Sobolev regularity for the Monge-Ampère equation, with application to the semigeostrophic equations

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Abstract

In this note we review some recent results on the Sobolev regularity of solutions to the Monge-Ampère equation, and show how these estimates can be used to prove some global existence results for the semigeostrophic equations.

1 The Monge-Ampère equation

The Monge-Ampère equation arises in connections with several problems from geometry and analysis (regularity for optimal transport maps, the Minkowski problem, the affine sphere problem, etc.) The regularity theory for this equation has been widely studied. In particular, Caffarelli developed in [4, 6, 5] a regularity theory for Alexandrov/viscosity solutions, showing that convex solutions of

$$\begin{cases} \det(D^2 u) = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{1.1}$$

are locally $C^{1,\alpha}$ provided $0 < \lambda \leq f \leq \Lambda$ for some $\lambda, \Lambda \in \mathbb{R}$. Moreover, for any $p > 1$ there exists $\delta = \delta(p) > 0$ such that $u \in W^{2,p}_{\rm loc}(\Omega)$ provided $|f - 1| \leq \delta$.

Then, few years later, Wang [17] showed that for any $p > 1$ there exists a function f satisfying $0 < \lambda \leq f \leq \Lambda$ such that $u \notin W^{2,p}_{\rm loc}(\Omega)$. This counterexample shows that the results of Caffarelli were more or less optimal. However, an important question which remained open was whether solutions of (1.1) with $0 < \lambda \leq f \leq \Lambda$ could be at least $W^{2,1}_{\rm loc}$, or even $W^{2,1+\epsilon}_{\rm loc}$ for some $\epsilon = \epsilon(n, \lambda, \Lambda) > 0$.

In the next section we motivate this $W^{2,1}_{\rm loc}$ question, showing how a positive answer to this question can be used to obtain some global existence results for the semigeostrophic equations [1, 2]. Then, following [11, 12], in Section 3 we prove that solutions to (1.1) are actually $W^{2,1+\varepsilon}_{\rm loc}$, and we show how the very same proof can be used to obtain Caffarelli's $W^{2,p}_{\rm loc}$ estimates.

2 The semigeostrophic equations

A motivation for being interested in the $W^{2,1}_{\rm loc}$ regularity of solutions to (1.1) comes from the semigeostrophic equations: The semigeostrophic equations are a simple model used in meteorology

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to describe large scale atmospheric flows. As explained for instance in [3, Section 2.2] (see also [9] for a more complete exposition), these equations can be derived from the 3-d incompressible Euler equations, with Boussinesq and hydrostatic approximations, subject to a strong Coriolis force. Since for large scale atmospheric flows the Coriolis force dominates the advection term, the flow is mostly bi-dimensional. For this reason, the study of the semigeostrophic equations in 2-d or 3-d is pretty similar, and in order to simplify our presentation we focus here on the 2-dimentional periodic case.

The semigeostrophic system can be written as

$$\begin{cases} \partial_t \nabla p_t + (\boldsymbol{u}_t \cdot \nabla)\nabla p_t + \nabla^\perp p_t + \boldsymbol{u}_t = 0 \\ \nabla \cdot \boldsymbol{u}_t = 0 \\ p_0 = \bar{p} \end{cases} \tag{2.1}$$

where $\boldsymbol{u}_t : \mathbb{R}^2 \to \mathbb{R}^2$ and $p_t : \mathbb{R}^2 \to \mathbb{R}$ are periodic functions corresponding respectively the velocity and the pressure.

As shown in [9], energetic considerations show that it is natural to assume that p_t is (-1)-convex, i.e., the function $P_t(x) := p_t(x) + |x|^2/2$ is convex on $\mathbb{R}^2$. If we denote with $\mathscr{L}_{\mathbb{T}^2}$ the Lebesgue measure on the 2-dimensional torus, then formally $\rho_t := (\nabla P_t)_\sharp \mathscr{L}_{\mathbb{T}^2}$ satisfies the following *dual problem*:

$$\begin{cases} \partial_t \rho_t + \nabla \cdot (\boldsymbol{U}_t \rho_t) = 0 \\ \boldsymbol{U}_t(x) = \big(x - \nabla P_t^*(x)\big)^\perp \\ \rho_t = (\nabla P_t)_\sharp \mathscr{L}_{\mathbb{T}^2} \\ P_0(x) = \bar{p}(x) + |x|^2/2, \end{cases} \tag{2.2}$$

where P_t^* is the convex conjugate of P_t, namely

$$P_t^*(y) := \sup_{x \in \mathbb{R}^2} \big\{ y \cdot x - P_t(x) \big\}.$$

The dual problem (2.2) is nowadays pretty well understood. In particular, Benamou and Brenier proved in [3] existence of weak solutions to (2.2). On the contrary, much less is known about the original system (2.1). Formally, given a solution ρ_t of (2.2) and defining P_t through the relation $\rho_t = (\nabla P_t)_\sharp \mathscr{L}_{\mathbb{T}^2}$ (namely the optimal transport map from ρ_t to $\mathscr{L}_{\mathbb{T}^2}$ for the quadratic cost on the torus), the pair $(p_t, \boldsymbol{u}_t)$ given by

$$\begin{cases} p_t(x) := P_t(x) - |x|^2/2 \\ \boldsymbol{u}_t(x) := \partial_t \nabla P_t^*(\nabla P_t(x)) + D^2 P_t^*(\nabla P_t(x)) \big(\nabla P_t(x) - x\big)^\perp \end{cases} \tag{2.3}$$

solves (2.1).

