Existence, uniqueness and regularity of optimal transport maps

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Abstract

Adapting some techniques and ideas of McCann [8], we extend a recent result with Fathi [6] to yield existence and uniqueness of a unique transport map in very general situations, without any integrability assumption on the cost function.
In particular this result applies for the optimal transportation problem on a n-dimensional non-compact manifold M with a cost function induced by a C^2-Lagrangian, provided that the source measure vanishes on sets with σ-finite $(n-1)$-dimensional Hausdorff measure. Moreover we prove that, in the case $c(x,y) = d^2(x,y)$, the transport map is approximatively differentiable a.e. with respect to the volume measure, and we extend some results of [4] about concavity estimates and displacement convexity.

1 Introduction and main result

Let M be a n-dimensional manifold (Hausdorff and with a countable basis), N a Polish space, $c : M \times N \to \mathbb{R}$ a cost function, μ and ν two probability measures on M and N respectively.
In a recent work with Fathi [6], we proved, under general assumption on the cost function, existence and uniqueness of optimal transport maps for the Monge-Kantorovich problem. More precisely, the result is:

Theorem 1.1. *Assume that $c : M \times N \to \mathbb{R}$ is lower semicontinuous, bounded from below, and such that*

$$\int_{M\times N} c(x,y)\, d\mu(x)\, d\nu(y) < +\infty.$$

If

(i) $x \mapsto c(x,y) = c_y(x)$ is locally semi-concave in x locally uniformly in y,

(ii) $\frac{\partial c}{\partial x}(x,\cdot)$ is injective on its domain of definition,

(iii) the measure μ gives zero mass to sets with σ-finite $(n-1)$-dimensional Hausdorff measure,

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then there exists a measurable map $T : M \to N$ such that any plan γ optimal for the cost c is concentrated on the graph of T.
More precisely, there exists a sequence of Borel subsets $B_n \subset M$, with $B_n \subset B_{n+1}$, $\mu(B_n) \nearrow 1$, and a sequence locally semi-concave functions $\varphi_n : M \to \mathbb{R}$, with φ_n is differentiable on B_n, such that, thanks to assumption (ii), the map $T : M \to N$ is uniquely defined on B_n by

$$\frac{\partial c}{\partial x}(x, T(x)) = d_x \varphi_n. \tag{1}$$

This implies both existence of an optimal transport map and uniqueness for the Monge problem.

Now we want to generalize this existence and uniqueness result for optimal transport maps without any integrability assumption on the cost function, adapting the ideas of [8]. We observe that, without the hypothesis

$$\int_{M\times N} c(x,y)\, d\mu(x)\, d\nu(y) < +\infty,$$

denoting with $\Pi(\mu,\nu)$ the set of probability measures on $M \times N$ whose marginals are μ and ν, in general the minimization problem

$$C(\mu,\nu) := \inf_{\gamma\in\Pi(\mu,\nu)} \left\{ \int_{M\times N} c(x,y)\, d\gamma(x,y) \right\}. \tag{2}$$

is ill-posed, as it may happen that $C(\mu,\nu) = +\infty$. Howewer, it is known that the optimality of a transport plan γ is equivalent to the c-cyclical monotonicity of the measure-theoretic support of γ whenever $C(\mu,\nu) < +\infty$ (see [2], [11], [13]), and so one may ask whether the fact that the measure-theoretic support of γ is c-cyclically monotone implies that γ is supported on a graph. Moreover one can also ask whether this graph is unique, that is it does not depends on γ, which is the case when the cost is $\mu \otimes \nu$ integrable, as Theorem 1.1 tells us. In that case, uniqueness follows by the fact that the functions φ_n are constructed using a pair of function (φ,ψ) which is optimal for the dual problem, and so they are independent of γ (see [6] for more details). The result we now want to prove is the following:

Theorem 1.2. *Assume that $c : M \times N \to \mathbb{R}$ is lower semicontinuous and bounded from below, and let γ be a plan concentrated on a c-cyclically monotone set. If*

(i) *the family of maps $x \mapsto c(x,y) = c_y(x)$ is locally semi-concave in x locally uniformly in y,*

(ii) *$\frac{\partial c}{\partial x}(x,\cdot)$ is injective on its domain of definition,*

(iii) *the measure μ gives zero mass to sets with σ-finite $(n-1)$-dimensional Hausdorff measure,*

then γ is concentrated on the graph of a measurable map $T : M \to N$ (existence). Moreover, if $\tilde\gamma$ is another plan concentrated on a c-cyclically monotone set, then $\tilde\gamma$ is concentrated on the same graph (uniqueness).

