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ON THE MONGE-AMPÈRE EQUATION

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1. INTRODUCTION

The Monge-Ampère equation is a nonlinear partial differential equation arising in several problems from analysis and geometry, such as the prescribed Gaussian curvature equation, affine geometry, optimal transportation, etc.

In its classical form, this equation is given by

$$\det D^2 u = f(x, u, \nabla u) \qquad \text{in } \Omega, \tag{1}$$

where $\Omega \subset \mathbb{R}^n$ is some open set, $u : \Omega \to \mathbb{R}$ is a convex function, and $f : \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^+$ is given. In other words, the Monge-Ampère equation prescribes the product of the eigenvalues of the Hessian of u, in contrast with the "model" elliptic equation $\Delta u = f$ which prescribes their sum. As we shall explain later, the convexity of the solution u is a necessary condition to make the equation degenerate elliptic, and therefore to hope for regularity results.

The goal of this note is to give first a general overview of the classical theory, and then discuss some recent important developments on this beautiful topic.

2. HISTORICAL BACKGROUND

The Monge-Ampère equation draws its name from its initial formulation in two dimensions by the French mathematicians Monge [52] and Ampère [9].

The first notable results on the existence and regularity for the Monge-Ampère equation are due to Minkowski [50, 51]: by approximating a general bounded convex set with convex polyhedra with given faces areas, he proved the existence of a weak solution to the "prescribed Gaussian curvature equation" (now called "Minkowski problem"). Later on, using convex polyhedra with given generalized curvatures at the vertices, Alexandrov also proved the existence of a weak solution in all dimensions, as well as the C^1 smoothness of solutions in two dimensions [3, 4, 5].

In higher dimension, based on his earlier works, Alexandrov [6] (and also Bakelman [10] in two dimensions) introduced a notion of generalized solution to the Monge-Ampère equation and proved the existence and uniqueness of solutions to the Dirichlet problem (see Section 3.2). The notion of weak solutions introduced by Alexandrov (now called "Alexandrov solutions") has continued to be frequently used in recent years, and a lot of attention has been drawn to prove smoothness of Alexandrov solutions under suitable assumptions on the right hand side and the boundary data.

The regularity of weak solutions in high dimensions is a very delicate problem. For $n \geq 3$, Pogorelov found a convex function in $\mathbb{R}^n$ which is not of class C^2 but satisfies the Monge-Ampère equation in a neighborhood of the origin with positive analytic right hand side (see (15) below). It became soon clear that the main issue in the lack of regularity was the presence of a line segment in the graph of u. Indeed, Calabi [19] and Pogorelov [57] were able to prove a priori interior second and third derivative estimate for strictly convex solutions, or for solutions which do not contain a line segment with both endpoints on the boundary. However, in order to perform the computations needed to deduce these a priori estimates, C^4 regularity of the solution had to be assumed. Hence, a natural way to prove existence of smooth solutions was to approximate the Dirichlet problem with nicer problems for which C^4 solutions exist, apply Pogorelov and Calabi's estimates to get C^2/C^3 a priori bounds, and then take the limit in the approximating problems. This argument was successfully implemented by Cheng and Yau [20] and Lions [46] to obtain the interior smoothness of solutions.

Concerning boundary regularity, thanks to the regularity theory developed by Ivochkina [39], Krylov [44], and Caffarelli-Nirenberg-Spruck [22], one may use the continuity method and Evans-Krylov's estimates [28, 43] to obtain globally smooth solutions to the Dirichlet problem (see Section 3.3). In particular, Alexandrov solutions are smooth up to the boundary provided all given data are smooth.

In all the situations mentioned above, one assumes that f is positive and sufficiently smooth. When f is merely bounded away from zero and infinity, Caffarelli proved the $C^{1,\alpha}$ regularity of strictly convex solutions [14]. Furthermore, when f is continuous (resp. $C^{0,\alpha}$), using perturbation arguments Caffarelli proved interior $W^{2,p}$ estimate for any $p > 1$ (resp. interior $C^{2,\alpha}$ estimates) [13].

As explained in Section 3.5, these results can be applied to obtain both the regularity in the Minkowski problem and in the optimal transportation problem. Of course, these are just some examples of possible applications of the regularity theory for Monge-Ampère. For instance, as described in the survey paper [64, Sections 5 and 6], Monge-Ampère equations play a crucial role in affine geometry, for example in the study of affine spheres and affine maximal surfaces.

3. CLASSICAL THEORY

In this section we give a brief overview of some relevant results on the Monge-Ampère equation. Before entering into the concept of weak solutions and their regularity, we first discuss convexity of solutions and the terminology "degenerate ellipticity" associated to this equation.

3.1. On the degenerate ellipticity of the Monge-Ampère equation

Let $u : \Omega \to \mathbb{R}$ be a smooth solution of (1) with $f = f(x) > 0$ smooth. A standard technique to prove regularity of solutions to nonlinear PDEs consists in differentiating the equation solved by u to obtain a linear second-order equation for its first derivatives. More precisely, let us fix a direction $e \in \mathbb{S}^{n-1}$ and differentiate (1) in the direction e. Then, using the formula

$$\frac{d}{d\varepsilon}\Big|_{\varepsilon=0} \det(A + \varepsilon B) = \det(A)\,\mathrm{tr}(A^{-1}B) \qquad \forall\, A, B \in \mathbb{R}^{n\times n} \text{ with } A \text{ invertible},$$

we obtain the equation

$$\det(D^2 u)\, u^{ij}\partial_{ij} u_e = f_e \qquad \text{in } \Omega. \tag{2}$$

Here u^{ij} denotes the inverse matrix of $u_{ij} := (D^2u)_{ij}$, lower indices denotes partial derivatives (thus $u_e := \partial_e u$), and we are summing over repeated indices. Since $\det D^2 u = f > 0$, the above equation can be rewritten as

$$a_{ij}\partial_{ij} u_e = \frac{f_e}{f} \qquad \text{in } \Omega, \qquad \text{where } a_{ij} := u^{ij}. \tag{3}$$

Thus, to obtain some regularity estimates on u_e, we would like the matrix a_{ij} to be positive definite in order to apply elliptic regularity theory for linear equations. But for the matrix $a_{ij} = u^{ij}$ to be positive definite we need $D^2 u$ to be positive definite, which is exactly the convexity assumption on u.[(1)]

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(1) Of course the theory would be similar if one assumes u to be concave. The real difference arises if the Hessian of u is indefinite, since (3) becomes hyperbolic (and the equation is then called "hyperbolic Monge-Ampère"). This is still a very interesting problem, but the theory for such equation is completely different from the one of the classical Monge-Ampère equation and it would go beyond the scope of this note.

We also observe that, without any a priori bound on D^2u, the matrix a_{ij} may have arbitrarily small eigenvalues and this is why one says that (1) is "degenerate elliptic".

Notice that if one can show that

$$c_0\mathrm{Id} \le D^2u \le C_0\mathrm{Id} \qquad \text{inside } \Omega \tag{4}$$

for some positive constants $c_0, C_0 > 0$, then $C_0^{-1}\mathrm{Id} \le (a_{ij})_{1\le i,j\le n} \le c_0^{-1}\mathrm{Id}$ and the linearized equation (3) becomes uniformly elliptic. For this reason, proving (4) is one of the key steps for the regularity of solutions to (1).

In this regard we observe that, under the assumption $f(x) \ge \lambda > 0$, the product of the eigenvalues of D^2u (which are positive) is bounded from below. Thus, if one can prove that $|D^2u| \le C$, one easily concludes that (4) holds (see [31, Remark 1.1] for more details).

In conclusion, the key step towards the smoothness of solutions consists in proving that D^2u is bounded.

3.2. Alexandrov solutions

In his study of the Minkowski problem, Alexandrov introduced a notion of weak solution to the Monge-Ampère equation that allowed him to give a meaning to the Gaussian curvature of non-smooth convex sets. We now introduce this fundamental concept.