Being P_t^* just a convex function, *a priori* $D^2 P_t^*$ is a matrix-valued measure, thus it is not clear the meaning to give to the previous formula. However, since ρ_t solves a continuity equation with a divergence free vector field (notice that $\boldsymbol{U}_t$ is the rotated gradient of the function $|x|^2/2 - P_t^*(x)$, see (2.2)), we know that

$$0 < \lambda \le \rho_t \le \Lambda \qquad \forall\, t > 0 \tag{2.4}$$

provided this bound holds at $t = 0$.

In addition, the relation $\rho_t = (\nabla P_t)_\sharp \mathscr{L}_{\mathbb{T}^2}$ implies that $(\nabla P_t^*)_\sharp \rho_t = \mathscr{L}_{\mathbb{T}^2}$ (since ∇P_t^* is the inverse of ∇P_t), from which it follows [8] that P_t^* solves in the Alexandrov sense the Monge-Ampère equation

$$\det(D^2 P_t^*) = \rho_t$$

(see Section 3.1 for the definition of Alexandrov solution). Hence, it becomes clear now our initial question on the $W^{2,1}$ regularity of solutions to the Monge-Ampère equation: if we can prove that under (2.4) we have $D^2 P_t^* \in L^1$, then we have hopes to give a meaning to the velocity field $\boldsymbol{u}_t$ defined in (2.3), and then prove that $(p_t, \boldsymbol{u}_t)$ solve (2.1).

2.1 Space-time Sobolev regularity of ∇P_t^*

In [11] we proved not only that solutions to (1.1) with $0 < \lambda \leq f \leq \Lambda$ are $W^{2,1}_{\mathrm{loc}}$, but that actually, for any $k > 0$,

$$\int_{\Omega'} |D^2 u| \log^k(2 + |D^2 u|) < \infty \qquad \forall\, \Omega' \subset\subset \Omega. \tag{2.5}$$

The proof of this estimate strongly exploits the affine invariance of Monge-Ampère, and can actually be pushed forward to show that solutions are $W^{2,1+\varepsilon}_{\mathrm{loc}}$ for some $\varepsilon = \varepsilon(n, \lambda, \Lambda) > 0$ [12, 16].

As shown in [1, Theorem 2.2], this estimate immediately extends to solutions on the torus, so in particular it applies to P_t^*. Thanks to this fact, it is easy to see that the second term in the definition of $\boldsymbol{u}_t$ (see (2.3)) is well-defined and belongs to L^1.

To deal with the term $\partial_t \nabla P_t^*$, we need a second argument. We use $\log_+$ to denote the positive part of the logarithm, i.e., $\log_+(t) = \max\{\log(t), 0\}$. The following estimate is proved in [1, Proposition 3.3], following an idea introduced in [15, Theorem 5.1]:

Proposition 2.1. *For every $k \in \mathbb{N}$ there exists a constant C_k such that, for almost every $t \geq 0$,*

$$\begin{aligned}
&\int_{\mathbb{T}^2} \rho_t |\partial_t \nabla P_t^*| \log_+^k(|\partial_t \nabla P_t^*|)\, dx \\
&\qquad \leq C_k \left(\int_{\mathbb{T}^2} \rho_t |D^2 P_t^*| \log_+^{2k}(|D^2 P_t^*|)\, dx + \big\| \rho_t |\boldsymbol{U}_t|^2 \big\|_{L^\infty(\mathbb{T}^2)} \int_{\mathbb{T}^2} |D^2 P_t^*|\, dx \right).
\end{aligned} \tag{2.6}$$

Remark 2.2. Let us mention that, by the $W^{2,1+\varepsilon}_{\mathrm{loc}}$ regularity of P_t^*, one could actually prove that

$$\int_{\mathbb{T}^2} \rho_t |\partial_t \nabla P_t^*|^\kappa\, dx \leq C, \qquad \kappa := \frac{2 + 2\varepsilon}{2 + \varepsilon} > 1.$$

Although this estimate is stronger, it is less suited when one investigates the problem in the whole space [2]: indeed, in that case one would obtain that, for any $R > 0$, there exist $\kappa_R > 1$ and $C_R > 0$ such that

$$\int_{B(0,R)} \rho_t |\partial_t \nabla P_t^*|^{\kappa_R}\, dx \leq C_R$$

(i.e., the integrability exponent depends on R), while the estimates with the logarithm reads

$$\int_{B(0,R)} \rho_t |\partial_t \nabla P_t^*| \log_+^k(|\partial_t \nabla P_t^*|)\, dx \leq C_{k,R},$$

and this makes it simpler to use.

Sketch of the proof of (2.6). In order to justify the following computations one needs to perform a careful regularization argument. Here we show just the formal argument, referring to [1] for a detailed proof.

