can establish well-posedness results for the Cauchy problem (3.1), (3.2) under the assumption that the initial data \bar{u}_1 and \bar{u}_2 are merely bounded and nonnegative.*

4. Related open problems. For a recent discussion on open problems concerning the continuity equation (1.1) and its extensions, see the contribution by Ambrosio [1] to this volume. Here we focus on the connec-

tion between the issues discussed in Section 2 and the following compactness conjecture, proposed by Bressan [13].

CONJECTURE 4.1. *Let $b_k : [0, +\infty[\times \mathbb{R}^d \rightarrow \mathbb{R}^d$, $k \in \mathbb{N}$, be a sequence of smooth vector fields and denote by X_k the solution of the Cauchy problem*

$$\begin{cases} \frac{dX_k}{dt} = b_k[t, X_k(t, x)] \\ X_k(0, x) = x \end{cases}$$

Assume that $\|b_k\|_\infty + \|\nabla b_k\|_{L^1}$ is uniformly bounded and that the flows X_k satisfy, for some constant $C > 0$,

$$\frac{1}{C} \leq \det(\nabla_x X_k(t, x)) \leq C.$$

Then the sequence $\{X_k\}$ is precompact with respect to the strong topology of $L^1_{\text{loc}}([0, +\infty[\times \mathbb{R}^d)$.

As a matter of fact, uniqueness results for the Cauchy problem (1.1), (1.2) are strongly related to the so-called renormalization properties: indeed, these properties can be used (see e.g. Di Perna and Lions [23] and Ambrosio [2]) to establish a comparison principle. See also Bouchut and Crippa [11] for a characterization of the relation between uniqueness and renormalization. Ambrosio, Bouchut and De Lellis [4] proposed the following renormalization conjecture (note that no regularity assumption is imposed on the function ρ).

CONJECTURE 4.2. *Let $b \in L^\infty_{\text{loc}} \cap BV_{\text{loc}}([0, +\infty[\times \mathbb{R}^d; \mathbb{R}^d)$ be a nearly incompressible vector field, and let ρ be the function appearing in Definition 2.1. Then, for every bounded function u solving*

$$\partial_t(\rho u) + \text{div}_x(b\rho u) = 0$$

in the sense of distributions and for every $\beta \in \mathcal{C}^1(\mathbb{R})$, the function $\beta(u)$ satisfies

$$\partial_t[\rho\beta(u)] + \text{div}_x[b\rho\beta(u)] = 0$$

in the sense of distributions.

Moreover, in [4] the authors show that, if Conjecture 4.2 is true, then also Conjecture 4.1 is verified. A proof of Conjecture 4.2, under additional assumptions, is due to Ambrosio, De Lellis and Malý [8]. See also Crippa and De Lellis [19] for results concerning Bressan's conjecture.

REFERENCES

- [1] L. AMBROSIO, *The flow associated to weakly differentiable vector fields: recent results and open problems*, in *Nonlinear Conservation Laws and Applications*, IMA Vol. Math. Appl., Springer. To appear. Available at <http://cvgmt.sns.it/cgi/get.cgi/papers/amb09/>.