Given an open convex domain Ω, the subdifferential of a convex function $u : \Omega \to \mathbb{R}$ is given by

$$\partial u(x) := \{p \in \mathbb{R}^n \,:\, u(y) \ge u(x) + p\cdot(y-x) \quad \forall\, y \in \Omega\}.$$

One then defines the *Monge-Ampère measure* of u as follows:

$$\mu_u(E) := |\partial u(E)| \qquad \text{for every Borel set } E \subset \Omega,$$

where

$$\partial u(E) := \bigcup_{x\in E} \partial u(x)$$

and $|\cdot|$ denotes the Lebesgue measure. It is possible to show that μ_u is a Borel measure (see [31, Theorem 2.3]). Note that, in the case $u \in C^2(\Omega)$, the change of variable formula gives

$$|\partial u(E)| = |\nabla u(E)| = \int_E \det D^2u(x)\,dx \qquad \text{for every Borel set } E \subset \Omega,$$

therefore

$$\mu_u = \det D^2u(x)\,dx$$

(see [31, Example 2.2]).

This discussion motivates the following definition:

Definition 3.1 (Alexandrov solutions). — *Given an open convex set* Ω *and a function* $f : \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^+$, *a convex function* $u : \Omega \to \mathbb{R}$ *is called an* Alexandrov solution *to the Monge-Ampère equation*

$$\det D^2 u = f(x, u, \nabla u) \qquad \text{in } \Omega$$

if $\mu_u = f(x, u, \nabla u)\, dx$ *as Borel measures, namely*

$$\mu_u(A) = \int_A f(x, u, \nabla u)\, dx \qquad \forall\, A \subset \Omega \text{ Borel.}$$

Note that because convex functions are locally Lipschitz, they are differentiable a.e. Thus $f(x, u, \nabla u)$ is defined a.e. and the above definition makes sense.

To simplify the presentation, we shall discuss only the case $f = f(x)$, although all the arguments can be extended to the case $f = f(x, u, \nabla u)$ under the assumption that $\partial_u f \geq 0$ (this is needed to ensure that the maximum principle holds, see [35, Chapter 17]).

Actually, even if one is interested in solving the Monge-Ampère equation with a smooth right hand side, in order to prove existence of solutions it will be useful to consider also Borel measures as right hand sides. So, given a nonnegative Borel measure ν inside Ω, we shall say that u is an Alexandrov solution of $\det D^2 u = \nu$ if $\mu_u = \nu$.

A fundamental property of the Monge-Ampère measure is that it is stable under uniform convergence (see [31, Proposition 2.6]):

Proposition 3.2. — *Let* $u_k : \Omega \to \mathbb{R}$ *be a sequence of convex functions converging locally uniformly to* u. *Then the associated Monge-Ampère measures* μ_{u_k} *weakly* converge to* μ_u, *i.e.*

$$\int_\Omega \varphi\, d\mu_{u_k} \to \int_\Omega \varphi\, d\mu_u \qquad \forall\, \varphi \in C_c(\Omega).$$

Another crucial property of this definition is the validity of a comparison principle (see [31, Theorem 2.10]):

Proposition 3.3. — *Let* $\mathcal{U} \subset \Omega$ *be an open bounded set, and let* $u, v : \Omega \to \mathbb{R}$ *be two convex functions satisfying*

$$\begin{cases} \mu_u \leq \mu_v & \text{in } \mathcal{U} \\ u \geq v & \text{on } \partial\mathcal{U}. \end{cases}$$

Then

$$u \geq v \qquad \text{in } \mathcal{U}.$$

A direct consequence of this result is the uniqueness and stability of solutions (see [31, Corollaries 2.11 and 2.12]):

Corollary 3.4. — *Let Ω be an open bounded set, and $\nu_k : \Omega \to \mathbb{R}$ a family of nonnegative Borel measures satisfying* $\sup_k \nu_k(\Omega) < \infty$. *Then, for any k there exists at most one convex function $u_k : \Omega \to \mathbb{R}$ solving the Dirichlet problem*

$$\begin{cases} \mu_{u_k} = \nu_k & \text{in } \Omega \\ u_k = 0 & \text{on } \partial\Omega. \end{cases}$$

In addition, if $\nu_k \rightharpoonup^* \nu_\infty$ and the solutions u_k exist, then $u_k \to u_\infty$ locally uniformly, where u_∞ is the unique solution of

$$\begin{cases} \mu_{u_\infty} = \nu_\infty & \text{in } \Omega \\ u_\infty = 0 & \text{on } \partial\Omega. \end{cases}$$

Finally, exploiting these results, one can prove existence of solutions (see [31, Theorem 2.13]):

Theorem 3.5. — *Let Ω be an open bounded convex set, and let ν be a nonnegative Borel measure with $\nu(\Omega) < \infty$. Then there exists a unique convex function $u : \Omega \to \mathbb{R}$ solving the Dirichlet problem*

$$\begin{cases} \mu_u = \nu & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{5}$$

Sketch of the proof. — Since uniqueness follows from Corollary 3.4, one only needs to show existence.

By the stability in Corollary 3.4, since any finite measure can be approximated in the weak* topology by a finite sum of Dirac deltas, it suffices to solve the Dirichlet problem (5) when $\nu = \sum_{i=1}^N \alpha_i \delta_{x_i}$ with $x_i \in \Omega$ and $\alpha_i > 0$.

To prove existence of a solution, one uses the so-called "Perron Method": one defines

$$\mathcal{S}[\nu] := \{v : \Omega \to \mathbb{R} \text{ convex} : v|_{\partial\Omega} = 0,\ \mu_v \geq \nu \text{ in } \Omega\},$$

and shows that this set is nonempty and that it is closed under maximum (namely, $v_1, v_2 \in \mathcal{S}[\nu] \Rightarrow \max\{v_1, v_2\} \in \mathcal{S}[\nu]$). Thanks to these properties, one obtains that $u := \sup_{v \in \mathcal{S}[\nu]} v$ is still an element of $\mathcal{S}[\nu]$, and then one exploits the maximality of u to deduce that $\mu_u = \nu$.

We refer to [31, Proof of Theorem 2.13] for more details. □

Actually, if Ω is strictly convex, a similar argument combined with the existence of suitable barriers allows one to prove the existence of solutions for any continuous boundary datum (see for instance [31, Theorem 2.14]):

Theorem 3.6. — *Let Ω be an open bounded strictly convex set, let ν be a nonnegative Borel measure with $\nu(\Omega) < \infty$, and let $g : \partial\Omega \to \mathbb{R}$ be a continuous function. Then there exists a unique convex function $u : \Omega \to \mathbb{R}$ solving the Dirichlet problem*

$$\begin{cases} \mu_u = \nu & \text{in } \Omega \\ u = g & \text{on } \partial\Omega. \end{cases} \tag{6}$$

3.3. Existence of smooth solutions and global regularity

As shown in the previous section, uniqueness of solutions to the Dirichlet problem holds even at the level of weak solutions. So, the main issue is existence.

Existence of smooth solutions to the Monge-Ampère equation dates back to the work of Pogorelov [57]. This is obtained through the well-celebrated *method of continuity* that we now briefly describe (we refer to [35, Chapter 17] and [31, Section 3.1] for a more detailed exposition).

Assume that Ω is a smooth uniformly convex domain,(2) and consider $\bar{u} : \Omega \to \mathbb{R}$ a smooth uniformly convex function that vanishes on $\partial\Omega$. Then, if we set $\bar{f} := \det D^2\bar{u}$, we have that $\bar{f} > 0$ in $\overline{\Omega}$ and $\bar{u}$ solves

$$\begin{cases} \det D^2\bar{u} = \bar{f} & \text{in } \Omega \\ \bar{u} = 0 & \text{on } \partial\Omega. \end{cases}$$

Now, assume we want to solve

$$\begin{cases} \det D^2 u = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{7}$$

for some given $f : \Omega \to \mathbb{R}$ with $f > 0$. Define $\{f_t := (1-t)\bar{f} + tf\}_{t\in[0,1]}$, and consider the 1-parameter family of problems

$$\begin{cases} \det D^2 u_t = f_t & \text{in } \Omega \\ u_t = 0 & \text{on } \partial\Omega. \end{cases} \tag{8}$$

The method of continuity consists in showing that the set of $t \in [0,1]$ such that (8) is smoothly solvable is both open and closed. Since the problem is solvable for $t = 0$ (because $\bar{u}$ is a solution), this implies the existence of a smooth solution to (7).