First of all, by differentiating in time the relation $\det(D^2P_t^*) = \rho_t$ we get

$$\sum_{i,j=1}^{2} M_{ij}(D^2P_t^*(x))\,\partial_t\partial_{ij}P_t^*(x) = \partial_t\rho_t,$$

where $M_{ij}(A) := \frac{\partial \det(A)}{\partial A_{ij}}$ is the cofactor matrix of A. Taking into account (2.2) and the well-known divergence-free property of the cofactor matrix

$$\sum_i \partial_i M_{ij}(D^2{P_t}^*(x)) = 0, \qquad j = 1, 2,$$

we can rewrite the above equation as

$$\sum_{i,j=1}^{2} \partial_i\big(M_{ij}(D^2P_t^*(x))\,\partial_t\partial_j P_t^*(x)\big) = -\nabla\cdot(\boldsymbol{U}_t\rho_t),$$

and recalling the well-known identity $M(A) = \det(A)\,A^{-1}$ we get

$$\nabla\cdot\big(\rho_t(D^2P_t^*)^{-1}\partial_t\nabla P_t^*\big) = -\nabla\cdot(\rho_t\boldsymbol{U}_t), \tag{2.7}$$

where we used again the relation $\det(D^2P_t^*) = \rho_t$.

We now multiply (2.7) by $\partial_t P_t^*$ and integrate by parts to obtain

$$\begin{aligned}\int_{\mathbb{T}^2} \rho_t|(D^2P_t^*)^{-1/2}\partial_t\nabla P_t^*|^2\,dx &= \int_{\mathbb{T}^2} \rho_t\partial_t\nabla P_t^*\cdot(D^2P_t^*)^{-1}\partial_t\nabla P_t^*\,dx \\ &= -\int_{\mathbb{T}^2} \rho_t\partial_t\nabla P_t^*\cdot\boldsymbol{U}_t\,dx.\end{aligned} \tag{2.8}$$

(Since the matrix $D^2{P_t}^*$ is positive definite, both its square root and the square root of its inverse are well-defined.) From Cauchy-Schwartz inequality, the right-hand side of (2.8) can be estimated as

$$\begin{aligned}&-\int_{\mathbb{T}^2} \rho_t\partial_t\nabla P_t^*\cdot(D^2P_t^*)^{-1/2}(D^2P_t^*)^{1/2}\boldsymbol{U}_t\,dx \\ &\le \left(\int_{\mathbb{T}^2} \rho_t|(D^2P_t^*)^{-1/2}\partial_t\nabla P_t^*|^2\,dx\right)^{1/2}\left(\int_{\mathbb{T}^2} \rho_t|(D^2P_t^*)^{1/2}\boldsymbol{U}_t|^2\,dx\right)^{1/2}.\end{aligned} \tag{2.9}$$

Moreover, the second term in the right-hand side of (2.9) can be bounded by

$$\int_{\mathbb{T}^2} \rho_t\boldsymbol{U}_t\cdot D^2P_t^*\boldsymbol{U}_t\,dx \le \sup_{\mathbb{T}^2}\big(\rho_t|\boldsymbol{U}_t|^2\big)\int_{\mathbb{T}^2}|D^2P_t^*|\,dx. \tag{2.10}$$

Hence, it follows from (2.8), (2.9), and (2.10) that

$$\int_{\mathbb{T}^2} \rho_t|(D^2P_t^*)^{-1/2}\partial_t\nabla P_t^*|^2\,dx \le \sup_{\mathbb{T}^2}\big(\rho_t|\boldsymbol{U}_t|^2\big)\int_{\mathbb{T}^2}|D^2P_t^*|\,dx. \tag{2.11}$$

We now apply the elementary estimate (see [1, Lemma 3.4])

$$ab\log_+^k(ab) \le C_k\left(a^2\log_+^{2k}(a^2) + b^2\right) \qquad \forall\,(a,b)\in\mathbb{R}^+\times\mathbb{R}^+$$

with $a = |(D^2P_t^*)|^{1/2}$ and $b = |(D^2P_t^*)^{-1/2}\partial_t\nabla {P_t}^*(x)|$, to deduce that

$$|\partial_t\nabla P_t^*|\log_+^k(|\partial_t\nabla P_t^*|) \le C_k\left(|D^2P_t^*|\log_+^{2k}(|D^2P_t^*|) + |(D^2P_t^*)^{-1/2}\partial_t\nabla P_t^*|^2\right).$$

Multiplying the above inequality by ρ_t and integrating it over $\mathbb{T}^2$, using (2.11) we obtain

$$\begin{aligned}
&\int_{\mathbb{T}^2}\rho_t|\partial_t\nabla P_t^*|\log_+^k(|\partial_t\nabla P_t^*|)\,dx\\
&\le C_k\left(\int_{\mathbb{T}^2}\rho_t|D^2P_t^*|\log_+^{2k}(|D^2P_t^*|)\,dx + \int_{\mathbb{T}^2}\rho_t|(D^2P_t^*)^{-1/2}\partial_t\nabla P_t^*|^2\,dx\right)\\
&\le C_k\left(\int_{\mathbb{T}^2}\rho_t|D^2P_t^*|\log_+^{2k}(|D^2P_t^*|)\,dx + \sup_{\mathbb{T}^2}\left(\rho_t|\boldsymbol{U}_t|^2\right)\int_{\mathbb{T}^2}|D^2P_t^*|\,dx\right),
\end{aligned}$$

which proves (2.6). $\square$

Thanks to (2.5) applied to P_t^* and (2.6), we deduce easily that the velocity field $\boldsymbol{u}_t$ in (2.3) belongs to $L^1(\mathbb{T}^2)$.

