A NOTE ON CHEEGER SETS

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Abstract. Starting from the quantitative isoperimetric inequality [21, 17], we prove a sharp quantitative version of the Cheeger inequality.

A Cheeger set E for an open subset $\Omega \subset \mathbb{R}^n$, $n \geq 2$, is any minimizer of the variational problem

$$c_m(\Omega) = \inf \left\{ \frac{P(E)}{|E|^m} : E \subset \Omega \,, 0 < |E| < \infty \right\} , \tag{1}$$

where $|E|$ is the Lebesgue measure of E, and $P(E)$ denotes its distributional perimeter, see [3, Chapter 3]. In order to avoid trivial situations, it is assumed that Ω has finite measure and that the parameter m satisfies the constraints

$$m > \frac{1}{n'} , \qquad \text{where} \quad n' = \frac{n}{n-1} . \tag{2}$$

Under these assumptions on Ω and m, it is not difficult to show that Cheeger sets always exist. The study of qualitative properties of Cheeger sets has received particular attention in recent years, see for example [1, 9, 10, 11, 28, 29, 27]. Another interesting question is how to provide lower bounds on $c_m(\Omega)$ in terms of geometric properties of Ω. The basic estimate in this direction is the *Cheeger inequality*,

$$|\Omega|^{m-(1/n')} c_m(\Omega) \geq |B|^{m-(1/n')} c_m(B) , \tag{3}$$

where B is the Euclidean unit ball. The bound is sharp, in the sense that equality holds in (3) if and only if $\Omega = x_0 + rB$ for some $x_0 \in \mathbb{R}^n$ and $r > 0$. In this note we strengthen this lower bound in terms of the *Fraenkel asymmetry* of Ω, defined as

$$A(\Omega) = \inf \left\{ \frac{|\Omega \Delta (x_0 + r\,B)|}{|\Omega|} : |r\,B| = |\Omega| \,, x_0 \in \mathbb{R}^n \right\} ,$$

where Δ denotes the symmetric difference between sets. Note that $A(\Omega) = 0$ if and only if Ω is a ball.

Theorem. *Let Ω be an open set in $\mathbb{R}^n$, $n \geq 2$, with $|\Omega| < \infty$, and let m satisfy* (2)*. Then*

$$|\Omega|^{m-(1/n')} c_m(\Omega) \geq |B|^{m-(1/n')} c_m(B) \left\{ 1 + \left(\frac{A(\Omega)}{C(n,m)} \right)^2 \right\} , \tag{4}$$

where $C(n,m)$ is a constant depending only on n and m.

As will be seen from the proof, a possible value for $C(m,n)$ is given by

$$C(n,m) = \frac{2}{m-(1/n')} + \frac{61\, n^7}{(2-2^{1/n'})^{3/2}} .$$

This kind of improvement on a given sharp geometric-functional inequality has been extensively considered in the literature, e.g. concerning the isoperimetric inequality [4, 7, 32, 20, 24, 25, 21, 30, 18, 2], Sobolev inequalities [8, 12, 13, 22, 14], Faber-Krahn and isocapacitary inequalities [31, 26, 5, 6, 23, 18, 19], the Gaussian isoperimetric inequality [15] and the Wulff inequality [16, 17]. In particular, inequality (4) improves an analogous result contained in [23], where the exponent 3 is found in place of the exponent 2; in turn, the exponent 2 is sharp as we will notice below.

In the proof of the theorem we will use the quantitative isoperimetric inequality

$$P(E) \geq n|B|^{1/n}|E|^{1/n'}\left\{1+\left(\frac{A(E)}{C_0(n)}\right)^2\right\}, \tag{5}$$

where the exponent 2 is sharp, see [21, 17, 30] (here, $C_0(n)$ is a constant depending only on the dimension n, which can be chosen equal to $\frac{61\,n^7}{(2-2^{1/n'})^{3/2}}$, see [17]). The strategy consists in showing that, if E is the Cheeger set of an almost optimal Ω in (3), then, first, $|\Omega\setminus E|$ is correspondingly small and, secondly, E is almost optimal in the isoperimetric inequality (and thus, by (5), it is close to a ball).

