A GLOBAL EXISTENCE RESULT
FOR THE SEMIGEOSTROPHIC EQUATIONS
IN THREE DIMENSIONAL CONVEX DOMAINS

LUIGI AMBROSIO, MARIA COLOMBO, GUIDO DE PHILIPPIS, AND ALESSIO FIGALLI

ABSTRACT. Exploiting recent regularity estimates for the Monge-Ampère equation, under some suitable assumptions on the initial data we prove global-in-time existence of Eulerian distributional solutions to the semigeostrophic equations in 3-dimensional convex domains.

1. INTRODUCTION

A simplified model for the motion of large scale atmospheric/oceanic flows inside a domain $\Omega \subset \mathbb{R}^3$ is given by the semigeostrophic equations.

Let $\Omega \subseteq \mathbb{R}^3$ be bounded open set with Lipschitz boundary. Then the semigeostrophic equations inside Ω are:

$$(1.1) \quad \begin{cases} \partial_t u_t^g(x) + \big(u_t(x) \cdot \nabla\big) u_t^g(x) + \nabla p_t(x) = -J u_t(x) + m_t(x) e_3 & (x,t) \in \Omega \times (0,\infty) \\ \partial_t m_t(x) + \big(u_t(x) \cdot \nabla\big) m_t(x) = 0 & (x,t) \in \Omega \times (0,\infty) \\ u_t^g(x) = J \nabla p_t(x) & (x,t) \in \Omega \times [0,\infty) \\ \nabla \cdot u_t(x) = 0 & (x,t) \in \Omega \times [0,\infty) \\ u_t(x) \cdot \nu_\Omega(x) = 0 & (x,t) \in \partial\Omega \times [0,\infty) \\ p_0(x) = p^0(x) & x \in \Omega. \end{cases}$$

Here p^0 is the initial condition for p,[1] ν_Ω is the unit outward normal to $\partial\Omega$, $e_3 = (0,0,1)^T$ is the third vector of the canonical basis in $\mathbb{R}^3$, J is the matrix given by

$$J := \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

and the functions u_t, p_t, and m_t represent respectively the *velocity*, the *pressure* and the *buoyancy* of the atmosphere, while u_t^g is the so-called *semi-geostrophic* wind.[2] Clearly the pressure is defined up to a (time-dependent) additive constant.

Substituting the relation $u_t^g = J \nabla p_t$ and introducing the function

$$(1.2) \quad P_t(x) := p_t(x) + \frac{1}{2}(x_1^2 + x_2^2),$$

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[1] As it will be clear from the discussion later, we do not need to specify any initial condition for u and m.

[2] We are using the notation f_t to denote the function $f(t, \cdot)$.

the system (1.1) can be rewritten in $\Omega \times [0, \infty)$ as

(1.3)
$$\begin{cases} \partial_t \nabla P_t(x) + \nabla^2 P_t(x) u_t(x) = J(\nabla P_t(x) - x) \\ \nabla \cdot u_t(x) = 0 \\ u_t(x) \cdot \nu_\Omega(x) = 0 \\ P_0(x) = p^0(x) + \frac{1}{2}(x_1^2 + x_2^2). \end{cases}$$

Notice that, given a solution (P, u) of (1.3), one easily recovers a solution of (1.1): indeed p_t can be obtained from P_t through (1.2) and the density m_t is given by $m_t = \partial_3 P_t$ (in particular, the third component of the first equation in (1.3) tells us that $\partial_t m_t + (u_t \cdot \nabla) m_t = 0$ is satisfied).

Energetic considerations (see [11, Section 3.2]) show that it is natural to assume that the function P_t is convex on Ω. This condition, first introduced by Cullen and Purser, is related in [14, 24] to a physical stability required for the semigeostrophic approximation to be appropriate. If we denote with $\mathscr{L}_\Omega$ the (normalized) Lebesgue measure on Ω, then formally $\rho_t := (\nabla P_t)_\sharp \mathscr{L}_\Omega$ (see, for example, [1, Appendix A]) satisfies the following *dual problem*

(1.4)
$$\begin{cases} \partial_t \rho_t + \nabla \cdot (U_t \rho_t) = 0 \\ U_t(x) = J(x - \nabla P_t^*(x)) \\ \rho_t = (\nabla P_t)_\sharp \mathscr{L}_\Omega \\ P_0(x) = p^0(x) + \frac{1}{2}(x_1^2 + x_2^2). \end{cases}$$

Here P_t^* is the convex conjugate of P_t, namely

$$P_t^*(y) := \sup_{x \in \Omega} (y \cdot x - P_t(x)) \qquad \forall y \in \mathbb{R}^3$$

and $(\nabla P_t)_\sharp \mathscr{L}_\Omega$ is the push-forward of the measure $\mathscr{L}_\Omega$ through the map $\nabla P_t : \Omega \to \mathbb{R}^3$ defined as

$$[(\nabla P_t)_\sharp \mathscr{L}_\Omega](A) = \mathscr{L}_\Omega\big((\nabla P_t)^{-1}(A)\big) \qquad \text{for all } A \subset \mathbb{R}^3 \text{ Borel.}$$

The dual problem is pretty well understood, and admits a solution obtained via time discretization (see [5, 13]). Moreover, at least formally, given a solution P_t of the dual problem (1.4) and setting

(1.5) $$u_t(x) := [\partial_t \nabla P_t^*](\nabla P_t(x)) + [\nabla^2 P_t^*](\nabla P_t(x)) J(\nabla P_t(x) - x),$$

the couple (P_t, u_t) solves the semi-geostrophic problem (1.1). However, because of the low regularity of the function P_t the previous velocity field may a priori not be well defined, and this creates serious difficulties for recovering a "real solution" from a "dual solution".

Still, a recent regularity result [15] can be applied to show that the map P_t is $W^{2,1}$ in space, so that we can give a meaning to the second term in the definition of u_t. More precisely, in [15] it is shown that $|D^2 u| \log_+^k |D^2 u| \in L^1_{\text{loc}}$ for any k, and following ideas developed in [1, 20], we will be able to show that the function P_t is regular enough also in time, so that the couple (P_t, u_t) is a true distributional solution of (1.1).

Let us point out that the regularity result in [15] has been recently extended, independently in [17] and [23], to $|D^2 u| \in L^\gamma$, where $\gamma > 1$ depends on the local L^∞ norm of $\log \rho_t$. However, as we will better explain in Remark 3.6, in our situation there is no advantage in using this improvement, since the fact that γ depends on $\| \log \rho_t \|_{\infty, \text{loc}}$ makes the estimates less readable. For this reason we will rely only on the $L \log L$ integrability given by [15], as we previously did in [1].

The first existence result about distributional solutions to the semigeostrophic equation is presented in [1], where the analysis is carried out on the 2-dimensional torus (see also [21] where a short time

existence result of smooth solutions is proved in dual variables, and because of smoothness the existence can be easily transferred to the physical variables).

The 3-dimensional case on the whole space $\mathbb{R}^3$, which is more physically relevant, presents additional difficulties. First, the equation (1.1) is much less symmetric compared to its 2-dimensional counterpart, because the action of Coriolis force Ju_t regards only the first and the second space components. Moreover, even considering regular initial data and velocities, regularity results require a finer regularization scheme, due to the non-compactness of the ambient space.

Our proofs are also based on some additional hypotheses on the decay of the probability measure $\rho_0 = (\nabla P_0)_\sharp \mathscr{L}_\Omega$. This decay condition happens to be stable in time on solutions of the dual equation (1.4), and allows us to perform a regularization scheme.

It would be extremely interesting to consider compactly supported initial data $\rho_0 = (\nabla P_0)_\sharp \mathscr{L}_\Omega$. However the nontrivial evolution of the support of the solution ρ_t under (1.4) prevents us to apply the results in [15] (which actually would be false in this situation), so at the moment this case seems to require completely new ideas and ingredients.

