AN EXCESS-DECAY RESULT FOR A CLASS OF DEGENERATE ELLIPTIC EQUATIONS

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Abstract. We consider a family of degenerate elliptic equations of the form $\mathrm{div}(\nabla F(\nabla u)) = f$, where $F \in C^{1,1}$ is a convex function which is elliptic outside a ball. We prove an excess-decay estimate at points where ∇u is close to a nondegenerate value for F. This result applies to degenerate equations arising in traffic congestion, where we obtain continuity of ∇u outside the degeneracy, and to anisotropic versions of the p-laplacian, where we get Hölder regularity of ∇u.

1. Introduction

We study the local regularity of minimizers of the functional

$$\int_\Omega F(\nabla u) + fu \tag{1.1}$$

where $\Omega \subseteq \mathbb{R}^n$ is an open set, $F : \mathbb{R}^n \to \mathbb{R}$, $f : \Omega \to \mathbb{R}$, and $u : \Omega \to \mathbb{R}$. When a uniform ellipticity condition on F holds true, the regularity results are classical. Even in the vectorial case, partial regularity of minimizers was proved under the uniform strict quasiconvexity assumption in [13, 1] (see also the references quoted therein).

To understand regularity for more degenerate elliptic problems, a natural idea is to prove Hölder regularity at points where the gradient is close to a value where the function F is C^2 and uniformly convex. This scheme has been carried out by Anzellotti and Giaquinta in [3] under the uniform convexity assumption for elliptic systems and in [2] if uniform strict quasiconvexity is assumed. In the latter paper it is proved that, if $u : \mathbb{R}^n \to \mathbb{R}^N$ (with $N \geq 1$) and

$$\lim_{r\to 0} ⨍_{B_r(x_0)} |\nabla u(y) - \xi_0|^2 \, dy = 0 \tag{1.2}$$

for some $\xi_0 \in \mathbb{R}^{nN}$ and $x_0 \in \mathbb{R}^n$, F is C^2 in a neighborhood of ξ_0, and a uniform strict quasiconvexity holds true around ξ_0, then u is of class $C^{1,\alpha}$ in a neighborhood of x_0 for every $\alpha < 1$. Their proof is based on a linearization argument. They differentiate the Euler equation

$$\partial_i(\partial_i \mathcal{F}(\nabla u)) = f \qquad \text{in } \Omega$$

(here and in the following we use the Einstein's summation convention) with respect to a direction $e \in S^{n-1}$ to obtain

$$\partial_i[\partial_{ij}\mathcal{F}(\nabla u(x))\partial_j(\partial_e u(x))] = \partial_e f(x) \qquad \text{in } \Omega.$$

Then, using (1.2), they prove that the solution of the differentiated operator is close, on smaller scales, to the solution v of a differential operator with constant coefficients

$$\partial_i[\partial_{ij}\mathcal{F}(\xi_0)\partial_j v(x)] = 0 \qquad \text{in } \Omega.$$

Since F is strictly quasiconvex in ξ_0, this equation is in turn nondegenerate. In this way, they obtain regularity of u from the regularity of the linearized operator.

In this paper we study the regularity of minimizers of the function (1.1) in the scalar case assuming that F is $C^{1,1}$ and uniformly elliptic outside a ball, and ellipticity may degenerate inside. Basic examples which fall under these assumptions are $F(x) = \boldsymbol{n}(x)^p$ for some $p > 1$ with $\boldsymbol{n}$ an elliptic norm (see Definition 3.1), and $F(x) = (|x| - 1)_+^p$ for some $p > 1$ (notice that, since we consider Lipschitz minimizers, the behavior of F at infinity is not relevant). The first example arises as an anisotropic generalization of the p-laplacian, whereas the second example is related to some recent problems of traffic dynamic. In the following we assume that $F \in C^{1,1}$ outside the degeneracy region to prove that every locally Lipschitz minimizer is $C^{1,\alpha}$ at nondegenerate points. In a previous paper [6] we already addressed this problem when $F \in C^2$, using techniques of Wang [22] and Savin [18]. However, for $F \in C^{1,1}$ new techniques are needed. In this respect we mention a De Giorgi type approach in a work of De Silva and Savin [7]; it looks possible to us that also their technique may lead to prove our result, but we believe that our approach in this setting has its own interest. On the contrary, the results in [2, 6] described above assumed $F \in C^2$ and this assumption cannot be easily removed with their technique, since their proof is based on a linearization argument which cannot work if the second derivatives of F are not continuous, because the linearized operator has no reason to stay close to the nonlinear one. Our approach is still based on a blow-up argument; however, we prove that the operator can be linearized, up to subsequence, around a limit operator which is uniformly elliptic thanks to the fact that the gradient is assumed to be mainly outside the degeneracy. To obtain strong compactness of a rescaled sequence, we use an idea of De Silva and Savin [7] presented in Lemma 4.4.

The paper is organized as follows. In section 2 we present the basic estimate of decay of the excess function around nondegenerate points. Then we see that this estimate can be iterated at every scale to obtain the $C^{1,\alpha}$ regularity. Finally, we see that the smallness assumption is satisfied if u is close to a linear nondegenerate function in a certain sense, which in turn can be verified in the applications. In section 3 we see how the estimate allows to prove $C^{1,\alpha}$ regularity for the solutions of the anisotropic p-laplacian and regularity outside the degeneracy for some equations arising in the context of traffic congestion. In section 4 we collect all the proofs.

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2. Main result

First we introduce the excess function, which measures the distance of the gradient of a solution ∇u from its average. In terms of this quantity we express the smallness condition which guarantees regularity. The $C^{0,\alpha}$ regularity for ∇u is expressed in terms of the decay of the excess itself, through Campanato's Theorem.

We denote by $B_r(x)$ the open ball of center $x \in \mathbb{R}^n$ and radius $r > 0$, often shortened as B_r if $x = 0$. Given $g : \Omega \to \mathbb{R}^n$, with the notation $\fint_{B_r(x)} g$ we mean the average of g on the ball $B_r(x)$. We refer to the same quantity also with the notation $(g)_{B_r(x)}$.

Let Ω be an open set and let $f \in L^q(\Omega)$ for some $q > n$. For every $u \in W^{1,2}(\Omega)$, $x \in \Omega$, $r < d(x, \Omega)$ we consider the excess

$$U(u, x, r) := \Big(\fint_{B_r(x)} |\nabla u(y) - (\nabla u)_{B_r(x)}|^2 \, dy\Big)^{1/2} + r^{(q-n)/(2q)} \|f\|_{L^q(B_1)}.$$

The following Theorem provides an excess-decay estimate for local minimizers of the functional (1.1) at points where ∇u is nondegenerate. As we shall show in the corollaries below, the result can be iterated on smaller scales to provide Hölder regularity for the gradient around nondegenerate points.

Theorem 2.1. *Let $f \in L^q(B_1)$ for some $q > n \geq 2$. Let $F : \mathbb{R}^n \to \mathbb{R}$ be a convex function such that $F \in C^{1,1}(\mathbb{R}^n \setminus B_{1/4}(0))$ and*

$$\lambda \, \mathrm{Id} \leq \nabla^2 F(x) \leq \Lambda \, \mathrm{Id} \qquad \text{for a.e. } x \in \mathbb{R}^n \setminus B_{1/4}(0). \tag{2.1}$$

Let $u \in W^{1,\infty}(B_1)$ be a minimizer of the functional (1.1) and let us assume that $|\nabla u| \leq 1$ in B_1.

Then there exist $\tau_0, \alpha > 0$, depending only on $n, q, \lambda, \Lambda, \|\nabla F\|_{L^\infty(B_1)}$, such that for every $\tau \leq \tau_0$ there exists $\varepsilon = \varepsilon(\tau)$ for which the following property holds true: If for some $x \in B_{1/2}$ and $r < 1/4$ we have

$$\frac{3}{4} \leq |(\nabla u)_{B_r(x)}| \leq 1, \qquad U(u, x, r) \leq \varepsilon,$$

then

$$U(u, x, \tau r) \leq \tau^\alpha U(u, x, r).$$

Theorem 2.1 can be iterated to obtain the decay of the excess at every scale.

Corollary 2.2. *Let q, f, F, and u be as in Theorem 2.1. Then there exist $\tau_0, \alpha > 0$, depending only on $n, q, \lambda, \Lambda, \|\nabla F\|_{L^\infty(B_1)}$, such that for every $\tau \leq \tau_0$ there exists $\varepsilon = \varepsilon(\tau)$ for which the following property holds true: If for some $x \in B_{1/2}$ and $r < 1/4$ we have*

$$\frac{7}{8} \leq |(\nabla u)_{B_r(x)}| \leq 1, \qquad U(u, x, r) \leq \varepsilon, \tag{2.2}$$

then

$$U(u, x, \tau^k r) \leq \tau^{\alpha k} U(u, x, r) \qquad \forall \, k \in \mathbb{N}. \tag{2.3}$$

The assumption in Corollary 2.3 is satisfied in a ball if the gradient of u is aligned in a fixed direction, as the following corollary states. This will be in turn useful to obtain $C^{1,\alpha}$ regularity at nondegenerate points in the applications of Section 3.

Corollary 2.3. *Let q, f, F, and u be as in Theorem 2.1. Then there exist $\eta, \alpha, C, \tau, r_0 > 0$, depending only on $n, q, \lambda, \Lambda, \|f\|_{L^q(B_1)}, \|\nabla F\|_{L^\infty(B_1)}$, such that if $|\nabla u(x)| \leq 1$ for every $x \in B_1$ and*

$$|\{x \in B_1 : \partial_{\mathbf{v}} u(x) \geq 1 - \eta\}| \geq (1 - \eta)|B_1| \tag{2.4}$$

for some $\mathbf{v} \in S^{n-1}$, then

$$U(u, x, \tau^k r_0) \leq \tau^{\alpha k} U(u, x, r_0) \qquad \forall \, k \in \mathbb{N} \qquad \forall \, x \in B_{1/2}. \tag{2.5}$$

In particular, we have

$$\|u\|_{C^{1,\alpha}(B_{1/2})} \leq C. \tag{2.6}$$

3. Applications

3.1. **The anisotropic p-Laplace equation.** The simplest example of degenerate elliptic equation is given by the p-Laplace equation

$$\partial_i(|\nabla u|^{p-2}\partial_i u) = f,$$

corresponding to the choice $F(x) = |x|^p/p$ in the minimization of the function (1.1); in this case the degeneracy consists in a single point, the origin, and it is possible to obtain $C^{1,\alpha}$ regularity of the solution. It has been proved by Uraltseva [21], Uhlenbeck [20], and Evans [10] for $p \geq 2$, and by Lewis [16] and Tolksdorff [19] for $p > 1$ (see also [8, 22]). In the following, we introduce a generalization of the p-laplacian which involves an anisotropic norm. We consider an open set $\Omega \subseteq \mathbb{R}^n$ and a local minimizer for the functional

$$\int_\Omega \frac{\boldsymbol{n}(\nabla u)^p}{p} + fu, \tag{3.1}$$

where $\boldsymbol{n} : \mathbb{R}^n \to \mathbb{R}^+$ is a positively 1-homogeneous convex function and $f \in L^q(\Omega)$ for some $q > n$.[1]

To ensure the equation to be elliptic outside the origin, we need to consider only norms which satisfy an ellipticity condition in the direction ortogonal to $\nabla\boldsymbol{n}$. For example, the p-norms (namely $\boldsymbol{n}(x) = (|x_1|^p + ... + |x_n|^p)^{1/p}$ for $x = (x_1, ..., x_n) \in \mathbb{R}^n$) are not included in the following definition and indeed the problem of regularity of minimizers is, to our knowledge, open.