Once the above result will be proven, it will follow as a simple corollary the uniqueness of Wasserstein geodesic between absolutely continuous measures (see Section 3). Finally, in Subsection 3.1, we will prove that, in the particular case $c(x, y) = \frac{1}{2}d^2(x, y)$, the optimal transport map is approximatively differentiable a.e. with respect to the volume measure, and we will obtain a concavity estimate on the Jacobian of the optimal transport map, that will allows us to generalize to non-compact manifolds a displacement convexity result proven in [4].

2 Proof of Theorem 1.2

Proof. **Existence.** We want to prove that γ is concentrated on a graph. First we recall that, since γ is concentrated on a c-cyclically monotone set, there exists a pair of function (φ, ψ), with φ μ-measurable and ψ ν-measurable, such that

$$\varphi(x) = \inf_{y \in N} \psi(y) + c(x, y) \quad \forall x \in M,$$

which implies

$$\varphi(x) - \psi(y) \leq c(x, y) \quad \forall (x, y) \in M \times N.$$

Moreover we have

$$\varphi(x) - \psi(y) = c(x, y) \quad \gamma\text{-a.e.} \tag{3}$$

and there exists a point $x_0 \in M$ such that $\varphi(x_0) = 0$ (see [13, Theorem 5.9]). In particular, this implies

$$\psi(y) \geq -c(x_0, y) > -\infty \quad \forall y \in N.$$

So, we can argue as in [6]. More precisely, taken a suitable increasing sequence of compact sets $(K_n) \subset N$ such that $\nu(K_n) \nearrow 1$ and $\psi \geq -n$ on K_n (it suffices to take an increasing sequence of compact sets $K_n \subset \{\psi \geq -n\}$ such that $\nu(\{\psi \geq -n\} \setminus K_n) \leq \frac{1}{n}$), we consider the locally semi-concave function

$$\varphi_n(x) := \inf_{y \in K_n} \psi(y) + c(x, y). \tag{4}$$

Then, thanks to (3), it is possible to find an increasing sequence of Borel sets $D_n \subset \mathrm{supp}(\mu)$, with $\mu(D_n) \nearrow 1$, such that φ_n is differentiable on D_n, $\varphi_n \equiv \varphi$ on D_n, the set $\{\varphi_n = \varphi\}$ has μ-density 1 at all the points of D_n, and γ is concentrated on the graph of the map T uniquely determined on D_n by

$$\frac{\partial c}{\partial x}(x, T(x)) = d_x\varphi_n \quad \text{for } x \in D_n.$$

Moreover one has

$$\varphi(x) = \psi(T(x)) + c(x, T(x)) \quad \forall x \in \bigcup_n D_n \tag{5}$$

(see [6] for more details).

Uniqueness. As we observed before, the difference here with the case of Theorem 1.1 is that the function φ_n depends on the pair (φ, ψ), which in this case depends on γ. Let so $(\tilde{\varphi}, \tilde{\psi})$ be a pair associated to $\tilde{\gamma}$ as above, and let $\tilde{\varphi}_n$ and $\tilde{D}_n$ be such that $\tilde{\gamma}$ is concentrated on the graph of the map $\tilde{T}$ determined on $\tilde{D}_n$ by

$$\frac{\partial c}{\partial x}(x, \tilde{T}(x)) = d_x \tilde{\varphi}_n \quad \text{for } x \in \tilde{D}_n.$$