Definition 1.1. Let $P : \Omega \times [0, \infty) \to \mathbb{R}$ and $u : \Omega \times [0, \infty) \to \mathbb{R}^3$. We say that (P, u) is a *weak Eulerian solution* of (1.3) if:

- $|u| \in L^\infty_{\mathrm{loc}}((0, \infty), L^1_{\mathrm{loc}}(\Omega))$, $P \in L^\infty_{\mathrm{loc}}((0, \infty), W^{1,\infty}_{\mathrm{loc}}(\Omega))$, and $P_t(x)$ is convex for any $t \geq 0$;
- For every $\phi \in C^\infty_c(\Omega \times [0, \infty))$, it holds

(1.6)
$$\int_0^\infty \int_\Omega \nabla P_t(x) \Big\{ \partial_t \phi_t(x) + u_t(x) \cdot \nabla \phi_t(x) \Big\} + J \Big\{ \nabla P_t(x) - x \Big\} \phi_t(x) \, dx \, dt + \int_\Omega \nabla P_0(x) \phi_0(x) \, dx = 0;$$

- For a.e. $t \in (0, \infty)$ it holds

(1.7)
$$\int_\Omega \nabla \psi(x) \cdot u_t(x) \, dx = 0 \qquad \text{for all } \psi \in C^\infty_c(\Omega).$$

Remark 1.2. This definition is the classical notion of distributional solution for (1.3) as introduced by Cullen and Feldman in [12, Definition 2.1] except for the fact that the boundary condition $u_t \cdot \nu_\Omega = 0$ is not taken into account. In this sense it may look natural to consider $\psi \in C^\infty(\overline{\Omega})$ in (1.7), but since we are only able to prove that the velocity u_t is locally in L^1, Equation (1.7) makes sense only with compactly supported ψ. On the other hand, as we shall explain in Remark 1.4, we will be able to prove that there exists a measure preserving Lagrangian flow $F_t : \Omega \to \Omega$ associated to u_t, and such existence result can be interpreted as a very weak formulation of the constraint $u_t \cdot \nu_\Omega = 0$.

As pointed out to us by Cullen, this weak boundary condition is actually very natural: indeed, the classical boundary condition would prevent the formation of "frontal singularities" (which are physically expected to occur), i.e. the fluid initially at the boundary would not be able to move into the interior of the fluid, while this is allowed by our weak version of the boundary condition.

We can now state our main result.

Theorem 1.3. *Let $\Omega \subseteq \mathbb{R}^3$ be a convex bounded open set, and let $\mathscr{L}_\Omega$ be the normalized Lebesgue measure restricted to Ω, that is $\mathscr{L}_\Omega(\Omega) = 1$. Let ρ_0 be a probability density on $\mathbb{R}^3$ such that $\rho_0 \in L^\infty(\mathbb{R}^3)$, $1/\rho_0 \in L^\infty_{\mathrm{loc}}(\mathbb{R}^3)$ and*

$$\limsup_{|x| \to \infty} \big(\rho_0(x)|x|^K \big) < \infty$$

for some $K > 4$. Let ρ_t be a solution of (1.4) *given by Theorem 3.1, $P_t^* : \mathbb{R}^3 \to \mathbb{R}$ the unique convex function such that*

$$P_t^*(0) = 0 \quad and \quad (\nabla P_t^*)_\sharp (\rho_t \mathscr{L}^3) = \mathscr{L}_\Omega,$$

and let $P_t : \mathbb{R}^3 \to \mathbb{R}$ be its convex conjugate.

Then the vector field u_t in (1.5) *is well defined, and the couple (P_t, u_t) is a weak Eulerian solution of* (1.3) *in the sense of Definition 1.1.*

Remark 1.4. Following Cullen and Feldman one can give also a notion of Lagrangian solution of the semigeostrophic equation. More precisely they show the existence of a measure preserving flow $F_t : \Omega \to \Omega$ which solves a sort of Lagrangian version of (1.1) (see [12] and [1, Section 5] for a more precise discussion). Actually the flow they constructed has the explicit expression $F_t = \nabla P_t^* \circ G_t \circ \nabla P_0$, where G_t is the regular Lagrangian flow associated to the BV vector field $U_t = J(x - \nabla P_t^*)$, in the sense of Ambrosio, Di Perna and Lions (see [3, 4, 18]). In [1, Section 5] we showed, in the two dimensional periodic setting, that for almost every x the map $t \mapsto F_t(x)$ is absolutely continuous with derivative given by $u_t(F_t(x))$. The proof of this fact can be almost verbatim extended to our contest, showing that, for almost every $x \in \Omega$, $t \mapsto F_t(x)$ is locally absolutely continuous in $[0, \infty)$ with derivative given by $u_t(F_t(x))$. We leave the proof of this fact to the interested reader. Finally we remark that the uniqueness of such a flow (both according to the definition given in [12] or in [1]) is unknown.

2. REGULARITY OF OPTIMAL TRANSPORT MAPS BETWEEN CONVEX SETS OF $\mathbb{R}^3$

Throughout this paper, $\Omega \subseteq \mathbb{R}^3$ is a bounded convex open set, $d_\Omega > 0$ is fixed in such a way that $\overline{\Omega} \subset B(0, d_\Omega)$, and $\mathscr{L}_\Omega$ denotes the normalized Lebesgue measure restricted to Ω.

In this section we recall some regularity results for optimal transport maps in $\mathbb{R}^3$ needed in the paper.

Theorem 2.1 (Space regularity of optimal maps between convex sets). *Let Ω_0, Ω_1 be open sets of $\mathbb{R}^3$, with Ω_1 bounded and convex. Let $\mu = \rho \mathscr{L}^3$ and $\nu = \sigma \mathscr{L}^3$ be probability densities such that $\mu(\Omega_0) = 1$, $\nu(\Omega_1) = 1$. Assume that the density ρ is locally bounded both from above and from below in Ω_0, namely that for every compact set $K \subset \Omega$ there exist $\lambda_0 = \lambda_0(K)$ and $\Lambda_0 = \Lambda_0(K)$ satisfying*

$$0 < \lambda_0 \leq \rho(x) \leq \Lambda_0 \quad \forall\, x \in K.$$

Futhermore, suppose that $\lambda_1 \leq \sigma(x) \leq \Lambda_1$ in Ω_1. Then the following properties hold true.

(i) *There exists a unique optimal transport map between μ and ν, namely a unique (up to an additive constant) convex function $P^* : \Omega_0 \to \mathbb{R}$ such that $(\nabla P^*)_\sharp \mu = \nu$. Moreover P^* is a strictly convex Alexandrov solution of*

$$\det \nabla^2 P^*(x) = f(x), \qquad with \ f(x) = \frac{\rho(x)}{\sigma(\nabla P^*(x))}.$$

(ii) *$P^* \in W^{2,1}_{\text{loc}}(\Omega_0) \cap C^{1,\beta}_{\text{loc}}(\Omega_0)$. More precisely, if $\Omega \Subset \Omega_0$ is an open set and $0 < \lambda \leq \rho(x) \leq \Lambda < \infty$ in Ω, then for any $k \in \mathbb{N}$ there exist constants $C_1 = C_1(k, \Omega, \Omega_1, \lambda, \Lambda, \lambda_1, \Lambda_1)$, $\beta = \beta(\lambda, \Lambda, \lambda_1, \Lambda_1)$, and $C_2 = C_2(\Omega, \Omega_1, \lambda, \Lambda, \lambda_1, \Lambda_1)$ such that*

$$\int_\Omega |\nabla^2 P^*| \log_+^k |\nabla^2 P^*| \, dx \leq C_1,$$

and

$$\|P^*\|_{C^{1,\beta}(\Omega)} \leq C_2.$$

(iii) *Let us also assume that Ω_0, Ω_1 are bounded and uniformly convex, $\partial\Omega_0, \partial\Omega_1 \in C^{2,1}$, $\rho \in C^{1,1}(\Omega_0)$, $\sigma \in C^{1,1}(\Omega_1)$, and $\lambda_0 \leq \rho(x) \leq \Lambda_0$ in Ω_0. Then*

$$P^* \in C^{3,\alpha}(\Omega_0) \cap C^{2,\alpha}(\overline{\Omega}_0) \qquad \forall \, \alpha \in (0,1),$$

and there exists a constant C which depends only on $\alpha, \Omega_0, \Omega_1, \lambda_0, \lambda_1, \|\rho\|_{C^{1,1}}, \|\sigma\|_{C^{1,1}}$ such that

$$\|P^*\|_{C^{3,\alpha}(\Omega_0)} \leq C \qquad and \qquad \|P^*\|_{C^{2,\alpha}(\overline{\Omega}_0)} \leq C.$$

Moreover, there exist positive constants c_1 and c_2 and κ, depending only on $\lambda_0, \lambda_1, \|\rho\|_{C^{0,\alpha}}$, and $\|\sigma\|_{C^{0,\alpha}}$, such that

$$c_1 Id \leq \nabla^2 P^*(x) \leq c_2 Id \qquad \forall \, x \in \Omega_0$$

and

$$\nu_{\Omega_1}(\nabla P^*(x)) \cdot \nu_{\Omega_0}(x) \geq \kappa \quad \forall \, x \in \partial\Omega_0.$$

The first statement is standard optimal transport theory, see [10, 22], except for the fact that we are not assuming that the second moment of μ is finite, thus the classical Wasserstein distance from μ and ν can be infinite. Nevertheless the existence of an "optimal" map is provided by [22]. The $W^{2,1}$ part of the second statement follows from a recent regularity result about solutions of the Monge-Ampère equation [15], while the $C^{1,\beta}$ regularity was proven by Caffarelli in [6, 9, 10]. The regularity up to the boundary and the oblique derivative condition of the third statement have been proven by Caffarelli [7] and Urbas [25].

