A note on the regularity of the free boundaries in the optimal partial transport problem

Alessio Figalli*

Abstract

In this work we study the global regularity of the free boundaries arising in the optimal partial transport problem. Assuming the supports of both the source and the target measure to be convex, we show that the free boundaries of the active regions are globally $C^{0,1/2}$.

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1 Introduction

In [2] the author studied the following problem: given two densities f and g, one considers the problem of transporting a fraction $m \in [0, \min\{\|f\|_{L^1}, \|g\|_{L^1}\}]$ of the mass of f onto g minimizing a transportation cost. Assuming the cost per unit of mass to be given by $|x-y|^2$, the author proved existence and uniqueness of the optimal transport map when $m \in [\|\min\{f,g\}\|_{L^1}, \min\{\|f\|_{L^1}, \|g\|_{L^1}\}]$. This generalized the result of Caffarelli and McCann [1], where the authors considered two densities with disjoint supports.

Let us denote by Ω and Λ the supports of f and g, respectively, and for any fixed $m \in [\|\min\{f,g\}\|_{L^1}, \min\{\|f\|_{L^1}, \|g\|_{L^1}\}]$ let $f_m \leq f$ and $g_m \leq g$ denote the fraction of f and g that the optimal map choose as source and target respectively (i.e. the optimal map sends f_m onto g_m, and it does not move the remaining part of f, see [2, Section 2]). We define the active regions as the (measure theoretical) supports of f_m and g_m (see Section 2 below for a precise definition). The boundaries of the active regions consist of a "fixed" part contained in $\partial\Omega$ (in the case of f_m) or in $\partial\Lambda$ (in the case of g_m), and of a "free" part which is contained inside Ω (for f_m) or inside Λ (for g_m). In [2, Section 4] the following regularity results on the free boundaries of the active regions are shown:

1. If Ω and Λ have Lipschitz boundaries, then the free boundaries are $(n-1)$-rectifiable. Moreover under some weak regularity assumptions on the geometry of the supports they are locally semiconvex away from $\Omega \cap \Lambda$.

*Centre de Mathématiques Laurent Schwartz, UMR 7640, Ecole Polytechnique - 91128 Palaiseau, France. e-mail: figalli@math.polytechnique.fr

2. If Ω and Λ are two bounded strictly convex sets, and f and g are bounded away from zero and infinity on their respective supports, then the free boundaries are locally C^1 away from $\Omega \cap \Lambda$.

These results left open the natural issue: what is the regularity of the free boundaries near $\Omega \cap \Lambda$? Here we answer to this question: assuming Ω and Λ to be convex, the free boundaries are globally $C^{0,1/2}$, without any assumption on the densities f and g (see Theorem 1 below for a precise statement).

2 Preliminaries and main result

Let f, g be two nonnegative integrable functions, such that $\Omega = \{f > 0\}$ and $\Lambda = \{g > 0\}$ are two open and bounded sets. Define $\Gamma_{\leq}(f, g)$ as the set of nonnegative finite Borel measures on $\mathbb{R}^n \times \mathbb{R}^n$ whose first and second marginal are dominated by f and g respectively. Fix a certain amount $m \in [0, \min\{\|f\|_{L^1}, \|g\|_{L^1}\}]$ which represents the mass one wants to transport, and consider the following partial transport problem:

$$\text{minimize } C(\gamma) := \int_{\mathbb{R}^n \times \mathbb{R}^n} |x - y|^2 \, d\gamma(x, y)$$

among all $\gamma \in \Gamma_{\leq}(f, g)$ with $\int_{\mathbb{R}^n \times \mathbb{R}^n} d\gamma = m$.

In [2, Section 2] the author shows that for any $m \in [\| \min\{f, g\}\|_{L^1}, \min\{\|f\|_{L^1}, \|g\|_{L^1}\}]$ there exists a unique optimal plan γ_m. Moreover, denoting by f_m and g_m the first and the second marginal of γ_m respectively, it is shown that there exists a convex function $\psi_m : \mathbb{R}^n \to \mathbb{R}$ such that $\gamma_m = (Id \times \nabla \psi_m)_\# f_m$, i.e.

$$\int_{\mathbb{R}^n \times \mathbb{R}^n} \varphi(x, y) \, d\gamma_m(x, y) = \int_{\mathbb{R}^n \times \mathbb{R}^n} \varphi\big(x, \nabla \psi_m(x)\big) f_m(x) \, dx \qquad \forall \varphi \in C(\mathbb{R}^d \times \mathbb{R}^d),$$

which means that γ_m is concentrated on the graph of $\nabla \psi_m$, and $\nabla \psi_m$ sends f_m onto g_m. Furthermore, defining the active source and the active target as

$$F_m := \text{ set of density points of } \{f_m > 0\}, \qquad G_m := \text{ set of density points of } \{g_m > 0\},$$

the inclusions

$$F_m \supset \Omega \cap \Lambda \qquad \text{and} \qquad G_m \supset \Omega \cap \Lambda$$

hold. As proven in [2, Section 3], F_m and G_m coincide respectively with $U_m \cap \Omega$ and $V_m \cap \Lambda$, where