Definition 3.1. An "elliptic norm" $\boldsymbol{n} \in C^{1,1}_{\rm loc}(\mathbb{R}^n \setminus \{0\})$ is a convex positively 1-homogenous function with $\boldsymbol{n}(0) = 0$, positive outside the origin, for which there exist $\lambda, \Lambda > 0$ such that

$$\lambda \left| \tau - (\tau\cdot\nabla\boldsymbol{n}(v))\frac{\nabla\boldsymbol{n}(v)}{|\nabla\boldsymbol{n}(v)|^2}\right|^2 \leq \boldsymbol{n}(v)\partial_{ij}\boldsymbol{n}(v)\tau_i\tau_j \leq \Lambda\,|\tau|^2 \tag{3.2}$$

for a.e. $v \in \mathbb{R}^n,\ \tau \in \mathbb{R}^n$.[2]

In the following, we prove that every Lipschitz solution of the anisotropic p-Laplace equation is $C^{1,\alpha}$.

Theorem 3.2. *Let $p > 1$, Ω a bounded open subset of $\mathbb{R}^n$, $n \geq 2$, and $f \in L^q(B_1)$ for some $q > n$. Let $\boldsymbol{n} : \mathbb{R}^n \to \mathbb{R}$ be an elliptic norm and let $u \in W^{1,\infty}_{\rm loc}(\Omega)$ be a local minimizer of the functional* (3.1)*.*

Then there exists $\alpha \in (0, 1)$, which depends only on n, p, q, λ, Λ, $\|\nabla\boldsymbol{n}\|_\infty$ such that $\nabla u \in C^{0,\alpha}_{\rm loc}(\Omega)$, namely for every $\Omega' \Subset \Omega$ there exists a constant $C > 0$ such that

$$|\nabla u(x) - \nabla u(y)| \leq C|x - y|^\alpha \qquad \forall\, x, y \in \Omega'. \tag{3.3}$$

This constant C depends only on n, p, q, λ, Λ, $\|\nabla\boldsymbol{n}\|_\infty$, Ω', $\|f\|_q$, and $\|\nabla u\|_\infty$ in a neighborhood of Ω'.

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[1] Recall that a function $u \in W^{1,1}_{\rm loc}(\Omega)$ is said a local minimizer of a function of the form (1.1) if, for every $\Omega' \Subset \Omega$, we have

$$\int_{\Omega'} F(\nabla u + \nabla\phi) + f(u + \phi) \geq \int_{\Omega'} F(\nabla u) + fu \qquad \forall\,\phi \in W^{1,1}_0(\Omega').$$

[2] In this definition the term "norm" is used with a slight abuse of notation: indeed we are not requiring the symmetry of $\boldsymbol{n}$, namely $\boldsymbol{n}(v) = \boldsymbol{n}(-v)$. We also observe that an equivalent formulation for (3.2) is to ask that

$$\lambda'|\tau|^2 \leq \partial_{ij}\mathcal{H}(v)\tau_i\tau_j \leq \Lambda'|\tau|^2 \qquad \forall\, v, \tau \in \mathbb{R}^n$$

for some $0 < \lambda' \leq \Lambda'$, where $\mathcal{H}(v) := \big(\boldsymbol{n}(v)\big)^2$.

In the theorem above we assume Lipschitz regularity of the solution to prove $C^{1,\alpha}$ regularity; notice that the Lipschitz regularity follows, for instance, from [9, 4, 11]. To avoid annoying details about a regularization argument, we prove the result in terms of an a-priori estimate; hence we assume that u is smooth, and so is $\boldsymbol{n}$ outside the origin (For more details about the regularization, see for instance [6, Proof of Theorem 1.1]).

The key idea to prove Theorem 3.2 is a lemma which provides a separation between degeneracy and nondegeneracy. It says that the gradient of the solution ∇u is either close to a nonzero constant, or it decays on a smaller ball. When the first case happens at some scale, we obtain $C^{1,\alpha}$ regularity of u through Corollary 2.3. Otherwise, the decay of ∇u at every scale provides $C^{1,\alpha}$ regularity of u.

As we show now the dichotomy, stated at scale one in Lemma 4.5, is based on the construction of suitable subsolutions to a uniformly elliptic equation, namely $(\partial_e u(x) - 1/2)_+$ for every $e \in S^{n-1}$. Indeed, let $u : B_1 \to \mathbb{R}$ be a Lipschitz local minimizer of (1.1) with Lipschitz constant 1; then it solves the Euler equation

$$\partial_i\Big[\boldsymbol{n}\big(\nabla u(x)\big)^{p-1}\partial_i\boldsymbol{n}\big(\nabla u(x)\big)\Big] = f(x) \qquad x \in B_1. \tag{3.4}$$

Let us introduce the coefficients

$$A_{ij}(x) := \boldsymbol{n}(x)^{p-2}\Big((p-1)\partial_i\boldsymbol{n}(x)\partial_j\boldsymbol{n}(x) + \boldsymbol{n}(x)\partial_{ij}\boldsymbol{n}(x)\Big) \qquad \forall\, x \in \mathbb{R}^n. \tag{3.5}$$

Given $e \in S^{n-1}$, we differentiate (3.4) in the direction $e \in S^{n-1}$ to obtain

$$\partial_i\Big[A_{ij}\big(\nabla u(x)\big)\partial_j\big(\partial_e u(x)\big)\Big] = \partial_e f(x).$$

We notice that, setting

$$a_{ij}(x) := (p-1)\partial_i\boldsymbol{n}(x)\partial_j\boldsymbol{n}(x) + \boldsymbol{n}(x)\partial_{ij}\boldsymbol{n}(x) \qquad \forall\, x \in \mathbb{R}^n, \tag{3.6}$$

the coefficients a_{ij} are uniformly elliptic. Indeed, $\nabla\boldsymbol{n}$ is 0-homogeneous and since $\boldsymbol{n} \in C^{1,1}_{\mathrm{loc}}(\mathbb{R}^n \setminus \{0\})$ we have that $0 < c \le |\nabla\boldsymbol{n}| \le C < \infty$; therefore for every $\tau \in \mathbb{R}^n$ we obtain that

$$\begin{aligned} a_{ij}\tau_i\tau_j &\ge (p-1)|\nabla\boldsymbol{n}(v)|^2\left|\tau \cdot \frac{\nabla\boldsymbol{n}(v)}{|\nabla\boldsymbol{n}(v)|}\right|^2 + \lambda\left|\tau - (\tau\cdot\nabla\boldsymbol{n}(v))\frac{\nabla\boldsymbol{n}(v)}{|\nabla\boldsymbol{n}(v)|^2}\right|^2 \\ &\ge \min\{c^2(p-1), \lambda\}|\tau|^2, \end{aligned}$$

and analogously from above. Hence the coefficients A_{ij} are uniformly elliptic in every compact region which does not contain the origin.

Since the function $t \mapsto (t - 1/2)_+$ is convex and Lipschitz with derivative $\mathbf{1}_{\{t>1/2\}}$, it follows that the function

$$v_e(x) := (\partial_e u(x) - 1/2)_+ \qquad e \in S^{n-1} \tag{3.7}$$

is a subsolution of the equation

$$\partial_i\Big[A_{ij}\big(\nabla u(x)\big)\partial_j v_e(x)\Big] = \partial_e f(x)1_{\{\partial_e u > 1/2\}}(x).$$

Notice that the values of the coefficients $A_{ij}(\nabla u(x))$ are only relevant when $1/2 \le |\nabla u(x)| \le 1$. Indeed the solution satisfies $|\nabla u(x)| \le 1$ (by assumption), and when $|\nabla u(x)| \le 1/2$ we have that $v_e(x) = 0$. Therefore, thanks to the ellipticity assumption on $\boldsymbol{n}$, the equation might be assumed to be uniformly elliptic.

The idea of the proof now follows a paper by Wang [22], where Theorem 3.2 is presented for the classical p-laplacian. In this case, however, the author considers a different subsolution, namely $\boldsymbol{n}(\nabla u)^p$, which solves an elliptic equation with nondegenerate coefficients. Indeed, given a locally lipschitz minimizer of (1.1) with $f = 0$, the coefficients a_{ij} (introduced in (3.6)) are uniformly elliptic and the function $\boldsymbol{n}(\nabla u)^p$ formally solves

$$\partial_i\Big[a_{ij}\big(\nabla u(x)\big)\partial_j\Big(\boldsymbol{n}\big(\nabla u(x)\big)^p\Big)\Big] \geq 0.$$

The choice of the subsolution in [22] leads to additional difficulties to pass from a nondegenerate slope of u in modulus to closeness to a linear function. Moreover, the regularity at nondegenerate points is carried out in [22] through the analysis of the equation in nondivergence form, proving as a key lemma that any solution of the p-laplace equation is close to the solution of the linearized problem at nondegeneracy points. Wang's scheme can be carried out for a general elliptic norm $\boldsymbol{n}$ only assuming better regularity on $\boldsymbol{n}$, namely $\boldsymbol{n} \in C^2(\mathbb{R}^n \setminus \{0\})$. Hence, as we shall see in Section 4, the proof of Theorem 3.2 requires the use of our Theorem 2.1.

3.2. **Degenerate elliptic equations and traffic models.** Corollary 2.3 can be used to prove local $C^{0,\alpha}$ regularity of the gradient of the solution of a degenerate elliptic equation outside the degeneracy region. We refer to [6, 5] and the references quoted therein for a detailed presentation of the model and for the physical meaning of the continuity of the gradient at nondegenerate points. We also refer to [15] for a nonvariational analysis of the same kind of degenerate elliptic equations.

The following result is a generalization of [6, Theorem 1.1] (see also [17] where the result is proved in dimension $n = 2$) to more general functions F (we do not require C^2 regularity of F). The degeneracy region is a convex set containing the origin, described as the unit ball of a convex positively 1-homogenous function which does not need to be elliptic. The variational proof is based on Corollary 2.3, which in turn uses a different technique with respect to the proof presented in [6] that is based on some ideas of Savin [18] and Wang [22].

Theorem 3.3. *Let Ω a bounded open subset of $\mathbb{R}^n$, $n \geq 2$, $f \in L^q(\Omega)$ for some $q > n$. Let $\boldsymbol{m} : \mathbb{R}^n \to \mathbb{R}$ be a convex positively 1-homogenous function with $\boldsymbol{m}(0) = 0$ which is positive outside the origin. Let $F : \mathbb{R}^n \to \mathbb{R}$ be a convex nonnegative function such that $F \in C^{1,1}_{loc}(\mathbb{R}^n \setminus \overline{\{\boldsymbol{m} \leq 1\}})$, and assume that for every $\delta > 0$ there exist $\lambda_\delta, \Lambda_\delta > 0$ such that*

$$\lambda_\delta I \leq \nabla^2 F(x) \leq \Lambda_\delta I \qquad \text{for a.e. } x \text{ such that } 1 + \delta \leq \boldsymbol{m}(x) \leq 1/\delta. \tag{3.8}$$

Let $u \in W^{1,\infty}_{\mathrm{loc}}(\Omega)$ be a local minimizer of the functional (1.1)*. Then, for any continuous function $H : \mathbb{R}^n \to \mathbb{R}$ such that $\{\boldsymbol{m} \leq 1\} \subseteq \{H = 0\}$, we have*

$$H(\nabla u) \in C^0(\Omega).$$

More precisely, for every open set $\Omega' \Subset \Omega$ there exists a modulus of continuity $\omega : [0, \infty) \to [0, \infty)$ for $H(\nabla u)$ on Ω', which depends only on n, the modulus of continuity of H, the functions $\delta \to \lambda_\delta, \delta \to \Lambda_\delta$, $\|\nabla u\|_\infty$ in a neighborhood $\Omega'' \subset \Omega$ of Ω', and $\|\nabla F\|_\infty$ in a neighborhood of $\nabla u(\Omega'')$, such that

$$\big|H(\nabla u(x)) - H(\nabla u(y))\big| \leq \omega(|x - y|) \qquad \forall\, x, y \in \Omega'.$$

In particular, if $F \in C^1(\mathbb{R}^n)$ then $\nabla F(\nabla u) \in C^0(\Omega)$.