Remark 2.2. By compactness and a standard contradiction argument, the constants C_1 and C_2 in the statement (ii) of the previous theorem remain uniformly bounded if Ω_1 varies in a compact class (with respect, for instance, to the Hausdorff distance) of convex sets. In particular, let Ω_1^n be a sequence of open convex sets which converges to Ω_1 with respect to the Hausdorff distance and σ_n a sequence of densities supported on Ω_1^n with $\lambda_1 \leq \sigma_n \leq \Lambda_1$ on Ω_1^n which converge to σ in $L^1(\mathbb{R}^3)$. Then the estimates in Theorem 2.1(ii) hold true with constants independent of n.

Remark 2.3. As already mentioned in the introduction, in statement (ii) the optimal regularity is that for every $\Omega \Subset \Omega_0$ and $0 < \lambda \leq \rho(x) \leq \Lambda < \infty$ in Ω, there exist $\gamma(\lambda, \Lambda, \lambda_1, \Lambda_1) > 1$ and $C(\Omega, \Omega_1, \lambda, \Lambda, \lambda_1, \Lambda_1)$, such that

$$\int_\Omega |D^2 u|^\gamma \leq C.$$

However, as explained in Remark 3.6, this improvement does not give any advantage.

3. The dual problem and the regularity of the velocity field

In this section we recall some properties of solutions of (1.4), and we show the L^1 integrability of the velocity field u_t defined in (1.5).

We have the following result whose proof follows adapting the argument of [5, 13], where compactly supported initial data are considered. Since the velocity U_t has at most linear growth in space, the speed of propagation is locally finite and the proof readily extends to general probability densities.

Theorem 3.1 (Existence of solutions of (1.4)). *Let $P_0 : \mathbb{R}^3 \to \mathbb{R}$ be a convex function such that $(\nabla P_0)_\sharp \mathscr{L}_\Omega \ll \mathscr{L}^3$. Then there exist convex functions $P_t, P_t^* : \mathbb{R}^3 \to \mathbb{R}$ such that $(\nabla P_t)_\sharp \mathscr{L}_\Omega = \rho_t \mathscr{L}^3$, $(\nabla P_t^*)_\sharp \rho_t = \mathscr{L}_\Omega$, $U_t(x) = J(x - \nabla P_t^*(x))$, and ρ_t is a distributional solution to (1.4), namely*

$$(3.1) \qquad \int \int_{\mathbb{R}^3} \left\{ \partial_t \varphi_t(x) + \nabla \varphi_t(x) \cdot U_t(x) \right\} \rho_t(x) \, dx \, dt + \int_{\mathbb{R}^3} \varphi_0(x) \rho_0(x) \, dx = 0$$

for every $\varphi \in C_c^\infty(\mathbb{R}^3 \times [0, \infty))$.

Moreover, the following regularity properties hold:

(i) *$\rho_t \mathscr{L}^3 \in C([0, \infty), \mathcal{P}_w(\mathbb{R}^3))$, where $\mathcal{P}_w(\mathbb{R}^3)$ is the space of probability measures endowed with the* weak *topology induced by the duality with $C_0(\mathbb{R}^3)$;*

(ii) *$P_t^* - P_t^*(0) \in L_{\text{loc}}^\infty([0, \infty), W_{\text{loc}}^{1,\infty}(\mathbb{R}^3)) \cap C([0, \infty), W_{\text{loc}}^{1,r}(\mathbb{R}^3))$ for every $r \in [1, \infty)$;*

(iii) *$|U_t(x)| \le |x| + d_\Omega$ for almost every $x \in \mathbb{R}^3$, for all $t \ge 0$.*

Observe that, by Theorem 3.1(ii), $t \mapsto \rho_t \mathscr{L}^3$ is weakly continuous, so ρ_t is a well-defined function for every $t \ge 0$. Further regularity properties of P_t and P_t^* with respect to time will be proven in Proposition 3.5.

In the proof of Theorem 1.3 we will need to test with functions which are merely $W^{1,1}$ with compact support. This is made possible by a simple approximation argument which we leave to the reader, see [1, Lemma 3.2].

Lemma 3.2. *Let ρ_t and P_t be as in Theorem 3.1. Then (3.1) holds for every $\varphi \in W^{1,1}(\mathbb{R}^3 \times [0, \infty))$ which is compactly supported in time and space, where now $\varphi_0(x)$ has to be understood in the sense of traces.*

Lemma 3.3 (Space-time regularity of transport). *Let $\Omega \subseteq \mathbb{R}^3$ be a uniformly convex bounded domain with $\partial \Omega \in C^{2,1}$, let $R > 0$, and consider $\rho \in C^\infty(\overline{B(0,R)} \times [0, \infty))$ and $U \in C_c^\infty(B(0,R) \times [0, \infty); \mathbb{R}^3)$ satisfying*

$$\partial_t \rho_t + \nabla \cdot (U_t \rho_t) = 0 \qquad in \ B(0,R) \times [0, \infty).$$

Assume that $\int_{B(0,R)} \rho_0 \, dx = 1$, and that for every $T > 0$ there exist λ_T and Λ_T such that

$$0 < \lambda_T \le \rho_t(x) \le \Lambda_T < \infty \qquad \forall \, (x,t) \in B(0,R) \times [0,T].$$

Consider the convex conjugate maps P_t and P_t^* such that $(\nabla P_t)_\sharp \mathscr{L}_\Omega = \rho_t$ and $(\nabla P_t^*)_\sharp \rho_t = \mathscr{L}_\Omega$ (unique up to additive constants in Ω and $B(0,R)$ respectively). Then:

(i) *$P_t^* - \fint_{B(0,R)} P_t^* \in \text{Lip}_{\text{loc}}([0, \infty); C^{2,\alpha}(\overline{B(0,R)}))$.*

(ii) *The following linearized Monge-Ampère equation holds for every $t \in [0, \infty)$:*

$$(3.2) \qquad \begin{cases} \nabla \cdot \left(\rho_t (\nabla^2 P_t^*)^{-1} \partial_t \nabla P_t^* \right) = -\nabla \cdot (\rho_t U_t) & in \ B(0,R) \\ \rho_t (\nabla^2 P_t^*)^{-1} \partial_t \nabla P_t^* \cdot \nu = 0 & on \ \partial B(0,R). \end{cases}$$

Proof. Observe that because ρ_t solves a continuity equation with a smooth compactly supported vector field, $\int_{B(0,R)} \rho_t \, dx = 1$ for all t.

Let us fix $T > 0$. From the regularity theory for the Monge-Ampère equation (Theorem 2.1 applied to P_t and P_t^*) we obtain that $P_t \in C^{3,\alpha}(\Omega) \cap C^{2,\alpha}(\overline{\Omega})$ and $P_t^* \in C^{3,\alpha}(B(0,R)) \cap C^{2,\alpha}(\overline{B(0,R)})$ for every $\alpha \in (0,1)$, uniformly for $t \in [0,T]$, and there exist constants $c_1, c_2 > 0$ such that

$$(3.3) \qquad c_1 Id \le \nabla^2 P_t^*(x) \le c_2 Id \qquad \forall \, (x,t) \in B(0,R) \times [0,T].$$

Let $h \in C^{2,1}(\mathbb{R}^3)$ be a convex function such that $\Omega = \{y : h(y) < 0\}$ and $|\nabla h(y)| = 1$ on $\partial\Omega$, so that $\nabla h(y) = \nu_\Omega(y)$. Since $\nabla P_t^* \in C^{1,\alpha}(\overline{B(0,R)})$, it is a diffeomorphism onto its image, we have

$$(3.4) \qquad h(\nabla P_t^*(x)) = 0 \qquad \forall\, (x,t) \in \partial B(0,R) \times [0,T].$$

To prove (i) we need to investigate the time regularity of $P_t^* - \fint_{B(0,R)} P_t^*$.

