A SHARP FREIMAN TYPE ESTIMATE FOR SEMISUMS IN TWO AND THREE DIMENSIONAL EUCLIDEAN SPACES

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Abstract. Freiman's Theorem is a classical result in additive combinatorics concerning the approximate structure of sets of integers that contain a high proportion of their internal sums. As a consequence, one can deduce an estimate for sets of real numbers: "If $A \subset \mathbb{R}$ and $\left|\frac{1}{2}(A+A)\right| - |A| \ll |A|$, then A is close to its convex hull." In this paper we prove a sharp form of the analogous result in dimensions 2 and 3.

1. Introduction

Given a set $A \subset \mathbb{R}^n$, define the semisum by

$$\tfrac{1}{2}(A+A) := \left\{ \tfrac{x+y}{2} \,:\, x \in A,\, y \in A \right\}.$$

Evidently, $\frac{1}{2}(A+A) \supset A$, and for convex sets K, $\frac{1}{2}(K+K) = K$. Also, $\left|\frac{1}{2}(A+A)\right| = |A|$ implies that A is equal to its convex hull $\mathrm{co}(A)$ minus a set of measure zero (see [4, Théorème 6]).

The stability of this statement is a natural question that has already been extensively investigated in the one dimensional case. Indeed, by approximating sets in $\mathbb{R}$ with finite unions of intervals, one can translate the problem to $\mathbb{Z}$ and in the discrete setting the question becomes a well studied problem in additive combinatorics. More precisely, set

$$\delta(A) := \left|\tfrac{1}{2}(A+A)\right| - |A|,$$

where $|\cdot|$ denotes the outer Lebesgue measure. The following theorem can be obtained as a corollary of a result of G. Freiman [10] about the structure of additive subsets of $\mathbb{Z}$ (see [6] for more details, and also [3] and the references therein for more recent developments on this one dimensional problem):

Theorem 1.1. *Let $A \subset \mathbb{R}$ be a measurable set of positive Lebesgue measure, and assume that $\delta(A) < |A|/2$. Then*

$$|\mathrm{co}(A) \setminus A| \le 2\,\delta(A).$$

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Note that the assumption $\delta(A) < |A|/2$ is necessary, as can be seen by considering the set $A = [0,1] \cup [R, R+1]$ with $R \gg 1$.

In [6, Theorem 1.2] we extended Theorem 1.1 to every dimension, but with a dimensional dependence in the exponent (see also [7] for a stability result when one considers the semisum of two different sets). Our result was as follows.

Theorem 1.2. *Let $n \geq 2$. There exist computable dimensional constants $\delta_n, C_n > 0$ such that if $A \subset \mathbb{R}^n$ is a measurable set of positive Lebesgue measure with $\delta(A) \leq \delta_n |A|$, then*

$$\frac{|\mathrm{co}(A) \setminus A|}{|A|} \leq C_n \left(\frac{\delta(A)}{|A|}\right)^{\alpha_n}, \qquad \textit{where } \alpha_n := \frac{1}{8 \cdot 16^{n-2} n!(n-1)!}.$$

Note that the dimensional smallness assumption on $\delta(A)$ is necessary. Indeed, consider $t = 1/2$ and the set

$$A := B_1(0) \cup \{Re_1\}, \qquad R \gg 1.$$

Then $|\mathrm{co}(A) \setminus A| \approx R$ is arbitrarily large, while $\delta(A) = \left|B_{1/2}\left(\frac{R}{2}e_1\right)\right| = 2^{-n}|A|$, hence $\delta_n \leq 2^{-n}$.

The proof in [6] is based on induction on dimension and Fubini-type arguments, and it leads to a bad estimate for the exponent α_n. In fact, we believe that $\alpha_n = 1$, which we formulate more precisely in the following conjecture.

Conjecture 1.3. *Suppose that A is a measurable subset of $\mathbb{R}^n$, of positive Lebesgue measure. There exist computable constants C_n and $d_n > 0$, depending only on n, such that the following holds: if $\delta(A) \leq d_n |A|$, then*

$$|\mathrm{co}(A) \setminus A| \leq C_n \, \delta(A).$$

In this paper we introduce a completely new strategy that allows us to prove this *sharp* stability estimate in dimensions 2 and 3.

Theorem 1.4. *Conjecture 1.3 is valid for $n \leq 3$.*

The exponent $\alpha_n = 1$ may look surprising at first sight, as most sharp stability results for minimizers of geometric inequalities in dimension $n \geq 2$ hold with the exponent 1/2. In particular, the best possible stability exponent for the Brunn-Minkowski inequality on convex sets is 1/2, see [8, 9]. In contrast, our stability inequality with exponent 1 is affine invariant and additive under partitions of the set by convex tilings, and these properties are crucial to the proof. Even though we have stopped at $n = 3$, the proof is by induction on n and is organized with the hope that parts of it will ultimately apply to the case of general n. There is at least one other stability inequality in which the exponent 1 is optimal in all dimensions, namely the one proved in [5]. (Observe that the exponent 1 becomes natural when looking at critical points instead of minimizers, see for instance [2, Theorem 1.2], but this is a consequence of the different definition the "deficit" δ.)

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2. Proof of Theorem 1.4

As the reader will see, many of the arguments for the proof of Theorem 1.4 are valid in any dimension. For this reason we shall work with a generic n for most of the proof, and we shall use some geometric considerations specific to $n = 2$ and $n = 3$ only towards the end.

Basic considerations. Since Theorem 1.4 is known for $n = 1$ (see Theorem 1.1), we can assume that $n \geq 2$ and, by induction on dimension, we can also assume that Theorem 1.4 holds in dimension $n - 1$.

Denote the convex hull of A by $K := \mathrm{co}(A)$. Since the theorem is affine invariant, after dilation we can assume, with no loss of generality, that $|A| = 1$. Assuming that $\delta(A) \ll 1$, it follows by [1] and/or [6, Theorem 1.2] that[1]

$$\mu := |K \setminus A| \ll 1. \tag{2.1}$$

In particular, $1 \leq |K| \leq 2$. Therefore, using the lemma of F. John [11], up to an affine transformation with Jacobian bounded from above and below by a dimensional constant, we can assume that K satisfies

$$B_{1/\sqrt{n}} \subset K \subset B_{\sqrt{n}} \tag{2.2}$$

for balls of radius $1/\sqrt{n}$ and $\sqrt{n}$ centered at the origin.

By approximation,[2] we can assume the set A is compact and that ∂K consists of finitely many polygonal faces. In particular, $\frac{1}{2}(A+A)$ is compact, hence measurable. Furthermore, since all vertices of the faces are extreme points, they belong to A. Finally, we may divide

[1] Although this estimate can be deduced as a consequence of [1], that result does not provide computable constants, as the proof is based on a contradiction argument relying on compactness.

[2] One way to define a suitable approximation is to consider a sequence of finite sets $V_k \subset V_{k+1} \subset A$ such that the polyhedra $P_k = \mathrm{co}(V_k)$ satisfy $|P_k| \to |\mathrm{co}(A)|$ as $k \to \infty$ and a sequence of compact subsets $A'_k \subset A$ such that $|A'_k| \to |A|$. Then let $A_k := V_k \cup [A'_k \cap (1 - 1/k)P_k]$. Since $|A_k| \to |A|$, it suffices to prove the estimate of Theorem 1.4 for A_k and then let $k \to \infty$.

each face into simplices without adding any vertices, so that ∂K can be seen as a finite union of simplices, all of whose vertices belong to A.

Reduction to a set A that contains $(1-C\mu^{1/n})K$. We get started with the proof by showing that points of K that are sufficiently far from the boundary of K are in $\frac{1}{2}(A+A)$. Indeed, since $\|f * g\|_{L^\infty} \leq \|f\|_{L^\infty}\|g\|_{L^1}$ for any pair of functions f and g,

$$\begin{aligned} |\chi_{K/2} * \chi_{K/2}(x) - \chi_{A/2} * \chi_{A/2}(x)| &\leq |\chi_{A/2} * (\chi_{K/2} - \chi_{A/2})|(x) \\ &\quad + |\chi_{A/2} * (\chi_{A/2} - \chi_{A/2})|(x) \\ &\leq 2\|\chi_{(K\setminus A)/2}\|_{L^1} = 2^{1-n}|K \setminus A| \\ &\leq |K \setminus A| = \mu \qquad \forall\, x \in \mathbb{R}^n. \end{aligned} \tag{2.3}$$

Because K satisfies (2.2), there is a dimensional constant $\hat{c} > 0$ such that

$$\chi_{K/2} * \chi_{K/2}(x) \geq \hat{c}\,\mathrm{dist}(x, \partial K)^n \qquad \forall\, x \in K, \tag{2.4}$$

therefore

$$\{x \in K \,:\, \hat{c}\,\mathrm{dist}(x, \partial K)^n > \mu\} \subset \{\chi_{K/2} * \chi_{K/2} > \mu\}. \tag{2.5}$$