More precisely, assuming that Ω is a uniformly convex domain of class $C^{2,\alpha}$ for some $\alpha \in (0,1)$, it follows that the function $\bar{f} = \det D^2\bar{u}$ belongs to $C^{0,\alpha}(\overline{\Omega})$. Then, assuming that $f \in C^{0,\alpha}(\overline{\Omega})$, we can consider the set of functions

$$\mathcal{C} := \{v : \overline{\Omega} \to \mathbb{R} \text{ convex functions of class } C^{2,\alpha}(\overline{\Omega}),\ v = 0 \text{ on } \partial\Omega\},$$

and define the nonlinear map

$$\begin{aligned} \mathcal{F} \colon \mathcal{C} \times [0,1] &\longrightarrow C^{0,\alpha}(\overline{\Omega}) \\ (v,t) &\longmapsto \det D^2 v - f_t. \end{aligned}$$

The goal is show that the set

$$\mathcal{T} := \{t \in [0,1] :\ \text{there exists a } u_t \in \mathcal{C} \text{ such that } \mathcal{F}(u_t,t) = 0\}$$

(2) We say that a domain is uniformly convex if there exists a radius R such that

$$\Omega \subset B_R(x_0 + R\nu_{x_0}) \qquad \text{for every } x_0 \in \partial\Omega,$$

where ν_{x_0} is the interior normal to Ω at x_0. Note that, for a smooth domain, this is equivalent to asking that the second fundamental form of $\partial\Omega$ is uniformly positive definite.

is nonempty, and it is both open and closed inside $[0,1]$. We now explain the main steps of the argument.

• Nonemptyness follows from the fact that $\mathcal{F}(\bar{u},0)=0$, thus $0\in\mathcal{T}$.

• Openness follows from the Implicit Function Theorem in Banach spaces (see [35, Theorem 17.6]). Indeed, the Fréchet differential of $\mathcal{F}$ with respect to v is given by the linearized Monge-Ampère operator (compare with (2))

$$D_u\mathcal{F}(v,t)[h]=\det(D^2u)u^{ij}h_{ij},\qquad h=0\ \text{on}\ \partial\Omega, \tag{9}$$

where we set $h_{ij}:=\partial_{ij}h$, u^{ij} is the inverse of $u_{ij}:=\partial_{ij}u$, and we are summing over repeated indices. Notice that if a function v is bounded in $C^{2,\alpha}$ and $\det D^2v$ is bounded from below, then the smallest eigenvalue of D^2v is bounded uniformly away from zero and the linearized operator becomes uniformly elliptic with $C^{0,\alpha}$ coefficients (cp. Section 3.1). Therefore, classical Schauder's theory gives the invertibility of $D_u\mathcal{F}(u_t,t)$ whenever u_t solves $\mathcal{F}(u_t,t)=0$ (see for instance [35, Chapter 6]).

• The proof of closedness is done via global a priori estimates. More precisely, the following fundamental a priori bound holds (see [31, Theorem 3.2]):(3)

Theorem 3.7. — *Let Ω be a uniformly convex domain of class C^3, and let $u\in C^4(\Omega)$ be a solution of* (7) *with $f\in C^2(\overline{\Omega})$ and $0<\lambda\le f\le 1/\lambda$. Then there exists a constant C, depending only on Ω, λ, $\|f\|_{C^2(\overline{\Omega})}$, such that*

$$\|D^2u\|_{C^0(\overline{\Omega})}\le C.$$

As already noticed in Section 3.1, once a uniform bound on D^2u inside $\overline{\Omega}$ holds, the Monge-Ampère equation becomes uniformly elliptic and classical elliptic regularity theory yields $C^{2,\alpha}$ estimates for solutions of $\mathcal{F}(u_t,t)=0$, proving the desired closedness of $\mathcal{T}$.

Thanks to this argument, one concludes the validity of the following existence result:

Theorem 3.8. — *Let Ω be a uniformly convex domain of class C^3. Then, for all $f\in C^2(\overline{\Omega})$ with $0<\lambda\le f\le 1/\lambda$, there exists a unique $u\in C^{2,\alpha}(\overline{\Omega})$ solution to* (7).

Recalling that uniqueness holds also at the level of Alexandrov solutions, this proves the $C^{2,\alpha}$ regularity (for any $\alpha<1$) of Alexandrov solutions in C^3 uniformly convex domains with C^2 right hand side. It is interesting to remark that the C^3 regularity assumption on the boundary is necessary, as shown by Wang in [67].

(3) The assumption $u\in C^4(\Omega)$ in Theorem 3.7 is not essential, as it is needed only to justify the computations in the proof.

3.4. Caffarelli's regularity theory

We now investigate the regularity of Alexandrov solutions under weaker smoothness assumptions on the right hand side.

In the 90's Caffarelli developed a regularity theory for Alexandrov solutions, showing that strictly convex solutions of (1) are locally $C^{1,\gamma}$ provided $\lambda \leq f \leq 1/\lambda$ for some $\lambda > 0$ [12, 14, 15]. We emphasize that, for weak solutions, strict convexity is not implied by the positivity of f (unless $n = 2$) and it is actually necessary for regularity, see Section 4.1 below.

The following result is proved in [14]:

THEOREM 3.9. — *Let $u : \Omega \to \mathbb{R}$ be a strictly convex Alexandrov solution of $\mu_u = f\,dx$ with $0 < \lambda \leq f \leq 1/\lambda$. Then $u \in C^{1,\gamma}_{\mathrm{loc}}(\Omega)$ for some $\gamma = \gamma(n, \lambda) > 0$.*

To explain the idea behind the proof of the above theorem, let us point out the following simple property of solutions to the Monge-Ampère equation (this is another manifestation of its degenerate ellipticity): if $A : \mathbb{R}^n \to \mathbb{R}^n$ is an affine transformation with $\det A = 1$,[4] and u is a solution of the Monge-Ampère equation with right hand side f, then $u \circ A$ is a solution to the Monge-Ampère equation with right hand side $f \circ A$. This affine invariance creates serious obstructions to obtain a local regularity theory. Indeed, for instance, the functions

$$u_\varepsilon(x_1, x_2) = \frac{\varepsilon x_1^2}{2} + \frac{x_2^2}{2\varepsilon} - 1$$

are solutions to $\det D^2 u_\varepsilon = 1$ inside the convex set $\{u_\varepsilon < 0\}$. Thus, unless the level set $\{u_\varepsilon = 0\}$ is sufficiently "round", there is no hope to obtain a priori estimates on u. The intuition of Caffarelli was to use the so-called John's Lemma [42]:

LEMMA 3.10. — *Let $\mathcal{K} \subset \mathbb{R}^n$ be a bounded convex set with non-empty interior. Then there exists an ellipsoid E satisfying*

$$E \subset \mathcal{K} \subset nE, \tag{10}$$

where nE denotes the dilation of E by a factor n with respect to its center.

We say that a convex set $\mathcal{K}$ is *normalized* if

$$B_1 \subset \mathcal{K} \subset nB_1.$$

Then Lemma 3.10 states that, for every bounded open convex set $\mathcal{K}$, there is an affine transformation $A : \mathbb{R}^n \to \mathbb{R}^n$ such that $A(\mathcal{K})$ is normalized.

Note that, if u is strictly convex, given a point $x \in \Omega$ and $p \in \partial u(x)$ one can choose $t > 0$ small enough so that the convex set

$$S(x, p, t) := \{z \in \Omega \,:\, u(z) - u(x) - p \cdot (z - x) < t\} \tag{11}$$

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(4) Given an affine transformation $Ax := Mx + v$, by abuse of notation we write $\det A$ in place of $\det M$.

is strictly contained inside Ω, namely $\overline{S(x,p,t)} \subset \Omega$. Then, if we replace $u(z)$ with $u_x(z) := u(z) - u(x) - p \cdot (z - x) - t$, it follows that

$$\lambda\, dx \le \mu_{u_x} \le \frac{1}{\lambda}\, dx \quad \text{in } S(x,p,t), \qquad u_x = 0 \quad \text{on } \partial\Big(S(x,p,t)\Big).$$

Also, if $A : \mathbb{R}^n \to \mathbb{R}^n$ normalizes $S(x,p,t)$, then $v := (\det A)^{2/n}\, u_x \circ A^{-1}$ solves

$$\lambda\, dx \le \mu_v \le \frac{1}{\lambda}\, dx \quad \text{in } A(S(x,p,t)), \qquad v = 0 \quad \text{on } \partial\Big(A(S(x,p,t))\Big). \tag{12}$$

Thanks to the above discussion, it suffices to prove the result when u is a solution inside a normalized convex set. In other words, Theorem 3.9 is a direct consequence of the following result:

Theorem 3.11. — *Let Ω be a normalized convex set, and u be a solution of*

$$\mu_u = f\, dx \quad \textit{in } \Omega, \qquad u = 0 \quad \textit{on } \partial\Omega,$$

with $0 < \lambda \le f \le 1/\lambda$. Then u is strictly convex inside Ω, and $u \in C^{1,\gamma}_{\mathrm{loc}}(\Omega)$ for some $\gamma = \gamma(n,\lambda) > 0$.