Possibly adding a time dependent constant to P_t, we can assume without loss of generality that $\int_{B(0,R)} P_t^* = 0$ for all t. By the condition $(\nabla P_t^*)_\sharp \rho_t = \mathscr{L}_\Omega$ we get that for any $0 \leq s,t \leq T$ and $x \in B(0,R)$ it holds

$$
\begin{aligned}
(3.5) \qquad \frac{\rho_s(x) - \rho_t(x)}{s - t} &= \frac{\det(\nabla^2 P_s^*(x)) - \det(\nabla^2 P_t^*(x))}{s - t} \\
&= \sum_{i,j=1}^{3} \left(\int_0^1 \frac{\partial \det}{\partial \xi_{ij}}(\tau \nabla^2 P_s^*(x) + (1-\tau)\nabla^2 P_t^*(x))\, d\tau \right) \frac{\partial_{ij} P_s^*(x) - \partial_{ij} P_t^*(x)}{s - t}.
\end{aligned}
$$

Moreover, from (3.4) we obtain that on $\partial B(0,R)$

$$
\begin{aligned}
(3.6) \qquad 0 &= \frac{h(\nabla P_s^*(x)) - h(\nabla P_t^*(x))}{s - t} \\
&= \int_0^1 \nabla h(\tau \nabla P_s^*(x) + (1-\tau)\nabla P_t^*(x))\, d\tau \cdot \frac{\nabla P_s^*(x) - \nabla P_t^*(x)}{s - t}.
\end{aligned}
$$

Now, given a matrix $A = (\xi_{ij})$, we denote by $M(A)$ the cofactor matrix of A. We recall that

$$(3.7) \qquad \frac{\partial \det(A)}{\partial \xi_{ij}} = M_{ij}(A),$$

and if A is invertible then $M(A)$ satisfies the identity

$$(3.8) \qquad M(A) = \det(A)\, A^{-1}.$$

Moreover, if A is symmetric and satisfies $c_1 Id \leq A \leq c_2 Id$ for some positive constants c_1, c_2, then

$$(3.9) \qquad c_1^2 Id \leq M(A) \leq c_2^2 Id.$$

Hence, from (3.5), (3.7), (3.3) and (3.9) it follows that

$$(3.10) \qquad \frac{\rho_s - \rho_t}{s - t} = \sum_{i,j=1}^{3} \left(\int_0^1 M_{ij}(\tau \nabla^2 P_s^* + (1-\tau)\nabla^2 P_t^*)\, d\tau \right) \partial_{ij}\left(\frac{P_s^* - P_t^*}{s - t} \right),$$

with

$$c_1^2 Id \leq \int_0^1 M(\tau \nabla^2 P_s^* + (1-\tau)\nabla^2 P_t^*)\, d\tau \leq c_2^2 Id.$$

Also, from Theorem 2.1(iii) the oblique derivative condition holds, namely there exists $\kappa > 0$ such that

$$\nabla h(\nabla P_t^*(x)) \cdot \nu_{B(0,R)}(x) \geq \kappa \qquad \forall\, x \in \partial B(0,R).$$

Thus, since

$$\lim_{s \to t} \int_0^1 \nabla h(\tau \nabla P_s^*(x) + (1-\tau)\nabla P_t^*(x))\, d\tau = \nabla h(\nabla P_t^*(x))$$

uniformly in t and x, we have that

$$\int_0^1 \nabla h(\tau \nabla P_s^*(x) + (1-\tau)\nabla P_t^*(x))\, d\tau \cdot \nu_{B(0,R)}(x) \geq \frac{\kappa}{2}$$

for $s - t$ small enough.

Hence, from the regularity theory for the oblique derivative problem [19, Theorem 6.30] we obtain that for any $\alpha \in (0, 1)$ there exists a constant C depending only on Ω, T, α, $\|(\rho_s - \rho_t)/(s - t)\|_{C^{0,\alpha}(B(0,R))}$, such that

$$\left\| \frac{P_s^*(x) - P_t^*(x)}{s - t} \right\|_{C^{2,\alpha}(\overline{B(0,R)})} \leq C.$$

Since $\partial_t \rho_t \in L^\infty([0, T], C^{0,\alpha}(B(0, R)))$, this proves point (i) in the statement. To prove the second part, we let $s \to t$ in (3.10) to obtain

$$(3.11) \qquad \partial_t \rho_t = \sum_{i,j=1}^3 M_{ij}(\nabla^2 P_t^*(x)) \, \partial_t \partial_{ij} P_t^*(x).$$

Taking into account the continuity equation and the well-known divergence-free property of the cofactor matrix

$$\sum_{i=1}^3 \partial_i M_{ij}(\nabla^2 P_t^*(x)) = 0, \qquad j = 1, 2, 3,$$

we can rewrite (3.11) as

$$-\nabla \cdot (U_t \rho_t) = \sum_{i,j=1}^3 \partial_i \big(M_{ij}(\nabla^2 P_t^*(x)) \, \partial_t \partial_j P_t^*(x) \big).$$

Hence, using (3.8) and the Monge-Ampère equation $\det(\nabla^2 P_t^*) = \rho_t$, we get equation (3.2).

In order to obtain the boundary condition in (3.2), we take to the limit as $s \to t$ in (3.6) to get

$$(3.12) \qquad \nabla h(\nabla P_t^*(x)) \cdot \partial_t \nabla P_t^*(x) = 0.$$

Since h satisfies $\Omega = \{y : h(y) < 0\}$ and ∇P_t^* maps $B(0, R)$ in Ω, we have that $B(0, R) = \{y : h(\nabla P_t^*(y)) < 0\}$. Hence $\nu_{B(0,R)}(x)$ is proportional to $\nabla[h \circ \nabla P_t^*](x) = \nabla^2 P_t^*(x) \nabla h(\nabla P_t^*(x))$, which implies that the exterior normal to Ω at point $\nabla P_t^*(x)$, which is $\nabla h(\nabla P_t^*(x))$, is collinear with $\rho_t (\nabla^2 P_t^*)^{-1} \nu_{B(0,R)}$. Hence from (3.12) it follows that

$$\rho_t (\nabla^2 P_t^*)^{-1} \nu_{B(0,R)} \cdot \partial_t \nabla P_t^* = 0,$$

as desired. $\qquad \square$

Lemma 3.4 (Decay estimates on ρ_t). *Let $v_t : \mathbb{R}^3 \times [0, \infty) \to \mathbb{R}^3$ be a C^∞ velocity field and suppose that*

$$\sup_{x,t} |\nabla \cdot v_t(x)| \leq N, \qquad |v_t(x)| \leq A|x| + D \quad \forall (x, t) \in \mathbb{R}^3 \times [0, \infty)$$

for suitable constants N, A, D. Let ρ_0 be a probability density, and let ρ_t be the solution of the continuity equation

$$(3.13) \qquad \partial_t \rho_t + \nabla \cdot (v_t \rho_t) = 0 \qquad in \ \mathbb{R}^3 \times (0, \infty)$$

starting from ρ_0. Then:

(i) *For every $r > 0$ and $t \in [0, \infty)$ it holds*

$$(3.14) \qquad \|\rho_t\|_\infty \leq e^{Nt} \|\rho_0\|_\infty,$$

$$(3.15) \qquad \rho_t(x) \geq e^{-Nt} \inf \left\{ \rho_0(y) : \ y \in B\left(0, re^{At} + D\frac{e^{At} - 1}{A}\right) \right\} \qquad \forall x \in B(0, r).$$

(ii) *Let $K \geq 0$ and let us assume that there exist $d_0 \in [0, \infty)$ and $M \in [0, \infty)$ such that*

$$(3.16) \qquad \rho_0(x) \leq \frac{d_0}{|x|^K} \qquad whenever \quad |x| \geq M.$$