Since

$$\begin{aligned} 0 < \chi_{A/2} * \chi_{A/2}(x) = \int_{\mathbb{R}^n} &\chi_{A/2}(y)\chi_{A/2}(x-y)\,dy \\ \Rightarrow \quad &\exists\, y \in A \text{ s.t. } y \in A/2,\ x - y \in A/2 \\ \Rightarrow \quad &x \in \tfrac{1}{2}(A+A), \end{aligned} \tag{2.6}$$

it follows from (2.3) and (2.5) that

$$(1-\hat{C}\mu^{1/n})K \subset \{\hat{c}\,\mathrm{dist}(\cdot, \partial K)^n > \mu\} \subset \{\chi_{K/2} * \chi_{K/2} > \mu\} \subset \tfrac{1}{2}(A+A)$$

for some dimensional constant $\hat{C}$. Consequently,

$$\big|[(1-\hat{C}\mu^{1/n})K] \setminus A\big| \leq \delta(A). \tag{2.7}$$

Denote

$$\rho := 2\hat{C}\mu^{1/n}, \quad A' := [(1-\rho)K] \cup A.$$

Then, since $A \subset K$ and

$$\max\big\{\tfrac{1}{2}(1-\rho) + \tfrac{1}{2}, 1-\rho\big\} = 1 - \rho/2,$$

we have

$$\begin{aligned} \tfrac{1}{2}(A' + A') &= \big[\tfrac{1}{2}(A+A)\big] \cup \big[\tfrac{1}{2}\big((1-\rho)K + A\big)\big] \cup (1-\rho)K \\ &\subset \big[\tfrac{1}{2}(A+A)\big] \cup \big[\tfrac{1}{2}\big((1-\rho)K + K\big)\big] \cup (1-\rho)K \\ &= \big[\tfrac{1}{2}(A+A)\big] \cup (1-\rho/2)K. \end{aligned}$$

Therefore, since $\rho/2 = \hat{C}\mu^{1/n}$, thanks to (2.7) we get

$$\delta(A') \leq \delta(A) + \big|[(1-\rho/2)K] \setminus A\big| \leq 2\delta(A).$$

Also, again by (2.7),

$$|K \setminus A| \le |K \setminus A'| + \delta(A).$$

Since $\mathrm{co}(A') = K$, if we prove the theorem with A' in place of A, then the result for A will follow immediately. Thus, after replacing A with A', we can assume that

$$A \supset (1-\rho)K \qquad \text{with} \quad \rho := 2\hat{C}\mu^{1/n}. \tag{2.8}$$

Recall that, by choosing d_n small enough, we can ensure that μ (and hence ρ) is arbitrarily small.

Splitting A into "simpler" sets. Denote by $\{\Sigma_i\}_{i=1}^M$ the simplices whose union is ∂K, let K_i be the convex hull of Σ_i with the origin, and define

$$A_i := A \cap K_i.$$

Note that (2.8) implies

$$(1-\rho)K_i \subset A_i, \qquad \rho = 2\hat{C}\mu^{1/n} \ll 1. \tag{2.9}$$

Also,

$$\sum_i |K_i \setminus A_i| = |K \setminus A|. \tag{2.10}$$

Moreover, since the sets $\{K_i\}_{i=1}^M$ are convex and disjoint, also the sets $\left\{\frac{1}{2}(A_i + A_i)\right\}_{i=1}^M$ are disjoint, therefore

$$\sum_i \left|\tfrac{1}{2}(A_i + A_i)\right| = \left|\bigcup_i \tfrac{1}{2}(A_i + A_i)\right| \le \left|\tfrac{1}{2}(A + A)\right|.$$

Since $\sum_i |A_i| = |A|$, this proves that

$$\sum_i \delta(A_i) \le \delta(A). \tag{2.11}$$

Main Lemma and conclusion. Our main lemma is the following.

Lemma 2.1. *Let A_i, K_i, and ρ be as above. Then, for $n \le 3$, there exist dimensional constants $\bar{C}_n \ge 1$ and $\rho_n > 0$ such that*

$$|K_i \setminus A_i| \le \bar{C}_n \delta(A_i). \tag{2.12}$$

provided $\rho \le \rho_n$.

Assuming Lemma 2.1 has been proved, Theorem 1.4 follows immediately. Indeed, choosing d_n sufficiently small, it follows by [6, Theorem 1.2] and the definitions of ρ and μ (see (2.8) and (2.1)) that $\rho \le \rho_n$ provided $\delta(A) \le d_n$. Then, adding the inequalities (2.12), (2.10), and (2.11), we find

$$|K \setminus A| = \sum_i |K_i \setminus A_i| \le \bar{C}_n \sum_i \delta(A_i) \le \bar{C}_n\, \delta(A),$$

as desired. Thus, we are left with proving Lemma 2.1.

Proof of Lemma 2.1. We begin by writing the lemma in a different normalized form. Fix an index i. Since inequality (2.12) is invariant under affine transformations, we may take Σ_i to be an equilateral simplex of $(n-1)$-Hausdorff measure 1, centered on the x_n-axis and contained in the hyperplane $\{x_n = 0\}$. Moreover, we may move the vertex of K_i from the origin to the point $\bigl(0, \ldots, 0, \frac{1}{2\rho}\bigr)$, so that (2.8) implies that $K_i \cap \bigl\{x_n \geq \frac{1}{2}\bigr\} \subset A_i$. It suffices to prove (2.12) in this normalized situation.

To simplify the notation further, we remove the subscript i, renaming Σ_i, K_i, A_i, with the letters Σ, K, A, respectively. With these changes, we can rewrite Lemma 2.1 as follows. (Note that, in this new normalization, $|K|$ is comparable to $1/\rho$ and (2.2) is not satisfied anymore.) Here and in the sequel, $\mathcal{H}^s$ denotes the s-dimensional Hausdorff measure.

Lemma 2.2. *Let Σ be an equilateral $(n-1)$-simplex centered on the x_n-axis satisfying*

$$\mathcal{H}^{n-1}(\Sigma) = 1, \qquad \Sigma \subset \{x_n = 0\}.$$

Let K be the n-simplex with one vertex at $\bigl(0, \ldots, 0, \frac{1}{2\rho}\bigr)$ and base Σ. Suppose that A is a compact set satisfying

$$K \cap \bigl\{x_n \geq \tfrac{1}{2}\bigr\} \subset A \subset K,$$

and that all of the vertices of Σ belong to A. Then, for $n \leq 3$, there exist dimensional constants $\bar{C}_n \geq 1$ and $\rho_n > 0$ such that

$$|K \setminus A| \leq \bar{C}_n \, \delta(A)$$

provided $\rho \leq \rho_n$.

Proof of Lemma 2.2. The rough idea of the proof is to start with the set

$$K \cap \{1 \leq x_n \leq 2\} \subset A$$

and use the fact that the vertices of Σ belong to A in order to apply the sum operation repeatedly to generate more points of A up to errors estimated by $\delta(A)$. As we shall see, a more refined argument involving several steps will be needed. The first five steps, proving (2.15), are valid in all dimensions, but the sixth step is restricted to dimensions 2 and 3.

Step 1: *Setting up an iteration.* Let $\epsilon > 0$ be a small dimensional constant to be fixed later, set $\gamma := \frac{1}{2} + \epsilon$, and define

$$K_j := K \cap \{\gamma^j \leq x_n \leq 2\gamma^j\} \qquad \forall\, j \geq 0. \tag{2.13}$$

Note that, with this definition, consecutive sets K_j are not completely disjoint but rather overlap in a fraction of order ϵ of their total volume.

Let the vertices of Σ be denoted by $\{\hat{x}_k\}_{k=1}^n$. We define the following sets iteratively:

$$E_0 := K_0, \qquad E_{j+1} := K_{j+1} \cap \left(\bigcup_{k=1}^n \tfrac{1}{2}(\hat{x}_k + E_j) \cup E_j \cup (1-\epsilon)K \right). \tag{2.14}$$

Here $(1-\epsilon)K$ denotes the dilation of K with respect to the origin, namely the n-simplex with one vertex at $\big(0,\dots,0,\frac{1-\epsilon}{2\rho}\big)$ and base $(1-\epsilon)\Sigma$. We note that $E_j = K_j$ when $n=2$.

Set $E := \cup_{j\geq 0} E_j$. We claim that there exists a dimensional constant C_0 such that

$$|E \setminus A| \leq C_0\, \delta(A). \tag{2.15}$$

The proof of this claim will be carried out in Steps 2–5 below.