- [2] ———, *Transport equation and Cauchy problem for BV vector fields*, Invent. Math., 158 (2004), pp. 227–260.
- [3] ———, *Transport equation and Cauchy problem for non-smooth vector fields*, in Calculus of variations and nonlinear partial differential equations, vol. 1927 of Lecture Notes in Math., Springer, Berlin, 2008, pp. 1–41.
- [4] L. AMBROSIO, F. BOUCHUT, AND C. DE LELLIS, *Well-posedness for a class of hyperbolic systems of conservation laws in several space dimensions*, Comm. Partial Differential Equations, 29 (2004), pp. 1635–1651.
- [5] L. AMBROSIO AND G. CRIPPA, *Existence, uniqueness, stability and differentiability properties of the flow associated to weakly differentiable vector fields*, In: Transport Equations and Multi-D Hyperbolic Conservation Laws, Lecture Notes of the Unione Matematica Italiana, 5 (2008).
- [6] L. AMBROSIO, G. CRIPPA, A. FIGALLI, AND L. V. SPINOLO, *Some new well-posedness results for continuity and transport equations, and applications to the chromatography system*, SIAM J. Math. Anal., 41 (2009), pp. 1890–1920.
- [7] L. AMBROSIO AND C. DE LELLIS, *Existence of solutions for a class of hyperbolic systems of conservation laws in several space dimensions*, Int. Math. Res. Not., (2003), pp. 2205–2220.
- [8] L. AMBROSIO, C. DE LELLIS, AND J. MALÝ, *On the chain rule for the divergence of BV-like vector fields: applications, partial results, open problems*, in Perspectives in nonlinear partial differential equations, vol. 446 of Contemp. Math., Amer. Math. Soc., Providence, RI, 2007, pp. 31–67.
- [9] P. BAITI AND A. BRESSAN, *The semigroup generated by a Temple class system with large data*, Differential Integral Equations, 10 (1997), pp. 401–418.
- [10] S. BIANCHINI, *Stability of L^∞ solutions for hyperbolic systems with coinciding shocks and rarefactions*, SIAM J. Math. Anal., 33 (2001), pp. 959–981 (electronic).
- [11] F. BOUCHUT AND G. CRIPPA, *Uniqueness, renormalization, and smooth approximations for linear transport equations*, SIAM J. Math. Anal., 38 (2006), pp. 1316–1328 (electronic).
- [12] A. BRESSAN, *Hyperbolic systems of conservation laws. The one-dimensional Cauchy problem*, vol. 20 of Oxford Lecture Series in Mathematics and its Applications, Oxford University Press, Oxford, 2000.
- [13] ———, *An ill posed Cauchy problem for a hyperbolic system in two space dimensions*, Rend. Sem. Mat. Univ. Padova, 110 (2003), pp. 103–117.
- [14] A. BRESSAN AND P. GOATIN, *Stability of L^∞ solutions of Temple class systems*, Differential Integral Equations, 13 (2000), pp. 1503–1528.
- [15] A. BRESSAN AND W. SHEN, *Uniqueness for discontinuous ODE and conservation laws*, Nonlinear Anal., 34 (1998), pp. 637–652.
- [16] G.-Q. CHEN AND M. RASCLE, *Initial layers and uniqueness of weak entropy solutions to hyperbolic conservation laws*, Arch. Ration. Mech. Anal., 153 (2000), pp. 205–220.
- [17] G. CRIPPA, *The flow associated to weakly differentiable vector fields*, vol. 12 of Tesi. Scuola Normale Superiore di Pisa (Nuova Serie) [Theses of Scuola Normale Superiore di Pisa (New Series)], Edizioni della Normale, Pisa, 2009.
- [18] G. CRIPPA AND C. DE LELLIS, *Oscillatory solutions to transport equations*, Indiana Univ. Math. J., 55 (2006), pp. 1–13.
- [19] ———, *Estimates and regularity results for the DiPerna-Lions flow*, J. Reine Angew. Math., 616 (2008), pp. 15–46.
- [20] C. M. DAFERMOS, *Hyperbolic Conservation Laws in Continuum Physics*, Springer-Verlag, Berlin, Second ed., 2005.
- [21] C. DE LELLIS, *Notes on hyperbolic systems of conservation laws and transport equations*, Handbook of Differential Equations: Evolutionary Differential Equations, III (2006), pp. 277–383.
- [22] N. DEPAUW, *Non unicité des solutions bornées pour un champ de vecteurs BV en dehors d’un hyperplan*, C. R. Math. Acad. Sci. Paris, 337 (2003), pp. 249–252.

- [23] R. J. DiPerna and P.-L. Lions, *Ordinary differential equations, transport theory and Sobolev spaces*, Invent. Math., 98 (1989), pp. 511–547.
- [24] H. Freistühler, *On the Cauchy problem for a class of hyperbolic systems of conservation laws*, J. Differential Equations, 112 (1994), pp. 170–178.
- [25] B. L. Keyfitz and H. C. Kranzer, *A system of nonstrictly hyperbolic conservation laws arising in elasticity theory*, Arch. Rational Mech. Anal., 72 (1979/80), pp. 219–241.
- [26] S. N. Kružkov, *First order quasilinear equations with several independent variables.*, Mat. Sb. (N.S.), 81 (123) (1970), pp. 228–255.
- [27] O. A. Oleĭnik, *Discontinuous solutions of non-linear differential equations*, Amer. Math. Soc. Transl. (2), 26 (1963), pp. 95–172.
- [28] E. Y. Panov, *On the theory of generalized entropy solutions of the Cauchy problem for a class of non-strictly hyperbolic systems of conservation laws*, SB MATH, 191 (2000), pp. 121–150.
- [29] E. Y. Panov, *Generalized Solutions of the Cauchy Problem for a Transport Equation with Discontinuous Coefficients*, in Instability in Models Connected with Fluid Flows II. Edited by Claude Bardos and Andrei Fursikov, Springer New York, 2008, pp. 23–84.
- [30] D. Serre, *Solutions à variations bornées pour certains systèmes hyperboliques de lois de conservation*, J. Differential Equations, 68 (1987), pp. 137–168.
- [31] B. Temple, *Systems of conservation laws with invariant submanifolds*, Trans. Amer. Math. Soc., 280 (1983), pp. 781–795.