2.2 $(p_t, \boldsymbol{u}_t)$ solves the semigeostrophic system

In order to prove that the couple $(p_t, \boldsymbol{u}_t)$ defined in (2.3) is a distributional solution of (2.1) we need to find some suitable test functions to use in (2.2).

More precisely, we first write (2.2) in distributional form:

$$\int\int_{\mathbb{T}^2}\Big\{\partial_t\varphi_t(x) + \nabla\varphi_t(x)\cdot\boldsymbol{U}_t(x)\Big\}\rho_t(x)\,dx\,dt + \int_{\mathbb{T}^2}\varphi_0(x)\rho_0(x)\,dx = 0 \tag{2.12}$$

for every $\varphi\in W^{1,1}(\mathbb{R}^2\times[0,\infty))$ $\mathbb{Z}^2$-periodic in the space variable.

We now take $\phi\in C_c^\infty(\mathbb{R}^2\times[0,\infty))$ a function $\mathbb{Z}^2$-periodic in space, and we consider the test function $\varphi:\mathbb{R}^2\times[0,\infty)\to\mathbb{R}^2$ defined as

$$\varphi_t(y) := J(y-\nabla P_t^*(y))\,\phi_t(\nabla P_t^*(y)), \tag{2.13}$$

where $J := \begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}$ denotes the rotation by $\pi/2$. We compute the derivatives of φ:

$$\begin{cases}
\partial_t\varphi_t(y) = -J[\partial_t\nabla P_t^*](y)\,\phi_t(\nabla P_t^*(y)) + J\big(y-\nabla P_t^*(y)\big)\,\partial_t\phi_t(\nabla P_t^*(y)) +\\
\qquad\qquad + J\big(y-\nabla P_t^*(y)\big)\big[\nabla\phi_t(\nabla P_t^*(y))\cdot\partial_t\nabla P_t^*(y)\big],\\
\nabla\varphi_t(y) = J\big(Id - D^2P_t^*(y)\big)\,\phi_t(\nabla P_t^*(y))\\
\qquad\qquad + J\big(y-\nabla P_t^*(y)\big)\otimes\big(\nabla^T\phi_t(\nabla P_t^*(y))D^2P_t^*(y)\big).
\end{cases} \tag{2.14}$$

Taking into account that $(\nabla P_t)_\sharp \mathscr{L}_{\mathbb{T}^2} = \rho_t$ and that $\nabla P_t^*(\nabla P_t(x)) = x$ a.e., we can rewrite the boundary term in (2.12) as

$$\int_{\mathbb{T}^2} \varphi_0(y)\rho_0(y)\,dy = \int_{\mathbb{T}^2} J\big(\nabla P_0(x) - x\big)\phi_0(x)\,dx = \int_{\mathbb{T}^2} J\nabla p_0(x)\phi_0(x)\,dx. \tag{2.15}$$

In the same way, since $\boldsymbol{U}_t(y) = J(y - \nabla P_t^*(y))$, we can use (2.14) to rewrite the other term as

$$\begin{aligned}
&\int_0^\infty \int_{\mathbb{T}^2} \Big\{\partial_t\varphi_t(y) + \nabla\varphi_t(y)\cdot \boldsymbol{U}_t(y)\Big\}\rho_t(y)\,dy\,dt \\
&= \int_0^\infty \int_{\mathbb{T}^2} \Big\{ - J[\partial_t \nabla P_t^*](\nabla P_t(x))\,\phi_t(x) + J\big(\nabla P_t(x) - x\big)\,\partial_t\phi_t(x) \\
&\quad + J\big(\nabla P_t(x) - x\big)\big[\nabla\phi_t(x)\cdot\partial_t\nabla P_t^*(\nabla P_t(x))\big] \\
&\quad + \Big[J\big(Id - D^2P_t^*(\nabla P_t(x))\big)\,\phi_t(x) \\
&\qquad + J\big(\nabla P_t(x) - x\big)\otimes\big(\nabla^T\phi_t(x)D^2P_t^*(\nabla P_t(x))\big)\Big]J\big(\nabla P_t(x) - x\big)\Big\}\,dx\,dt
\end{aligned} \tag{2.16}$$

which, taking into account the formula (2.3) for $\boldsymbol{u}_t$, after rearranging the terms turns out to be equal to

$$\int_0^\infty \int_{\mathbb{T}^2} \Big\{J\nabla p_t(x)\big(\partial_t\phi_t(x) + \boldsymbol{u}_t(x)\cdot\nabla\phi_t(x)\big) + \big(-\nabla p_t(x) - J\boldsymbol{u}_t(x)\big)\phi_t(x)\Big\}\,dx\,dt. \tag{2.17}$$

Hence, combining (2.15), (2.16), (2.17), and (2.12), we obtain that $(p_t, \boldsymbol{u}_t)$ solve the first equation in (2.1). The fact that $\boldsymbol{u}_t$ is divergence free is proved in a similar way, using the test function

$$\varphi_t(y) := \phi(t)\psi(\nabla {P_t}^*(y)).$$

Therefore, we obtain the following result [1, Theorem 1.2]:

Theorem 2.3. *Let $\bar p : \mathbb{R}^2 \to \mathbb{R}$ be a $\mathbb{Z}^2$-periodic function such that $\bar p(x) + |x|^2/2$ is convex, and assume that the measure $(Id + \nabla\bar p)_\sharp \mathscr{L}_{\mathbb{T}^2}$ is absolutely continuous with respect to the Lebesgue measure with density $\bar\rho$, namely*

$$(Id + \nabla\bar p)_\sharp \mathscr{L}_{\mathbb{T}^2} = \bar\rho.$$

Moreover, let us assume that both $\bar\rho$ and $1/\bar\rho$ belong to $L^\infty(\mathbb{R}^2)$.