Step 2: *Setting the notation.* Define the numbers

$$\nu_j := |E_j \setminus A|, \qquad \delta_j := \Big|\big(\big[\tfrac{1}{2}(A+A)\big] \setminus A\big) \cap K_j\Big|, \tag{2.16}$$

and

$$\sigma_j := \big|\big[(1-\epsilon)K \cap K_j \cap K_{j+1}\big] \setminus A\big|. \tag{2.17}$$

Note that

$$|K_j| \geq |E_j| \geq |(1-\epsilon)K \cap \{\gamma^j \leq x_n \leq 2\gamma^j\}| = (1-\epsilon)^{n-1}|K_j|. \tag{2.18}$$

We claim that there exist dimensional constants $M, N \geq 1$, with N integer, such that

$$\nu_{j+1} \leq \frac{8}{9}\nu_j + \sigma_j + M\sum_{i=0}^N \delta_{j+i} \qquad \forall\, j \geq 0. \tag{2.19}$$

The proof of (2.19) will be split over Step 3 and Steps 4(a)-4(e) below.

Step 3: *The case ν_j large.* Consider first the case

$$\nu_j \geq \frac{2}{3}|E_j|. \tag{2.20}$$

Note that, for $\rho \ll 1$, the sets K_j are almost vertical cylinders of height γ^j, and more precisely (recalling that $\mathcal{H}^{n-1}(\Sigma) = \mathcal{H}^{n-1}(K \cap \{x_n = 0\}) = 1$)

$$\gamma^j \geq |K_j| \geq (1 - C\rho)\gamma^j, \tag{2.21}$$

where $C > 0$ is a dimensional constant. This implies that $|K_{j+1}| = \big(1 + O(\rho)\big)\gamma|K_j|$, so it follows by (2.18) that

$$\nu_j \geq \frac{2}{3}|E_j| \geq \frac{2}{3}(1-\epsilon)^{n-1}|K_j| = \frac{2(1-\epsilon)^{n-1}}{3\gamma\big(1+O(\rho)\big)}|K_{j+1}| \geq \frac{2(1-\epsilon)^{n-1}}{3\gamma\big(1+O(\rho)\big)}\nu_{j+1},$$

which proves (2.19) because $\frac{3\gamma}{2(1-\epsilon)^{n-1}}\big(1+O(\rho)\big) \leq \frac{8}{9}$ provided ϵ and ρ are sufficiently small.

Step 4: *The case ν_j not too large.* We now consider the case

$$\nu_j \leq \frac{2}{3}|E_j|. \tag{2.22}$$

Step 4(a): *Finding some nontrivial fraction of A near the vertices.* Using (2.18), it follows that

$$|A\cap K_j| \geq |A\cap E_j| = |E_j| - \nu_j \geq \frac{1}{3}|E_j| \geq \frac{1}{3}(1-\epsilon)^{n-1}|K_j| \geq \frac{1}{4}|K_j|. \tag{2.23}$$

Now, for any $k=1,\ldots,n$, consider the sets

$$A^k_{j,\ell} := (1-2^{-\ell})\hat{x}_k + 2^{-\ell}(A\cap K_j) \qquad \forall\, \ell \geq 0,$$

and note that, because of (2.23),

$$|A^k_{j,\ell}| = 2^{-n\ell}|A\cap K_j| \geq 2^{-n\ell-2}|K_j|. \tag{2.24}$$

Our goal is to show that, provided the numbers δ_{j+i} are small enough for sufficiently many indices i, then $A^k_{j,\ell}\cap A$ has almost the same measure as $A^k_{j,\ell}$. To prove this, for convenience we define the auxiliary numbers

$$\delta^k_{j,\ell} := \Big|\Big(\big[\tfrac12(A+A)\big]\setminus A\Big)\cap A^k_{j,\ell}\Big| \qquad \forall\, \ell \geq 0.$$

Also, we iteratively define

$$B^k_{j,0} := A\cap K_j, \qquad B^k_{j,\ell+1} := \tfrac12\big(\hat{x}_k + (A\cap B^k_{j,\ell})\big).$$

Since $\hat{x}_k\in A$, one can easily see by induction on ℓ that the following inclusion holds:

$$\tfrac12\big(\hat{x}_k + (A\cap B^k_{j,\ell})\big) \subset \big[\tfrac12(A+A)\big]\cap A^k_{j,\ell+1} \qquad \forall\, \ell\geq 0.$$

Therefore

$$|A\cap B^k_{j,\ell+1}| \geq \big|\tfrac12\big(\hat{x}_k + A\cap B^k_{j,\ell}\big)\big| - \delta^k_{j,\ell+1} = \frac{1}{2^n}|A\cap B^k_{j,\ell}| - \delta^k_{j,\ell+1} \qquad \forall\, \ell\geq 0.$$

Since $|A\cap B^k_{j,0}| = |A\cap K_j| = 2^{n\ell}|A^k_{j,\ell}|$ and $A\cap B^k_{j,\ell}\subset A^k_{j,\ell}$, we deduce that

$$|A\cap A^k_{j,\ell}| \geq |A\cap B^k_{j,\ell}| \geq |A^k_{j,\ell}| - \sum_{r=1}^{\ell}\delta^k_{j,r} \qquad \forall\, \ell\geq 1. \tag{2.25}$$

We now start to fix some parameters. Choose an integer m such that

$$\epsilon^2 \leq 2^{-m} \leq 2\epsilon^2, \tag{2.26}$$

and then choose N large enough so that

$$2^{-(m+1)} \leq \gamma^N \leq 2^{-m}. \tag{2.27}$$

With these definitions, it follows that $\cup_{r=1}^m A_{j,r}^k \subset \cup_{i=0}^N K_{j+i}$. Therefore, since the sets $\{A_{j,r}^k\}_{1\le r\le m}$ are disjoint, it follows that

$$\sum_{r=1}^m \delta_{j,r}^k \le \sum_{i=1}^N \delta_{j+i}.$$

Hence, by (2.25) applied with $\ell = m$, we get

$$|A\cap A_{j,m}^k| \ge |A_{j,m}^k| - \sum_{i=1}^N \delta_{j+i}. \tag{2.28}$$

We are now ready to prove (2.19). Consider first the case in which

$$\sum_{i=1}^N \delta_{j+i} \ge \epsilon |A_{j,m}^k|.$$

Then, since $\nu_{j+1} \le |K_{j+1}| \le |K_j|$ for ρ small enough (see (2.21)), recalling (2.24) and that $\gamma^{-N} \le 2^{-m}$ (see (2.27)), we deduce that

$$\sum_{i=1}^N \delta_{j+i} \ge \frac{\epsilon}{4}\gamma^{nN}|K_j| \ge \frac{\epsilon}{4}\gamma^{nN}\nu_{j+1},$$

so (2.19) follows immediately with $M = 4\gamma^{-nN}\epsilon^{-1}$.

Next, we must consider the case in which $\sum_{i=1}^N \delta_{j+i} \le \epsilon|A_{j,m}^k|$. In that case, (2.28) gives

$$|A\cap A_{j,m}^k| \ge (1-\epsilon)|A_{j,m}^k| \qquad \forall\, k=1,\dots,n. \tag{2.29}$$

In other words, we proved that A covers almost all the sets $\{A_{j,m}^k\}_{k=1}^n$, which are small rescaled copies of $A\cap K_j$ that live in a ϵ^2 neighborhood of the n vertices $\hat{x}_k$ (recall (2.26)).

Note that whereas the sets $A_{j,m}^k$ for different k are translates of each other, the sets $A\cap A_{j,m}^k$ are not. To enforce this additional property, we first translate them to the same point, intersect them, and then move them back. More precisely, we set

$$\hat{A}_{j,m} := \bigcap_{k=1}^n \big((A\cap A_{j,m}^k) - (1-2^{-m})\hat{x}_k\big), \qquad \hat{A}_{j,m}^k := \hat{A}_{j,m} + (1-2^{-m})\hat{x}_k.$$

Now, thanks to (2.29),

$$\hat{A}_{j,m}^k \subset A\cap A_{j,m}^k, \qquad |\hat{A}_{j,m}^k| \ge (1-n\epsilon)|A_{j,m}^k| \qquad \forall\, k=1,\dots,n, \tag{2.30}$$

and $\hat{A}_{j,m}^k$ and $\hat{A}_{j,m}^{k'}$ are the same set for any $k,k'\in\{1,\dots,n\}$, up to a translation orthogonal to the x_n axis. Also, it follows by (2.30), (2.24), and (2.21), that

$$\begin{aligned} |\hat{A}_{j,m}^k| &\le |A_{j,m}^k| = 2^{-nm}|A\cap K_j| \le 2^{-nm}|K_j| \le 2^{-nm}\gamma^j,\\ |\hat{A}_{j,m}^k| &\ge (1-n\epsilon)|A_{j,m}^k| \ge (1-n\epsilon)2^{-nm-2}|K_j| \ge 2^{-nm-3}\gamma^j, \end{aligned} \tag{2.31}$$

provided ϵ and ρ are sufficiently small.