In the proof of the above theorem, a key step consists in showing that solutions of (7) inside normalized domains have a *universal* modulus of strict convexity. A fundamental ingredient to prove this fact is the following important result of Caffarelli [12] (see also [31, Theorem 4.10]):

Proposition 3.12. — *Let u be a solution of*

$$\lambda\, dx \le \mu_u \le \frac{1}{\lambda}\, dx$$

inside a convex set Ω, $x \in \Omega$, and $p \in \partial u(x)$. Let $\ell(z) := u(x) + p \cdot (z - x)$. If the convex set

$$W := \{z \in \Omega \,:\, u(z) = \ell(z)\}$$

contains more than one point, then it cannot have extremal points inside Ω.

This statement says that if a solution coincides with one of its supporting planes on more than one point (that is, it is not strictly convex), then the contact set has to cross the domain. In particular this is not possible if $u|_{\partial\Omega} = 0$ (as otherwise one would deduce that $u \equiv 0$ by the convexity of u), proving that solutions to (12) are strictly convex.

Sketch of the proof of Theorem 3.11. — As mentioned above, Proposition 3.12 implies that u is strictly convex. Also, by compactness, one can prove that the modulus of strict convexity of u is universal (see [31, Section 4.2.2]).

We then apply this information at all scales. More precisely, given any point $x \in \Omega$, $p \in \partial u(x)$, and $t > 0$ small, we consider $u_x(z) := u(z) - u(x) - p \cdot (z - x) - t$. Then, if $A : \mathbb{R}^n \to \mathbb{R}^n$ normalizes $S(x,p,t)$, the function $v := (\det A)^{2/n}\, u_x \circ A^{-1}$ enjoys the same strict convexity properties as u. Using this fact at all points x and for all small values of t, a careful iteration argument proves the validity of Theorem 3.11 (see the proof of [31, Theorem 4.20] for more details). □

Note that Theorem 3.8 is unsatisfactory from a PDE viewpoint: indeed, it requires the C^2 regularity of f to prove the $C^{2,\alpha}$ regularity of the solution, while the usual elliptic regularity theory would suggest that $f \in C^{0,\alpha}$ should be enough. This is indeed true, as proved by Caffarelli in [13] (again, it suffices to consider normalized convex sets):

Theorem 3.13. — *Let Ω be a normalized convex set, and u be a solution of*

$$\mu_u = f\,dx \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega,$$

with $0 < \lambda \le f \le 1/\lambda$ and $f \in C^{0,\alpha}_{\mathrm{loc}}(\Omega)$. Then $u \in C^{2,\alpha}_{\mathrm{loc}}(\Omega)$.

The proof of the above theorem is based on the property that, under the assumption that f is almost constant (say, very close to 1), u is very close to a solution of $\mu_v = dx$. Since this latter function is smooth (by Theorem 3.8), an iteration argument permits to show that the $C^{2,\alpha}$ norm of u remains bounded (see also [31, Theorem 4.42]).

With this line of reasoning one can also prove the following theorem [13]:

Theorem 3.14. — *Let Ω be a normalized convex set, and u be a solution of*

$$\mu_u = f\,dx \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega.$$

Then, for every $p > 1$ there exists a positive constant $\delta(p)$ such that if $\|f - 1\|_\infty \le \delta(p)$ then $u \in W^{2,p}_{\mathrm{loc}}(\Omega)$.

Since any continuous function is arbitrarily close to a constant at small scales, one obtains the following:

Corollary 3.15. — *Let Ω be a normalized convex set, and u be a solution of*

$$\mu_u = f\,dx \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega,$$

with $f > 0$ continuous. Then $u \in W^{2,p}_{\mathrm{loc}}(\Omega)$ for any $p < \infty$.

Remark 3.16. — As shown in [30], exploiting the ideas introduced in [24, 26] one can find an explicit estimate for $\delta(p)$ in terms of p in Theorem 3.14, namely $\delta(p) \simeq e^{-C\,p}$ for some dimensional constant $C > 0$.

3.5. Some applications

We now briefly describe two applications of the regularity theory developed before.

3.5.1. *The Minkowski problem.* — A classical problem in convex geometry is to prescribe some geometric quantity (the surface area, the Gaussian curvature, etc.) and find necessary and sufficient conditions ensuring that such a quantity comes from a convex domain. In this section we briefly discuss the "prescribed Gaussian curvature" problem.

Let $K \subset \mathbb{R}^n$ be an open bounded convex domain containing the origin, and parameterize ∂K in polar coordinates as follows:

$$\partial K = \Big\{ \rho(x)\, x \,:\, x \in \mathbb{S}^{n-1},\, \rho : \mathbb{S}^{n-1} \to \mathbb{R}^+ \Big\}.$$

Then, to any point $z \in \partial K$ we associated the normal mapping

$$N_K(z) := \Big\{ y \in \mathbb{S}^{n-1} \,:\, K \subset \{y \,:\, \langle y, w - z\rangle \le 0\} \Big\}.$$

Geometrically, the normal mapping finds the normals of all supporting hyperplanes at z, and we can think of N_K as an analogue of the subdifferential map.

Finally, we consider the (multivalued) Gauss map $G_K : \mathbb{S}^{n-1} \to \mathbb{S}^{n-1}$ defined by

$$G_K(x) := N_K\big(\rho(x)\, x\big),$$

and define the Gaussian curvature measure

$$\mu_K(E) := \mathcal{H}^{n-1}\big(G_K(E)\big) \qquad \forall\, E \subset \mathbb{S}^{n-1} \text{ Borel},$$

where $\mathcal{H}^{n-1}$ denotes the $(n-1)$-dimensional Hausdorff measure on $\mathbb{S}^{n-1}$.

As for the Monge-Ampère measure, one can show that μ_K is a Borel measure. One then asks the following question: *Given a Borel measure ν on $\mathbb{S}^{n-1}$, can we find an open bounded convex set K containing the origin and such that $\mu_K = \nu$?*

In [2, 3], Alexandrov found necessary and sufficient conditions to ensure the existence of a solution to this problem. As for the existence of Alexandrov solutions to the Monge-Ampère equation, the existence of K is first proved when ν is a finite sum of Dirac deltas, and then one obtains the general case by approximation. The original existence proof of Alexandrov when ν is discrete was based on a topological argument relying on the Invariance of Domain Theorem [2] (see also [5]).