Then for every $t \in [0, \infty)$ we have that

$$(3.17) \qquad \rho_t(x) \leq \frac{d_0 2^K e^{(N+AK)t}}{|x|^K} \qquad whenever \quad |x| \geq 2Me^{At} + 2D\frac{e^{At} - 1}{A}.$$

(iii) *Let us assume that there exists $R > 0$ such that ρ_0 is smooth in $\overline{B(0, R)}$, vanishes outside $\overline{B(0, R)}$, and that v_t is compactly supported inside $B(0, R)$ for all $t \geq 0$. Then ρ_t is smooth inside $\overline{B(0, R)}$ and vanishes outside $\overline{B(0, R)}$ for all $t \geq 0$. Moreover if $0 < \lambda \leq \rho_0 \leq \Lambda < \infty$ inside $B(0, R)$, then*

$$(3.18) \qquad \lambda e^{-tN} \leq \rho_t \leq \Lambda e^{tN} \quad inside \ B(0, R) \ for \ all \ t \geq 0.$$

Proof. Let $X_t(x) \in C^\infty(\mathbb{R}^3 \times [0, \infty))$ be the flow associated to the velocity field v_t, namely the solution to

$$(3.19) \qquad \begin{cases} \frac{d}{dt} X_t(x) = v_t(X_t(x)) \\ X_0(x) = x. \end{cases}$$

For every $t \geq 0$ the map $t \mapsto X_t(x)$ is invertible in $\mathbb{R}^3$, with inverse denoted by X_t^{-1}.

The solution to the continuity equation (3.13) is given by $\rho_t = X_{t\sharp}\rho_0$, and from the well-known theory of characteristics it can be written explicitly using the flow:

$$(3.20) \qquad \rho_t(x) = \rho_0(X_t^{-1}(x))e^{\int_0^t \nabla \cdot v_s(X_s(X_t^{-1}(x))) \, ds} \qquad \forall (x, t) \in \mathbb{R}^3 \times [0, \infty).$$

Since the divergence is bounded, we therefore obtain

$$(3.21) \qquad \rho_0(X_t^{-1}(x))e^{-Nt} \leq \rho_t(x) \leq \rho_0(X_t^{-1}(x))e^{Nt}$$

Now we deduce the statements of the lemma from the properties of the flow X_t.

(i) From (3.21) we have that

$$\rho_t(x) \leq e^{Nt}\rho_0(X_t^{-1}(x)) \leq e^{Nt} \sup_{x \in \mathbb{R}^3} \rho_0(x),$$

which proves (3.14). From the equation (3.19) we obtain

$$\left| \frac{d}{dt} |X_t(x)| \right| \leq |\partial_t X_t(x)| \leq A|X_t(x)| + D$$

which can be rewritten as

$$(3.22) \qquad -A|X_t(x)| - D \leq \frac{d}{dt}|X_t(x)| \leq A|X_t(x)| + D.$$

From the first inequality we get

$$|X_t(x)| \geq |x|e^{-At} - D\frac{1 - e^{-At}}{A},$$

which implies

$$|x|e^{At} + D\frac{e^{At} - 1}{A} \geq |X_t^{-1}(x)|,$$

or equivalently

$$(3.23) \qquad X_t^{-1}\big(\{|x| \le r\}\big) \subseteq \Big\{|x| \le re^{At} + D\frac{e^{At} - 1}{A}\Big\}.$$

Hence from (3.21) and (3.23) we obtain that, for every $x \in B(0, r)$,

$$
\begin{aligned}
\rho_t(x) \quad &\ge \quad e^{-Nt}\rho_0(X_t^{-1}(x)) \\
&\ge \quad e^{-Nt}\inf\{\rho_0(y) : y \in X_t^{-1}(B_r(0))\} \\
&\ge \quad e^{-Nt}\inf\Big\{\rho_0(y) : |y| \le re^{At} + D\frac{e^{At} - 1}{A}\Big\},
\end{aligned}
$$

which proves (3.15).

(ii) From the second inequality in (3.22), we infer

$$|X_t(x)| \le |x|e^{At} + D\frac{e^{At} - 1}{A},$$

which implies

$$(3.24) \qquad |x| \le |X_t^{-1}(x)|e^{At} + D\frac{e^{At} - 1}{A}.$$

Thus, if $|x| \ge 2Me^{At} + 2D\frac{e^{At}-1}{A}$, we easily deduce from (3.24) that $|X_t^{-1}(x)| \ge M + |x|e^{-At}/2$, so by (3.16)

$$\rho_t(x) \le e^{Nt}\rho_0(X_t^{-1}(x)) \le \frac{d_0 e^{Nt}}{|X_t^{-1}(x)|^K} \le \frac{d_0 2^K e^{(N+AK)t}}{|x|^K},$$

which proves (3.17).

(iii) If $v_t = 0$ in a neighborhood of $\partial B(0, R)$ it can be easily verified that the flow maps $X_t : \mathbb{R}^3 \to \mathbb{R}^3$ leave both $B(0, R)$ and its complement invariant. Moreover the smoothness of v_t implies that also X_t is smooth. Therefore all the properties of ρ_t follow directly from (3.20). $\qquad\square$

We are now ready to prove the regularity of ∇P_t^*.

Proposition 3.5 (Time regularity of optimal maps). *Let $\Omega \subseteq \mathbb{R}^3$ be a bounded, convex, open set and let d_Ω be such that $\overline{\Omega} \subset B(0, d_\Omega)$. Let ρ_t and P_t be as in Theorem 3.1, in addition let us assume that there exist $K > 4$, $M \ge 0$ and $c_0 > 0$ such that*

$$(3.25) \qquad \rho_0(x) \le \frac{c_0}{|x|^K} \qquad \text{whenever } |x| \ge M.$$

Then $\nabla P_t^* \in W_{\mathrm{loc}}^{1,1}(\mathbb{R}^3 \times [0, \infty); \mathbb{R}^3)$. Moreover for every $k \in \mathbb{N}$ and $T > 0$ there exists a constant $C = C(k, T, M, c_0, \|\rho_0\|_\infty, d_\Omega)$ such that, for almost every $t \in [0, T]$ it holds

$$(3.26)\quad \int_{B(0,r)} \rho_t|\partial_t \nabla P_t^*|\log_+^k(|\partial_t \nabla P_t^*|)\,dx \le 2^{3(k-1)}\int_{B(0,r)} \rho_t|\nabla^2 P_t^*|\log_+^{2k}(|\nabla^2 P_t^*|)\,dx + C \qquad \forall\, r > 0.$$

Proof. Step 1: The smooth case. In the first part of the proof we assume that Ω is a convex smooth domain, and, besides (3.25), that for some $R > 0$ the following additional properties hold:

(3.27) $\qquad \rho_t \in C^\infty(\overline{B(0,R)} \times \mathbb{R}), \ U_t \in C_c^\infty(B(0,R) \times \mathbb{R}; \mathbb{R}^3), \ |\nabla \cdot U_t| \leq N$

(3.28) $\qquad \lambda 1_{B(0,R)}(x) \leq \rho_0(x) \leq \Lambda 1_{B(0,R)}(x) \qquad \forall\, x \in \mathbb{R}^3,$

(3.29) $\qquad \partial_t \rho_t + \nabla \cdot (U_t \rho_t) = 0 \qquad \text{in } \mathbb{R}^3 \times [0, \infty),$

(3.30) $\qquad (\nabla P_t^*)_\sharp \rho_t = \mathscr{L}_\Omega,$

(3.31) $\qquad |U_t(x)| \leq |x| + d_\Omega$

for some constants N, λ, Λ, and we prove that (3.26) holds for every $t \in [0, T]$. Notice that in this step we do not assume any coupling between the velocity U_t and the transport map ∇P_t^*. In the second step we prove the general case through an approximation argument.