Step 4(b): *Finding an almost full slice in A near $\{x_n = 0\}$ using Fubini and induction.* We look at the slab

$$S_{j,m} := K \cap \{2^{-m}\gamma^j \le x_n \le 2^{-m+1}\gamma^j\},$$

and define $\delta_{j,m} := \left|\left(\left[\frac{1}{2}(A+A)\right] \setminus A\right) \cap S_{j,m}\right|$. Note that $A^k_{j,m} \subset S_{j,m}$ for any $k = 1, \dots, n$.

Recall that d_{n-1} is the dimensional constant corresponding to Theorem 1.4 in dimension $n-1$. It will suffice to prove the existence of suitable slice inside $S_{j,m}$ assuming

$$\delta_{j,m} \le \epsilon^{2n+8} d_{n-1} |\hat{A}^k_{j,m}| \tag{2.32}$$

(note that $|\hat{A}^k_{j,m}|$ is independent of k). Indeed, if not, since $S_{j,k} \subset K_{j+N-1} \cup K_{j+N}$ it holds

$$\delta_{j,m} \le \delta_{j+N-1} + \delta_{j+N}. \tag{2.33}$$

Hence, if (2.32) fails then (recall (2.31) and (2.27))

$$\delta_{j+N-1} + \delta_{j+N} \ge \epsilon^{2n+8} d_{n-1} |\hat{A}^k_{j,m}| \ge \epsilon^{2n+8} d_{n-1} (1 - n\epsilon) 2^{-nm-2} |K_j| \ge \frac{\epsilon^{2n+8} d_{n-1} \gamma^{nN}}{8} \nu_{j+1},$$

which proves (2.19) with $M = 8\gamma^{-nN}\epsilon^{-2n-8} d_{n-1}^{-1}$.

Now we can proceed under the additional assumption (2.32). Define

$$A_t := A \cap \{x_n = t\} \supset \left(\cup_{k=1}^n (\hat{A}^k_{j,m})\right) \cap \{x_n = t\} =: \hat{A}_t,$$

and consider $\delta(A_t) = \mathcal{H}^{n-1}\left(\frac{1}{2}(A_t + A_t) \setminus A_t\right)$. Since $\hat{A}^k_{j,m} \subset A$, it follows by (2.32) and (2.31) that

$$\int_{2^{-m}\gamma^j}^{2^{-m+1}\gamma^j} \delta(A_t)\, dt \le \delta_{j,m} \le \epsilon^{2n+8} d_{n-1} |\hat{A}^k_{j,m}| \le \epsilon^{2n+8} d_{n-1} 2^{-nm} \gamma^j. \tag{2.34}$$

Also, recalling (2.31), it follows that

$$\frac{1}{2^{-m}\gamma^j} \int_{2^{-m}\gamma^j}^{2^{-m+1}\gamma^j} \mathcal{H}^{n-1}(\hat{A}_t)\, dt \ge \frac{1}{2^{-m}\gamma^j} \sum_{k=1}^n |\hat{A}^k_{j,m}| \ge n 2^{-(n+1)m-3}.$$

Hence, since $\mathcal{H}^{n-1}(\hat{A}_t) \le \mathcal{H}^{n-1}(A_t) \le 1$, there exists a set $J \subset [2^{-m}\gamma^j, 2^{-m+1}\gamma^j]$ such that[3]

$$\mathcal{H}^1(J) \ge n2^{-(n+2)m-4}\gamma^j, \quad \text{with} \quad \mathcal{H}^{n-1}(\hat{A}_t) \ge n2^{-(n+1)m-4} \quad \forall\, t \in J.$$

Combining this estimate with (2.34), we deduce that

$$\begin{aligned}\frac{1}{\mathcal{H}^1(J)}\int_J \delta(A_t) &\le \frac{\delta_{j,m}}{n2^{-(n+2)m-4}\gamma^j} \le \frac{\epsilon^{2n+8}d_{n-1}2^{-nm}\gamma^j}{n2^{-(n+2)m-4}\gamma^j}\\ &\le \frac{\epsilon^{2n+8}d_{n-1}2^{(n+3)m+8}}{n^2}\mathcal{H}^{n-1}(\hat{A}_t) \le \frac{\epsilon^{2n+8}d_{n-1}2^{(n+3)m+8}}{n^2}\mathcal{H}^{n-1}(A_t) \qquad \forall\, t \in J.\end{aligned}$$

Recalling (2.26), this proves that

$$\frac{1}{\mathcal{H}^1(J)}\int_J \delta(A_t) \le \frac{2^{n+6}}{n\epsilon^{n+2}\gamma^j}\delta_{j,m} \le \frac{2^{n+11}}{n^2}\epsilon^2 d_{n-1}\mathcal{H}^{n-1}(\hat{A}_t) \qquad \forall\, t \in J.$$

In particular, choosing ϵ sufficiently small, by the Mean Value Theorem we can find $t \in [2^{-m}\gamma^j, 2^{1-m}\gamma^j]$ such that

$$\delta(A_t) \le \frac{2^{n+6}}{n\epsilon^{n+2}\gamma^j}\delta_{j,m} \le \epsilon^{3/2}d_{n-1}\mathcal{H}^{n-1}(\hat{A}_t), \qquad \mathcal{H}^{n-1}(\hat{A}_t) > 0.$$

Hence, since $\mathcal{H}^{n-1}(\hat{A}_t) \le \mathcal{H}^{n-1}(A_t)$, we can apply Theorem 1.4 to A_t and we deduce that

$$\mathcal{H}^{n-1}(\mathrm{co}(A_t) \setminus A_t) \le C_{n-1}\delta(A_t) \le C_{n-1}\frac{2^{n+6}}{n\epsilon^{n+2}\gamma^j}\delta_{j,m} \le C_{n-1}\epsilon^{3/2}d_{n-1}. \tag{2.35}$$

Also, because $\mathcal{H}^{n-1}(\hat{A}_t) > 0$, it follows that $\mathrm{co}(A_t)$ contains at least one point in $\hat{A}^k_{j,m} \cap \{x_n = t\}$ for any $k = 1, \ldots, n$. Recalling that $\hat{A}^k_{j,m} \subset (1-2^{-m})\hat{x}_k + 2^{-m}K_j$ and that $2^{-m} \le 2\epsilon^2$ (see (2.26)), it follows that $\mathrm{co}(A_t)$ contains n points $\{\hat{x}^k_t\}_{k=1}^n$ such that $|\hat{x}^k_t - \hat{x}^k| \le C\epsilon^2$, thus

$$\mathrm{co}(A_t) \supset \big((1-\epsilon)K\big) \cap \{x_n = t\}. \tag{2.36}$$

In the next steps we use the slice A_t and semisum to control a large fraction of ν_{j+1}. Because the argument in dimension $n = 2$ is much easier than in higher dimensions, for convenience of the reader we first treat this case.

Step 4(c): *Use the slice from Step 4(b) and semisum to control a large fraction of ν_{j+1}: the case $n = 2$.* Thanks to (2.35) and (2.36), there exists a point $z = (z_1, t) \in A \subset \mathbb{R}^2$

[3] This estimate follows by the following general simple fact: If $f : I \subset \mathbb{R} \to [0,1]$ satisfies $\frac{1}{\mathcal{H}^1(I)}\int_I f(t)\,dt \ge \eta > 0$, then there exists $J \subset I$ such that

$$\mathcal{H}^1(J) \ge \frac{\eta}{2}\mathcal{H}^1(I) \qquad \text{and} \qquad f(t) \ge \frac{\eta}{2} \quad \forall\, t \in J.$$

Indeed, if this was false, we would have that $f \le \eta/2$ on a set $I' \subset I$ of measure larger than $(1-\eta/2)\mathcal{H}^1(I)$, therefore (recall that $0 \le f \le 1$)

$$\int_I f(t)\,dt \le \int_{I'} f(t)\,dt + \int_{I\setminus I'} f(t)\,dt \le \mathcal{H}^1(I')\frac{\eta}{2} + \mathcal{H}^1(I \setminus I') \le \Big(1 - \frac{\eta}{2}\Big)\frac{\eta}{2} + \frac{\eta}{2} < \eta,$$

a contradiction.

with $|z_1| \le C\epsilon^{3/4}$. In particular, recalling that $t \sim \epsilon^2\gamma^j$ and that $\gamma = \frac{1}{2} + \epsilon$, we have, for ϵ sufficiently small,

$$\left(\tfrac{1}{4}K\right) \cap (K_{j+1} \setminus K_j) \subset \left(\tfrac{1}{4}K\right) \cap \left\{\tfrac{\gamma^j+t}{2} \le x_2 \le \tfrac{2\gamma^j+t}{2}\right\} \subset \tfrac{1}{2}(z + K_j) \subset K_{j+1} \cup K_{j+2}$$

where $\frac{1}{4}K$ denotes the dilation of K by a factor $\frac{1}{4}$ with respect to the origin. Finally, since $K_j = E_j$ for $n = 2$, the definition of ν_j and δ_j (see (2.16)) yields

$$\begin{aligned}\left|\left[\left(\tfrac{1}{4}K\right) \cap (K_{j+1} \setminus K_j)\right] \setminus A\right| &\le \left|\tfrac{1}{2}\big(z + (K_j \setminus A)\big)\right| + \delta_{j+1} + \delta_{j+2} \\ &\le \frac{1}{4}\nu_j + \delta_{j+1} + \delta_{j+2}. \quad (2.37)\end{aligned}$$