4. Proofs

4.1. **Proof of Theorem 2.1.** Before proving the result, we state some simple lemmas. The proof of the first lemma is an easy computation which is left to the reader.

Lemma 4.1. *Let $p > 1$, $X \in \mathbb{R}^n$, and let $\mathbf{v}_1, ..., \mathbf{v}_n \in \mathbb{R}^n$ be a family of vectors satisfying $|\mathbf{v}_i| = 1$ for any $i = 1, ..., n$ and $\left|\det\left(\mathbf{v}_1|\dots|\mathbf{v}_n\right)\right| > c_0 > 0$ (here $\left(\mathbf{v}_1|\dots|\mathbf{v}_n\right)$ denotes the matrix whose columns are given by the vectors $\mathbf{v}_1, ..., \mathbf{v}_n \in \mathbb{R}^n$). Then there exists a constant $c > 0$, which depends only on n and c_0, such that*

$$|X \cdot \mathbf{v}_j| \le |X| \le \frac{1}{c}\sum_{i=1}^{n} |X \cdot \mathbf{v}_i| \qquad \forall\, j = 1, ..., n. \tag{4.1}$$

From Lemma 4.1 we deduce that, given independent unit vectors $\mathbf{v}_1, ..., \mathbf{v}_n \in \mathbb{R}^n$ and $X \in L^2(\Omega; \mathbb{R}^n)$, we have

$$\|X \cdot \mathbf{v}_j\|_{L^2(\Omega)} \le \|X\|_{L^2(\Omega;\mathbb{R}^n)} \le \frac{1}{c}\sum_{i=1}^{n} \|X \cdot \mathbf{v}_i\|_{L^2(\Omega)} \qquad \forall\, j = 1, ..., n.$$

This implies the following result:

Lemma 4.2. *Let $\Omega \subseteq \mathbb{R}^n$ be an open set. Let $\{X_h\}_{h\in\mathbb{N}} \subseteq L^2(\Omega; \mathbb{R}^n)$, $X_\infty \in L^2(\Omega; \mathbb{R}^n)$, and let $\{\mathbf{v}_1, ..., \mathbf{v}_n\}$ be a basis of $\mathbb{R}^n$. Then $\{X_h\}_{h\in\mathbb{N}}$ is precompact in $L^2(\Omega; \mathbb{R}^n)$ if and only if $\{X_h \cdot \mathbf{v}_i\}_{h\in\mathbb{N}}$ is precompact in $L^2(\Omega)$ for every $i = 1, .., n$. If this happens then we have that*

$$\begin{aligned} &\lim_{h\to\infty} X_h = X_\infty \quad \text{in } L^2(\Omega;\mathbb{R}^n) \qquad \text{if and only if} \\ &\qquad\qquad \lim_{h\to\infty} X_h \cdot \mathbf{v}_i = X_\infty \cdot \mathbf{v}_i \quad \text{in } L^2(\Omega)\ \forall\, i = 1, .., n. \end{aligned} \tag{4.2}$$

Another useful result is the following:

Lemma 4.3. *Let $\Lambda > \lambda > 0$ and $r > 0$. For every $h \in \mathbb{N}$ let $A^h : B_r \to \mathbb{R}^{n\times n}$ be a sequence of measurable functions such that $A^h(x)$ is a nonnegative symmetric matrix for a.e. $x \in B_r$, $A^h \le \Lambda\,\mathrm{Id}$ and*

$$\lim_{h\to\infty} \left|\{A^h \le \lambda\,\mathrm{Id}\}\right| = 0. \tag{4.3}$$

Then there exists a measurable function $A : B_r \to \mathbb{R}^{n\times n}$ such that $A(x)$ is a nonnegative symmetric matrix for a.e. $x \in B_r$,

$$\lambda\,\mathrm{Id} \le A(x) \le \Lambda\,\mathrm{Id} \qquad \text{for a.e. } x \in B_r, \tag{4.4}$$

and, up to subsequences,

$$A^h \to A \qquad \text{weakly in } L^2(B_r; \mathbb{R}^{n\times n}). \tag{4.5}$$

Proof. Since $0 \le A^h \le \Lambda\,\mathrm{Id}$ for every $h \in \mathbb{N}$ we have that there exists a function $A : B_r \to \mathbb{R}^{n\times n}$ with $0 \le A \le \Lambda\,\mathrm{Id}$ and such that, up to a subsequence, (4.5) holds. By (4.3), up to a further subsequence we may assume that

$$\sum_{h=1}^{\infty} \left|\{A^h \le \lambda\,\mathrm{Id}\}\right| < \infty. \tag{4.6}$$

Setting

$$I_k = \bigcup_{k\le h} \{A^h \le \lambda\,\mathrm{Id}\} \qquad \forall\, k \in \mathbb{N}$$

we have that $|I_k| \to 0$ by (4.6) and that, by (4.5), $A^h \to A$ weakly in $L^2(B_r \setminus I_k; \mathbb{R}^{n\times n})$ for every $k \in \mathbb{N}$. The set $\{A \in \mathbb{R}^{n\times n} : \lambda\,\mathrm{Id} \le A \le \Lambda\,\mathrm{Id}\}$ is convex and closed in $\mathbb{R}^{n\times n}$. Since $\lambda\,\mathrm{Id} \le A^h(x) \le \Lambda\,\mathrm{Id}$ for every $x \in B_r \setminus I_k$ and for every $h > k$, we take the limit in the weak convergence as $h \to \infty$ and we obtain that $\lambda\,\mathrm{Id} \le A(x) \le \Lambda\,\mathrm{Id}$ for a.e. $x \in B_r \setminus I_k$. Since k is arbitrary, we obtain (4.4). □

The following lemma is a Caccioppoli inequality for a subsolution of an elliptic differental operator in terms of an a priori estimate. The proof follows an idea in [7, Proposition 2.3] and it is based on the variational structure of the equation (4.8).

Lemma 4.4. *Let $\mathbf{v} \in S^{n-1}$, $\lambda > 0$, $c > 0$, and $f \in C^1(B_1)$. Let $F \in C^2(\mathbb{R}^n)$ be a convex function such that*

$$\lambda\,\mathrm{Id} \le \nabla^2 F(x) \qquad \text{for all } x \in \mathbb{R}^n \text{ such that } x \cdot \mathbf{v} \ge c. \tag{4.7}$$

Let $u \in C^2(B_1)$ be a solution of

$$\partial_i(\partial_i F(\nabla u)) = f \qquad \text{in } B_1 \tag{4.8}$$

which is Lipschitz with constant 1 in B_1. Let $G : \mathbb{R} \to \mathbb{R}$ be a nondecreasing 1-Lipschitz function which is constant on the set $\{t \le c\}$. Then there exists $C > 0$, depending only on n and λ, such that for every $\eta \in \mathbb{R}^n$

$$\big\|\nabla[G(\partial_{\mathbf{v}} u)]\big\|_{L^2(B_{3/4})} \le C\big(\|G(\partial_{\mathbf{v}} u)\|_{L^2(B_1)} + \|f\|_{L^2(B_1)} + \|\nabla F(\nabla u) - \eta\|_{L^2(B_1)}\big). \tag{4.9}$$

Proof. By approximation, it suffices to prove the result when $G \in C^1$.

We differentiate the equation (4.8) in the direction $\mathbf{v}$ to get

$$\partial_i(\partial_{ij} F(\nabla u)\partial_{j\mathbf{v}} u) = \partial_{\mathbf{v}} f \qquad \text{in } B_1.$$

Let $\zeta \in C^\infty_c(B_1)$ be a nonnegative and smooth cutoff function which is 1 in $B_{3/4}$. We test the above equation with the test function $G(\partial_{\mathbf{v}} u)\,\zeta^2$, which is Lipschitz and compactly supported, and we integrate by parts:

$$\begin{aligned} &\int_{B_1} \partial_{ij} F(\nabla u)\,\partial_{j\mathbf{v}} u\,\partial_i[G(\partial_{\mathbf{v}} u)]\,\zeta^2 \\ &= -2\int_{B_1} \partial_{ij} F(\nabla u)\,\partial_{j\mathbf{v}} u\, G(\partial_{\mathbf{v}} u)\,\zeta\partial_i\zeta + \int_{B_1} f\,\partial_{\mathbf{v}}[G(\partial_{\mathbf{v}} u)]\,\zeta^2 + 2\int_{B_1} f\, G(\partial_{\mathbf{v}} u)\,\zeta\partial_{\mathbf{v}}\zeta. \end{aligned} \tag{4.10}$$

We estimate each term of (4.10). As regards the left-hand side we notice that $G'(\partial_{\mathbf{v}} u) = 0$ on the set $\{\partial_{\mathbf{v}} u \le c\}$. Hence we apply (4.7) and the fact that $0 \le G' \le 1$ to get

$$\int_{B_1} \partial_{ij} F(\nabla u)\,\partial_{j\mathbf{v}} u\, G'(\partial_{\mathbf{v}} u)\,\partial_{i\mathbf{v}} u\,\zeta^2 \ge \lambda \int_{B_1} G'(\partial_{\mathbf{v}} u)\,|\nabla\partial_{\mathbf{v}} u|^2\zeta^2 \ge \lambda \int_{B_1} |\nabla[G(\partial_{\mathbf{v}} u)]|^2\zeta^2 \tag{4.11}$$

To estimate the first term in the right-hand side of (4.10) we integrate by parts and, for some ε to be chosen later, we have

$$\begin{aligned}
&-2\int_{B_1}\partial_{ij}F(\nabla u)\,\partial_{j\mathbf{v}}u\,G(\partial_{\mathbf{v}}u)\,\zeta\partial_i\zeta = -2\int_{B_1}\partial_{\mathbf{v}}[\partial_iF(\nabla u)-\eta_i]\,G(\partial_{\mathbf{v}}u)\,\zeta\partial_i\zeta\\
&= 2\int_{B_1}[\partial_iF(\nabla u)-\eta_i]\,\partial_{\mathbf{v}}[G(\partial_{\mathbf{v}}u)]\,\zeta\partial_i\zeta + 2\int_{B_1}[\partial_iF(\nabla u)-\eta_i]\,G(\partial_{\mathbf{v}}u)\,\partial_{\mathbf{v}}[\zeta\partial_i\zeta]\\
&\le \varepsilon\int_{B_1}|\nabla[G(\partial_{\mathbf{v}}u)]|^2\zeta^2 + \frac{\|\nabla\zeta\|_\infty^2}{\varepsilon}\int_{B_1}|\nabla F(\nabla u)-\eta|^2\\
&\quad + \|\nabla[\zeta\nabla\zeta]\|_\infty^2\int_{B_1}|G(\partial_{\mathbf{v}}u)|^2 + \int_{B_1}|\nabla F(\nabla u)-\eta|^2
\end{aligned}\tag{4.12}$$

As regards the last two terms in (4.10) we have

$$\begin{aligned}
&\int_{B_1} f\,\partial_{\mathbf{v}}[G(\partial_{\mathbf{v}}u)] + 2\int_{B_1} f\,G(\partial_{\mathbf{v}}u)\,\zeta\partial_{\mathbf{v}}\zeta\\
&\le \frac{\varepsilon}{2}\int_{B_1}|\nabla[G(\partial_{\mathbf{v}}u)]|^2 + \frac{1}{2\varepsilon}\int_{B_1}f^2 + \|\nabla\zeta\|_\infty^2\int_{B_1}|G(\partial_{\mathbf{v}}u)|^2 + \int_{B_1}f^2.
\end{aligned}\tag{4.13}$$