Thanks to the regularity theory developed by Caffarelli, one obtains the following regularity result:

Theorem 3.17. — *Let $K \subset \mathbb{R}^n$ be an open bounded convex domain containing the origin, and assume that $\mu_K = f\, d\mathcal{H}^{n-1}$ for some $f : \mathbb{S}^{n-1} \to \mathbb{R}$, with $0 < \lambda \le f \le 1/\lambda$. Then ∂K is strictly convex and of class $C^{1,\gamma}$. If in addition $f \in C^{0,\alpha}$ for some $\alpha \in (0,1)$, then $\partial K \in C^{2,\alpha}$.*

Sketch of the proof. — Since K is convex, one can locally parameterize the boundary as the graph of a convex function $u : \Omega \subset \mathbb{R}^{n-1} \to \mathbb{R}$. It is a classical fact that the Gaussian curvature of the graph of a C^2 function $v : \Omega \subset \mathbb{R}^{n-1} \to \mathbb{R}$ is given by

$$\frac{\det D^2 v}{(1 + |\nabla v|^2)^{\frac{n+1}{2}}}.$$

Then, by the assumption $\mu_K = f\, d\mathcal{H}^{n-1}$, an approximation argument based on Proposition 3.2 yields the validity of the equation

$$\mu_u = f(x)\Big(1 + |\nabla u|^2\Big)^{\frac{n+1}{2}}\, dx,$$

where ∇u exists at almost every point since u is locally Lipschitz (being convex). In particular, μ_u is locally bounded. Applying Proposition 3.12 and Theorem 3.9, one deduces that ∂K is strictly convex and of class $C^{1,\gamma}$. Finally, the $C^{2,\alpha}$ regularity when $f \in C^{0,\alpha}$ follows from Theorem 3.13. □

Remark 3.18. — The regularity theory for Monge-Ampère plays a crucial role in many other variants of the Minkowski problem. For instance, it appears in the proof of existence and uniqueness of convex domains with prescribed harmonic measure [41].

3.5.2. *The optimal transport problem.* — Let μ and ν denote two probability measures on $\mathbb{R}^n$. The optimal transport problem (with quadratic cost) consists in finding the "optimal" way of transporting μ onto ν given that the transportation cost to move a point from x to y is $|x-y|^2$. Hence, one is naturally led to minimize

$$\int_{\mathbb{R}^n} |S(x) - x|^2\, d\mu(x)$$

among all maps S that "transport μ onto ν". Mathematically, this corresponds to saying that $S_\# \mu = \nu$, that is, for any bounded Borel function $\varphi : \mathbb{R}^n \to \mathbb{R}$,

$$\int_{\mathbb{R}^n} \varphi(y)\, d\nu(y) = \int_{\mathbb{R}^n} \varphi\Big(S(x)\Big)\, d\mu(x).$$

By a classical theorem of Brenier [11] (see also [21, 58, 49]), existence and uniqueness of optimal maps hold provided that μ is absolutely continuous. Moreover, such a map is given by the gradient of a convex function. This is summarized in the next theorem:

THEOREM 3.19. — *Let μ, ν be probability measures on $\mathbb{R}^n$ with $\mu = f\, dx$ and $\nu = g\, dy$. Then:*

- *There exists a μ-a.e. unique optimal transport map T.*
- *There exists a lower semicontinuous convex function $u : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ such that $T = \nabla u$ μ-a.e. and*

$$\det(D^2 u) = \frac{f}{g \circ \nabla u} \qquad \mu\text{-a.e.}$$

The above theorem shows that optimal transport maps solve a Monge-Ampère equation in a weak sense, that is usually referred to as "Brenier sense".

While for Alexandrov solutions one may apply the regularity theory developed in the previous sections, Caffarelli observed in [15] that even for smooth densities one cannot expect any general regularity result for Brenier solutions without making some geometric assumptions on the support of the target measure. Indeed, let $n = 2$ and suppose that $X = B_1$ is the unit ball centered at the origin and $Y = \Big(B_1^+ + e_1\Big) \cup \Big(B_1^- - e_1\Big)$ is the union of two half-balls, where

$$B_1^+ := \Big(B_1 \cap \{x_1 > 0\}\Big), \qquad B_1^- := \Big(B_1 \cap \{x_1 < 0\}\Big),$$

and (e_1, e_2) denotes the canonical basis of $\mathbb{R}^2$. Then if $f = \frac{1}{|X|}\mathbf{1}_X$ and $g = \frac{1}{|Y|}\mathbf{1}_Y$, the optimal transport map is given by

$$T(x) := \begin{cases} x + e_1 & \text{if } x_1 > 0 \\ x - e_1 & \text{if } x_1 < 0, \end{cases}$$

which corresponds to the gradient of the convex function $u(x) = |x|^2/2 + |x_1|$.

Thus, in order to hope for a regularity result for u we need at least to assume the connectedness of Y. However, starting from the above construction and considering a sequence of domains Y_ε where one adds a small strip of width $\varepsilon > 0$ to glue together $\Big(B_1^+ + e_1\Big) \cup \Big(B_1^- - e_1\Big)$, one can also show that for $\varepsilon > 0$ small enough the optimal map will still be discontinuous (see [15, 29]). Hence, connectedness is not enough to ensure regularity. As shown by Caffarelli [15, 17], convexity of Y is the right assumption to ensure that a Brenier solution is also an Alexandrov solution, so that the general regularity theory from the previous sections apply (see also [27, 65]):

Theorem 3.20. — *Let $X, Y \subset \mathbb{R}^n$ be two bounded open sets, let $f, g : \mathbb{R}^n \to \mathbb{R}^+$ be two probability densities that are zero outside X, Y and are bounded away from zero and infinity on X, Y, respectively. Denote by $T = \nabla u : X \to Y$ the optimal transport map provided by Theorem 3.19, and assume that Y is convex. Then there exists $\gamma > 0$ such that $T \in C^{0,\gamma}_{\mathrm{loc}}(X)$. Furthermore, if $f \in C^{k,\alpha}(\overline{X})$ and $g \in C^{k,\alpha}(\overline{Y})$ for some integer $k \geq 0$ and some $\alpha \in (0, 1)$, and if both X and Y are smooth and uniformly convex, then $T : \overline{X} \to \overline{Y}$ is a global diffeomorphism of class $C^{k+1,\alpha}$.*

As shown for instance in [34], the convexity of the target is necessary for the continuity of the optimal transport map. Even worse, as recently shown in [40], even with constant densities one can construct a discontinuous optimal transport map from a smooth convex domain to a small Lipschitz deformation of itself.

All these facts motivate the following very natural question: *What can one say when the convexity assumption on the target is removed?* As shown in [33, 25] (see also [29] for a more precise description of the singular set in two dimensions, and [36] for a recent variational proof of the result in [33]), one can always prove that the optimal transport map is smooth outside a closed set of measure zero.

4. RECENT DEVELOPMENTS I: INTERIOR REGULARITY

In [66] Wang showed that for any $p > 1$ there exists a function f satisfying $0 < \lambda(p) \leq f \leq 1/\lambda(p)$ such that $u \notin W^{2,p}_{\mathrm{loc}}$. This counterexample shows that the results of Caffarelli are more or less optimal. However, an important question which remained open was whether strictly convex solutions of $\mu_u = f\,dx$ with $0 < \lambda \le f \le 1/\lambda$ could be at least $W^{2,1}_{\rm loc}$, or even $W^{2,1+\varepsilon}_{\rm loc}$ for some $\varepsilon = \varepsilon(n,\lambda) > 0$. The question of $W^{2,1}_{\rm loc}$ regularity has been recently solved by De Philippis and Figalli in [24]. Following the ideas introduced there, the result has been refined to $u \in W^{2,1+\varepsilon}_{\rm loc}$ for some $\varepsilon > 0$ (see [26, 63]).

Theorem 4.1. — *Let Ω be a normalized convex set, and u be a solution of*

$$\mu_u = f\,dx \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega,$$

with $0 < \lambda \le f \le 1/\lambda$. Then there exists $\varepsilon = \varepsilon(n,\lambda) > 0$ such that $u \in W^{2,1+\varepsilon}_{\rm loc}(\Omega)$.

Again, as in Section 3.4, the previous result holds for strictly convex solutions of $\mu_u = f\,dx$ with $0 < \lambda \le f \le 1/\lambda$.

Sketch of the proof. — Given $x \in \Omega$ and $t > 0$ small, we consider the family $\{S(x,\nabla u(x),t)\}_{x\in\Omega,\,t>0}$ as defined in (11). Thinking of $S_t(x) := S(x,\nabla u(x),t)$ as the "ball centered at x with radius t", any subdomain $\Omega' \subset\subset \Omega$ endowed with this family of "balls" is a space of homogeneous type in the sense of Coifman and Weiss, see [18, 37, 1]. In particular Stein's Theorem implies that if

$$\mathcal{M}(D^2u)(x) := \sup_{t>0} \fint_{S(x,\nabla u(x),t)} |D^2u| \in L^1_{\rm loc}(\Omega),$$

then $|D^2u| \in L\log L_{\rm loc}$, that is $\int_{\Omega'} |D^2u|\log(2+|D^2u|) \le C(\Omega')$ for any $\Omega' \subset\subset \Omega$. The key estimate in [24] consists in showing that

$$\|\mathcal{M}(D^2u)\|_{L^1_{\rm loc}(\Omega)} \le C\|D^2u\|_{L^1_{\rm loc}(\Omega)},$$

for some constant $C = C(n,\lambda)$.