Step 4(d): *Use the slice from Step 4(b) and semisum to control a large fraction of ν_{j+1}: the case $n \ge 3$.* Given $s \ge 0$, define $K_{s,\epsilon} := \big((1-\epsilon)K\big) \cap \{x_n = s\}$ and $A_{s,\epsilon} := A \cap K_{s,\varepsilon}$. Then

$$\tfrac{1}{2}\big(A_{s,\epsilon} + A_{t,\epsilon}\big) \setminus A_{\frac{s+t}{2},\epsilon} \subset \left[\tfrac{1}{2}(A + A) \setminus A\right] \cap \big((1-\epsilon)K\big) \cap \left\{x_n = \tfrac{s+t}{2}\right\}.$$

Using the above equation for $s \in [\gamma^j, 2\gamma^j]$, and noticing that

$$K_{j+1} \setminus K_j \subset K \cap \left\{\tfrac{\gamma^j+t}{2} \le x_n \le \tfrac{2\gamma^j+t}{2}\right\} \subset K_{j+1} \cup K_{j+2}$$

for ϵ sufficiently small, we get

$$\begin{aligned}\left|\left[(1-\epsilon)K \cap (K_{j+1} \setminus K_j)\right] \setminus A\right| &\le \left|(1-\epsilon)K \cap \left\{\tfrac{\gamma^j+t}{2} \le x_n \le \tfrac{2\gamma^j+t}{2}\right\} \setminus A\right| \\ &\le \left|(1-\epsilon)K \cap \left\{\tfrac{\gamma^j+t}{2} \le x_n \le \tfrac{2\gamma^j+t}{2}\right\} \setminus \tfrac{1}{2}(A+A)\right| \\ &\quad + \left|\left(\left[\tfrac{1}{2}(A+A)\right] \setminus A\right) \cap \left\{\tfrac{\gamma^j+t}{2} \le x_n \le \tfrac{2\gamma^j+t}{2}\right\}\right| \\ &\le \int_{\frac{\gamma^j+t}{2}}^{\frac{2\gamma^j+t}{2}} \mathcal{H}^{n-1}\big(K_{\tau,\epsilon} \setminus \tfrac{1}{2}(A+A)\big)\, d\tau + \delta_{j+1} + \delta_{j+2} \\ &= \frac{1}{2}\int_{\gamma^j}^{2\gamma^j} \mathcal{H}^{n-1}\big(K_{\frac{s+t}{2},\epsilon} \setminus \tfrac{1}{2}(A+A)\big)\, ds + \delta_{j+1} + \delta_{j+2} \\ &\le \frac{1}{2}\int_{\gamma^j}^{2\gamma^j} \mathcal{H}^{n-1}\big(K_{\frac{s+t}{2},\epsilon} \setminus \tfrac{1}{2}(A_{s,\epsilon}+A_{t,\epsilon})\big)\, ds + \delta_{j+1} + \delta_{j+2}.\end{aligned}$$

Define the "vertical" semisum of two sets F_s and F_t contained respectively in two levels $\{x_n = s\}$ and $\{x_n = t\}$ by

$$\tfrac{1}{2}(F_s +_v F_t) := \left\{\tfrac{1}{2}(z + w, s + t) : (z, s) \in F_s,\ (w, t) \in F_t,\ (1 - 2\rho s)w = (1 - 2\rho t)z\right\}.$$

Note that if $\rho = 0$ this is just the semisum in the vertical variable (since in that case $z = w$). In our case, since K is not quite a vertical cylinder but instead has a small angle

2ρ, we are asking that the points (z,s) and (w,t) be collinear with the vertex $\big(0,\frac{1}{2\rho}\big)$ of K. One can easily check that

$$\mathcal{H}^{n-1}\big(K_{\frac{s+t}{2},\epsilon}\setminus\tfrac{1}{2}(A_{s,\epsilon}+_v A_{t,\epsilon})\big)\le\big(1+O(\rho)\big)\Big(\mathcal{H}^{n-1}(K_{s,\epsilon}\setminus A_{s,\epsilon})+\mathcal{H}^{n-1}(K_{t,\epsilon}\setminus A_{t,\epsilon})\Big).$$

Indeed, the vertical semisum can be viewed as a semisum of suitable $1+O(\rho)$ dilates of $(n-1)$-dimensional sets and the inequality follows from the $(n-1)$-dimensional Brunn-Minkowski inequality. Also, we observe that

$$\tfrac{1}{2}(A_{s,\epsilon}+A_{t,\epsilon})\supset\tfrac{1}{2}(A_{s,\epsilon}+_v A_{t,\epsilon}).$$

Combining together all these bounds, and recalling (2.35), (2.36), and (2.33), we get

$$\begin{aligned}\big|\big[(1-\epsilon)K\cap(K_{j+1}\setminus K_j)\big]\setminus A\big| &\le\frac{1+O(\rho)}{2}\int_{\gamma^j}^{2\gamma^j}\mathcal{H}^{n-1}\big(K_{s,\epsilon}\setminus A_{s,\epsilon}\big)\,ds\\ &\quad+\frac{1+O(\rho)}{2}\int_{\gamma^j}^{2\gamma^j}\mathcal{H}^{n-1}\big(K_{t,\epsilon}\setminus A_{t,\epsilon}\big)\,ds+\delta_{j+1}+\delta_{j+2}\\ &\le\frac{1+O(\rho)}{2}\big|\big((1-\epsilon)K\big)\cap K_j\big]\setminus A\big|\\ &\quad+\frac{1+O(\rho)}{2}\int_{\gamma^j}^{2\gamma^j}C_{n-1}\frac{2^{n+6}}{n\epsilon^{n+2}\gamma^j}\delta_{j,m}\,ds+\delta_{j+1}+\delta_{j+2}\\ &\le\frac{1+O(\rho)}{2}\big|\big((1-\epsilon)K\big)\cap K_j\big]\setminus A\big|\\ &\quad+C_{n-1}\frac{2^{n+6}}{n\epsilon^{n+2}}(\delta_{j+N-1}+\delta_{j+N})+\delta_{j+1}+\delta_{j+2}\end{aligned}$$

Recalling the definitions of E_j, ν_j, and σ_j (see (2.14), (2.16), and (2.17)), this proves that

$$\big|\big[(1-\epsilon)K\cap K_{j+1}\big]\setminus A\big|\le\frac{1+O(\rho)}{2}\nu_j+\sigma_j+C_{n-1}\frac{2^{n+6}}{n\epsilon^{n+2}}(\delta_{j+N-1}+\delta_{j+N})+\delta_{j+1}+\delta_{j+2}. \tag{2.38}$$

Step 4(e): *Use semisum to control the remaining fraction of* ν_{j+1}. Since $\hat{x}_k\in A$, we see that

$$\Big(\bigcup_{k=1}^n\tfrac{1}{2}(\hat{x}_k+E_j)\Big)\setminus\tfrac{1}{2}(A+A)\subset\Big(\bigcup_{k=1}^n\tfrac{1}{2}(\hat{x}_k+(E_j\setminus A))\Big),$$

therefore, since $\cup_{k=1}^n\frac{1}{2}(\hat{x}_k+E_j)\subset K_{j+1}$, recalling the definition of ν_j and δ_j (see (2.16)) we get

$$\Big|\Big(\bigcup_{k=1}^n\tfrac{1}{2}(\hat{x}_k+E_j)\Big)\setminus A\Big|\le\frac{n}{2^n}|E_j\setminus A|+\delta_{j+1}=\frac{n}{2^n}\nu_j+\delta_{j+1}. \tag{2.39}$$

Noticing that for $n = 2$ we have

$$E_j = K_j \qquad \text{and} \qquad \left(\bigcup_{k=1}^{2} \tfrac{1}{2}(\hat{x}_k + K_j)\right) \cup \left[\left(\tfrac{1}{4}K\right) \cap K_{j+1}\right] \supset K_{j+1},$$

combining (2.39) with (2.37) and (2.38), and noticing that $n2^{-n} \leq 3/8$ for $n \geq 3$, we obtain

$$\nu_{j+1} \leq \left(\frac{1+O(\rho)}{2} + \frac{3}{8}\right)\nu_j + \sigma_j + M\sum_{i=0}^{N} \delta_{j+i}$$

for some dimensional constant M, concluding the proof of (2.19).