We choose $\varepsilon \le 2\lambda/3$ and we obtain from (4.10), (4.11), (4.12), (4.13) that there exists a constant C, depending only on n and λ, such that

$$\begin{aligned}
\int_{B_{3/4}}|\nabla[G(\partial_{\mathbf{v}}u)]|^2 &\le \int_{B_1}|\nabla[G(\partial_{\mathbf{v}}u)]|^2\zeta^2\\
&\le C\Big(\int_{B_1}|\nabla[G(\partial_{\mathbf{v}}u)]|^2 + \int_{B_1}f^2 + \int_{B_1}|\nabla F(\nabla u)-\eta|^2\Big),
\end{aligned}$$

proving (4.9). ☐

Proof of Theorem 2.1. With a standard regularization, presented in detail in an analogous situation in [6, Proof of Theorem 1.1], we may assume without loss of generality that $F \in C^2(B_1)$, $f \in C^1(B_1)$, and that $u \in C^2(B_1)$ is a solution of

$$\partial_i(\partial_iF(\nabla u)) = f \qquad \text{in } B_1. \tag{4.14}$$

By contradiction, let $\tau, \alpha > 0$ to be chosen later and let us consider sequences $\{x_h\}_{h\in\mathbb{N}} \subseteq B_{1/2}$, $\{r_h\}_{h\in\mathbb{N}} \subseteq (0,1/4)$, and $\{u_h\}_{h\in\mathbb{N}} \subseteq C^2(B_1)$ such that u_h are solutions to (4.14) and

$$|\nabla u_h| \le 1 \qquad \text{in } B_1 \quad \forall\, h\in\mathbb{N}, \tag{4.15}$$

$$U(u_h, x_h, r_h) = \lambda_h \to 0 \qquad \text{as } h\to\infty, \tag{4.16}$$

$$U(u_h, x_h, \tau r_h) > \tau^\alpha U(u_h, x_h, r_h) \qquad \forall\, h\in\mathbb{N}, \tag{4.17}$$

$$(\nabla u_h)_{B_{r_h}(x_h)} \to \gamma_\infty \qquad \text{as } h\to\infty, \qquad \gamma_\infty\in\mathbb{R}^n,\ \frac{3}{4}\le|\gamma_\infty|\le 1. \tag{4.18}$$

Let us define the rescaled functions

$$\tilde u_h(x) := \frac{u_h(x_h + r_h x)}{r_h} \qquad x\in B_1;$$

since u_h are solutions to (4.14) we have

$$\partial_i(\partial_iF(\nabla\tilde u_h)) = \tilde f_h \qquad \text{in } B_1, \tag{4.19}$$

where $\tilde f_h(x) := r_h f(x_h + r_h x)$ for $x \in B_1$. Moreover, setting $\gamma_h := (\nabla u_h)_{B_{r_h}(x_h)}$, we have that $\gamma_h = (\nabla \tilde u_h)_{B_1}$. We remark that, by a change of variables,

$$\|\tilde f_h\|_{L^q(B_1)} = r_h^{(q-n)/q}\Big(\int_{B_{r_h}(x_h)} |f(y)|^q\,dy\Big)^{1/q} = r_h^{(q-n)/q}\|f\|_{L^q(B_{r_h}(x_h))} \tag{4.20}$$

By the change of variable formula we rewrite (4.15), (4.16), (4.17), and (4.18) in terms of $\tilde u_h$:

$$|\nabla \tilde u_h| \le 1 \qquad \forall\, h \in \mathbb{N}, \tag{4.21}$$

$$\Big(\fint_{B_1} |\nabla \tilde u_h(y) - \gamma_h|^2\,dy\Big)^{1/2} + r_h^{(q-n)/(2q)}\|f\|_{L^q(B_1)} = \lambda_h \to 0 \qquad \text{as } h \to \infty, \tag{4.22}$$

(which implies that $r_h \to 0$ as $h \to \infty$),

$$\Big(\fint_{B_\tau} \frac{|\nabla \tilde u_h - (\nabla \tilde u_h)_{B_\tau}|^2}{\lambda_h^2}\Big)^{1/2} + \frac{(\tau r_h)^{(q-n)/(2q)}}{\lambda_h}\|f\|_{L^q(B_1)} > \tau^\alpha \qquad \forall\, h \in \mathbb{N}, \tag{4.23}$$

$$(\nabla \tilde u_h)_{B_1} = \gamma_h \to \gamma_\infty \qquad \text{as } h \to \infty, \qquad \gamma_\infty \in \mathbb{R}^n,\ \frac{3}{4} \le |\gamma_\infty| \le 1. \tag{4.24}$$

By Poincaré inequality and (4.22) we have that

$$\|\tilde u_h(x) - \tilde u_h(0) - \gamma_h \cdot x\|_{L^2(B_1)} \lesssim \lambda_h; \tag{4.25}$$

therefore the functions

$$\frac{\tilde u_h(x) - \tilde u_h(0) - \gamma_h \cdot x}{\lambda_h}$$

are bounded in $W^{1,2}(B_1)$. Hence there exists $u_\infty \in W^{1,2}(B_1)$ such that, up to a subsequence,

$$\frac{\tilde u_h(x) - \tilde u_h(0) - \gamma_h \cdot x}{\lambda_h} \to u_\infty(x) \qquad \text{in } L^2(B_1), \tag{4.26}$$

$$\frac{\nabla \tilde u_h(x) - \gamma_h}{\lambda_h} \to \nabla u_\infty(x) \qquad \text{weakly in } L^2(B_1). \tag{4.27}$$

Step 1. Let $\mathbf{v} \in S^{n-1}$ be such that $5/8 < \gamma_\infty \cdot \mathbf{v}$ (so that $1/2 < \gamma_h \cdot \mathbf{v} \le 1$ for h large enough), and set

$$v_h(x) := \Big(\partial_{\mathbf{v}} \tilde u_h(x) - \frac{\gamma_h \cdot \mathbf{v}}{2}\Big)_+ - \frac{\gamma_h \cdot \mathbf{v}}{2},$$

$$w_h(x) := \partial_{\mathbf{v}} \tilde u_h(x) - \gamma_h \cdot \mathbf{v}. \tag{4.28}$$

From the fact that

$$v_h = w_h \qquad \text{on } \Big\{x \in B_1 : \partial_{\mathbf{v}} \tilde u_h(x) \ge \frac{\gamma_h \cdot \mathbf{v}}{2}\Big\}$$

and

$$0 > v_h = -\frac{\gamma_h \cdot \mathbf{v}}{2} > w_h \qquad \text{on } \Big\{x \in B_1 : \partial_{\mathbf{v}} \tilde u_h(x) < \frac{\gamma_h \cdot \mathbf{v}}{2}\Big\}$$

we obtain

$$\|v_h\|_{L^2(B_1)} \le \|w_h\|_{L^2(B_1)} \le C_0 \lambda_h, \tag{4.29}$$

which implies

$$\lim_{h\to\infty} \|v_h\|_{L^2(B_1)} = 0, \qquad \lim_{h\to\infty} \|w_h\|_{L^2(B_1)} = 0. \tag{4.30}$$

Let $\sigma := 2n/(n-1)$. We claim that there exist constants $C_1, C_2, C_3 > 0$ such that

$$\|\nabla v_h\|_{L^2(B_{3/4})} \le C_1\big(\|v_h\|_{L^2(B_1)} + \|\tilde f_h\|_{L^2(B_1)} + \|\nabla F(\nabla \tilde u_h) - \nabla F(\gamma_h)\|_{L^2(B_1)}\big) \le C_2\lambda_h, \tag{4.31}$$

$$\|v_h\|_{L^\sigma(B_{3/4})} + \|w_h\|_{L^\sigma(B_{3/4})} \le C_3\lambda_h, \tag{4.32}$$

$$\lim_{h\to\infty}\Big\|\frac{v_h - w_h}{\lambda_h}\Big\|_{L^2(B_{3/4})} = 0. \tag{4.33}$$

Notice that from (4.31) and (4.29) we obtain that the sequence $\{v_h/\lambda_h\}_{h\in\mathbb{N}}$ is bounded in $W^{1,2}(B_{3/4})$ and therefore it is precompact in $L^2(B_{3/4})$; from (4.33) we also obtain that

$$\text{the sequence } \{w_h/\lambda_h\}_{h\in\mathbb{N}} \text{ is precompact in } L^2(B_{3/4}). \tag{4.34}$$

We now prove (4.31), (4.32), and (4.33).

By Lemma 4.4 applied with $u = \tilde u_h$, $f = \tilde f_h$, $c = \gamma_h\cdot\mathbf{v}/2 > 1/4$, $\eta = \nabla F(\gamma_h)$, and $G(x) = (x - \gamma_h\cdot\mathbf{v}/2)_+ - \gamma_h\cdot\mathbf{v}/2$, we obtain that

$$\|\nabla v_h\|_{L^2(B_{3/4})} \le C_1\Big(\|v_h\|_{L^2(B_1)} + \|\tilde f_h\|_{L^2(B_1)} + \|\nabla F(\nabla\tilde u_h) - \nabla F(\gamma_h)\|_{L^2(B_1)}\Big)$$

We claim that the three terms in the right-hand side can be estimated by the excess λ_h up to a constant. Indeed by (4.29) we estimate the first term; from (4.20) we deduce that

$$\|\tilde f_h\|_{L^2(B_1)} \lesssim \|\tilde f_h\|_{L^q(B_1)} \le r_h^{-(q-n)/(2q)}\|\tilde f_h\|_{L^q(B_1)} \lesssim \lambda_h.$$

Finally, for the last term we remember that $|\gamma_h| \ge 3/4$, F is Lipschitz in B_1 (by convexity) and $F \in C^{1,1}(\mathbb{R}^n\setminus B_{1/4})$. Hence, $|\nabla F(\nabla\tilde u_h) - \nabla F(\gamma_h)|$ can be estimated thanks to the Lipschitz regularity of ∇F on the set $\{|\nabla\tilde u_h| \ge 1/4\}$; on the complement $\{|\nabla\tilde u_h| < 1/4\}$ we estimate the quantity $|\nabla F(\nabla\tilde u_h) - \nabla F(\gamma_h)|$ by $2\|\nabla F\|^2_{L^\infty(B_1)}$ and we notice that on that set $|\nabla\tilde u_h - \gamma_h| \ge 1/8$. We therefore obtain

$$\int_{B_1}|\nabla F(\nabla\tilde u_h) - \nabla F(\gamma_h)|^2 \le C\Big(\|\nabla^2 F\|^2_{L^\infty(\mathbb{R}^n\setminus B_{1/4})} + \|\nabla F\|^2_{L^\infty(B_1)}\Big)\int_{B_1}|\nabla\tilde u_h - \gamma_h|^2$$

and we conclude the proof of the second inequality in (4.31).

Since $W^{1,2}(B_{3/4})$ embeds into $L^\sigma(B_{3/4})$, by (4.31) we have that

$$\|v_h\|_{L^\sigma(B_{3/4})} \le C_4\lambda_h;$$

from the higher integrability of v_h and the fact that $\gamma_h\cdot\mathbf{v}/2 \ge 1/4$ we obtain

$$\begin{aligned}\Big|\Big\{x\in B_{3/4} : \partial_{\mathbf{v}}\tilde u_h(x) < \frac{\gamma_h\cdot\mathbf{v}}{2}\Big\}\Big| &\le \Big|\Big\{x\in B_{3/4} : \partial_{\mathbf{v}}\tilde u_h(x) < \frac{\gamma_h\cdot\mathbf{v}}{2}\Big\}\Big| 4^\sigma\Big(\frac{\gamma_h\cdot\mathbf{v}}{2}\Big)^\sigma \\ &\le 4^\sigma\|v_h\|^\sigma_{L^\sigma(B_{3/4})} \le C_5\lambda_h^\sigma.\end{aligned} \tag{4.35}$$

Then, from (4.35) and since $\tilde u_h$ is Lipschitz with constant 1 (see (4.21)) we get

$$\Big\|\frac{v_h - w_h}{\lambda_h}\Big\|^2_{L^2(B_{3/4})} \le \frac{4}{\lambda_h^2}\Big|\Big\{x\in B_{3/4} : \partial_{\mathbf{v}}\tilde u_h(x) < \frac{\gamma_h\cdot\mathbf{v}}{2}\Big\}\Big| \le 4C_5\lambda_h^{\sigma-2},$$

which converges to 0 by (4.22) and proves (4.33).