Let ρ_t be a solution of (2.2) starting from $\bar\rho$, and let $P_t : \mathbb{R}^2 \to \mathbb{R}$ be the (unique up to an additive constant) convex function such that $(\nabla P_t)_\sharp \mathscr{L}_{\mathbb{T}^2} = \rho_t$ and $P_t(x) - |x|^2/2$ is $\mathbb{Z}^2$-periodic. Denote by $P_t^* : \mathbb{R}^2 \to \mathbb{R}$ its convex conjugate.

Then the couple $(p_t, \boldsymbol{u}_t)$ defined in (2.3) is a distributional solution of (2.1).

Although the vector field $\boldsymbol{u}_t$ provided by the previous theorem is only L^1, in [1] we also showed how to associate to it a measure-preserving Lagrangian flow. In particular we recovered (in the particular case of the 2-dimensional periodic setting) the result of Cullen and Feldman [10] on the existence of Lagrangian solutions to the semigeostrophic equations in physical space.

The 3-dimensional case on a bounded convex domain $\Omega \subset \mathbb{R}^3$ presents additional difficulties [2]. Indeed, first of all, in 3-d, Equation (2.1) becomes much less symmetric compared to its

2-d counterpart, because the action of Coriolis force regards only the first and the second space components. Moreover, even considering regular initial data and velocities, regularity results require a finer regularization scheme, due to the non-compactness of the ambient space. Still, under suitable assumptions on the initial data, we can prove the existence of distributional solutions (see [2] for more details).

Let us mention that a key assumption made in [2] is that the initial data $\rho_0 = (\nabla P_0)_\sharp \mathscr{L}_\Omega$ is supported on the whole $\mathbb{R}^3$ (here $\mathscr{L}_\Omega$ denotes the Lebesgue measure restricted to Ω). It would be extremely interesting to remove this assumption in order to deal with the case when ρ_0 is compactly supported (which is the most interesting case from a physical point of view). However, the nontrivial evolution of the support of the solution ρ_t does not allow to apply the regularity results in [11, 12, 16], which are actually expected to fail in this situation, so completely new ideas need to be introduced in order to prove existence of distributional solutions in this case.

3 Sobolev regularity for the Monge-Ampère equation

In this section we prove that solutions to (1.1) are $W^{2,1+\varepsilon}_{\rm loc}$. For this, we follow the arguments in [11, 12]. In addition, we show that the very same proof can be used to obtain the $W^{2,p}_{\rm loc}$ estimates from [5].

3.1 Notation and Preliminaries

We say that a convex function $u : \Omega \to \mathbb{R}$ is an Alexandrov solution of the Monge-Ampère equation (1.1) if

$$\Big| \bigcup_{x\in E} \partial u(x) \Big| = \int_E f \qquad \text{for all } E \subset \Omega \text{ Borel},$$

where $\partial u(x)$ denotes the subdifferential of u at x, and $|E|$ denotes the Lebesgue measure of a set E.

Given $u : \Omega \to \mathbb{R}$ a C^1 convex function, we define the section $S_h(x_0)$ centered at x_0 at height h as

$$S_h(x_0) := \big\{ x \in \Omega \,:\, u(x) < u(x_0) + \nabla u(x_0)\cdot(x-x_0) + h \big\}.$$

If $u : \Omega \to \mathbb{R}$ solves (1.1) then $u \in C^{1,\alpha}_{\rm loc}(\Omega)$ [4, 6], and sections well contained inside Ω (say, $S_h(x_0) \subset \Omega' \subset\subset \Omega$) enjoy several nice geometric properties (see [4, 6, 7, 14]).

Indeed, first of all, there exists a universal constant $\sigma > 1$ such that the following holds: for any section $S_h(x_0) \subset \Omega'$, there exists an affine transformation A with $\det A = 1$ such that

$$B_{\sqrt{h/\sigma}} \subset A(S_h(x_0) - x_0) \subset B_{\sqrt{\sigma h}}, \tag{3.1}$$

which implies in particular that

$$|S_h(x_0)| \simeq h^{n/2}. \tag{3.2}$$

In addition, there exists $\eta \in (0, 1)$ universal such that:

(a) If $h_1 \le h_2$ and $S_{\eta h_1}(x_1) \cap S_{\eta h_2}(x_2) \neq \emptyset$ then

$$S_{\eta h_1}(x_1) \subset S_{h_2}(x_2).$$

(b) If $h_1 \le h_2$ and $x_1 \in \overline{S_{h_2}(x_2)}$ then we can find a point z such that

$$S_{\eta h_1}(z) \subset S_{h_1}(x_1) \cap S_{h_2}(x_2).$$

(c) If $x_1 \in S_h(x_2)$ then

$$S_{\eta h}(x_1) \subset S_{2h}(x_2).$$

Because of Property (a) above, sections are well suited for covering lemmas (see for instance [12, Lemma 2.2]):