Step 5: *Proof of* (2.15). Since $\nu_0 = 0$ (because $K_0 \subset A$ by assumption), by summing (2.19) with respect to j we obtain

$$\sum_{j\geq 0} \nu_j \leq \frac{8}{9}\left(\sum_{j\geq 0} \nu_j\right) + \sum_{j\geq 0} \sigma_j + M\sum_{j\geq 1}\sum_{i=0}^{N} \delta_{j+i}.$$

Moreover, the last term can be bounded by

$$MN\sum_{j\geq 0} \delta_j = MN\sum_{j\geq 0} \delta_{2j} + MN\sum_{j\geq 0} \delta_{2j+1}.$$

Noticing that the sets $\{K_{2j}\}_{j\geq 0}$ and the sets $\{K_{2j+1}\}_{j\geq 0}$ are disjoint, it follows that

$$\sum_{j\geq 0} \delta_{2j} \leq \delta(A), \qquad \sum_{j\geq 0} \delta_{2j+1} \leq \delta(A).$$

Hence, combining these estimates together, we proved that

$$\frac{1}{9}\left(\sum_{j\geq 0} \nu_j\right) \leq \sum_{j\geq 0} \sigma_j + 2MN\,\delta(A).$$

Since $\sum_{j\geq 0} \nu_j \geq |E \setminus A|$, we get

$$\frac{1}{9}|E \setminus A| \leq \sum_{j\geq 0} \sigma_j + 2MN\,\delta(A).$$

Note that this would prove (2.15) if we did not have the additional term $\sum_{j\geq 0} \sigma_j$. The idea to get rid of this additional term is the following: since the volume of $K_j \cap K_{j+1}$ is only a fraction ϵ of the volume of K_j and K_{j+1}, if A were uniformly distributed inside the sets K_j, then we would have

$$\sigma_j \leq C\epsilon(\nu_j + \nu_{j+1}),$$

from which we would conclude easily. Although A need not be uniformly distributed, we can prove analogous inequalities starting our iteration at many levels, and then add them up so that the average overlap of A with $K_j \cap K_{j+1}$ is sufficiently uniform.

Thus, to handle the terms σ_j, we take $\tau \in [\gamma, 1]$ and define the sets

$$K_j^\tau := K \cap \{\tau\gamma^j \le x_n \le 2\tau\gamma^j\}$$

$$E_0^\tau := K_0^\tau, \qquad E_{j+1}^\tau := \left(\bigcup_{k=1}^n \tfrac{1}{2}(\hat{x}_k + E_j^\tau)\right) \cup \big((1-\epsilon)K \cap \{-2\tau\gamma^j \le x_n \le -\tau\gamma^j\}\big),$$

and $E^\tau := \cup_{j\ge 0} E_j^\tau$, and the numbers

$$\nu_j^\tau := |E_j^\tau \setminus A|, \qquad \delta_j^\tau := \Big|\big(\big[\tfrac{1}{2}(A+A)\big] \setminus A\big) \cap K_j^\tau\Big|,$$

and

$$\sigma_j^\tau := \big|\big[(1-\epsilon)K \cap K_j^\tau \cap K_{j+1}^\tau\big] \setminus A\big|.$$

Now, if we repeat the very same proof as above with these new sets, we obtain

$$\frac{1}{9}|E^\tau \setminus A| \le \sum_{j\ge 0} \sigma_j^\tau + 2MN\,\delta(A)$$

(note that we still have $K_0^\tau \subset A$, therefore $\nu_j^\tau = 0$). Noticing that $E = E^1 \subset E^\tau$ for all $\tau \in (\gamma, 1)$ (in other words, the sets E^τ are monotonically decreasing in τ), this proves that

$$\frac{1}{9}|E \setminus A| \le \sum_{j\ge 0} \sigma_j^\tau + 2MN\,\delta(A). \tag{2.40}$$

We now observe that, since $\gamma = \frac{1}{2} + \epsilon$,

$$K_j^\tau \cap K_{j+1}^\tau = K \cap \{\tau\gamma^{j+1} \le x_n \le 2\tau\gamma^j\} = K \cap \{\tau\gamma^j \le x_n \le (1+2\epsilon)\tau\gamma^j\},$$

hence the sets

$$\big\{K_j^{\tau_m} \cap K_{j+1}^{\tau_m} \,:\, j \ge 0,\, \tau_m = 1 - 2m\epsilon,\, m = 0, \ldots, \lfloor \tfrac{\gamma}{4\epsilon} \rfloor\big\}$$

are disjoint. This implies that

$$\sum_{m=0}^{\lfloor \frac{\gamma}{4\epsilon} \rfloor} \sum_{j\ge 0} \sigma_j^{\tau_m} \le |E \setminus A|,$$

that combined with (2.40) gives

$$\lfloor \tfrac{\gamma}{4\epsilon} \rfloor\, |E \setminus A| \le 9 \sum_{m=0}^{\lfloor \frac{\gamma}{4\epsilon} \rfloor} \sum_{j\ge 0} \sigma_j^{\tau_m} + 18 \cdot MN \lfloor \tfrac{\gamma}{4\epsilon} \rfloor\, \delta(A) \le 9|E \setminus A| + 18 \cdot MN \lfloor \tfrac{\gamma}{4\epsilon} \rfloor\, \delta(A).$$

Choosing ϵ sufficiently small that $\lfloor \frac{\gamma}{4\epsilon} \rfloor \ge 10$ proves the desired result (2.15).

Step 6: *Getting control on A on all of K.* Note that (2.15) provides control on the measure of A inside E. In particular, since $E_j = K_j$ when $n = 2$, this already proves Lemma 2.1 (and therefore Theorem 1.4) in the case $n = 2$. Thus for the remainder of the proof we may assume $n = 3$. In this case, we will enlarge the set E on which we control the measure of A to all of K.

For $0 \le t < 1/2\rho$, set
$$\Sigma(t) = K \cap \{x_3 = t\}.$$
By hypothesis, $\Sigma(t) \cap A = \Sigma(t)$ for $t \ge 1/2$. Our approach to estimating $\Sigma(t) \setminus A$ for $0 \le t < 1/2$ will be to intersect $\Sigma(t) \setminus E$ with segments parallel to sides of the triangle $\Sigma(t)$ near the boundary and show that these missing parts are sufficiently small and atomized that we can apply the following one-dimensional lemma.

Lemma 2.3. *Let $J \subset \mathbb{R}$ be an interval. Suppose that $A \subset J$ and $E \subset J$, and*
$$\chi_{E/2} * \chi_{E/2}(x) \ge \frac{1}{10}\mathrm{dist}(x, \partial J) \quad \text{for all } x \in J. \tag{2.41}$$
Then
$$|J \setminus A| \le |\tfrac{1}{2}(A + A) \setminus A| + 20|E \setminus A|.$$

Proof. The proof of (2.3) applies with K replaced by E and shows that
$$|\chi_{E/2} * \chi_{E/2}(x) - \chi_{A/2} * \chi_{A/2}(x)| \le |E \setminus A|.$$
Therefore, if $x \in J$ and $\mathrm{dist}(x, \partial J) > 10\,|E \setminus A|$, we can use (2.41) to obtain
$$\chi_{A/2} * \chi_{A/2}(x) \ge \chi_{E/2} * \chi_{E/2}(x) - |E \setminus A| > |E \setminus A| - |E \setminus A| = 0,$$
thus $x \in \frac{1}{2}(A + A)$. Since
$$|\{x \in J \,:\, \mathrm{dist}(x, \partial J) > 10\,|E \setminus A|\}| \le 20\,|E \setminus A|,$$
it follows that $|J \setminus \frac{1}{2}(A + A)| \le 20\,|E \setminus A|$, and consequently
$$|J \setminus A| \le |\tfrac{1}{2}(A + A) \setminus A| + |J \setminus \tfrac{1}{2}(A + A)| \le |\tfrac{1}{2}(A + A) \setminus A| + 20\,|E \setminus A|.$$
□

To describe the complement of E in $\Sigma(t)$, we introduce several more notations. Recall that the vertices of $\Sigma = \Sigma(0)$ are $\hat{x}_i$, $i = 1, 2, 3$, so that the vertices of $\Sigma(t)$ are given by $\hat{x}_i(t) = (1 - 2\rho t)\hat{x}_i + (0, 0, \frac{1}{2\rho})$. Denote the sides of $\Sigma(t)$ by $\Sigma_i(t)$, with the convention that the endpoints of $\Sigma_1(t)$ are $\hat{x}_2(t)$ and $\hat{x}_3(t)$, and likewise for permutations of the indices. Since Σ has sidelength
$$s_0 := 2 \cdot 3^{-1/4},$$
the length of the sides of $\Sigma_i(t)$ is given by
$$s(t) := \mathcal{H}^1(\Sigma_i(t)), \qquad s(t) = (1 - 2\rho t)s_0.$$
Let $m \ge 1$ be such that $2^{-m} \le t < 2^{-m+1}$. We will define, iteratively, the set of open subintervals $I_{j,k}(t)$ of $\Sigma_1(t)$, with $j = 1, \dots m$ and $k = 1, \dots, 2^{j-1}$, whose union is the complement of E in $\Sigma_1(t)$. To begin, set
$$I_{1,1}(t) := \Sigma_1(t) \setminus \big(\tfrac{1}{2}(\hat{x}_2 + \Sigma_1(2t)) \cup \tfrac{1}{2}(\hat{x}_3 + \Sigma_1(2t))\big)$$