Finally, by (4.21), (4.35), and (4.29) we have

$$\begin{aligned}\|w_h\|^\sigma_{L^\sigma(B_{3/4})} &\leq \int_{B_{3/4}\cap\{\partial_{\mathbf{v}}\tilde{u}_h\geq\frac{\gamma_h\cdot\mathbf{v}}{2}\}}|\partial_{\mathbf{v}}\tilde{u}_h(x)-\gamma_h\cdot\mathbf{v}|^\sigma+\Big|\Big\{x\in B_{3/4}:\partial_{\mathbf{v}}\tilde{u}_h(x)<\frac{\gamma_h\cdot\mathbf{v}}{2}\Big\}\Big|2^\sigma\\ &\leq \|v_h\|^\sigma_{L^\sigma(B_{3/4})}+C_5 2^\sigma\lambda_h^\sigma\leq(1+2^\sigma C_5)\lambda_h^\sigma,\end{aligned}$$

which proves (4.32).

Step 2. We claim that

$$\lim_{h\to\infty}\frac{\nabla\tilde{u}_h-\gamma_h}{\lambda_h}=\nabla u_\infty\qquad\text{in } L^2(B_{3/4}) \tag{4.36}$$

and

$$\|\nabla\tilde{u}_h-\gamma_h\|_{L^\sigma(B_{3/4})}\leq C_6\lambda_h. \tag{4.37}$$

Indeed, let $\mathbf{v}_1,...,\mathbf{v}_n\in S^{n-1}$ be n linearly independent vectors such that $\gamma_\infty\cdot\mathbf{v}_i>5/8$ and $\big|\det\big(\mathbf{v}_1|...|\mathbf{v}_n\big)\big|\geq C(n)>0$. First, we prove that the sequence $(\nabla\tilde{u}_h-\gamma_h)/\lambda_h$ is precompact in $L^2(B_{3/4};\mathbb{R}^n)$. Thanks to Lemma 4.2 it is enough to show that $\mathbf{v}_i\cdot(\nabla\tilde{u}_h-\gamma_h)/\lambda_h$ is precompact for every $i=1,...,n$, which in turn follows from (4.34), applied with $w_h=\partial_{\mathbf{v}_i}\tilde{u}_h(x)-\gamma_h\cdot\mathbf{v}_i$. The characterization of the limit of a subsequence of $(\nabla\tilde{u}_h-\gamma_h)/\lambda_h$ follows from (4.27). As a consequence, it is not necessary to consider a subsequence. Finally, from Lemma 4.1 and (4.32) we obtain that

$$\|\nabla\tilde{u}_h-\gamma_h\|_{L^\sigma(B_{3/4})}\lesssim\sum_{i=1}^n\|\mathbf{v}_i\cdot(\nabla\tilde{u}_h-\gamma_h)\|_{L^\sigma(B_{3/4})}\leq nC_3\lambda_h,$$

which proves (4.37).

Step 3. Given a function $f:B_1\to\mathbb{R}$, $\mathbf{v}\in S^{n-1}$, and $\varepsilon>0$, we define the discrete derivative of f as

$$[\partial^\varepsilon_{\mathbf{v}}f](x):=\frac{f(x+\varepsilon\mathbf{v})-f(x)}{\varepsilon}\qquad x\in B_{1-\varepsilon}$$

and the discrete gradient as

$$\nabla^\varepsilon f(x):=\Big([\partial^\varepsilon_{e_1}f](x),...,[\partial^\varepsilon_{e_n}f](x)\Big)\qquad x\in B_{1-\varepsilon}.$$

We claim that, for ε sufficiently small,

$$\|\nabla^\varepsilon u_\infty\|_{L^2(B_{3/4})}\leq\|\nabla u_\infty\|_{L^2(B_1)}\lesssim 1, \tag{4.38}$$

$$\fint_{B_\tau}|\nabla u_\infty-(\nabla u_\infty)_{B_\tau}|^2\,dx\geq\frac{\tau^{2\alpha}}{4}, \tag{4.39}$$

$$\fint_{B_\tau}|\nabla^\varepsilon u_\infty-(\nabla^\varepsilon u_\infty)_{B_\tau}|^2\,dx>\frac{\tau^{2\alpha}}{8}. \tag{4.40}$$

We notice that the second inequality in (4.38) follows from (4.27) and the lower semicontinuity of the norm. To prove (4.39), we first take the limit in the second term of the left-hand side of (4.23) to get

$$\limsup_{h\to\infty}\frac{(\tau r_h)^{(q-n)/(2q)}}{\lambda_h}\|f\|_{L^q(B_1)}\leq\tau^{(q-n)/(2q)}\leq\frac{\tau^\alpha}{2}, \tag{4.41}$$

where in the last inequality we have assumed that $\alpha<(q-n)/(2q)$ and τ is sufficiently small (depending on q,n,α).

We notice now, as a general remark, that if $r \in (0,1]$, $f_h, f \in L^2(B_r)$, and $\lim_{h\to\infty} f_h = f$ in $L^2(B_r)$, then

$$\lim_{h\to\infty} \fint_{B_r} |f_h - (f_h)_{B_r}|^2 = \fint_{B_r} |f - (f)_{B_r}|^2. \tag{4.42}$$

Applying (4.42) to $f_h := (\nabla u_h - \gamma_h)/\lambda_h$ and $r := \tau < 3/4$ (so that by (4.36) we have that $(\nabla u_h - \gamma_h)/\lambda_h \to \nabla u_\infty$ in $L^2(B_\tau)$), letting $h \to \infty$ in (4.23) and taking (4.41) into account we obtain

$$\begin{aligned}
\Big(\fint_{B_\tau} |\nabla u_\infty - (\nabla u_\infty)_{B_\tau}|^2\,dx\Big)^{1/2} &= \lim_{h\to\infty} \Big(\fint_{B_\tau} \frac{|\nabla \tilde u_h - (\nabla \tilde u_h)_{B_\tau}|^2}{\lambda_h^2}\Big)^{1/2} \\
&\geq \liminf_{h\to\infty} \Big(\tau^\alpha - \frac{(\tau r_h)^{(q-n)/(2q)}}{\lambda_h} \|f\|_{L^q(B_1)}\Big) \\
&\geq \tau^\alpha - \frac{\tau^\alpha}{2} = \frac{\tau^\alpha}{2},
\end{aligned}$$

which proves (4.39).

Finally, since $\lim_{\varepsilon\to 0} \nabla^\varepsilon u_\infty = \nabla u_\infty$ in $L^2(B_\tau)$, we apply (4.42) to $\nabla^\varepsilon u_\infty$ and $r = \tau$ to deduce from (4.39) that (4.40) holds true for ε sufficiently small.

Step 4. Let $\mathbf{v} \in S^{n-1}$ and for every $h \in \mathbb{N}$ let $\mathbf{w}_h = \gamma_h/|\gamma_h|$. We claim that the function $\partial^\varepsilon_{\mathbf{v}} \tilde u_h$ solves

$$\int_{B_{3/4}} A^{h,\varepsilon}_{ij}(x)\, \partial_i \partial^\varepsilon_{\mathbf{v}} \tilde u_h(x)\, \partial_j \phi(x)\,dx + \int_{B_{3/4}} \partial^\varepsilon_{\mathbf{v}} \tilde f_h(x)\, \phi(x)\,dx = 0 \tag{4.43}$$

for every $\phi \in W^{1,2}_0(B_{3/4})$, $h \in \mathbb{N}$, and $\varepsilon \in (0,1/4)$, for some measurable coefficients $A^{h,\varepsilon}_{ij} : B_{3/4} \to \mathbb{R}$ with the property that $A^{h,\varepsilon}(x)$ is a nonnegative symmetric matrix for every $x \in B_{3/4}$ and that

$$\lambda\,\mathrm{Id} \leq (A^{h,\varepsilon}_{ij}(y)) \leq \Lambda\,\mathrm{Id} \qquad \forall\, y \in B_{3/4} \cap \Big\{\partial_{\mathbf{w}_h} \tilde u_h \geq \frac{1}{4}\Big\} \cap \Big\{\partial_{\mathbf{w}_h} \tilde u_h(\cdot + \varepsilon \mathbf{v}) \geq \frac{1}{4}\Big\}. \tag{4.44}$$

Indeed, since $\tilde u_h$ are solutions of (4.19), for every $\phi \in W^{1,2}_0(B_{3/4})$ and $\varepsilon < 1/4$ we have

$$\int_{B_{3/4}} \partial_i F(\nabla \tilde u_h(x))\, \partial_i \phi(x) = -\int_{B_{3/4}} \tilde f_h(x)\, \phi(x),$$

$$\int_{B_{3/4}} \partial_i F(\nabla \tilde u_h(x + \varepsilon \mathbf{v}))\, \partial_i \phi(x) = -\int_{B_{3/4}} \tilde f_h(x + \varepsilon \mathbf{v})\, \phi(x).$$

Subtracting the two equations and dividing by ε we obtain

$$\begin{aligned}
-\int_{B_{3/4}} \frac{\tilde f_h(x+\varepsilon\mathbf{v}) - \tilde f_h(x)}{\varepsilon}\,\phi(x) &= \int_{B_{3/4}} \frac{\partial_i F(\nabla \tilde u_h(x+\varepsilon\mathbf{v})) - \partial_i F(\nabla \tilde u_h(x))}{\varepsilon}\,\partial_i\phi(x) \\
&= \int_{B_{3/4}} A^{h,\varepsilon}_{ij}(x)\,\partial_j \partial^\varepsilon_{\mathbf{v}} \tilde u_h(x)\,\partial_i \phi(x),
\end{aligned}$$

where

$$A^{h,\varepsilon}_{ij}(x) := \int_0^1 \partial_{ij} F\big((1-t)\nabla \tilde u_h(x+\varepsilon\mathbf{v}) + t\nabla \tilde u_h(x)\big)\,dt \qquad \forall\, x \in B_{3/4}. \tag{4.45}$$

Notice that, if $x \in B_{3/4}$ is a point such that $\partial_{\mathbf{w}_h} \tilde u_h(x) \geq 1/4$ and $\partial_{\mathbf{w}_h} \tilde u_h(x+\varepsilon\mathbf{v}) \geq 1/4$ then for every $t \in [0,1]$

$$|(1-t)\nabla \tilde u_h(x+\varepsilon\mathbf{v}) + t\nabla \tilde u_h(x)| \geq (1-t)\partial_{\mathbf{w}_h} \tilde u_h(x+\varepsilon\mathbf{v}) + t\partial_{\mathbf{w}_h} \tilde u_h(x) \geq \frac{1}{4},$$

therefore (4.44) holds true thanks to (2.1).