Let us assume that the regularity assumptions (3.27) through (3.31) hold. By Lemma 3.4 we infer that, for any $T > 0$, there exist positive constants $\lambda_T, \Lambda_T, c_T, M_T$, with $M_T \geq 1$, such that

(3.32) $\qquad \lambda_T 1_{B(0,R)}(x) \leq \rho_t(x) \leq \Lambda_T 1_{B(0,R)}(x),$

(3.33) $\qquad \rho_t(x) \leq \dfrac{c_T}{|x|^K} \quad \text{for } |x| \geq M_T, \qquad \text{for all } t \in [0, T].$

By Lemma 3.3 we have that $\partial_t P_t^* \in C^2(\overline{B(0,R)})$, and it solves

(3.34) $\qquad \begin{cases} \nabla \cdot \left(\rho_t (\nabla^2 P_t^*)^{-1} \partial_t \nabla P_t^* \right) = -\nabla \cdot (\rho_t U_t) & \text{in } B(0,R) \\ \rho_t (\nabla^2 P_t^*)^{-1} \partial_t \nabla P_t^* \cdot \nu = 0 & \text{in } \partial B(0,R). \end{cases}$

Multiplying (3.34) by $\partial_t P_t^*$ and integrating by parts, we get

(3.35)
$$\int_{B(0,R)} \rho_t |(\nabla^2 P_t^*)^{-1/2} \partial_t \nabla P_t^*|^2 \, dx = \int_{B(0,R)} \rho_t \partial_t \nabla P_t^* \cdot (\nabla^2 P_t^*)^{-1} \partial_t \nabla P_t^* \, dx$$
$$= -\int_{B(0,R)} \rho_t \partial_t \nabla P_t^* \cdot U_t \, dx.$$

(Notice that, thanks to the boundary condition in (3.34), we do not have any boundary term in (3.35).) From Cauchy-Schwartz inequality it follows that the right-hand side of (3.35) can be rewritten and estimated by

(3.36)
$$-\int_{B(0,R)} \rho_t \partial_t \nabla P_t^* \cdot (\nabla^2 P_t^*)^{-1/2} (\nabla^2 P_t^*)^{1/2} U_t \, dx$$
$$\leq \left(\int_{B(0,R)} \rho_t |(\nabla^2 P_t^*)^{-1/2} \partial_t \nabla P_t^*|^2 \, dx \right)^{1/2} \left(\int_{B(0,R)} \rho_t |(\nabla^2 P_t^*)^{1/2} U_t|^2 \, dx \right)^{1/2}.$$

Moreover, the second term in the right-hand side of (3.36) is controlled by

(3.37) $\qquad \displaystyle\int_{B(0,R)} \rho_t U_t \cdot \nabla^2 P_t^* U_t \, dx \leq \max_{B(0,R)} \left(\rho_t^{1/2} |U_t|^2 \right) \int_{B(0,R)} \rho_t^{1/2} |\nabla^2 P_t^*| \, dx.$

Hence from (3.35), (3.36), and (3.37) we obtain

(3.38) $\qquad \displaystyle\int_{B(0,R)} \rho_t |(\nabla^2 P_t^*)^{-1/2} \partial_t \nabla P_t^*|^2 \, dx \leq \max_{B(0,R)} \left(\rho_t^{1/2} |U_t|^2 \right) \int_{B(0,R)} \rho_t^{1/2} |\nabla^2 P_t^*| \, dx.$

From (3.31), (3.32), and (3.33) we estimate the first factor as follows:

$$(3.39) \qquad \max_{|x| \leq M_T} \left(\rho_t^{1/2}(x) |U_t(x)|^2 \right) \leq \Lambda_T^{1/2} (M_T + d_\Omega)^2,$$

$$(3.40) \qquad \max_{M_T \leq |x|} \left(\rho_t^{1/2}(x) |U_t(x)|^2 \right) \leq \max_{M_T \leq |x|} \left\{ \frac{\sqrt{c_T}}{|x|^{K/2}} (|x| + d_\Omega)^2 \right\},$$

and the latter term is finite because $M_T \geq 1$ and $K > 4$.

In order to estimate the second factor, we observe that since $\nabla^2 P_t^*$ is a nonnegative matrix the estimate $|\nabla^2 P_t^*| \leq \Delta P_t^*$ holds (here we are using the operator norm on matrices). Hence, by (3.32) and (3.33) we obtain

$$\int_{B(0,R)} \rho_t^{1/2} |\nabla^2 P_t^*| \, dx \leq \int_{\{|x| \leq M_T\}} \rho_t^{1/2} |\nabla^2 P_t^*| \, dx + \int_{\{|x| > M_T\}} \rho_t^{1/2} |\nabla^2 P_t^*| \, dx$$
$$\leq \int_{\{|x| \leq M_T\}} \Lambda_T^{1/2} \Delta P_t^* \, dx + \int_{\{|x| > M_T\}} \frac{\sqrt{c_T}}{|x|^{K/2}} \Delta P_t^* \, dx.$$

The second integral can be rewritten as

$$\int_0^\infty \int_{\{|x| > M_T\} \cap \{|x|^{-K/2} > s\}} \Delta P_t^* \, dx \, ds,$$

which is bounded by

$$\int_0^{[M_T]^{-K/2}} ds \int_{\{|x| \leq s^{-2/K}\}} \Delta P_t^* \, dx.$$

From the divergence formula, since $|\nabla P_t^*(x)| \leq d_\Omega$ (because $\nabla P_t^*(x) \in \Omega$ for every $x \in \mathbb{R}^3$) and $M_T \geq 1$ (so $[M_T]^{-K/2} \leq 1$) we obtain

(3.41)

$$\int_{B(0,R)} \rho_t^{1/2} |\nabla^2 P_t^*| \, dx \leq \Lambda_T^{1/2} \int_{\{|x| = M_T\}} |\nabla P_t^*| \, d\mathcal{H}^2 + \sqrt{c_T} \int_0^{[M_T]^{-K/2}} ds \int_{\{|x| = s^{-2/K}\}} |\nabla P_t^*| \, d\mathcal{H}^2$$
$$\leq 4\pi \Lambda_T^{1/2} M_T^2 d_\Omega + 4\pi \sqrt{c_T} d_\Omega \int_0^1 s^{-4/K} \, ds$$

for all $t \in [0, T]$. Since $K > 4$ the last integral is finite, so the right-hand side is bounded and we obtain a global-in-space estimate on the left-hand side.

Thus, from (3.38), (3.39), (3.40), and (3.41), it follows that there exists a constant $C_1 = C_1(T, M, c_0, \Lambda, d_\Omega)$ (notice that the constant does not depend on the lower bound on the density) such that

$$(3.42) \qquad \int_{B(0,R)} \rho_t |(\nabla^2 P_t^*)^{-1/2} \partial_t \nabla P_t^*|^2 \, dx \leq C_1.$$

Applying now the inequality

$$(3.43) \qquad ab \log_+^k(ab) \leq 2^{k-1} \left[\left(\frac{k}{e} \right)^k + 1 \right] b^2 + 2^{3(k-1)} a^2 \log_+^{2k}(a) \qquad \forall (a, b) \in \mathbb{R}^+ \times \mathbb{R}^+,$$

(see [1, Lemma 3.4]) with $a = |(\nabla^2 P_t^*)^{1/2}|$ and $b = |(\nabla^2 P_t^*)^{-1/2}\partial_t\nabla P_t^*(x)|$ we deduce the existence of a constant $C_2 = C_2(k)$ such that

$$|\partial_t\nabla P_t^*| \log_+^k(|\partial_t\nabla P_t^*|) \le 2^{3(k-1)}|(\nabla^2 P_t^*)^{1/2}|^2 \log_+^{2k}(|(\nabla^2 P_t^*)^{1/2}|^2) + C_2|(\nabla^2 P_t^*)^{-1/2}\partial_t\nabla P_t^*|^2$$

$$= 2^{3(k-1)}|\nabla^2 P_t^*|\log_+^{2k}(|\nabla^2 P_t^*|) + C_2|(\nabla^2 P_t^*)^{-1/2}\partial_t\nabla P_t^*|^2.$$

Integrating the above inequality over $B(0,r)$ and using (3.42), we finally obtain

$$
\begin{aligned}
& \int_{B(0,r)} \rho_t|\partial_t\nabla P_t^*|\log_+^k(|\partial_t\nabla P_t^*|)\,dx \\
(3.44)\quad & \le 2^{3(k-1)}\int_{B(0,r)} \rho_t|\nabla^2 P_t^*|\log_+^{2k}(|\nabla^2 P_t^*|)\,dx + C_2\int_{B(0,R)} \rho_t|(\nabla^2 P_t^*)^{-1/2}\partial_t\nabla P_t^*|^2\,dx \\
& \le 2^{3(k-1)}\int_{B(0,r)} \rho_t|\nabla^2 P_t^*|\log_+^{2k}(|\nabla^2 P_t^*|)\,dx + C_1\cdot C_2,
\end{aligned}
$$

for all $0 < r \le R$.