To begin with, we notice that $c_m(B)=\frac{P(B)}{|B|^m}$. Indeed, if $F\subset B$ has finite and positive measure, and $r\in(0,1]$ is such that $|r\,B|=|F|$, then $P(F)\geq P(r\,B)$ by the isoperimetric inequality. Therefore,

$$\frac{P(F)}{|F|^m}\geq\frac{P(r\,B)}{|r\,B|^m}=\frac{n|B|r^{n-1}}{|B|^m r^{nm}}\geq n|B|^{1-m}=\frac{P(B)}{|B|^m},$$

where in the last inequality we have used (2) and $r\leq 1$. This ensures that $c_m(B)=\frac{P(B)}{|B|^m}$ and, by the well-known characterization of the equality cases in the isoperimetric inequality, B is the only Cheeger set for B. A similar argument proves in fact the validity of (3). Indeed, assume without loss of generality that $|\Omega|=|B|$ and consider $E\subset\Omega$, with finite and positive measure. If $r\in(0,1]$ is such that $|E|=|r\,B|$, then, again by the isoperimetric inequality,

$$\frac{P(E)}{|E|^m}\geq r^{n-1-nm}\frac{P(B)}{|B|^m}\geq\frac{P(B)}{|B|^m}=c_m(B),$$

and (3) follows.

We notice that inequality (4) is sharp in the decay rate of $A(\Omega)$. Indeed, by (1) we know that $c_m(\Omega)\leq\frac{P(\Omega)}{|\Omega|^m}$, and, from $c_m(B)=\frac{P(B)}{|B|^m}=n|B|^{1-m}$, we immediately get

$$|\Omega|^{m-(1/n')}c_m(\Omega)-|B|^{m-(1/n')}c_m(B)\leq n|B|^{1/n}\left(\frac{P(\Omega)}{n|B|^{1/n}|\Omega|^{1/n'}}-1\right).$$

Then, being the exponent 2 sharp in (5), it is *a fortiori* sharp in (4).

We can now prove our result.

Proof of the theorem. Without loss of generality, we can assume that $|\Omega|=|B|$. Since we always have $A(\Omega)\leq 2$, if $c_m(\Omega)\geq 2\,c_m(B)$, then (4) is verified as soon as we take $C(n,m)\geq 4$. We are therefore going to assume that $c_m(\Omega)\leq 2\,c_m(B)$.

Let $E\subset\Omega$ a Cheeger set for Ω, so that

$$\frac{P(E)}{|E|^m}=c_m(\Omega). \tag{6}$$

Note that, up to a translation of E (and, correspondingly, of Ω), we can also assume that

$$A(E)=\frac{|E\Delta(r\,B)|}{|E|}, \tag{7}$$

for some $r \in (0,1]$. We now divide the argument in two steps.

Step one: We introduce the *isoperimetric deficit* $\delta(E)$ *of* E, defined as

$$\delta(E) = \frac{P(E)}{n|B|^{1/n}|E|^{1/n'}} - 1\,,$$

and prove the following inequalities concerning E:

$$|E| \;\geq\; |\Omega| \left(\frac{c_m(B)}{c_m(\Omega)}\right)^{\frac{1}{m-(1/n')}}\,, \tag{8}$$

$$\delta(E) \;\leq\; \frac{c_m(\Omega) - c_m(B)}{c_m(B)}\,. \tag{9}$$

In order to prove (8), note that, by the isoperimetric inequality,

$$\frac{P(E)}{|E|^m} \geq n|B|^{1/n}|E|^{(1/n')-m}\,.$$

Thus, by (6), recalling that $c_m(B) = n|B|^{1-m}$, we have

$$|E|^{m-(1/n')} \geq \frac{n|B|^{1/n}}{c_m(\Omega)} = |B|^{m-(1/n')}\frac{c_m(B)}{c_m(\Omega)}\,,$$