Step 5. Let $\mathbf{v} \in S^{n-1}$. We claim that, for every $\varepsilon > 0$ sufficiently small, the function $\partial^\varepsilon_{\mathbf{v}} \tilde u_\infty$ solves

$$\int_{B_{3/4}} A^\varepsilon_{ij}(x)\,\partial_i \partial^\varepsilon_{\mathbf{v}} \tilde u_\infty(x)\,\partial_j\varphi(x)\,dx = 0 \tag{4.46}$$

for every $\varphi \in W^{1,2}_0(B_{3/4-\varepsilon})$, for some measurable coefficients $A^\varepsilon_{ij} : B_{3/4} \to \mathbb{R}$ with the property that

$$\lambda\,\mathrm{Id} \le (A^\varepsilon_{ij}) \le \Lambda\,\mathrm{Id} \qquad \forall\, x \in B_{3/4}. \tag{4.47}$$

Indeed, let us consider the function $\phi(x) := \varphi(x)\chi(\partial_{\mathbf{w}_h}\tilde u_h(x))\chi(\partial_{\mathbf{w}_h}\tilde u_h(x+\varepsilon\mathbf{v}))$ where $\chi \in C^\infty(\mathbb{R})$ is such that $\chi((-\infty, 1/2]) = 0$ and $\chi([5/8,\infty)) = 1$. By the identity

$$\chi(\partial_{\mathbf{w}_h}\tilde u_h) = \chi\Big(\Big(\partial_{\mathbf{w}_h}\tilde u_h(x) - \frac{|\gamma_h|}{2}\Big)_+ - \frac{|\gamma_h|}{2}\Big)$$

and (4.31) applied to $v_h = \big(\partial_{\mathbf{w}_h}\tilde u_h(x) - |\gamma_h|/2\big)_+ + |\gamma_h|/2$ we have that $\chi(\partial_{\mathbf{w}_h}\tilde u_h(x)) \in W^{1,2}\cap L^\infty(B_{3/4})$ with derivative

$$\partial_j[\chi(\partial_{\mathbf{w}_h}\tilde u_h)] = \chi'(\partial_{\mathbf{w}_h}\tilde u_h)\partial_j\Big[\Big(\partial_{\mathbf{w}_h}\tilde u_h(x) - \frac{|\gamma_h|}{2}\Big)_+\Big].$$

Similarly $\chi(\partial_{\mathbf{w}_h}\tilde u_h(x+\varepsilon\mathbf{v})) \in W^{1,2}\cap L^\infty(B_{3/4})$ with derivative

$$\partial_j[\chi(\partial_{\mathbf{w}_h}\tilde u_h(x+\varepsilon\mathbf{v})))] = \chi'(\partial_{\mathbf{w}_h}\tilde u_h(x+\varepsilon\mathbf{v}))\partial_j\Big[\Big(\partial_{\mathbf{w}_h}\tilde u_h(x+\varepsilon\mathbf{v}) - \frac{|\gamma_h|}{2}\Big)_+\Big].$$

Hence $\phi(x) \in W^{1,2}_0 \cap L^\infty(B_{5/8})$. Notice also that from (4.31) it follows that

$$\Big\|\nabla\Big(\partial_{\mathbf{w}_h}\tilde u_h(x) - \frac{|\gamma_h|}{2}\Big)_+\Big\|_{L^2(B_{3/4})} \lesssim \lambda_h. \tag{4.48}$$

Moreover we have that, since $|\gamma_h \ge 3/4$,

$$\begin{aligned}\frac{\Big|\Big\{x \in B_{3/4} : \partial_{\mathbf{w}_h}\tilde u_h(x) < \frac58\Big\}\Big|^{1/2}}{|B_{3/4}|^{1/2}}\cdot\frac18 &\le \Big(\fint_{B_{3/4}}\Big|\partial_{\mathbf{w}_h}\tilde u_h(y) - |\gamma_h|\Big|^2 dy\Big)^{1/2}\\ &\le \Big(\fint_{B_1}|\nabla\tilde u_h(y) - \gamma_h|^2\,dy\Big)^{1/2} \le \lambda_h\end{aligned}$$

and therefore

$$\lim_{h\to\infty}\Big|\Big\{x \in B_{3/4} : \partial_{\mathbf{w}_h}\tilde u_h(x) \ge \frac58\Big\}\Big| = |B_{3/4}|. \tag{4.49}$$

Similarly

$$\lim_{h\to\infty}\Big|\Big\{x \in B_{3/4} : \partial_{\mathbf{w}_h}\tilde u_h(x+\varepsilon\mathbf{v}) \ge \frac58\Big\}\Big| = |B_{3/4}|. \tag{4.50}$$

Using ϕ as a test function in (4.43) and dividing by λ_h we obtain

$$
\begin{aligned}
0 &= \int_{B_{3/4}} \frac{\partial^\varepsilon_{\mathbf{v}} \tilde f_h(x)}{\lambda_h}\,\phi(x) + \int_{B_{3/4}} A^{h,\varepsilon}_{ij}(x)\,\partial_i\partial^\varepsilon_{\mathbf{v}}\tilde u_h(x)\,\partial_j\Big[\varphi(x)\chi(\partial_{\mathbf{w}_h}\tilde u_h(x))\chi(\partial_{\mathbf{w}_h}\tilde u_h(x+\varepsilon\mathbf{v}))\Big] \\
&= \int_{B_{3/4}} \frac{\partial^\varepsilon_{\mathbf{v}} \tilde f_h(x)}{\lambda_h}\,\phi(x) + \int_{B_{3/4}} A^{h,\varepsilon}_{ij}(x)\,\frac{\partial_i\partial^\varepsilon_{\mathbf{v}}\tilde u_h(x)}{\lambda_h}\,\partial_j\varphi(x)\,\chi(\partial_{\mathbf{w}_h}\tilde u_h(x))\,\chi(\partial_{\mathbf{w}_h}\tilde u_h(x+\varepsilon\mathbf{v})) \\
&\quad + \int_{B_{3/4}} A^{h,\varepsilon}_{ij}(x)\,\frac{\partial_i\partial^\varepsilon_{\mathbf{v}}\tilde u_h(x)}{\lambda_h}\,\varphi(x)\,\partial_j\Big(\chi(\partial_{\mathbf{w}_h}\tilde u_h(x))\chi(\partial_{\mathbf{w}_h}\tilde u_h(x+\varepsilon\mathbf{v}))\Big).
\end{aligned}
\tag{4.51}
$$

We want to take the limit as $h\to\infty$ in (4.51). As regards the first term in the right-hand side, by Hölder inequality and (4.20) we have

$$
\begin{aligned}
\Big|\int_{B_{3/4}} \frac{\partial^\varepsilon_{\mathbf{v}} \tilde f_h(x)}{\lambda_h}\phi(x)\Big| &\le \frac{\|\partial^\varepsilon_{\mathbf{v}}\tilde f_h(x)\|_{L^1(B_{3/4})}}{\lambda_h}\|\phi(x)\|_{L^\infty(B_{3/4})} \\
&\lesssim \frac{\|\tilde f_h(x)\|_{L^q(B_1)}}{\varepsilon r_h^{(q-n)/(2q)}\|f\|_{L^q(B_1)}}\|\phi(x)\|_{L^\infty(B_{3/4})} \\
&\le \frac{r_h^{(q-n)/(2q)}}{\varepsilon}\|\phi(x)\|_{L^\infty(B_{3/4})},
\end{aligned}
$$

therefore

$$
\lim_{h\to\infty}\int_{B_{3/4}} \frac{\partial^\varepsilon_{\mathbf{v}}\tilde f_h(x)}{\lambda_h}\phi(x) = 0. \tag{4.52}
$$

Then, we want to apply Lemma 4.3 to $A^h(x) := A^{h,\varepsilon}(x)\chi(\partial_{\mathbf{w}_h}\tilde u_h(x))\chi(\partial_{\mathbf{w}_h}\tilde u_h(x+\varepsilon\mathbf{v}))$. For this, notice that the assumption (4.3) of the lemma is satisfied thanks to (4.44), (4.49), (4.50), and the fact that

$$
\chi(\partial_{\mathbf{w}_h}\tilde u_h(x)) = \chi(\partial_{\mathbf{w}_h}\tilde u_h(x+\varepsilon\mathbf{v})) = 1
$$

on the set

$$
\Big\{x\in B_{3/4} : \partial_{\mathbf{w}_h}\tilde u_h(x) \ge \frac{5}{8}\Big\} \cap \Big\{x\in B_{3/4} : \partial_{\mathbf{w}_h}\tilde u_h(x+\varepsilon\mathbf{v}) \ge \frac{5}{8}\Big\}.
$$

Moreover, for every $x\in B_{3/4}$ such that $\chi(\partial_{\mathbf{w}_h}\tilde u_h(x))\chi(\partial_{\mathbf{w}_h}\tilde u_h(x+\varepsilon\mathbf{v})) > 0$ we have that $\partial_{\mathbf{w}_h}\tilde u_h(x) > 1/2$ and $\partial_{\mathbf{w}_h}\tilde u_h(x+\varepsilon\mathbf{v}) > 1/2$ and therefore

$$
\lambda\,\mathrm{Id} \le A^{h,\varepsilon}(x) \le \Lambda\,\mathrm{Id}.
$$

This implies that

$$
0 \le A^{h,\varepsilon}(x)\,\chi(\partial_{\mathbf{w}_h}\tilde u_h(x))\,\chi(\partial_{\mathbf{w}_h}\tilde u_h(x+\varepsilon\mathbf{v})) \le \Lambda\,\mathrm{Id} \qquad \forall\, x\in B_{3/4}.
$$

Hence, we obtain that there exist $A^\varepsilon : B_{3/4}\to\mathbb{R}^{n\times n}$ such that $\lambda\,\mathrm{Id}\le A^\varepsilon\le\Lambda\,\mathrm{Id}$ and, up to subsequences,

$$
A^{h,\varepsilon}(x)\,\chi(\partial_{\mathbf{w}_h}\tilde u_h(x))\,\chi(\partial_{\mathbf{w}_h}\tilde u_h(x+\varepsilon\mathbf{v})) \to A^\varepsilon(x) \qquad \text{weakly in } L^2(B_{3/4};\mathbb{R}^{n\times n}). \tag{4.53}
$$

From the equality

$$
\frac{\partial_i\partial^\varepsilon_{\mathbf{v}}\tilde u_h(x)}{\lambda_h} = \frac{1}{\varepsilon}\left(e_i\cdot\Big(\frac{\nabla\tilde u_h(x+\varepsilon\mathbf{v})-\gamma_h}{\lambda_h}\Big) - e_i\cdot\Big(\frac{\nabla\tilde u_h(x)-\gamma_h}{\lambda_h}\Big)\right)
$$

and by (4.36) we have

$$\lim_{h\to\infty}\frac{\partial_i\partial^\varepsilon_{\mathbf{v}}\tilde u_h(x)}{\lambda_h}=\frac{\partial_i u_\infty(x+\varepsilon\mathbf{v})-\partial_i u_\infty(x)}{\varepsilon}=\partial_i\partial^\varepsilon_{\mathbf{v}}u_\infty(x)\qquad\text{in } L^2(B_{3/4-\varepsilon}),\tag{4.54}$$

so by (4.53), (4.54), and the fact that $\partial_j\varphi\in L^\infty(B_{3/4})$, we obtain

$$\begin{aligned}\lim_{h\to\infty}\int_{B_{3/4}}A^{h,\varepsilon}_{ij}(x)\,\frac{\partial_i\partial^\varepsilon_{\mathbf{v}}\tilde u_h(x)}{\lambda_h}\,\partial_j\varphi(x)\,\chi(\partial_{\mathbf{w}_h}\tilde u_h(x))\,\chi(\partial_{\mathbf{w}_h}\tilde u_h(x+\varepsilon\mathbf{v}))\\=\int_{B_{3/4}}A^\varepsilon_{ij}(x)\,\partial_i\partial^\varepsilon_{\mathbf{v}}\tilde u_\infty(x)\,\partial_j\varphi(x).\end{aligned}\tag{4.55}$$

To estimate the last term we notice that, since

$$\frac12+\frac1\sigma+\frac1{2n}=1,$$

by Hölder inequality we have that

$$\begin{aligned}&\left|\int_{B_{3/4}}A^{h,\varepsilon}_{ij}(x)\,\frac{\partial_i\partial^\varepsilon_{\mathbf{v}}\tilde u_h(x)}{\lambda_h}\,\varphi(x)\,\chi'(\partial_{\mathbf{w}_h}\tilde u_h(x))\,\partial_j\Big(\partial_{\mathbf{w}_h}\tilde u_h(x)-\frac{|\gamma_h|}{2}\Big)_+\chi(\partial_{\mathbf{w}_h}\tilde u_h(x+\varepsilon\mathbf{v}))\right|\\&\le\|\varphi\|_{L^\infty(B_{3/4})}\cdot\Big\|\frac{\nabla\partial^\varepsilon_{\mathbf{v}}\tilde u_h(x)}{\lambda_h}\Big\|_{L^\sigma(B_{3/4})}\cdot\Big\|\nabla\Big(\partial_{\mathbf{w}_h}\tilde u_h(x)-\frac{|\gamma_h|}{2}\Big)_+\Big\|_{L^2(B_{3/4})}\\&\quad\cdot\Big\|A^{h,\varepsilon}_{ij}(x)\,\chi'(\partial_{\mathbf{w}_h}\tilde u_h(x))\,\chi(\partial_{\mathbf{w}_h}\tilde u_h(x+\varepsilon\mathbf{v}))\Big\|_{L^{2n}(B_{3/4})}.\end{aligned}$$