Once this estimate is proved, it follows by the convexity of u that the $L^1_{\rm loc}$ norm of D^2u is locally bounded (see [31, Equation (4.74)]), thus[(5)]

$$|D^2u|\log(2+|D^2u|) \in L^1_{\rm loc}(\Omega). \tag{13}$$

By this a priori estimate and an approximation argument with smooth solutions, as shown in [24] one easily deduces that D^2u is an L^1 function, and therefore $u \in W^{2,1}_{\rm loc}$.

We now explain how this argument actually implies that $u \in W^{2,1+\varepsilon}_{\rm loc}$. In view of (13), the measure of the set where $|D^2u|$ is large decays in a quantitative way:

$$|\{|D^2u| \ge M\}| \le \frac{1}{M\log M}\int_{\{|D^2u|\ge M\}|} |D^2u|\log(2+|D^2u|) \le \frac{C}{M\log M},$$

(5) Here the reader may be confused by the sentence "Since u is convex, the $L^1_{\rm loc}$ norm of D^2u is locally bounded". Indeed, this seems to say that the $W^{2,1}_{\rm loc}$ regularity of u is trivial since the integral of $|D^2u|$ is locally finite. This is not the case because, for a convex function, D^2u may be a measure and so $\int_E |D^2u|$ denotes the integral over a set E of the measure $|D^2u|$. So, to prove that $u \in W^{2,1}_{\rm loc}$, it is not enough to show that $\int |D^2u|$ is locally finite but one needs to show that $|D^2u|$ is absolutely continuous with respect to the Lebesgue measure.

for any M large. In particular, choosing first M sufficiently large and then taking $\varepsilon > 0$ small enough, we deduce (a localized version of) the bound

$$|\{|D^2u| \geq M\}| \leq \frac{1}{M^{1+2\varepsilon}}|\{|D^2u| \geq 1\}|.$$

Applying this estimate at all scales (cp. the sketch of the proof of Theorem 3.11) together with a covering lemma yields

$$|\{|D^2u| \geq M^k\}| \leq \frac{1}{M^{(1+2\varepsilon)k}}|\{|D^2u| \geq 1\}| \qquad \forall\, k \geq 1,$$

and the local $L^{1+\varepsilon}$ integrability for $|D^2u|$ follows (see for instance [31, Section 4.8.4] for more details). □

4.0.1. *An application: the semigeostrophic equations.* — The semigeostrophic equations are a simple model used in meteorology to describe large scale atmospheric flows, and can be derived from the 3-d Euler equations, with Boussinesq and hydrostatic approximations, subject to a strong Coriolis force [23]. Since for large scale atmospheric flows the Coriolis force dominates the advection term, the flow is mostly bi-dimensional. For this reason, the study of the semigeostrophic equations in 2-d or 3-d is pretty similar, and in order to simplify our presentation we focus here on the 2-dimensional periodic case.

The semigeostrophic system can be written as

$$\text{(14)} \qquad \begin{cases} \partial_t \nabla p_t + (\boldsymbol{u}_t \cdot \nabla)\nabla p_t + \nabla^{\perp} p_t + \boldsymbol{u}_t = 0 \\ \nabla \cdot \boldsymbol{u}_t = 0 \\ p_0 = \bar{p} \end{cases}$$

where $\boldsymbol{u}_t : \mathbb{R}^2 \to \mathbb{R}^2$ and $p_t : \mathbb{R}^2 \to \mathbb{R}$ are periodic functions corresponding respectively to the velocity and the pressure, and $\nabla^{\perp} p_t$ is the $\pi/2$ counterclockwise rotation of ∇p.

As shown in [23], energetic considerations show that it is natural to assume that p_t is (-1)-convex, i.e., the function $P_t(x) := p_t(x) + |x|^2/2$ is convex on $\mathbb{R}^2$. Let P_t^* be the convex conjugate of P_t, namely

$$P_t^*(y) := \sup_{x \in \mathbb{R}^2} \Big\{ y \cdot x - P_t(x) \Big\}.$$

Then, assuming that $0 < \lambda \leq \det(D^2 P_0^*) \leq 1/\lambda$, one can prove that

$$0 < \lambda \leq \det(D^2 P_t^*) \leq 1/\lambda \qquad \forall\, t > 0$$

in the Alexandrov sense (see [7] for more details). Thanks to Theorem 4.1 this implies that $P_t^* \in W^{2,1+\varepsilon}_{\mathrm{loc}}$, which is one of the key ingredients to prove the global existence of distributional solutions to (14) on the 2-dimentional torus [7] and in three dimensional domains [8].

Remark 4.2. — From a physical point of view, the lower bound on $\det(D^2 P_0^*)$ is not natural and it would be very useful if the $W^{2,1}$ regularity of solutions to $\mu_u \leq \frac{1}{\lambda}\, dx$ was true, at least in two dimensions. Unfortunately this is false, as shown by Mooney in [55].

On a different direction, one would like to prove global existence of smooth solutions of (14) when the initial datum is smooth. Motivated by the analogous result for the 2d incompressible Euler equation, a possible strategy to prove this result would be to show that strictly convex solutions of $\mu_u = f\,dx$ with $f \in C^{0,\alpha}$ such that $0 < \lambda \le f \le 1/\lambda$ satisfy $\|D^2u\|_{C^{0,\alpha}} \le C(n,\lambda,\alpha)\|f\|_{C^{0,\alpha}}$ (namely, the control is linear with respect to the norm of f). As shown in [32] this is false, and the global existence of smooth solutions is still an open problem.

4.1. On the strict convexity of weak solutions

As already mentioned, strict convexity is not just a technical assumption but it is necessary to obtain regularity. Indeed, as discovered by Pogorelov, there exist Alexandrov solutions to the Monge-Ampère equation with smooth positive right-hand side which are not C^2. For instance, the function

$$u(x_1,x') := |x'|^{2-2/n}(1+x_1^2), \qquad (x_1,x') \in \mathbb{R}\times\mathbb{R}^{n-1}, \qquad n \ge 3, \tag{15}$$

is $C^{1,1-2/n}$ and solves $\det D^2u = c_n(1+x_1^2)^{n-2}(1-x_1^2) > 0$ inside $B_{1/2}$. Furthermore, the bound $0 < \lambda \le \det D^2u \le 1/\lambda$ is not even enough for C^1 regularity: the function

$$u(x_1,x') := |x'| + |x'|^{n/2}(1+x_1^2), \qquad (x_1,x') \in \mathbb{R}^n, \qquad n \ge 3,$$

is merely Lipschitz and solves $0 < \lambda \le \det D^2u \le 1/\lambda$ in a small convex neighborhood of the origin.(6)

Alexandrov showed in [4] that, in contrast with the above counterexamples, every two dimensional solution of $\mu_u \ge \lambda\,dx > 0$ is strictly convex. In [16], Caffarelli generalized these examples to solutions that degenerate along subspaces, and he proved that solutions can degenerate only on subspaces of dimension less than $n/2$.

Since one cannot hope for C^1 regularity of non-strictly convex solutions, it is natural to ask whether one can obtain some integrability estimates for the second derivatives.