Lemma 3.1. *Let $D \subset \Omega' \subset\subset \Omega$ be a compact set, and assume that to each $x \in D$ we associate a corresponding section $S_h(x) \subset \Omega'$. Then we can find a finite number of these sections $S_{h_i}(x_i)$, $i = 1, \dots, m$, such that*

$$D \subset \bigcup_{i=1}^{m} S_{h_i}(x_i), \qquad \text{with } S_{\eta h_i}(x_i) \text{ disjoint.}$$

In the proof of our result we will use the "normalized size" of a section to measure the size of D^2u: we say that $S_h(x_0)$ has *normalized size* α if

$$\alpha := \|A\|^2$$

for some matrix A as in (3.1). (Notice that, although A may not be unique, this definition fixes the value of α up to multiplicative universal constants.) It is not difficult to check that if u is C^2 in a neighborhood of x_0, then as $h \to 0$ the normalized size of $S_h(x_0)$ converges $\|D^2u(x_0)\|$, up to dimensional constants.

Given a transformation A as in (3.1), we define $\tilde{u}$ to be the rescaling of u

$$\tilde{u}(\tilde{x}) := h^{-1}u(x), \qquad \tilde{x} = Tx := h^{-1/2}A\,(x - x_0). \tag{3.3}$$

Then $\tilde{u}$ solves an equation of the same form:

$$\det D^2\tilde{u} = \tilde{f}, \qquad \text{with} \quad \tilde{f}(\tilde{x}) := f(x).$$

In particular,

$$\lambda \le f \le \Lambda \quad (\text{resp. } |f-1| \le \delta) \quad \Longrightarrow \quad \lambda \le \tilde{f} \le \Lambda \quad (\text{resp. } |\tilde{f}-1| \le \delta).$$

In addition the section $\tilde{S}_1(0)$ of $\tilde{u}$ at height 1 is normalized, that is,

$$B_{1/\sigma} \subset \tilde{S}_1(0) \subset B_\sigma, \qquad \tilde{S}_1(0) := T(S_h(x_0)).$$

Also $D^2u(x) = A^T D^2\tilde{u}(\tilde{x})A$, which implies $\|D^2u(x)\| \le \|A\|^2\|D^2\tilde{u}(\tilde{x})\|$, and

$$\gamma_1 I \le D^2\tilde{u}(\tilde{x}) \le \gamma_2 I \quad \Rightarrow \quad \gamma_1\|A\|^2 \le \|D^2u(x)\| \le \gamma_2\|A\|^2.$$

3.2 $W^{2,1+\varepsilon}$ and $W^{2,p}$ regularity

We assume throughout that u is a normalized solution of (1.1), that is

$$\det D^2u = f \quad \text{in } \Omega, \qquad \lambda \le f \le \Lambda, \qquad S_2(0) \subset \Omega' \subset\subset \Omega, \qquad B_{1/\sigma} \subset S_1(0) \subset B_\sigma.$$

Our goal is to show that

$$\int_{S_1(0)} \|D^2u\|^{1+\varepsilon} dx \le C, \tag{3.4}$$

for some universal constants $\varepsilon, C > 0$. In addition, we will also prove that for any $p > 1$ there exists $\delta = \delta(p) \simeq e^{-Cp}$ such that

$$\int_{S_1(0)} \|D^2u\|^p dx \le C \qquad \text{provided } |f-1| \le \delta. \tag{3.5}$$

Once these estimates are proved, the desired interior regularity follows by a standard scaling/covering argument, see for instance [11].

Without loss of generality we may assume that $u \in C^2$, since the general case follows by approximation (for instance, one may convolve f to have a smooth solution, prove the estimates in this case, and then pass to the limit).

Lemma 3.2. *Assume $S_2(0) \subset \Omega'$, and $0 \in \overline{S_t(y)} \subset \Omega'$ for some $t \ge 1$. Then there exists a large universal constant $K > 0$ such that:*

(i)

$$\int_{S_1(0)} \|D^2u\| dx \le K \left|\{\mathrm{Id}/K \le D^2u \le K\mathrm{Id}\} \cap S_\eta(0) \cap S_t(y)\right|.$$

(ii) If in addition $|f-1| \le \delta$ then

$$\left|S_1(0) \cap \{\|D^2u\| \ge K\}\right| \le K\,\delta^\gamma \left|\{\mathrm{Id}/K \le D^2u \le K\mathrm{Id}\} \cap S_\eta(0) \cap S_t(y)\right|$$

for some $\gamma > 0$ universal.

Proof. Since u is convex we have

$$\int_{S_1(0)} \|D^2u\| dx \le \int_{S_1(0)} \Delta u \, dx \le \int_{\partial S_1(0)} u_\nu \le C_1, \tag{3.6}$$

where the last inequality follows from the interior Lipschitz estimate of u in $S_2(0)$.