$$U_m := (\Omega \cap \Lambda) \cup \bigcup_{(\bar{x}, \bar{y}) \in \Gamma_m} B_{|\bar{x} - \bar{y}|}(\bar{y}), \qquad V_m := (\Omega \cap \Lambda) \cup \bigcup_{(\bar{x}, \bar{y}) \in \Gamma_m} B_{|\bar{x} - \bar{y}|}(\bar{x}),$$

and $\Gamma_m \subset \mathbb{R}^n \times \mathbb{R}^n$ is a suitably chosen set on which γ_m is concentrated. In particular the free boundaries of the active regions are given by $\partial U_m \cap \Omega$ and $\partial V_m \cap \Lambda$. Finally, assuming the existence of an open convex set C such that $\Lambda \subset C$ and $\Omega \setminus \Lambda \subset \mathbb{R}^n \setminus C$ (for instance, take $C = \Lambda$

if Λ is convex), [2, Proposition 4.5] shows that ∂U_m can be locally written as a semiconvex graph away from $\Omega \cap \Lambda$.

We now want to prove the global regularity of ∂U_m, assuming as above that there exists an open convex set C such that $\Lambda \subset C$ and $\Omega \setminus \Lambda \subset \mathbb{R}^n \setminus C$. Since ∂U_m is locally semiconvex (and thus locally Lipschitz) away from $\partial(\Omega \cap \Lambda)$, we only need to study its regularity near $\partial(\Omega \cap \Lambda)$.

Fix $x_0 \in \partial(\Omega \cap \Lambda)$. We now show the following result, which gives the desired global $C^{0,1/2}$-regularity:

Theorem 1 *There exists a bi-Lipschitz homeomorphism of a neighborhood $\mathcal{O}$ of x_0 which transforms ∂U_m into the graph of a $C^{0,1/2}$ function, whose Hölder constant depends only on* $\mathrm{diam}(\Omega \cup \Lambda)$ *and the $C^{0,1/2}$-norm of ∂C inside $\mathcal{O}$ (i.e. the $C^{0,1/2}$-norm of a local parametrization of ∂C inside $\mathcal{O}$). More precisely, the $C^{0,1/2}$-norm can be bounded by $\sqrt{2[\mathrm{diam}(\Omega \cup \Lambda)]} + \|\partial C\|_{C^{0,1/2}(\mathcal{O})}$. Moreover the Lipschitz constant of the homeomorphism and of its inverse are both bounded by $1 + \|\partial C\|_{\mathrm{Lip}(\mathcal{O})}$.*

Proof. Since C is convex, there exists an halfspace H such that $C \subset H$ and $x_0 \in \partial H$. Without loss of generality we can assume that $H = \{x_n < 0\}$ and $x_0 = 0$. Let $f : \mathbb{R}^{n-1} \to (-\infty, 0]$ be a concave function such that ∂C is locally parameterized near 0 by $(x', f(x'))$, and perform the change of variable $(x', x_n) \mapsto (x', x_n - f(x'))$, so that ∂C becomes flat near the origin. We want to show that, in this system of coordinates, ∂U_m is given by the graph of a $C^{0,1/2}$ function.
We remark that before such a change of coordinates we had

$$U_m = (\Omega \cap \Lambda) \cup \bigcup_{(x,r) \in \Gamma'} B_r(x)$$

for some Borel set $\Gamma' \subset \{x_n < 0\} \times [0, R']$, with $R' \leq \mathrm{diam}(\Omega \cup \Lambda)$. Hence, in these new coordinates, U_m coincides near the origin with the union of $\{x_n < 0\}$ with open sets $\{E_\alpha\}$ of the form

$$\left\{(x', x_n) \, : \, \bigl(x_n - (x_\alpha)_n - f(x')\bigr)^2 + \left|x' - x'_\alpha\right|^2 < R_\alpha^2 \qquad \text{for some } x_\alpha = (x'_\alpha, (x_\alpha)_n),\ R_\alpha > 0\right\},$$

where $(x_\alpha)_n + f(x') \leq 0$ (these sets E_α are just the images of the balls $B_r(x)$ through the change of variable). Thus, near the origin, the boundary of U_m is given by the graph of the function $u : \mathbb{R}^{n-1} \to [0, +\infty)$, with

$$u := \sup_\alpha u_\alpha, \qquad u_\alpha(x') := \max\left\{\sqrt{R_\alpha^2 - |x'|^2} + (x_\alpha)_n + f(x'), 0\right\}.$$

We now observe that the function $x' \mapsto \sqrt{R_\alpha^2 - |x'|^2}$ is $C^{0,1/2}$ on the set $\{|x'| \leq R_\alpha\}$, with Hölder constant bounded by $\sqrt{2R_\alpha}$. Therefore, recalling that $(x_\alpha)_n + f(x') \leq 0$ and that f is concave (and thus Lipschitz), it is easy to see that $u_\alpha(x')$ is $C^{0,1/2}$, with Hölder constant bounded by $\sqrt{2R_\alpha} + \|f\|_{C^{0,1/2}}$. Since the radii R_α are bounded by R', we obtain that u is $C^{0,1/2}$, with Hölder constant bounded by $\sqrt{2R'} + \|f\|_{C^{0,1/2}}$. □

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