Step 2: The approximation argument. We now consider the velocity field U given by Theorem 3.1, we take a sequence of smooth convex domains Ω_n which converges to Ω in the Hausdorff distance, and a sequence $(\psi^n) \subset C_c^\infty(B(0,n))$ of cut off functions such that $0 \le \psi_n \le 1$, $\psi^n(x) = 1$ inside $B(0,n/2)$, $|\nabla\psi^n| \le 2/n$ in $\mathbb{R}^3$. Let us also consider a sequence of space-time mollifiers σ^n with support contained in $B(0,1/n)$ and a sequence of space mollifiers φ^n. We extend the function U_t for $t \le 0$ by setting $U_t = 0$ for every $t < 0$.

Let us consider a compactly supported space regularization of ρ_0 and a space-time regularization of U, namely

$$\rho_0^n := \frac{(\rho_0 * \varphi^n)}{c_n}1_{B(0,n)}, \qquad U_t^n(x) := (U * \sigma^n)\psi^n,$$

where $c_n \uparrow 1$ is chosen so that ρ_0^n is a probability measure on $\mathbb{R}^3$. Let ρ_t^n be the solution of the continuity equation

$$\partial_t\rho_t^n + \nabla\cdot(U_t^n\rho_t^n) = 0 \qquad \text{in } \mathbb{R}^3 \times [0,\infty)$$

with initial datum ρ_0^n. From the regularity of the velocity field U_t^n and of the initial datum ρ_0^n we have that $\rho^n \in C^\infty(B(0,n) \times [0,\infty))$.

Since U_t is divergence-free and satisfies the inequality $|U_t(x)| \le |x| + d_\Omega$, we get

$$|U_t^n|(x) \le |U * \sigma^n|(x) \le \|U_t\|_{L^\infty(B(x,1/n))} \le |x| + d_\Omega + \frac{1}{n} \le |x| + d_\Omega + 1,$$

$$|\nabla\cdot U_t^n|(x) = |(U_t * \sigma^n)\cdot\nabla\psi^n|(x) \le \frac{2(n+1+d_\Omega)}{n} \le 3$$

for n large enough. Moreover, from the properties of ρ_0 we obtain that, for n large enough,

$$\rho_0^n(x) \le \frac{2c_0}{(|x|-1/n)^K} \le \frac{4c_0}{|x|^K} \qquad \forall\, |x| \ge M + 2,$$

$$\|\rho_0^n\|_\infty \le 2\|\rho_0\|_\infty \qquad \text{and} \qquad \left\|\frac{1}{\rho_0^n}\right\|_{L^\infty(B(0,n))} \le \left\|\frac{1}{\rho_0}\right\|_{L^\infty(B(0,n+1))}.$$

Hence the hypotheses of Lemma 3.4 are satisfied with $N = 3$, $A = 1$, $D = d_\Omega+1$, $d_0 = 4c_0$. Moreover ρ_t^n vanishes outside $B(0,n)$, and by (3.18) there exist constants $\lambda_n := e^{-3T}\left\|\frac{1}{\rho_0}\right\|_{L^\infty(B(0,n+1))}^{-1} > 0$,

$\Lambda := 2e^{3T}\|\rho_0\|_\infty$, and M_1, c_1 depending on T, M, c_0, d_Ω only, such that

$$\lambda_{n,T} \le \rho_t^n(x) \le \Lambda \qquad \forall\, (x,t) \in B(0,n) \times [0,T],$$

$$\rho_t^n(x) \le \frac{c_1}{|x|^K} \qquad \text{whenever } |x| \ge M_1.$$

(Observe that λ_n depends on n, but the other constants are all independent of n.) Thus, from Statement (ii) of Lemma 3.4 we get that, for all $r > 0$,

$$(3.45) \quad \rho_t^n(x) \ge e^{-3T} \inf\left\{ \rho_0^n(y) : \ y \in B\Big(0, re^t + (d_\Omega + 1)[e^t - 1]\Big) \right\} \qquad \forall\, (x,t) \in B(0,r) \times [0,T].$$

If n is large enough, the right-hand side of (3.45) is different from 0, and can be estimated from below in terms of ρ_0 by

$$\lambda = \lambda(r, T, \rho_0, \Omega) := e^{-3T} \inf\left\{ \rho_0(y) : \ y \in B\Big(0, re^t + (d_\Omega + 1)[e^t - 1] + 1\Big) \right\} > 0.$$

Therefore, for any $r > 0$ we can bound the density ρ^n from below inside $B(0,r)$ with a constant *independent of n*:

$$(3.46) \qquad\qquad \lambda \le \rho_t^n(x) \le \Lambda \qquad \forall\, (x,t) \in B(0,r) \times [0,T].$$

Let now P_t^{n*} be the unique convex function such that $P_t^{n*}(0) = 0$ and $(\nabla P_t^{n*})_\sharp \mathscr{L}_{\Omega_n} = \rho_t^n$. From the stability of solutions to the continuity equation with BV velocity field, [3, Theorem 6.6], we infer that

$$(3.47) \qquad\qquad \rho_t^n \to \rho_t \qquad \text{in } L^1_{\mathrm{loc}}(\mathbb{R}^3), \text{ for any } t > 0,$$

where ρ_t is the unique solution of (1.4) corresponding to the velocity field U. Since Ω_n is converging to Ω, from standard stability results for optimal transport maps (see for instance [26, Corollary 5.23] and [16, Section 4]) it follows that

$$(3.48) \qquad\qquad \nabla P_t^{n*} \to \nabla P_t^* \qquad \text{in } L^1_{\mathrm{loc}}(\mathbb{R}^3)$$

for any $t > 0$. Moreover, by Theorem 2.1(ii), Remark 2.2, and (3.46), for every $k \in \mathbb{N}$

$$(3.49) \qquad \limsup_{n\to\infty} \int_{B(0,r)} \rho_t^n |\nabla^2 P_t^{n*}| \log_+^{2k}(|\nabla^2 P_t^{n*}|)\, dx < \infty \qquad \forall\, r > 0,$$

and by the stability theorem in the Sobolev topology estabilished in [16, Theorem 1.2] it follows that

$$(3.50) \quad \lim_{n\to\infty} \int_{B(0,r)} \rho_t^n |\nabla^2 P_t^{n*}| \log_+^{2k}(|\nabla^2 P_t^{n*}|)\, dx = \int_{B(0,r)} \rho_t |\nabla^2 P_t^*| \log_+^{2k}(|\nabla^2 P_t^*|)\, dx \qquad \forall\, r > 0.$$

Since (ρ_t^n, U_t^n) satisfy the assumptions (3.27) through (3.31), by Step 1 we can apply (3.44) to (ρ_t^n, U_t^n) to obtain

$$(3.51) \quad \int_{B(0,r)} \rho_t^n |\partial_t \nabla P_t^{n*}| \log_+^{k}(|\partial_t \nabla P_t^{n*}|)\, dx \le 2^{3(k-1)} \int_{B(0,r)} \rho_t |\nabla^2 P_t^{n*}| \log_+^{2k}(|\nabla^2 P_t^{n*}|)\, dx + C$$

for all $r < n$, where the constant C does not depend on n.