In addition, Property (b) in Subsection 3.1 gives

$$S_\eta(0) \cap S_t(y) \supset S_{\eta^2}(z)$$

for some point z, which by (3.2) implies that

$$|S_\eta(0) \cap S_t(y)| \ge c_1$$

for some $c_1 > 0$ universal. These two inequalities show that

$$\left|\left\{\|D^2u\| \le 2C_1/c_1\right\} \cap S_\eta(0) \cap S_t(y)\right| \ge c_1/2.$$

In addition, the lower bound $\det D^2u \ge \lambda$ implies that there exists a universal constant $C_2 > 0$ such that

$$\mathrm{Id}/C_2 \le D^2u \le 2C_1/c_1\mathrm{Id} \qquad \text{inside } \{\|D^2u\| \le 2C_1/c_1\},$$

from which we deduce that

$$\left|\left\{\mathrm{Id}/C_2 \le D^2u \le 2C_1/c_1\mathrm{Id}\right\} \cap S_\eta(0) \cap S_t(y)\right| \ge c_1/2. \tag{3.7}$$

- *Case (i).* By (3.6) and (3.7) we get the desired estimate choosing $K := \max\{C_2, 2C_1/c_1\}$.
- *Case (ii).* Thanks to (3.7), the desired estimate follows immediately from the bound

  $$\left|S_1(0) \cap \left\{\|D^2u\| \ge C_3\right\}\right| \le C_3\,\delta^\gamma, \qquad C_3, \gamma > 0 \text{ universal},$$

  (see for instance [5, Lemma 6 and Corollary 1] or [13, Theorem 6.1.1]), choosing $K := \max\{C_3, 2C_3/c_1, C_2, 2C_1/c_1\}$.

□

Applying Lemma 3.2 to the rescaling $\tilde{u}$ defined in Section 2 (see (3.3)) we obtain the following key estimates (see [12] for more details):

Lemma 3.3. *Let $S_{2h}(x_0) \subset \Omega'$, and assume $x_0 \in \overline{S_t(y)} \subset \Omega'$ for some $t \ge h$. If $S_h(x_0)$ has normalized size α, then:*

(i)

$$\int_{S_h(x_0)} \|D^2u\|\,dx \le K\alpha\,\left|\left\{\alpha/K \le \|D^2u\| \le K\alpha\right\} \cap S_{\eta h}(x_0) \cap S_t(y)\right|.$$

(ii) If in addition $|f-1| \le \delta$ then

$$\left|S_h(x_0) \cap \left\{\|D^2u\| \ge K\alpha\right\}\right| \le K\,\delta^\gamma\,\left|\left\{\alpha/K \le \|D^2u\| \le K\alpha\right\} \cap S_{\eta h}(x_0) \cap S_t(y)\right|.$$

Next, we denote by D_k the compact sets

$$D_k := \left\{x \in \overline{S_1(0)} : \|D^2u(x)\| \ge M^k\right\}, \tag{3.8}$$

where $M > 0$ is a large constant (to be chosen). As we show now, Lemma 3.3 combined with a covering argument gives a geometric decay for both $\int_{D_k}\|D^2u\|$ and $|D_k|$.

Lemma 3.4. *If M is sufficiently large (the largeness being universal), then:*

(i)

$$\int_{D_{k+1}} \|D^2u\|\,dx \le (1-\tau)\int_{D_k} \|D^2u\|\,dx, \qquad \tau := \frac{1}{1+K^2}.$$

(ii) If in addition $|f-1| \le \delta$ then

$$|D_{k+1}| \le K^2 \delta^\gamma |D_k|.$$

Proof. Let $M \gg K$ (to be fixed later), and for each $x \in D_{k+1}$ consider a section

$$S_h(x) \text{ of normalized size } \alpha := KM^k,$$

which is compactly included in $S_2(0)$. This is possible: indeed, as $h \to 0$ the normalized size of $S_h(x)$ converges up to a dimensional constant to $\|D^2u(x)\|$ (recall that $u \in C^2$), which by assumption is greater than M^{k+1} (since $x \in D_{k+1}$). On the other hand $S_\eta(x) \subset S_2(0)$ (by Property (c) in Subsection 3.1) and the normalized size of $S_{\eta/2}(x)$ is bounded above by a universal constant, and therefore by α. Hence, by continuity, there exists $h \in (0, \eta/2)$ such that $S_h(x)$ has normalized size $\alpha = KM^k$.

We now apply Lemma 3.1 to find a finite subfamily of sections $\{S_{h_i}(x_i)\}_{i=1,\dots,m}$ covering D_{k+1} such that the sections $S_{\eta h_i}(x_i)$ are disjoint. Then, by Lemma 3.3 applied with $y = 0$ and $t = 1$, for each $i = 1, \dots, m$ we get

$$\int_{S_{h_i}(x_i)} \|D^2u\| dx \le K^2 M^k \left| \left\{ M^k \le \|D^2u\| \le K^2 M^k \right\} \cap S_{\eta h_i}(x_i) \cap S_1(0) \right|,$$

and

$$\left| S_{h_i}(x_i) \cap \left\{ \|D^2u\| \ge K^2 M^k \right\} \right| \le K^2 \delta^\gamma \left| \left\{ M^k \le \|D^2u\| \le K^2 M^k \right\} \cap S_{\eta h_i}(x_i) \cap S_1(0) \right|$$

provided $|f-1| \le \delta$. Adding these inequalities over $i = 1, \dots, m$, and using that

$$D_{k+1} \subset \bigcup_{i=1}^m S_{h_i}(x_i), \qquad S_{\eta h_i}(x_i) \text{ disjoint},$$

we obtain:

- *Case (i).*

$$\begin{aligned} \int_{D_{k+1}} \|D^2u\| dx \le\ & K^2 M^k \left| \left\{ M^k \le \|D^2u\| \le K^2 M^k \right\} \cap S_1(0) \right| \\ \le\ & K^2 \int_{\left\{ M^k \le \|D^2u\| \le K^2 M^k \right\} \cap S_1(0)} \|D^2u\| \\ \le\ & K^2 \int_{D_k \setminus D_{k+1}} \|D^2u\| dx \end{aligned}$$

  provided $M > K^2$, so by adding $K^2 \int_{D_{k+1}} \|D^2u\|$ to both sides of the above inequality the conclusion follows with $\tau = 1/(1+K^2)$.