(6) Actually, for $n \ge 3$, one can even construct a Lipschitz Alexandrov solution of $\det D^2u = 1$ in a small ball $B_\rho(0)$. To see this, let $\eta > 0$ and set $v_\eta(x) := \eta\left(|x'| + |x'|^{n/2}(1+x_1^2)\right)$. Then, if $\eta > 0$ is large enough, it follows that $\det D^2v_\eta \ge 1$ inside $B_\rho(0)$ for some $\rho > 0$ small.
Let $w_\eta : B_\rho(0) \to \mathbb{R}$ be the convex envelope of $v_\eta|_{\partial B_\rho(0)}$. It is a classical fact that $\det D^2w_\eta = 0$ in the Alexandrov sense (see for instance [56]). Also, since $v_\eta \ge 0$ it follows that $w_\eta \ge 0$. Finally, since $v_\eta(x_1,0) = 0$ for $x_1 = \pm\rho$, we have $w_\eta(x_1,0) = 0$ for $|x_1| \le \rho$.
Now, let u be the Alexandrov solution of

$$\begin{cases} \det D^2u = 1 & \text{in } B_\rho \\ u = v_\eta & \text{on } \partial B_\rho \end{cases}$$

provided by Theorem 3.6. Then it follows by Proposition 3.3 that $v_\eta \le u \le w_\eta$ inside B_ρ. This implies in particular that $u(x_1,0) = 0$ for $|x_1| \le \rho$, that combined with

$$u(x_1,x') \ge v_\eta(x_1,x') \ge \eta|x'|$$

shows that u is merely Lipschitz continuous.

In the previous section we showed that strictly convex solutions of $0 < \lambda\, dx \le \mu_u \le \frac{1}{\lambda}\, dx$ are $W^{2,1+\varepsilon}_{\rm loc}$ for some $\varepsilon = \varepsilon(n,\lambda) > 0$. If one denotes by Σ the "singular set" of points where u is not strictly convex, that is

$$\Sigma := \{x \in \Omega \,:\, \exists\, z \in \Omega \setminus \{x\} \text{ and } p \in \partial u(x) \quad \text{s.t.} \quad u(z) = u(x) + \langle p, z - x\rangle\},$$

then one may wonder whether the second derivatives of u can concentrate on Σ. This fact has been recently ruled out by Mooney [53] who showed that the $(n-1)$-dimensional Hausdorff measure of Σ vanishes. From this, he deduced the $W^{2,1}$ regularity of solutions without any strict convexity assumptions. Actually, in a subsequent paper [54], he was able to strengthen this result by showing a small logarithmic integrality improvement and proving that such a result is optimal.

Theorem 4.3. — *Let $\Omega \subset \mathbb{R}^n$ be an open set, and $u : \Omega \to \mathbb{R}$ be a convex function satisfying $\mu_u = f\, dx$ for some $0 < \lambda \le f \le 1/\lambda$. Then $\mathcal{H}^{n-1}(\Sigma) = 0$ and $u \in W^{2,1}_{\rm loc}(\Omega)$. In addition, there exists $\eta = \eta(n) > 0$ such that*

$$\int_{\Omega'} |D^2 u| \log\Big(2 + |D^2 u|\Big)^{\eta} dx < \infty \qquad \forall\, \Omega' \subset\subset \Omega.$$

On the other hand, if $M > 0$ is sufficiently large, one can construct a solution u with $f \equiv 1$ such that

$$\int_{\Omega'} |D^2 u| \log\Big(2 + |D^2 u|\Big)^{M} dx = +\infty \qquad \textit{for some } \Omega' \subset\subset \Omega.$$

5. RECENT DEVELOPMENTS II: BOUNDARY REGULARITY

The interior regularity theory for Alexandrov solutions relies on several geometric properties of sections $S(x,p,y)$ of u that are strictly contained inside Ω (see (11)). In particular one can prove that, if u is a strictly convex solution of $\lambda\, dx \le \mu_u \le \frac{1}{\lambda}\, dx$ and $S_t(x) := S(x, \nabla u(x), t)$ is is compactly contained inside Ω, then $S_t(x)$ is comparable to an ellipsoid of volume $t^{n/2}$ (see for instance [31, Lemma 4.6]).

In order to develop a boundary regularity theory, it is crucial to understand the geometry of sections $S_t(x)$ when $x \in \partial\Omega$. This has been done by Savin in [59, 60, 61], where he recently introduced new techniques to obtain global versions of all the previous regularity results under suitable regularity assumptions on the boundary data. Let us describe the main results.

Assume that $0 \in \partial\Omega$, that $\Omega \subset \mathbb{R}^n$ is a bounded open convex set satisfying

$$B_\rho(\rho\, e_n) \subset \Omega \subset B_{1/\rho}(0) \cap \{x_n > 0\} \tag{16}$$

for some $\rho > 0$, and that $u : \Omega \to \mathbb{R}$ satisfies

$$\mu_u = f\, dx \qquad \text{in } \Omega \tag{17}$$

for some $0 < \lambda \le f \le 1/\lambda$. Extend u by letting it being equal to $+\infty$ in $\mathbb{R}^n \setminus \overline{\Omega}$, and up to subtracting a linear function assume that $\ell(x) \equiv 0$ is the tangent plane to u at 0, that is

$$u \ge 0, \qquad u(0) = 0, \qquad \text{and} \qquad u(x) \not\ge \varepsilon\, x_n \quad \forall\, \varepsilon > 0. \tag{18}$$

The main result in [59] shows that if $u \approx |x|^2$ along $\partial\Omega \cap \{x_n \le \rho\}$, then the sections $S_t(0) := \{x \in \Omega : u(x) < t\}$ are comparable to half-ellipsoids for t small. More precisely, the following holds:

Theorem 5.1. — *Let $\Omega \subset \mathbb{R}^n$ be a bounded open convex set satisfying* (16)*, and $u : \Omega \to \mathbb{R}$ be a convex function satisfying* (17) *for some $0 < \lambda \le f \le 1/\lambda$. Assume that* (18) *holds, and that*

$$\beta\, |x|^2 \le u(x) \le \frac{1}{\beta}\, |x|^2 \qquad \text{on } \partial\Omega \cap \{x_n \le \rho\} \tag{19}$$

for some $\beta > 0$. Then, for any $t > 0$ small, there exists an ellipsoid $\mathcal{E}_t$ of volume $t^{n/2}$ such that

$$\Big(\frac{1}{K}\, \mathcal{E}_t\Big) \cap \overline{\Omega} \subset S_t(0) \subset \Big(K\, \mathcal{E}_t\Big) \cap \overline{\Omega},$$

where $K > 1$ depends only on n, λ, ρ, and β. In addition, the ellipsoid $\mathcal{E}_t$ is comparable to a ball of radius $\sqrt{t}$, up to a possible translation along the x_n-direction of size $|\log t|$. Specifically, there exists a linear transformation $A_t : \mathbb{R}^n \to \mathbb{R}^n$ of the form

$$A_t(x) = x - \tau\, x_n, \qquad \tau = (\tau_1, \ldots, \tau_{n-1}, 0) \in \mathbb{R}^n, \qquad \text{and} \qquad |\tau| \le K\, |\log t|,$$

such that $\mathcal{E}_t = A_t\Big(B_{\sqrt{t}}(0)\Big)$.

The last part of the above result provides information about the behavior of the second derivatives near the origin. Indeed, heuristically, this result states that inside $S_t(0)$ the tangential second derivatives are uniformly bounded both from above and below, while the mixed second derivatives are bounded by $|\log t|$. This is very interesting given that μ_u is only bounded from above and below, and that the boundary data as well as the boundary are only $C^{1,1}$.

As a consequence of Theorem 5.1 and the interior estimates proved in Section 3.4, in [60, 61] Savin obtained the following global $C^{2,\alpha}$ and $W^{2,p}$ estimates.

Theorem 5.2. — *Let $\Omega \subset \mathbb{R}^n$ be a bounded open uniformly convex set, $u : \Omega \to \mathbb{R}$ be a convex function satisfying* (17) *for some $0 < \lambda \le f \le 1/\lambda$, and assume that both $u|_{\partial\Omega}$ and $\partial\Omega$ are of class $C^{1,1}$. Suppose also that u separates quadratically on $\partial\Omega$ from its tangent plane, that is*

$$u(z) - u(x) \ge \langle \nabla u(x), z - x\rangle + \beta\, |z - x|^2 \qquad \forall\, x, z \in \partial\Omega$$

for some $\beta > 0$. Then:

- *There exists $\varepsilon > 0$ such that $u \in W^{2,1+\varepsilon}(\overline{\Omega})$.*
- *For any $p > 1$, if $\|f - 1\|_{L^\infty(\overline{\Omega})} \le e^{-Cp}$ then $u \in W^{2,p}(\overline{\Omega})$.*

- *Assume that* $f \in C^{0,\alpha}(\overline{\Omega})$ *and that both* $u|_{\partial\Omega}$ *and* $\partial\Omega$ *are of class* $C^{2,\alpha}$*. Then* $u \in C^{2,\alpha}(\overline{\Omega})$*.*

As observed in [59], the assumption that u separates quadratically on $\partial\Omega$ from its tangent plane is verified, for instance, whenever $\partial\Omega$ and $u|_{\partial\Omega}$ are of class C^3 with Ω uniformly convex.

