STABILITY RESULTS FOR THE BRUNN-MINKOWSKI INEQUALITY

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1. Introduction

The Brunn-Miknowski inequality gives a lower bound on the Lebesgue measure of a sumset in terms of the measures of the individual sets. This classical inequality in convex geometry was inspired by issues around the isoperimetric problem and was considered for a long time to belong to geometry, where its significance is widely recognized. However, it is by now clear that the Brunn-Miknowski inequality has also applications in analysis, statistics, informations theory, etc. (we refer the reader to [29] for an extended exposition on the Brunn-Minkowski inequality and its relation to several other famous inequalities).

To focus more on the analytic side, we recall that Brunn-Minkowski **(BM)** is intimately connected to several other famous inequalities such as the isoperimetric **(Isop)** inequality, Sobolev **(Sob)** inequalities, and Gagliardo-Nirenberg **(GN)** inequalities. In particular, it is well-known that the following chain of implications holds, although in general one cannot obtain one inequality from the other with sharp constants (see for instance [20] for a more detailed discussion):

$$(\mathbf{BM}) \quad \Rightarrow \quad (\mathbf{Isop}) \quad \Rightarrow \quad (\mathbf{Sob}) \quad \Rightarrow \quad (\mathbf{GN}).$$

The issue of the sharpness of a constant, as well as the characterization of minimizers, is a classical and important question which is by now well understood (at least for the class of inequalities we are considering). More recently, a lot of attention has been given to the stability issue:

Suppose that a function almost attains the equality in one of the previous inequalities. Can we prove, if possible in some quantitative way, that such a function is close (in some suitable sense) to one of the minimizers?

In the latest years several results have been obtained in this direction, showing stability for isoperimetric inequalities [28, 23, 12, 17, 13], the Brunn-Minkowski inequality on convex sets [24], Sobolev [11, 25, 15] and Gagliardo-Nirenberg inequalities [3, 15]. We notice that, apart from their own interest, this kind of results have applications in the study of geometric problems (see for instance [21, 22, 9]) and can be used to obtain quantitative rates of convergence for diffusion equations (see for instance [3]).

Very recently, some quantitative stability results have been proved also for the Brunn-Minkowski inequality on general Borel sets [24, 18, 19]. The study of this problem involves an interplay between linear structure, analysis, and affine-invariant geometry of Euclidean spaces.

2. Setting and statement of the results

Given two sets $A, B \subset \mathbb{R}^n$, and $c > 0$, we define the set sum and scalar multiple by

$$A + B := \{a + b : a \in A,\ b \in B\}, \quad cA := \{ca : a \in A\} \tag{2.1}$$

Let $|E|$ denote the Lebesgue measure of a set E (if E is not measurable, $|E|$ denotes the outer Lebesgue measure of E). The Brunn-Minkowski inequality states that, given $A, B \subset \mathbb{R}^n$ measurable sets,

$$|A+B|^{1/n} \geq |A|^{1/n} + |B|^{1/n}. \tag{2.2}$$

In addition, if $|A|, |B| > 0$, then equality holds if and only if there exist a convex set $\mathcal{K} \subset \mathbb{R}^n$, $\lambda_1, \lambda_1 > 0$, and $v_1, v_2 \in \mathbb{R}^n$, such that

$$\lambda_1 A + v_1 \subset \mathcal{K}, \quad \lambda_2 B + v_2 \subset \mathcal{K}, \qquad |\mathcal{K} \setminus (\lambda_1 A + v_1)| = |\mathcal{K} \setminus (\lambda_2 B + v_2)| = 0.$$

Our aim is to investigate the stability of such a statement.

When $n = 1$, the following sharp stability result holds as a consequence of classical theorems in additive combinatorics (an elementary proof of this result can be given using Kemperman's theorem [7, 8]):

Theorem 2.1. *Let $A, B \subset \mathbb{R}$ be measurable sets. If $|A+B| < |A| + |B| + \delta$ for some $\delta \leq \min\{|A|, |B|\}$, then there exist two intervals $I, J \subset \mathbb{R}$ such that $A \subset I$, $B \subset J$, $|I \setminus A| \leq \delta$, and $|J \setminus B| \leq \delta$.*

Concerning the higher dimensional case, in [5, 6] M. Christ proved a *qualitative* stability result for (2.2), namely, if $|A+B|^{1/n} - \big(|A|^{1/n} + |B|^{1/n}\big) =: \delta \ll 1$ then A and B are close to homothetic convex sets. Since his result relies on compactness, it does not yield any explicit information about the dependence on the parameter δ.

On the *quantitative* side, in [23, 24] the author together with F. Maggi and A. Pratelli obtained a sharp stability result for the Brunn-Minkowski inequality on *convex sets*. After dilating A and B appropriately, we can assume $|A| = |B| = 1$ while replacing the sum $A + B$ by a convex combination $S := tA + (1-t)B$. It follows by (2.2) that $|S| = 1 + \delta$ for some $\delta \geq 0$.