Since $0\le A^{h,\varepsilon}_{ij}(x)\le\Lambda\,\mathrm{Id}$ for every x such that $\chi'(\partial_{\mathbf{w}_h}\tilde u_h(x))\chi(\partial_{\mathbf{w}_h}\tilde u_h(x+\varepsilon\mathbf{v}))>0$, it follows that

$$\Big\|A^{h,\varepsilon}_{ij}(x)\,\chi'(\partial_{\mathbf{w}_h}\tilde u_h(x))\,\chi(\partial_{\mathbf{w}_h}\tilde u_h(x+\varepsilon\mathbf{v}))\Big\|_{L^{2n}(B_{3/4})}\le C(\Lambda)\|\chi'\|_\infty.\tag{4.56}$$

Thus, from (4.56), (4.37), (4.48), (4.56) we have that

$$\lim_{h\to\infty}\int_{B_{3/4}}A^{h,\varepsilon}_{ij}(x)\,\frac{\partial_i\partial^\varepsilon_{\mathbf{v}}\tilde u_h(x)}{\lambda_h}\,\varphi(x)\,\chi'(\partial_{\mathbf{w}_h}\tilde u_h(x))\partial_j\Big(\partial_{\mathbf{w}_h}\tilde u_h(x)-\frac{|\gamma_h|}{2}\Big)_+\chi(\partial_{\mathbf{w}_h}\tilde u_h(x+\varepsilon\mathbf{v}))=0.\tag{4.57}$$

Similarly,

$$\lim_{h\to\infty}\int_{B_{3/4}}A^{h,\varepsilon}_{ij}(x)\,\frac{\partial_i\partial^\varepsilon_{\mathbf{v}}\tilde u_h(x)}{\lambda_h}\,\varphi(x)\,\chi(\partial_{\mathbf{w}_h}\tilde u_h(x))\chi'(\partial_{\mathbf{w}_h}\tilde u_h(x+\varepsilon\mathbf{v}))\,\partial_j\Big(\partial_{\mathbf{w}_h}\tilde u_h(x+\varepsilon\mathbf{v})-\frac{|\gamma_h|}{2}\Big)_+=0.\tag{4.58}$$

Hence, letting $h\to\infty$ in (4.43) and taking (4.52), (4.55), (4.57), and (4.58) into account, we obtain

$$0=\int_{B_{3/4}}A^\varepsilon_{ij}(x)\,\partial_i\partial^\varepsilon_{\mathbf{v}}\tilde u_\infty(x)\,\partial_j\varphi(x).$$

Step 6. We find a contradiction.

Since by (4.46) the functions $\partial^\varepsilon_{\mathbf{v}}\tilde u_\infty\in W^{1,2}(B_{3/4})$ solve a uniformly elliptic equation for $\varepsilon>0$ small enough, by De Giorgi-Nash-Moser Theorem there exists $\alpha>0$ such that for every $\mathbf{v}\in S^{n-1}$

$$\|\partial^\varepsilon_{\mathbf{v}}\tilde u_\infty\|_{C^{0,2\alpha}(B_{1/2})}\lesssim\|\partial^\varepsilon_{\mathbf{v}}\tilde u_\infty\|_{L^2(B_{3/4})}\le\|\nabla^\varepsilon\tilde u_\infty\|_{L^2(B_{3/4})}\lesssim1,\tag{4.59}$$

where the last inequality follows from (4.38); in particular, applying the previous inequality to $\mathbf{v} = e_1, ..., e_n$, we obtain that

$$\|\nabla^\varepsilon \tilde{u}_\infty\|_{C^{0,2\alpha}(B_{1/2})} \le C_7. \tag{4.60}$$

Hence, by Jensen inequality and (4.60), for τ sufficiently small we have that

$$\fint_{B_\tau} |\nabla^\varepsilon u_\infty(x) - (\nabla^\varepsilon u_\infty)_{B_\tau}|^2\, dx \le \fint_{B_\tau}\fint_{B_\tau} |\nabla^\varepsilon u_\infty(x) - \nabla^\varepsilon u_\infty(y)|^2\, dx\, dy \le C_7 \tau^{4\alpha} < \frac{\tau^{2\alpha}}{8},$$

which contradicts (4.40) and concludes the proof. □

4.2. **Proof of Corollaries 2.2 and 2.3.**

Proof of Corollary 2.2. Let $x \in B_{1/2}$ and $r < 1/4$; let $\tau_0, \alpha, \tau, \varepsilon(\tau)$ be as in Theorem 2.1. Let $\varepsilon \le \varepsilon(\tau)$ be a constant to be chosen later. We prove (2.3) by induction. For $k = 1$ we apply Theorem 2.1 and we obtain (2.3). Assuming as inductive assumption that

$$U(u, x, \tau^i r) \le \tau^{\alpha i} U(u, x, r) \qquad \forall\, i \le k-1, \tag{4.61}$$

we prove

$$U(u, x, \tau^k r) \le \tau^{\alpha k} U(u, x, r). \tag{4.62}$$

By (2.2) and (4.61) applied with $i = k-1$ we have that $U(u, x, \tau^{k-1} r) \le \varepsilon \le \varepsilon(\tau)$. In order to satisfy the assumptions of Theorem 2.1 at x with radius $\tau^{k-1} r$ we have to show that

$$\frac{3}{4} \le |(\nabla u)_{B_{\tau^{k-1} r}(x)}| \le 1. \tag{4.63}$$

For every $i \in \mathbb{N}$ let us set $\gamma_i = (\nabla u)_{B_{\tau^i r}(x)}$. For every $i = 1, ..., k-1$ by (4.61) we have that

$$\begin{aligned}
|\gamma_i - \gamma_{i-1}| &= \Big(\fint_{B_{\tau^i r}(x)} |\gamma_i - \gamma_{i-1}|\, dy\Big)^{1/2} \\
&\le \Big(\fint_{B_{\tau^i r}(x)} |\nabla u(y) - \gamma_i|^2\, dy\Big)^{1/2} + \Big(\fint_{B_{\tau^i r}(x)} |\nabla u(y) - \gamma_{i-1}|^2\, dy\Big)^{1/2} \\
&\le \Big(\fint_{B_{\tau^i r}(x)} |\nabla u(y) - \gamma_i|^2\, dy\Big)^{1/2} + \frac{1}{\tau^{n/2}}\Big(\fint_{B_{\tau^{i-1} r}(x)} |\nabla u(y) - \gamma_{i-1}|^2\, dy\Big)^{1/2} \\
&\le U(u, x, \tau^i r) + \frac{1}{\tau^{n/2}} U(u, x, \tau^{i-1} r) \le \Big(\tau^\alpha + \frac{1}{\tau^{n/2}}\Big)\tau^{\alpha(i-1)} U(u, x, r).
\end{aligned}$$

Hence, by the triangular inequality we obtain

$$\begin{aligned}
|(\nabla u)_{B_{\tau^{k-1} r(x)}} - (\nabla u)_{B_{r(x)}}| &= |\gamma_{k-1} - \gamma_0| \le \sum_{i=1}^{k-1} |\gamma_i - \gamma_{i-1}| \\
&\le \Big(\tau^\alpha + \frac{1}{\tau^{n/2}}\Big)\Big(\sum_{i=1}^{\infty} \tau^{\alpha(i-1)}\Big) U(u, x, r) \le C(\tau, n, \alpha)\, \varepsilon \le \frac{1}{8}
\end{aligned} \tag{4.64}$$

where in the last inequality we have chosen ε small (depending on n, τ, α). From (4.64) and (2.2) we obtain (4.63). So, we can apply Theorem 2.1 with radius $\tau^{k-1} r$ to obtain

$$U(u, x, \tau^k r) \le \tau^\alpha U(u, x, \tau^{k-1} r), \tag{4.65}$$

which, together with (4.61), implies (4.62). □

Proof of Corollary 2.3. Let $x \in B_{1/2}$; let $\tau = \tau_0, \alpha, \varepsilon = \varepsilon(\tau_0)$ be as in Corollary 2.2. First we prove that, if η and r_0 are chosen sufficiently small, then

$$\frac{7}{8} \leq |(\nabla u)_{B_{r_0}(x)}| \leq 1 \qquad U(u,x,r_0) \leq \varepsilon. \tag{4.66}$$

We choose $r_0 < 1/4$ sufficiently small so that

$$r_0^{(q-n)/(2q)} \|f\|_{L^q(B_1)} \leq \frac{\varepsilon}{2}. \tag{4.67}$$

We estimate the first term in the excess splitting the integral over $B_{r_0}(x) \cap \{\partial_{\mathbf{v}} u \geq 1-\eta\}$ and its complement. For every $y \in B_{r_0}(x) \cap \{\partial_{\mathbf{v}} u \geq 1-\eta\}$ we have that

$$|\nabla u(y) - \mathbf{v}|^2 = |\nabla u(y)|^2 + |\mathbf{v}|^2 - 2\nabla u(y)\cdot \mathbf{v} \leq 2(1 - \nabla u(y)\cdot \mathbf{v}) \leq 2\eta.$$

In the complement of $B_{r_0}(x) \cap \{\partial_{\mathbf{v}} u \geq 1-\eta\}$ we have that $|\nabla u - \mathbf{v}| \leq |\nabla u| + |\mathbf{v}| \leq 2$. Therefore we have

$$\begin{aligned} \fint_{B_{r_0}(x)} |\nabla u(y) - \mathbf{v}|^2\,dy &\leq \frac{|\{y \in B_{r_0}(x) : \partial_{\mathbf{v}} u(y) \geq 1-\eta\}|}{|B_{r_0}|} 4\eta^2 + 4\frac{|\{y \in B_{r_0}(x) : \partial_{\mathbf{v}} u(y) \leq 1-\eta\}|}{|B_{r_0}|} \\ &\leq 4\eta^2 + \frac{1}{|B_{r_0}|}|\{y \in B_1 : \partial_{\mathbf{v}} u(y) \leq 1-\eta\}|. \end{aligned} \tag{4.68}$$

Noticing that (2.4) implies that $|\{y \in B_1 : \partial_{\mathbf{v}} u(y) \leq 1-\eta\}| \leq \eta |B_1|$ we obtain

$$\fint_{B_{r_0}(x)} |\nabla u(y) - \mathbf{v}|^2\,dy \leq 4\eta^2 + \eta\frac{|B_1|}{|B_{r_0}|} \leq \frac{\varepsilon^2}{4}, \tag{4.69}$$

where in the last inequality we have chosen η sufficiently small. From (4.69) it follows that

$$\begin{aligned} \fint_{B_{r_0}(x)} |\nabla u(y) - (\nabla u)_{B_{r_0}(x)}|^2\,dy &= \inf_{\gamma\in\mathbb{R}^n} \fint_{B_{r_0}(x)} |\nabla u(y) - \gamma|^2\,dy \\ &\leq \fint_{B_{r_0}(x)} |\nabla u(y) - \mathbf{v}|^2\,dy \leq \frac{\varepsilon^2}{4} \end{aligned} \tag{4.70}$$

and therefore by (4.70) and (4.67) we get the second inequality in (4.66). From (4.69) we have

$$\Big|\fint_{B_{r_0}(x)} (\nabla u(y) - \mathbf{v})\,dy\Big| \leq \Big(\fint_{B_{r_0}(x)} |\nabla u(y) - \mathbf{v}|^2\,dy\Big)^{1/2} \leq \frac{\varepsilon}{2};$$

it implies

$$|(\nabla u)_{B_{r_0}(x)}| \geq |\mathbf{v}| - |(\nabla u)_{B_{r_0}(x)} - \mathbf{v}| \geq 1 - \frac{\varepsilon}{2} \geq \frac{7}{8},$$

which proves the first inequality in (4.66). Hence the assumptions of Corollary 2.2 are satisfied and we obtain (2.5).