- *Case (ii).*

$$|D_{k+1}| \le K^2 \delta^\gamma \left| \left\{ M^k \le \|D^2u\| \le K^2 M^k \right\} \cap S_1(0) \right| \le K^2 \delta^\gamma |D_k|,$$

  provided $M > K^2$.

$\square$

By the above result, the proof of (3.4) and (3.5) is immediate. Indeed Lemma 3.4(i), (3.6), and Chebyshev's inequality imply

$$|D_k| \leq \frac{1}{M^k}\int_{D_k}\|D^2u\| \leq \frac{(1-\tau)^k}{M^k}\int_{S_1(0)}\|D^2u\| \leq C_1\frac{(1-\tau)^k}{M^k} = \frac{C_1}{M^{k(1+2\varepsilon)}}, \qquad \varepsilon := \frac{|\log(1-\tau)|}{2\log(M)},$$

while Lemma 3.4(ii) gives

$$|D_k| \leq \left(K^2\delta^\gamma\right)^k \leq \frac{1}{M^{k(p+1)}} \qquad \text{provided } \delta \leq \frac{1}{\left(M^{p+1}K^2\right)^{1/\gamma}},$$

so both (3.4) and (3.5) follow by the classical layer-cake formula

$$\int_{S_1(0)}\|D^2u\|^q = q\int_0^\infty t^{q-1}|S_1(0)\cap\{\|D^2u\|\geq t\}|\,dt \lesssim \sum_{k=1}^\infty M^{kq}|D_k| \qquad \forall\, q\geq 1.$$

References

[1] L. Ambrosio, M. Colombo, G. De Philippis, A. Figalli: *Existence of Eulerian solutions to the semigeostrophic equations in physical space: the 2-dimensional periodic case*, Comm. Partial Differential Equations, to appear.

[2] L. Ambrosio, M. Colombo, G. De Philippis, A. Figalli: *A global existence result for the semigeostrophic equations in three dimensional convex domains*, Discrete Contin. Dyn. Syst., to appear.

[3] J.-D. Benamou, Y. Brenier: *Weak existence for the semigeostrophic equation formulated as a coupled Monge-Ampère/transport problem.* SIAM J. Appl. Math., **58** (1998), 1450–1461.

[4] L. Caffarelli: *A localization property of viscosity solutions to the Monge-Ampère equation and their strict convexity.* Ann. of Math. (2), **131** (1990), no. 1, 129–134.

[5] L. Caffarelli: *Interior $W^{2,p}$ estimates for solutions of the Monge-Ampère equation.* Ann. of Math. (2), **131** (1990), no. 1, 135–150.

[6] L. Caffarelli: *Some regularity properties of solutions to Monge-Ampère equations.* Comm. Pure Appl. Math., **44** (1991), 965–969.

[7] L. Caffarelli, C. Gutierrez: *Real analysis related to the Monge-Ampre equation.* Trans. Amer. Math. Soc., **348** (1996), no. 3, 1075–1092.

[8] D. Cordero Erausquin: *Sur le transport de mesures périodiques.* C. R. Acad. Sci. Paris Sér. I Math., **329** (1999), 199–202.

[9] M.Cullen: *A mathematical theory of large-scale atmosphere/ocean flow.* Imperial College Press (2006).

[10] M. Cullen, M. Feldman: *Lagrangian solutions of semigeostrophic equations in physical space.* SIAM J. Math. Anal., **37** (2006), 1371–1395.

[11] G. De Philippis, A. Figalli: $W^{2,1}$ *regularity for solutions of the Monge-Ampère equation*, Invent. Math., to appear.

[12] G. De Philippis, A. Figalli, O. Savin: *A note on interior* $W^{2,1+\epsilon}$ *estimates for the Monge-Ampère equation*, Math. Ann., to appear.

[13] C.Gutierrez: *The Monge-Ampére equation*. Progress in Nonlinear Differential Equations and their Applications, 44. Birkhäuser Boston, Inc., Boston, MA, 2001.

[14] C.Gutierrez, Q. Huang: *Geometric properties of the sections of solutions to the Monge-Ampre equation.* Trans. Amer. Math. Soc., **352** (2000), no. 9, 4381–4396.

[15] G. Loeper: *On the regularity of the polar factorization for time dependent maps.* Calc. Var. Partial Differential Equations, **22** (2005), 343–374.

[16] T. Schmidt: $W^{2,1+\epsilon}$ *estimates for the Monge-Ampère equation*, preprint (2012).

[17] X.-J Wang.: *Some counterexamples to the regularity of Monge-Ampère equations*, Proc. Amer. Math. Soc., **123** (1995), no. 3, 841–845.