In [23, 24] a sharp quantitative stability result is proved when A and B are both convex.

Theorem 2.2. *There is a computable dimensional constant $C_0(n)$ such that if $A, B \subset \mathbb{R}^n$ are convex sets satisfying $|A| = |B| = 1$, $|tA + (1-t)B| = 1 + \delta$ for some $t \in [\tau, 1-\tau]$, then, up to a translation,*

$$|A \Delta B| \leq C_0(n) \tau^{-1/2n} \delta^{1/2}$$

(Here and in the sequel, $E \Delta F$ denotes the symmetric difference between two sets E and F, that is $E \Delta F = (E \setminus F) \cup (F \setminus E)$.)

The main theorem in [19] is a quantitative version of Christ's result. Since the proof is by induction on the dimension, it is convenient to allow the measures of $|A|$ and $|B|$ not to be exactly equal, but just close in terms of δ. Here is the main result of that paper.

Theorem 2.3. *Let $n \geq 2$, let $A, B \subset \mathbb{R}^n$ be measurable sets, and define $S := tA + (1-t)B$ for some $t \in [\tau, 1-\tau]$, $0 < \tau \leq 1/2$. There are computable dimensional constants N_n and computable functions $M_n(\tau), \varepsilon_n(\tau) > 0$ such that if*

$$\big||A| - 1\big| + \big||B| - 1\big| + \big||S| - 1\big| \leq \delta \tag{2.3}$$

for some $\delta \leq e^{-M_n(\tau)}$, then there exists a convex set $\mathcal{K} \subset \mathbb{R}^n$ such that, up to a translation,

$$A, B \subset \mathcal{K} \qquad \text{and} \qquad |\mathcal{K} \setminus A| + |\mathcal{K} \setminus B| \leq \tau^{-N_n} \delta^{\varepsilon_n(\tau)}.$$

Explicitly, we may take

$$M_n(\tau) = \frac{2^{3^{n+2}} n^{3^n} |\log \tau|^{3^n}}{\tau^{3^n}}, \qquad \varepsilon_n(\tau) = \frac{\tau^{3^n}}{2^{3^{n+1}} n^{3^n} |\log \tau|^{3^n}}.$$

In particular, the measure of the difference between the sets A and B and their convex hull is bounded by a power δ^ϵ, confirming a conjecture of Christ [5].

In order to understand the above statement, it will be useful to go through the conceptual steps that led to its proof.

3. Conceptual path

The question we are trying to address is the following: Assume that (2.2) is almost an equality. Is it true that both A and B are almost convex, and that actually they are close to the same convex set?

Notice that this question has two statements in it. Indeed, we are wondering if:
- The error in the Brunn-Minkowski inequality controls how far A and B are from their convex hulls **(Convexity)**.
- The error in the Brunn-Minkowski inequality controls the difference between the shapes of A and B **(Homothety)** .

We will proceed by steps as follows: in Section 3.1 we will focus only on the **(Homothety)** issue. More precisely, we assume that A and B are already convex and we prove that, if equality almost holds in (2.2), then A and B have almost the same shape. Then, in Section 3.2 we will focus on the **(Convexity)** issue in the simpler case $A = B$, and we shall prove that A is close to its convex hull. Finally, in Section 3.3 we will deal with the general case.

3.1. **Stability on convex sets.** Let A, B be bounded convex set with $0 < \lambda \leq |A|, |B| \leq \Lambda$, and set

$$\delta(A,B) := \left|\frac{A+B}{2}\right|^{1/n} - \frac{|A|^{1/n} + |B|^{1/n}}{2}.$$

It follows from (2.2) that $\delta(A,B) \geq 0$, and we would like to show that $\delta(A,B)$ controls some kind of "distance" between the shape of A and the one of B.

In order to compare A and B, we first want them to have the same volume. Hence, we renormalize A so that it has the same measure of B: if $\gamma := \frac{|B|^{1/n}}{|A|^{1/n}}$ then

$$|\gamma A| = |B|.$$

We then define a "distance"[1] between A and B as follows:

$$d(A,B) := \min_{x \in \mathbb{R}^n} |B \Delta (x + \gamma A)|.$$

[1]Notice that d is not properly a distance since it is not symmetric. Still, it is a natural geometric quantity which measures, up to translations, the L^1-closeness between γA and B: indeed, observe that an equivalent formulation for d is

$$d(A,B) := \min_{x \in \mathbb{R}^n} \|\mathbf{1}_B - \mathbf{1}_{x+\gamma A}\|_{L^1(\mathbb{R}^n)}.$$

The following result has been obtained in [23, Section 4] (see also [24]):

Theorem 3.1. *Let A, B be bounded convex set with $0 < \lambda \leq |A|, |B| \leq \Lambda$. There exists $C = C(n, \lambda, \Lambda)$ such that*

$$d(A, B) \leq C\, \delta(A, B)^{1/2}.$$

As observed in [23, Section 4] the exponent 1/2 is optimal and the constant C is explicit. The proof of this theorem is obtained by carefully inspecting the proof of Brunn-Minkowski via optimal transport given in [30]. We refer the reader to [20, Section 3] for an idea of the proof.