Then $I_{1,1}(t)$ is the open interval centered at the midpoint of $\Sigma_1(t)$ of length $s(t)-s(2t) = 2\rho t s_0$. The set $\Sigma_1(t) \setminus I_{1,1}(t)$ consists of two closed segments. Define $I_{2,1}(t)$ and $I_{2,2}(t)$ to be the open intervals with the same length as $I_{1,1}(t)$ centered at the midpoints of these two closed intervals. Continue iteratively, given $2^\ell - 1$ open subintervals of $\Sigma_1(t)$

$$I_{j,k}(t), \qquad j = 1, \ldots, \ell, \quad k = 1, \ldots, 2^{j-1},$$

of equal length $2\rho t s_0$ and equal spacing. The intervals $\{I_{\ell+1,k}(t)\}_{1\le k\le 2^\ell}$ are of length $2\rho t s_0$ and centered at the midpoints of the closed intervals complementary to the intervals we have already defined.

Set

$$V_\ell := \{i 2^{-\ell}\hat{x}_2 + (2^\ell - i - 1)2^{-\ell}\hat{x}_3 : i = 0, \ldots, 2^\ell - 1\}$$

Then, by construction,

$$E \cap \Sigma_1(t) = \bigcup_{v\in V_m} \big(v + 2^{-m}\Sigma_1(2^m t)\big) = \Sigma_1(t) \setminus \bigcup_{j=1}^{m} \bigcup_{k=1}^{2^{j-1}} I_{j,k}(t) \tag{2.42}$$

There is, of course, a similar description of $E \cap \Sigma_2(t)$ and $E \cap \Sigma_3(t)$.

To describe the rest of $E \cap \Sigma(t)$, we introduce more notation. For any 1-dimensional segment I in $\mathbb{R}^3$, given $h > 0$ and $\alpha \ge 1$, define a “flared neighborhood" of I by

$$\mathcal{F}_{h,\alpha}(I) := \{x \in \mathbb{R}^3 \,:\, \operatorname{dist}(x, I^*) \le h, \quad \operatorname{dist}(x, I) \le \alpha \operatorname{dist}(x, I^*)\}$$

where I^* denotes the line containing I. Note that $\mathcal{F}_{h,\alpha}(I)$ is symmetric with respect to I^*, and consists of the union of two trapezoids with I as shorter base.

For $2^{-m} \le t < 2^{-m+1}$, set

$$\mathcal{S}_1^\alpha(t) := \bigcup_{j=1}^{m} \bigcup_{k=1}^{2^{j-1}} \mathcal{F}_{s_0 2^{-j+1}\epsilon,\alpha}(I_{j,k}(t)).$$

(See Figures 1 and 2.)

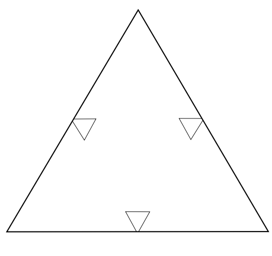

FIGURE 1. $\mathcal{S}_1^\alpha(t) \cup S_2^\alpha(t) \cup S_3^\alpha(t) \subset \Sigma(t)$ for $\alpha = \pi/3$ and $t \in [1/2, 1)$.

Define $\mathcal{S}_i^\alpha(t)$ as the image of $\mathcal{S}_1^\alpha(t)$ under any rigid motion of $\mathbb{R}^3$ that maps $\Sigma_1(t)$ to $\Sigma_i(t)$. With these notations we can now estimate the complement of E.

Lemma 2.4. *For ρ and ϵ sufficiently small and for all t, $0 < t < 1/2$,*

$$\Sigma(t) \setminus E \ \subset \ \bigcup_{i=1}^{3} \mathcal{S}_i^2(t)\,.$$

Before proving this lemma we will use it to finish the proof of Lemma 2.1 and hence Theorem 1.4.

Note that $\mathcal{S}_1^2(t) \cap \Sigma(t)$ is a union of "upward" trapezoids whose shorter bases are the $2^m - 1$ intervals $I_{j,k}(t)$ of length $2\rho t s_0$ ($2^{-m} \le t < 2^{-m+1}$, $1 \le j \le m$, $1 \le k \le 2^{j-1}$). The complements in $\Sigma_1(t)$ of these bases are 2^m intervals of equal length $\ell(t)$ given by

$$\ell(t) := \mathcal{H}^1(2^{-m}\Sigma_1(2^m t)) = 2^{-m}(1 - 2\rho 2^m t)s_0 > (1 - 4\rho)2^{-m}s_0\,.$$

For $i = 1, 2, 3$, let $T_i(t)$ be the isosceles triangle in $\Sigma(t)$ with base $\Sigma_i(t)$ whose equal sides are of slope 4ϵ relative to the base (and hence of height less than $2\epsilon s_0$). We claim that

$$\mathcal{S}_i^2(t) \cap \Sigma(t) \subset T_i(t), \quad i = 1,\, 2,\, 3,$$

see Figure 2.

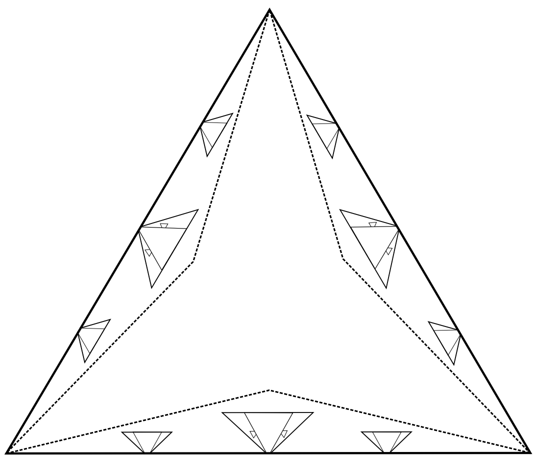

FIGURE 2. Some illustrative part of the fractal set appearing in the proof of Lemma 2.4. Note that the basis of the trapezoids have all the same length, given by $2\rho t s_0$. By widening the trapezoids from $\alpha = 2/\sqrt{3}$ to $\alpha = 2$, we ensure that even when later on in the iteration we may add some additional trapezoids to the sides of a previous one, these will still be included in the wider trapezoid. The dotted lines represent the triangles $T_i(t)$, $i = 1, 2, 3$.

To see this, suppose, without loss of generality, that $i = 1$ and call the direction of $\Sigma_1(t)$ horizontal. The left side of the smallest isosceles triangle with base $\Sigma_1(t)$ that encloses $\mathcal{S}_1^2(t) \cap \Sigma(t)$ starts at the left endpoint of $\Sigma_1(t)$ and passes through the upper left corner of the short trapezoid in $\mathcal{S}_1^2(t) \cap \Sigma(t)$ nearest that corner. That trapezoid has height

$h_m = 2^{-m+1}\epsilon s_0$, and horizontal distance from the endpoint of $\Sigma_1(t)$ given by $\ell(t) - \sqrt{3}h_m$. Thus the slope is

$$\frac{h_m}{\ell(t) - h_m} \le \frac{2^{-m+1}\epsilon s_0}{(1-4\rho)2^{-m}s_0 - \sqrt{3}\,2^{-m+1}\epsilon s_0} = \frac{2\epsilon}{1-4\rho-2\sqrt{3}\,\epsilon} \le 4\epsilon,$$

for ρ and ϵ less than $1/100$.