We are left to prove (2.6). From (2.5) and (4.66) it follows that for every $k \in \mathbb{N}$ and $x \in B_{1/2}$,

$$\Big(\fint_{B_{\tau^k r_0}(x)} |\nabla u(y) - (\nabla u)_{B_{\tau^k r_0}(x)}|^2\,dy\Big)^{1/2} \leq U(u,x,\tau^k r_0) \leq \tau^{\alpha k} U(u,x,r_0) \leq \varepsilon \tau^{\alpha k}.$$

From Campanato theorem [12, Theorem 1.3, section III] we obtain (2.6).

$\square$

4.3. **Proof of Theorem 3.2.**

Lemma 4.5. *Let $\eta \in (0,1)$. Let p, $\boldsymbol{n}$, λ, Λ, u, q, f be as in Theorem 3.2 with $\Omega = B_1$. Assume that $|\nabla u(x)| \le 1$ for every $x \in B_1$ and*

$$\sup_{e\in S^{n-1}} |\{x \in B_{1/2} : \partial_e u(x) \ge (1-\eta)\}| \le (1-\eta)|B_{1/2}|. \tag{4.71}$$

Then there exist constants $c := c(n,p,q,\lambda,\Lambda)$ and $C := C(n,\eta,p,q,\lambda,\Lambda)$ such that if $\|f\|_{L^q(B_1)} \le C$ then

$$|\nabla u| \le 1 - c\eta^2 \qquad \forall\, x \in B_{1/4}. \tag{4.72}$$

Proof. Let us fix $e \in S^{n-1}$ and let v_e be defined as in (3.7). We repeat the proof of [14, Theorem 8.18] (see also [17, Lemma 4]) applied to the function $1/2 - v_e(x)$, which is a nonnegative supersolution in B_1 of the equation

$$\partial_i\Big[A_{ij}(\nabla u(x))\partial_j\Big(\frac{1}{2} - v_e(x)\Big)\Big] = \partial_e f(x) 1_{\{\partial_e u \ge 1/2\}}$$

(the coefficients A_{ij} are defined in (3.5); as we mentioned before, to properly justify this computation one needs to perform a suitable regularization argument in the spirit of [6, Proof of Theorem 1.1] and [22]). This equation can be considered to be uniformly elliptic since the values of $A_{ij}(\nabla u(x))$ where $|\nabla u(x)| \le 1/2$ are not relevant. We obtain that there exists a constant $c_0 := c_0(n,p,q,\lambda,\Lambda)$ such that a weak Harnack inequality holds

$$c_0\|1/2 - v_e\|_{L^1(B_{1/2})} \le \inf_{x\in B_{1/4}(0)} \{1/2 - v_e(x)\} + \|f\|_{L^q(B_1)}.$$

On the set

$$\{x \in B_{1/2} : \partial_e u \le (1-\eta)\}$$

(whose measure is greater than $\eta|B_{1/2}|$ from (4.71)), the integrand is greater or equal to η and we obtain

$$\begin{aligned}\inf\{1/2 - v_e(x) : x \in B_{1/4}\} &\ge c_0 \int_{B_{1/2}} (1/2 - v_e(x))\,dx - \|f\|_{L^q(B_1)}\\ &\ge c_0\eta|\{x \in B_{1/2} : (\partial_e u(x) - 1/2)_+ \le 1-\eta\}| - \|f\|_{L^q(B_1)}\\ &\ge c_0\eta^2|B_{1/2}| - C.\end{aligned} \tag{4.73}$$

Therefore, setting $c := c_0|B_{1/2}|/2$ and $C := c_0\eta^2|B_{1/2}|/2$, we have

$$\inf\{1/2 - v_e(x) : x \in B_{1/4}\} \ge c\eta^2,$$

which in turn can be rewritten as

$$\partial_e u(x) \le 1 - c\eta^2 \qquad \forall\, x \in B_{1/4}.$$

Since this argument holds true for every direction $e \in S^{n-1}$ we obtain (4.72). □

Iterating the previous lemma on smaller scales and using the scale invariance of the anisotropic p-laplacian we obtain the following result.

Lemma 4.6. *Let p, $\boldsymbol{n}$, λ, Λ, u, q, f be as in Theorem 3.2. Let $\eta > 0$ be sufficiently small, c and C as in Lemma 4.5, $\delta = c\eta^2$, and $k \in \mathbb{N}$. If $|\nabla u(x)| \le 1$ for every $x \in B_1$,*

$$\sup_{e\in S^{n-1}} |\{x \in B_{2^{-2i-1}}(0) : \partial_e u \ge (1-\eta)(1-\delta)^i\}| \le (1-\eta)|B_{2^{-2i-1}}| \qquad \forall\, i = 1, ..., k, \tag{4.74}$$

and $\|f\|_{L^q(B_1)} \leq C$*, then we have that*

$$|\nabla u(x)| \leq (1-\delta)^i \qquad \forall\, x \in B_{2^{-2i}} \qquad \forall\, i = 1, ..., k+1. \tag{4.75}$$

Proof. We prove the result by induction on i. Assuming (4.74) with $i = 0$ we obtain (4.75) with $i = 1$ from Lemma 4.5. Let us assume that the result holds true for i and let us prove it for $i + 1$. Thanks to the homogeneity of the anisotropic p-laplacian, the function

$$v(x) := \frac{2^{2i} u(2^{-2i}x)}{(1-\delta)^i} \qquad x \in B_1$$

satisfies by inductive assumption $|\nabla v| \leq 1$ in B_1 and it is a minimizer of

$$\int_{B_1} \frac{\boldsymbol{n}(\nabla v)^p}{p} + \tilde{f} v, \tag{4.76}$$

where

$$\tilde{f}(x) := \frac{2^{-2i}}{(1-\delta)^{i(p-1)}} f(2^{-2i}x).$$

Hence the norm of $\tilde{f}$ is estimated by

$$\|\tilde{f}\|^q_{L^q(B_1)} = \frac{2^{-2i(p-n)}}{(1-\delta)^{i(p-1)q}} \int_{B_{2^{-2i}}} |f(y)|^q \, dy \leq \frac{2^{-2i(q-n)}}{(1-\delta)^{i(p-1)q}} \|f\|^q_{L^q(B_1)}.$$

Therefore, provided that δ is chosen small enough so that $2^{-2(q-n)/(pq-q)} \leq 1 - \delta$, we obtain that $\|\tilde{f}\|_{L^q(B_1)} \leq \|f\|_{L^q(B_1)} \leq C$. The assumption (4.74) can be rewritten as (4.71) applied to v instead of u; therefore, Lemma 4.5 gives us that

$$|\nabla v(x)| \leq 1 - \delta \qquad \forall\, x \in B_{1/4},$$

which implies (4.75) with $i + 1$ in place of i. $\square$

Proof of Theorem 3.2. By a covering argument, it is enough to show that, if $u : B_1 \to \mathbb{R}$ is Lipschitz, then

$$\sup_{x \in B_{2^{-2i}}} |\nabla u(x) - \nabla u(0)| \leq C_0 2^{-2\alpha i} \qquad \forall\, i \in \mathbb{N}, \tag{4.77}$$

for some $\alpha \in (0, 1)$, $C_0 > 0$ which depends only on d, p, λ, Λ to be chosen later. Let $\eta > 0$ to be fixed later; let $c, C, \delta = c\eta^2$ as in Lemma 4.6. Up to changing u with

$$\frac{u(r_0 x) - u(0)}{r_0 \|\nabla u\|_{L^\infty(B_1)}} \qquad \forall\, x \in B_1$$

thanks to the homogeneity of the anisotropic p-laplacian we can assume that

$$u(0) = 0, \quad |\nabla u(x)| \leq 1 \quad \forall\, x \in B_1, \quad \text{and} \quad \|f\|_{L^q(B_1)} \leq C.$$

Let $k \in \mathbb{N} \cup \{\infty\}$ be the largest index for which (4.74) holds true. Let $\alpha_1 \in (0, \infty)$ be such that $2^{-2\alpha_1} = 1 - \delta$. If $k = \infty$ we have that by Lemma 4.6

$$\sup_{x \in B_{2^{-2i}}} |\nabla u(x)| \leq (1-\delta)^i = 2^{-2\alpha_1 i} \qquad \forall\, i \in \mathbb{N};$$

hence (4.77) is satisfied. If $k < \infty$ set

$$v(x) := \frac{2^{2(k+1)} u(2^{-2(k+1)}x)}{(1-\delta)^{k+1}}.$$

By the maximality of k we have that there exists $e \in S^{n-1}$ such that

$$|\{x \in B_{1/2} : \partial_e v(x) \geq 1-\eta\}| \geq (1-\eta)|B_1|. \tag{4.78}$$

Thanks to Lemma 4.6 applied to u we obtain that

$$\sup_{x \in B_{2^{-2i}}} |\nabla u(x)| \leq (1-\delta)^i = 2^{-2\alpha_1 i} \qquad \forall\, i = 1, ..., k+1. \tag{4.79}$$

and

$$|\nabla v(x)| \leq 1 \qquad \forall\, x \in B_1.$$

We choose $\eta > 0$ so that Corollary 2.3 applies to v with $F(x) = \boldsymbol{n}(x)^p/p$ (notice that assumption (2.1) is not a restriction since $|\nabla v| \leq 1$); we obtain that there exist $\alpha_2, C_2 > 0$ such that for every $i \in \mathbb{N}$

$$\frac{1}{2^{-\alpha_1(k+1)}} \sup_{x \in B_{2^{-2i}}} |\nabla u(2^{-2(k+1)}x) - \nabla u(0)| = \sup_{x \in B_{2^{-2i}}} |\nabla v(x) - \nabla v(0)| \leq C_2 2^{-2\alpha_2 i},$$

which can be rewritten, setting $\alpha = \min\{\alpha_1, \alpha_2\}$, as

$$\sup_{x \in B_{2^{-2(i+k+1)}}} |\nabla u(x) - \nabla u(0)| \leq C_2 2^{-2\alpha_2 i + \alpha_1(k+1)} \leq C_2 2^{-2\alpha(i+k+1)}. \tag{4.80}$$

From (4.79) we deduce that for every $i = 1, ..., k+1$

$$\sup_{x \in B_{2^{-2i}}} |\nabla u(x) - \nabla u(0)| \leq 2 \sup_{x \in B_{2^{-2i}}} |\nabla u(x)| \leq 2 \cdot 2^{-2\alpha_1 i} \leq 2 \cdot 2^{-2\alpha i},$$

which, together with (4.80), proves (4.77) when $k < \infty$ with $C_0 = \max\{2, C_2\}$. □

4.4. **Proof of Theorem 3.3.** Since the proof of this theorem follows closely the lines of [6, Theorem 1.1], we just outline the argument.

First we remark that all results in Section 2 hold replacing B_1 and $B_{1/4}$ with sets $\{\boldsymbol{m} < M\}$ and $\{\boldsymbol{m} < m\}$ for some $0 \leq m < M$ (indeed, the statements and the proofs can easily be adapted to this setting with easy modifications).

Then we regularize the equation by approximation, reducing ourselves to prove an a-priori estimate on a regular solution as in [6, Theorem 1.4].

Finally, to prove regularity at nondegenerate points we use Corollary 2.3 instead of [6, Lemma 4.1 and Proposition 4.3]. □

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