Let $\phi \in C_c^\infty((0,T))$ be a nonnegative function. From the Dunford-Pettis Theorem, taking into account (3.47) and (3.48), it is clear that $\phi(t)\rho_t^n \partial_t \nabla P_t^{n*}$ converge weakly in $L^1(B(0,r) \times (0,T))$ to $\phi(t)\rho_t \partial_t \nabla P_t^*$. Moreover, since the function $w \mapsto |w| \log_+^k(|w|/r)$ is convex for every $r \in (0,\infty)$, we can

apply Ioffe lower semicontinuity theorem [2, Theorem 5.8] to the functions $\phi(t)\rho_t^n\partial_t\nabla P_t^{n*}$ and $\phi(t)\rho_t^n$ to infer

(3.52)
$$\int_0^T \phi(t)\int_{B(0,r)}\rho_t|\partial_t\nabla P_t^*|\log_+^k(|\partial_t\nabla P_t^*|)\,dx\,dt \leq \liminf_{n\to\infty}\int_0^T\phi(t)\int_{B(0,r)}\rho_t^n|\partial_t\nabla P_t^{n*}|\log_+^k(|\partial_t\nabla P_t^{n*}|)\,dx\,dt.$$

Taking (3.51), (3.50), (3.49), and (3.52) into account, by Lebesgue dominated convergence theorem we obtain

$$\int_0^T \phi(t)\int_{B(0,r)}\rho_t|\partial_t\nabla P_t^*|\log_+^k(|\partial_t\nabla P_t^*|)\,dx\,dt$$
$$\leq \int_0^T\phi(t)\left(2^{3(k-1)}\int_{B(0,r)}\rho_t|\nabla^2 P_t^*|\log_+^{2k}(|\nabla^2 P_t^*|)\,dx + C\right)dt.$$

Since this holds for every $\phi\in C_c^\infty((0,T))$ nonnegative, by a localization argument we obtain the desired result. $\qquad\square$

Remark 3.6. Thanks to Remark 2.3 one can prove that for every $T > 0$ and $r > 0$ there exist a constant $\kappa > 1$ and a constant C which depend on r, ρ_0, T, d_Ω such that, for almost every $t\in[0,T]$ we have that $\nabla P_t^*\in W^{1,\kappa}(B(0,r)\times[0,\infty);\mathbb{R}^3)$ and

$$\int_{B(0,r)}\rho_t|\partial_t\nabla P_t^*|^\kappa\,dx \leq C.$$

This estimate provides better local integrability of the time derivative of ∇P_t^*. The proof follows the same lines of Proposition 3.5 (see also [20, Proposition 5.1]). However the exponent κ is not universal, but depends in a nontrivial way from the local lower bounds on the density which are related to r, ρ_0 and T. Therefore we preferred to state Proposition 3.5 with a universal modulus of integrability.

We finally point out that in the compact setting studied in [1] the same argument provides a global L^κ estimate of $\partial_t\nabla P_t^*$ on the torus, with κ depending only on the upper and lower bound on ρ_0, which is also uniform in time.

4. Existence of an Eulerian solution

Proof of Theorem 1.3. First of all notice that by statement (ii) of Theorem 2.1 and Proposition 3.5, it holds $|\nabla^2 P_t^*|, |\partial_t\nabla P_t^*|\in L_{\mathrm{loc}}^\infty([0,\infty), L_{\mathrm{loc}}^1(\mathbb{R}^3))$. Moreover, since $(\nabla P_t)_\sharp\mathscr{L}_\Omega = \rho_t$, it is immediate to check the function u in (1.5) is well-defined and $|u|$ belongs to $L_{\mathrm{loc}}^\infty([0,\infty), L_{\mathrm{loc}}^1(\mathbb{R}^3))$.

Let $\phi\in C_c^\infty(\Omega\times[0,\infty))$ be a test function and let us consider $\varphi:\mathbb{R}^3\times[0,\infty)\to\mathbb{R}^3$ given by

(4.1)
$$\varphi_t(y) := y\phi_t(\nabla P_t^*(y)).$$

Clearly φ is compactly supported in time because so is ϕ; moreover P_t are Lipschitz on $\operatorname{supp}\phi_t$ as t varies in any compact subset of $[0,\infty)$ with bounded Lipschitz constants. Hence the set $\nabla P_t(\operatorname{supp}\phi_t)$, which contains $\operatorname{supp}\varphi_t$, is bounded in space. Therefore ϕ_t is compactly supported in $\mathbb{R}^3\times[0,\infty)$. Moreover, Proposition 3.5 implies that $\varphi\in W^{1,1}(\mathbb{R}^3\times[0,\infty))$. So, by Lemma 3.2, each component of the function $\varphi_t(y)$ is an admissible test function for (3.1). For later use, we write down explicitly the derivatives of φ:

(4.2)
$$\begin{cases}\partial_t\varphi_t(y) = y[\partial_t\phi_t](\nabla P_t^*(y)) + y\big([\nabla\phi_t](\nabla P_t^*(y))\cdot\partial_t\nabla P_t^*(y)\big),\\ \nabla\varphi_t(y) = Id\,\phi_t(\nabla P_t^*(y)) + y\otimes\big([\nabla^T\phi_t](\nabla P_t^*(y))\nabla^2 P_t^*(y)\big).\end{cases}$$

Taking into account that $(\nabla P_t)_\sharp \mathscr{L}_\Omega = \rho_t \mathscr{L}^3$ and that $[\nabla P_t^*](\nabla P_t(x)) = x$ almost everywhere, we can rewrite the boundary term in (3.1) as

$$(4.3) \qquad \int_{\mathbb{R}^3} \varphi_0(y)\rho_0(y)\, dy = \int_\Omega \nabla P_0(x)\phi_0(x)\, dx.$$

In the same way, since $U_t(y) = J(y - \nabla P_t^*(y))$, we can use (4.2) to rewrite the other term as

$$\int_0^\infty \int_{\mathbb{R}^3} \Big\{\partial_t\varphi_t(y) + \nabla\varphi_t(y) \cdot U_t(y)\Big\}\rho_t(y)\, dy\, dt$$

$$(4.4)$$

$$= \int_0^\infty \int_\Omega \Big\{ \nabla P_t(x)\partial_t\phi_t(x) + \nabla P_t(x)\big(\nabla\phi_t(x) \cdot [\partial_t \nabla P_t^*](\nabla P_t(x)))$$

$$+ \big[Id\, \phi_t(x) + \nabla P_t(x) \otimes \big(\nabla^T\phi_t(x)\nabla^2 P_t^*(\nabla P_t(x)))\big] J(\nabla P_t(x) - x) \Big\}\, dx\, dt$$

which, taking into account the formula (1.5) for u, after rearranging the terms turns out to be equal to

$$(4.5) \qquad \int_0^\infty \int_\Omega \nabla P_t(x)\Big\{\partial_t\phi_t(x) + u_t(x) \cdot \nabla\phi_t(x)\Big\} + J\Big\{\nabla P_t(x) - x\Big\}\phi_t(x)\, dx\, dt.$$

Hence, combining (4.3), (4.4), (4.5), and (3.1), we obtain the validity of (1.6).

Now we prove (1.7). Given $\phi \in C_c^\infty(0,\infty)$ and $\psi \in C_c^\infty(\Omega)$, let us consider $\varphi : \mathbb{R}^3 \times [0,\infty) \to \mathbb{R}$ defined by

$$(4.6) \qquad \varphi_t(y) := \phi(t)\psi(\nabla P_t^*(y)).$$

As in the previous case, $\varphi \in W^{1,1}(\mathbb{R}^3 \times [0,\infty))$ and is compactly supported in time and space, so we can use φ as a test function in (3.1). Then, identities analogous to (4.2) yield

$$0 = \int_0^\infty \int_{\mathbb{R}^3} \{\partial_t\varphi_t(y) + \nabla\varphi_t(y) \cdot U_t(y)\}\, \rho_t(y)\, dy\, dt$$

$$= \int_0^\infty \phi'(t) \int_\Omega \psi(x)\, dx\, dt$$

$$+ \int_0^\infty \phi(t) \int_\Omega \Big\{ \nabla\psi(x) \cdot \partial_t \nabla P_t^*(\nabla P_t(x)) + \nabla^T\psi(x)\nabla^2 P_t^*(\nabla P_t(x))J(\nabla P_t(x) - x) \Big\}\, dx\, dt$$

$$= \int_0^\infty \phi(t) \int_\Omega \nabla\psi(x) \cdot u_t(x)\, dx\, dt.$$

Since ϕ is arbitrary we obtain

$$\int_\Omega \nabla\psi(x) \cdot u_t(x)\, dx = 0 \qquad \text{for a.e. } t > 0.$$

By a standard density argument it follows that the above equation holds outside a negligible set of times independent of the test function ψ, thus proving (1.7). $\qquad \square$

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Scuola Normale Superiore, p.za dei Cavalieri 7, I-56126 Pisa, Italy
E-mail address: l.ambrosio@sns.it

Scuola Normale Superiore, p.za dei Cavalieri 7, I-56126 Pisa, Italy
E-mail address: maria.colombo@sns.it

Scuola Normale Superiore, p.za dei Cavalieri 7, I-56126 Pisa, Italy
E-mail address: guido.dephilippis@sns.it

Department of Mathematics, The University of Texas at Austin, 1 University Station C1200, Austin TX 78712, USA
E-mail address: figalli@math.utexas.edu