that is (8). We now prove (9). On dividing by $n|B|^{1/n}$ the inequality

$$c_m(\Omega) - c_m(B) \geq \frac{P(E)}{|E|^{1/n'}}\,|E|^{(1/n')-m} - n|B|^{1-m}\,,$$

we find that

$$\begin{aligned}\frac{c_m(\Omega) - c_m(B)}{n|B|^{1/n}} \;&\geq\; (1+\delta(E))|E|^{(1/n')-m} - |B|^{(1/n')-m}\\ &=\; \delta(E)|E|^{(1/n')-m} + (|E|^{(1/n')-m} - |B|^{(1/n')-m})\,.\end{aligned}$$

By (2) and $|E| \leq |\Omega| = |B|$, the second term on the right hand side is non negative, therefore we have proved that

$$\frac{c_m(\Omega) - c_m(B)}{n|B|^{1/n}} \geq \frac{\delta(E)}{|E|^{m-(1/n')}} \geq \frac{\delta(E)}{|B|^{m-(1/n')}}\,,$$

as desired.

Step two: Thanks to (7), we can estimate $A(\Omega)$ as follows:

$$\begin{aligned}|\Omega|A(\Omega) &\leq |\Omega\Delta B| \leq |\Omega\Delta E| + |E\Delta(r\,B)| + |B\Delta(r\,B)|\\ &= 2(|\Omega| - |E|) + |E|A(E) \leq 2(|\Omega| - |E|) + |\Omega|A(E)\,.\end{aligned} \tag{10}$$

By (8) we find that

$$|\Omega| - |E| \leq \frac{|\Omega|}{c_m(\Omega)^{\frac{1}{m-(1/n')}}}\left(c_m(\Omega)^{\frac{1}{m-(1/n')}} - c_m(B)^{\frac{1}{m-(1/n')}}\right)\,.$$

Since $t^a \leq s^a + at^{a-1}(t-s)$ whenever $a \geq 1$ and $0 < s \leq t$, and $t^a \leq s^a + as^{a-1}(t-s)$ whenever $0 < a \leq 1$ and $0 < s \leq t$, minding that $c_m(\Omega) \geq c_m(B)$ we get

$$|\Omega| - |E| \leq \frac{|\Omega|}{m-(1/n')}\,\frac{c_m(\Omega) - c_m(B)}{c_m(B)}\,. \tag{11}$$

On the other hand, by (9) and (5)

$$A(E) \le C_0(n)\sqrt{\frac{c_m(\Omega) - c_m(B)}{c_m(B)}}\,, \tag{12}$$

and combining (10), (11) and (12), we find

$$A(\Omega) \le \frac{2}{m-(1/n')}\left(\frac{c_m(\Omega) - c_m(B)}{c_m(B)}\right) + C_0(n)\sqrt{\frac{c_m(\Omega) - c_m(B)}{c_m(B)}}\,.$$

Since $c_m(\Omega) \le 2c_m(B)$, we finally get

$$A(\Omega) \le C(n,m)\sqrt{\frac{c_m(\Omega) - c_m(B)}{c_m(B)}}\,,$$

where $C(n,m)$ is defined as

$$C(n,m) = \frac{2}{m-(1/n')} + C_0(n)\,.$$

We have thus achieved the proof of the theorem. □

To conclude, let us remark that the above argument may be repeated in the case the Euclidean perimeter $P(E)$ in (1) is replaced by some anisotropic perimeter

$$P_\psi(E) = \int_{\partial E} \psi(\nu_E(x))d\mathcal{H}^{n-1}(x)$$

(here E has smooth boundary, ν_E is its outer unit normal vector field, and $\psi : \mathbb{R}^n \to [0,\infty)$ is a convex function with $\psi(t\nu) = t\psi(\nu) > 0$ for every $t > 0$ and $\nu \in \partial B$). The only relevant change consists in replacing (5) with the corresponding quantitative version of the Wulff inequality proved in [17].

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