Next, for $0 \le h \le \epsilon s_0$, we consider segments parallel to the side $\Sigma_1(t)$, excluding very short segments at the ends corresponding to the thin triangles $T_2(t)$ and $T_3(t)$, and then remove, in addition, $\mathcal{S}_1^2(t)$:

$$J_1^h(t) := \{x \in \Sigma(t) : \operatorname{dist}(x, \Sigma_1(t)) = h\} \setminus (T_2(t) \cup T_3(t)); \quad E_1^h(t) = J_1^h(t) \setminus \mathcal{S}_1^2(t),$$

(See Figure 3.) We define $E_i^h(t) \subset J_i^h(t)$ analogously for $i = 2$, 3. Lemma 2.4 implies that

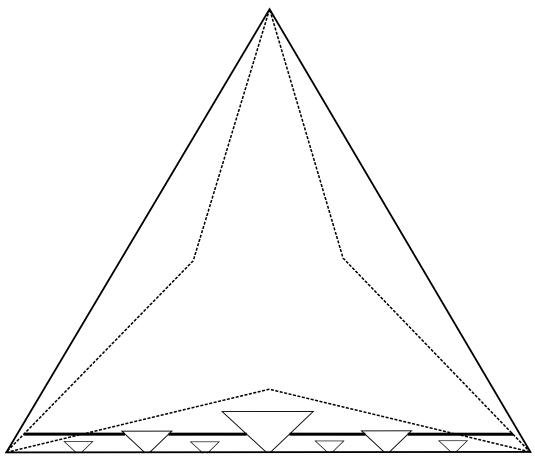

FIGURE 3. The bold line represents the set $E_1^h(t)$. This is obtained by taking an horizontal segment at height h connecting the triangles $T_2(t)$ and $T_3(t)$, and removing the part covered by $\mathcal{S}_1^2(t)$

$E_i^h(t) \subset E$, and hence

$$\int_0^1 \int_0^{\epsilon s_0} \mathcal{H}^1(E_i^h(t) \setminus A)\, dh\, dt \le |E \setminus A|, \quad i = 1,\, 2,\, 3. \tag{2.43}$$

To confirm that the one-dimensional Lemma 2.3 applies to $E_1^h(t)$ as a subset of the interval $J_1^h(t)$, observe that the set $J_1^h(t) \cap \mathcal{S}_1^2(t)$ that we excluded to form $E_1^h(t)$ consists of equally spaced intervals of equal length

$$\mathcal{H}^1(I_{j,k}) + 2\sqrt{3}\,h = 2\rho t s_0 + 2\sqrt{3}\,h, \quad s_0 2^{-j+1}\epsilon \ge h, \quad 1 \le k \le 2^{j-1}.$$

The value of j ranges from 1 to j^* with the maximum value determined by the constraints $j^* \le m$ and $2^{j^*} \le 2s_0\epsilon/h$. The total number of intervals is

$$1 + 2 + \cdots + 2^{j^*-1} \;=\; 2^{j^*} - 1 \;<\; 2^{j^*} \;\le\; \min\left(2^m, \frac{2s_0\epsilon}{h}\right).$$

The total length of these complementary intervals is less than

$$(2\rho t s_0 + 2\sqrt{3}\,h) \min\left(2^m, \frac{2s_0\epsilon}{h}\right) \le 2^{m+1}\rho t s_0 + 4\sqrt{3}\epsilon s_0 \le 10(\rho+\epsilon)s_0.$$

Note that $J_1^h(t)$ is $(1 - O(\epsilon+\rho))s_0$, and that $J_1^h(t) \cap \mathcal{S}_1^2(t)$ is at most an $O(\rho+\epsilon)$ fraction of $J_1^h(t)$. It follows that, for all $x \in J_1^h(t)$,

$$\chi_{E_1^h(t)/2} * \chi_{E_1^h(t)/2}(x) \ge (1 - O(\epsilon+\rho))\mathrm{dist}(x, \partial J_1^h(t)),$$

in which we abuse notation by identifying $J_1^h(t)$ with its isometric image in a real line and likewise the subset $E_1^h(t)$. (Note that although the $2^m - 2$ internal intervals of $E_1^h(t)$ have equal length, the two on the ends are slightly longer. This only improves the convolution inequality at the very ends. We excluded the triangles $T_2(t)$ and $T_3(t)$ from $J_1^h(t)$ in order to arrange this favorable situation at the ends: we do not want the interval on which we apply Lemma 2.3 to intersect $\mathcal{S}_2^2(t)$ and $\mathcal{S}_3^2(t)$.)

Having confirmed the hypothesis of Lemma 2.3, and likewise for the analogous sets $E_i^h(t) \subset J_i^h(t)$, we apply the lemma to conclude that

$$\mathcal{H}^1(J_i^h(t)\setminus A) \le \mathcal{H}^1(J_i^h(t) \cap \tfrac{1}{2}(A+A)\setminus A) + 20\,\mathcal{H}^1(E_i^h(t)\cap A), \quad i = 1,\ 2\ 3.$$

Finally, these three inequalities, along with (2.43) and Fubini's theorem imply

$$\begin{aligned} |K\setminus A| \ &\le |E\setminus A| + \sum_{i=1}^3 \int_0^1 \int_0^{\epsilon s_0} \mathcal{H}^1(J_i^h(t)\setminus A)\, dh dt \\ &\le |E\setminus A| + 3|\tfrac{1}{2}(A+A)\setminus A| + 60|E\setminus A| \le 64 C_0\,\delta(A) \end{aligned}$$

with the dimensional constant C_0 of (2.15). This ends the proof of Lemma 2.1 and Theorem 1.4, except for the proof of Lemma 2.4 that we now provide.

Proof of Lemma 2.4. The complementary set $\Sigma(t)\setminus E$ is a fractal built iteratively out of (occasionally truncated) trapezoids arising as the complements of sets of scaled equilateral triangles. Figure 1 shows the fractal in its simplest, starting layer $1/2 \le t < 1$. We will organize the description of a superset of the fractal. Figure 2 shows the widened trapezoids of the superset that we will use to enclose successive generations of smaller and smaller trapezoids in the fractal. Within $T_1(t)$, the triangle with base $\Sigma_1(t)$, defined in the end of the proof of main Lemma 2.1 just above, we will refer to the "first generation" of the complementary set as the set involving semisums with the endpoints $\hat{x}_2$ and $\hat{x}_3$ and trapezoids that touch $\Sigma_1(t)$ only. This first generation is a subset of $\mathcal{S}_1^\alpha(t)$ with $\alpha = \alpha_0 = 2/\sqrt{3}$, corresponding to the angle $\pi/3$. The second generation of points in

$T_1(t) \setminus E$ arise from first generation points in $T_2(2t)$ and $T_3(2t)$. Consider, for example, the semisum of $\hat{x}_3$ with points of the first generation in $T_3(2t)$. For any $\alpha < 2$,

$$\mathcal{S}_3^\alpha(t) \cap \Sigma(2t) \subset \mathcal{S}_3^2(2t) \cap \Sigma(2t) \subset T_3(2t).$$

Therefore,

$$\tfrac{1}{2}\left(\hat{x}_3 + \Sigma(2t) \cap \mathcal{S}_3^\alpha(2t)\right) \setminus (1-\epsilon)K \subset \tfrac{1}{2}\left(\hat{x}_3 + T_3(2t)\right) \setminus (1-\epsilon)K$$

is contained in a triangle of base size $O(\epsilon)$ and height $O(\epsilon^2)$. More precisely, the base is a non-parallel side of the trapezoid $\Sigma(t) \cap \mathcal{F}_{s_0\epsilon,\alpha_0}(I_{1,1}(t))$, and the other vertex is on the line parallel to $I_{1,1}(t)$ at distance $s_0\epsilon$. Note the very important shrinkage that comes from subtracting $(1-\epsilon)K$. The set we are translating is contained in a triangle of size $O(1)$ by $O(\epsilon)$ but the part of the translation that is outside of $(1-\epsilon)K$ has diameter $O(\epsilon)$ and width $O(\epsilon^2)$. The second generation exceptional set is covered by opening the neighborhood of $I_{1,1}(t)$ by changing the flare parameter from α_0 to $\alpha_1 = \alpha_0 + 10\epsilon$. The same widening eventually occurs, appropriately scaled, at all of the intervals $I_{j,k}(t)$ at least for sufficiently small t, but no other additions occur if we only use one step with a convex combination involving a vertex and an opposite side. In all, at the second generation, in which at most one such step is used, the exceptional set is contained in the set

$$\bigcup_{i=1}^{3} \mathcal{S}_i^{\alpha_1}(t), \quad \alpha_1 = \alpha_0 + 10\epsilon\,.$$

Repeating this argument, we find that the exceptional set generated using at most k steps involving a vertex and an opposite side is contained in

$$\bigcup_{i=1}^{3} \mathcal{S}_i^{\alpha_k}(t), \quad \alpha_k = \alpha_0 + (10\epsilon) + (10\epsilon)^2 + \cdots + (10\epsilon)^k\,.$$

Evidently, for sufficiently small ϵ, $\alpha_k < 2$ for all k. This covers the entire complement of E in $\Sigma(t)$ and concludes the proof of Lemma 2.4. $\square$

Remark 2.5. *In closing, we note that in our inductive argument for $n = 3$, we proved that the complement of E contains only relatively short one-dimensional segments at all appropriate scales near the boundary of K. When $n = 4$ the set E has nearly full $\mathcal{H}^4$ measure on many suitably scaled subsets, but its complement has too many segments of large diameter near ∂K. Therefore, further arguments are required to enlarge E enough to finish the case $n = 4$ and higher.*

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