3.2. **Stability when $A = B$.** As explained for instance at the beginning of [20, Section 4], the proof of the quantitative stability for Brunn-Minkowski exploiting optimal transportation works only if both A and B are convex. In particular, it cannot be used to solve the **(Convexity)** issue raised at the end of Section 2, and a completely new strategy is needed to address this issue.

3.2.1. *The case $n = 1$.* Already in the one dimensional case the problem is far from being trivial. Up to rescale A we can always assume that $|A| = 1$. Define

$$\delta_1(A) := |A + A| - 2|A|.$$

It is easy to see that $\delta_1(A)$ cannot control in general $|\mathrm{co}(A) \setminus A|$: indeed take

$$A := [0, 1/2] \cup [L, L + 1/2]$$

with $L \gg 1$. Then

$$A + A = [0, 1] \cup [L, L + 1] \cup [2L, 2L + 1],$$

which implies that $\delta_1(A) = 1 (= |A|)$ while $|\mathrm{co}(A) \setminus A| = L - 1/2$ is arbitrarily large. Luckily, as shown by the following theorem, this is essentially the only thing that can go wrong.

Theorem 3.2. *Let $A \subset \mathbb{R}$ be a measurable set with $|A| = 1$, and denote by $\mathrm{co}(A)$ its convex hull. If $\delta_1(A) < 1$ then*

$$|\mathrm{co}(A) \setminus A| \leq \delta_1(A).$$

This theorem can be obtained as a corollary of a result of G. Freiman [26] about the structure of additive subsets of $\mathbb{Z}$. (See [27] or [31, Theorem 5.11] for a statement and a proof.) However, it turns out that to prove of Theorem 3.2 one only needs weaker results, and a simple proof of the above theorem is given in [18, Section 2] (see also [20, Section 4.1]).

3.2.2. *The case $n \geq 2$.* Let us define the *deficit* of A as

$$\delta(A) := \frac{\left|\frac{1}{2}(A + A)\right|}{|A|} - 1 = \frac{|A + A|}{|2A|} - 1.$$

As mentioned above, one can obtain a precise stability result in one dimension by translating it into a problem on $\mathbb{Z}$. The main result in [18] is a quantitative stability result in arbitrary dimension, showing that a power of $\delta(A)$ dominates the measure of the difference between A and its convex hull $\mathrm{co}(A)$. The proof is done by induction on the dimension, combining several tools from analysis, measure theory, and affine-invariant geometry.

Theorem 3.3. *Let $n \geq 2$. There exist computable dimensional constants $\delta_n, c_n > 0$ such that if $A \subset \mathbb{R}^n$ is a measurable set of positive measure with $\delta(A) \leq \delta_n$, then*

$$\delta(A)^{\alpha_n} \geq c_n \frac{|\,\mathrm{co}(A) \setminus A|}{|A|}, \qquad \alpha_n := \frac{1}{8 \cdot 16^{n-2} n! (n-1)!}.$$

3.3. **Stability when $A \neq B$.** As already mentioned in Section 2, when $n = 1$ a sharp stability result holds as a consequence of classical theorems in additive combinatorics.

As in the case $A = B$ and $t = 1/2$ (see Theorem 3.3), the proof of Theorem 2.3 uses the one dimensional result from Theorem 2.1 together with an inductive argument. We want however to point out that, with respect to the one of Theorem 3.3, the proof of Theorem 2.3 is much more elaborate: it combines the techniques of M. Christ in [5, 6] with those developed in [18], as well as several new ideas (see [20, Section 5] for a sketch of the proof).

4. Concluding remarks

Although the stability results stated in this note look very much the same (in terms of the statements we want to prove), their proofs involve substantially different methods: Theorem 3.1 relies on optimal transportation techniques, Theorem 3.2 is based on additive combinatorics' arguments, and Theorems 3.3 and 2.3 involve an interplay between measure theory, analysis, and affine-invariant geometry.

We notice the our statements still leave space for improvements: for instance, the exponents α_n and $\beta_n(\tau)$ depend on the dimension, and it looks very plausible to us that they are both non-sharp. An important question in this direction would be to improve our exponents and, if possible, understand what the sharp exponents should be. Notice that this is not a merely academic question, as improving exponents in stability inequalities plays an important role in applications (see for instance [3] and [9]).

It is our belief that this line of research will continue growing in the next years, producing new and powerful stability results.

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