6. RECENT DEVELOPMENTS III: SMOOTHNESS OF THE FIRST EIGENFUNCTION

Let Ω be a smooth uniformly convex set. In the paper [48], P.-L. Lions investigated the existence and uniqueness of the first eigenvalue for the Monge-Ampère operator, namely the existence of a nontrivial convex function $\psi_1 \in C^{1,1}(\overline{\Omega}) \cap C^\infty(\Omega)$ and a positive constant λ_1 such that

$$(\det D^2\psi_1)^{1/n} = -\lambda_1\psi_1 \quad \text{in } \Omega, \qquad \psi_1 = 0 \quad \text{on } \partial\Omega. \tag{20}$$

As shown in [48], the couple (λ_1, ψ_1) is essentially unique. More precisely, if $\psi : \Omega \to \mathbb{R}$ is a nontrivial convex function and λ a positive constant such that

$$(\det D^2\psi)^{1/n} = -\lambda\psi \quad \text{in } \Omega, \qquad \psi = 0 \quad \text{on } \partial\Omega,$$

then $\lambda = \lambda_1$ and $\psi = \theta\psi_1$ for some positive constant θ.

Using the algebraic formula

$$(\det A)^{1/n} = \inf\Big\{\mathrm{tr}(AB) \,:\, B \text{ symm. pos. def., } \det B \geq \frac{1}{n^n}\Big\}, \tag{21}$$

one can prove that

$$\lambda_1 = \inf\Big\{\lambda_1(a_{ij}) \,:\, a_{ij} \in C(\overline{\Omega}),\, a_{ij} \text{ symm. pos. def., } \det(a_{ij}) \geq \frac{1}{n^n}\Big\},$$

where $\lambda_1(a_{ij})$ is the first eigenvalue of the linear elliptic operator $a_{ij}\partial_{ij}$. In addition, again using (21), one can approximate the Monge-Ampère equation with Hamilton-Jacobi-Bellman equations of the form

$$\mathcal{A}^\epsilon\psi := \inf\Big\{a_{ij}\partial_{ij}\psi \,:\, a_{ij} \in C(\overline{\Omega}),\, a_{ij} \text{ symm. pos. def., } \det(a_{ij}) \geq \frac{1}{n^n},\, \mathrm{tr}(a_{ij}) \leq \frac{1}{\epsilon}\Big\},$$

and deduce some interesting stochastic interpretation for λ_1 (see [47, 48] for more details).

As observed in Section 3.2, many results for the equation $\mu_u = f(x)\,dx$ can be extended to the general case $\mu_u = f(x, u, \nabla u)\,dx$ provided $\partial_u f \geq 0$, as this ensures the validity of the maximum principle. An interesting consequence of Lions' result is the validity of a maximum principle also when f is slightly decreasing with respect to u. More precisely, the equation $\mu_u = F(x, u)\,dx$ has a unique solution provided $\partial_u\Big(F(x, u)^{1/n}\Big) > -\lambda_1$ (see [48, Corollary 2]).

Note that, in view of the $C^{1,1}$ regularity of ψ_1, near the boundary of Ω one can write $|\psi_1(x)| = g(x)d_{\partial\Omega}(x)$, where $g : \Omega \to \mathbb{R}$ is a strictly positive Lipschitz function, and $d_{\partial\Omega}(x) = \mathrm{dist}(x, \partial\Omega)$ denotes the distance function to the boundary. In other words, ψ_1 solves a Monge-Ampère equation of the form

$$(22)\qquad \det D^2\psi_1(x) = G(x)\, d_{\partial\Omega}(x)^n \quad \text{in } \Omega, \qquad \psi_1 = 0 \quad \text{on } \partial\Omega,$$

where $G(x) \geq c_0 > 0$ is Lipschitz.

Because the right hand side vanishes of the equation on $\partial\Omega$, (22) is degenerate near the boundary and it has been an open problem for more than 30 years whether ψ_1 is smooth up to the boundary. The solution to this question has been given only recently, first by Hong, Huang, and Wang in two dimensions [38], and then by Savin [62] and by Le and Savin [45] in arbitrary dimensions.

More precisely, consider the general class of Monge-Ampère equations

$$(23)\qquad \mu_u = f\, dx \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega, \qquad f(x) = G(x)d_{\partial\Omega}(x)^s,$$

where $s > 0$ and G is a continuous strictly positive function. In [62] Savin proved the following C^2 regularity estimate at the boundary:

Theorem 6.1. — *Let $\Omega \subset \mathbb{R}^n$ be a bounded open convex set satisfying* (16)*, and $u : \Omega \to \mathbb{R}$ be a convex function satisfying* (23)*. Assume that* (18) *and* (19) *hold, and that $u|_{\partial\Omega\cap B_\rho(0)}$ is of class C^2 for some $\rho > 0$. Then u is C^2 at 0. More precisely, there exist a vector τ perpendicular to e_n, a quadratic polynomial $Q : \mathbb{R}^{n-1} \to \mathbb{R}$, and a constant $a > 0$, such that*

$$u(x + \tau x_n) = Q_0(x') + a x_n^{2+s} + o(|x'|^2 + x_n^{2+s}) \qquad \forall\, x = (x', x_n) \in B_\rho(0).$$

As a consequence of this result, since (20) is of the form (23) with $s = n$, Savin obtained the global C^2 regularity of the first eigenfunction:

Corollary 6.2. — *Let $\Omega \subset \mathbb{R}^n$ be a uniformly convex set of class C^2, and let ψ_1 be the first eigenfunction (see* (20)*). Then $\psi_1 \in C^2(\overline{\Omega})$.*

By a perturbative approach based on Theorem 6.1, Le and Savin improved the boundary C^2 regularity to $C^{2,\beta}$. More precisely, they showed the following pointwise estimate:

Theorem 6.3. — *Let $\Omega \subset \mathbb{R}^n$ be a bounded open convex set satisfying* (16)*, and $u : \Omega \to \mathbb{R}$ be a convex function satisfying* (23)*. Assume that* (18) *and* (19) *hold, and that $u|_{\partial\Omega\cap B_\rho(0)}$ is of class $C^{2,\beta}$ for some $\beta \in \left(0, \frac{2}{2+s}\right)$ and $\rho > 0$. Also, assume that $G \in C^{0,\gamma}(\overline{\Omega} \cap B_\rho(0))$ for some $\gamma \geq \frac{\beta(2+s)}{2}$. Then u is $C^{2,\beta}$ at 0. More precisely, there exist a vector τ perpendicular to e_n, a quadratic polynomial $Q : \mathbb{R}^{n-1} \to \mathbb{R}$, and a constant $a > 0$, such that*

$$u(x + \tau x_n) = Q_0(x') + a x_n^{2+s} + O(|x'|^2 + x_n^{2+s})^{1+\beta/2} \qquad \forall\, x = (x', x_n) \in B_\rho(0).$$

As a consequence of this result, one obtains the global $C^{2,\beta}$ regularity of the first eigenfunction for any $\beta < \frac{2}{2+n}$.

We note that usually, when dealing with elliptic questions, once one obtains $C^{2,\beta}$ regularity then the higher regularity follows easily by Schauder estimates. This is not the case in this situation because of the high degeneracy of the equation. The key idea in [45] consists in performing both an hodograph transform and a partial Legendre transform in order to deduce that (a suitable transformation of) the first eigenfunction satisfies a degenerate Grushin-type equation with Hölder coefficients. Once this is achieved, Le and Savin conclude the global smoothness of ψ_1 by applying Schauder estimates for Grushin-type operators:

Corollary 6.4. — *Let $\Omega \subset \mathbb{R}^n$ be a uniformly convex set of class C^∞, and let ψ_1 be the first eigenfunction (see* (20)*). Then $\psi_1 \in C^\infty(\overline{\Omega})$.*

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