We need to prove that $T = \tilde{T}$ μ-a.e.
Let us define $C_n := D_n \cap \tilde{D}_n$. Then $\mu(C_n) \nearrow 1$. We want to prove that, if x is a μ-density point of C_n for a certain n, then $T(x) = \tilde{T}(x)$ (we recall that, since $\mu(\cup_n C_n) = 1$, also the union of the μ-density points of C_n is of full μ-measure, see for example [5, Chapter 1.7]).
Let us assume by contradiction that $T(x) \neq \tilde{T}(x)$, that is

$$d_x \varphi_n \neq d_x \tilde{\varphi}_n.$$

Since $x \in \operatorname{supp}(\mu)$, each ball around x must have positive measure under μ. Moreover, the fact that the sets $\{\varphi_n = \varphi\}$ and $\{\tilde{\varphi}_n = \tilde{\varphi}\}$ have μ-density 1 in x implies that the set

$$\{\varphi = \tilde{\varphi}\}$$

has μ-density 0 in x. In fact, as φ_n and $\tilde{\varphi}_n$ are locally semi-concave, up to adding a C^1 function they are concave in a neighborhood of x and their gradients differ at x. So we can apply the non-smooth version of the implicit function theorem proven in [8], which tells us that $\{\varphi_n = \tilde{\varphi}_n\}$ is a set with finite $(n-1)$-dimensional Hausdorff measure in a neighborhood of x (see [8, Theorem 17 and Corollary 19]). So we have

$$\limsup_{r \to 0} \frac{\mu(\{\varphi = \tilde{\varphi}\} \cap B_r(x))}{\mu(B_r(x))} \leq \limsup_{r \to 0} \left[\frac{\mu(\{\varphi \neq \varphi_n\} \cap B_r(x))}{\mu(B_r(x))} + \frac{\mu(\{\varphi_n = \tilde{\varphi}_n\} \cap B_r(x))}{\mu(B_r(x))} + \frac{\mu(\{\tilde{\varphi}_n \neq \tilde{\varphi}\} \cap B_r(x))}{\mu(B_r(x))} \right] = 0.$$

Now, exchanging φ with $\tilde{\varphi}$ if necessary, we may assume that

$$\mu(\{\varphi < \tilde{\varphi}\} \cap B_r(x)) \geq \frac{1}{4} \mu(B_r(x)) \quad \text{for } r > 0 \text{ sufficiently small.} \tag{6}$$

Let us define $A := \{\varphi < \tilde{\varphi}\}$, $A_n := \{\varphi_n < \tilde{\varphi}_n\}$, $E_n := A \cap A_n \cap C_n$. Since the sets $\{\varphi_n = \varphi\}$ and $\{\tilde{\varphi}_n = \tilde{\varphi}\}$ have μ-density 1 in x, and x is a μ-density point of C_n, we have

$$\lim_{r \to 0} \frac{\mu((A \setminus E_n) \cap B_r(x))}{\mu(B_r(x))} = 0,$$

and so, by (6), we get

$$\mu(E_n \cap B_r(x)) \geq \frac{1}{5} \mu(B_r(x)) \quad \text{for } r > 0 \text{ sufficiently small.} \tag{7}$$

Now, arguing as in the proof of the Aleksandrov's lemma (see [8, Lemma 13]), we can prove that

$$X := \tilde{T}^{-1}(T(A)) \subset A$$

and $X \cap E_n$ lies a positive distance from x. In fact let us assume, without loss of generality, that

$$\varphi(x) = \varphi_n(x) = \tilde{\varphi}(x) = \tilde{\varphi}_n(x) = 0, \quad d_x\varphi_n \neq d_x\tilde{\varphi}_n = 0.$$

To obtain the inclusion $X \subset A$, let $z \in X$ and $y := \tilde{T}(z)$. Then $y = T(m)$ for a certain $m \in A$. For any $w \in M$, recalling (5), we have

$$\varphi(w) \leq c(w, y) - c(m, y) + \varphi(m),$$

$$\tilde{\varphi}(m) \leq c(m, y) - c(z, y) + \tilde{\varphi}(z).$$

Since $\varphi(m) < \tilde{\varphi}(m)$ we get

$$\varphi(w) < c(w, \tilde{T}(z)) - c(z, \tilde{T}(z)) + \tilde{\varphi}(z) \quad \forall w \in M.$$

In particular, taking $w = z$, we obtain $z \in A$, that proves the inclusion $X \subset A$.
Let us suppose now, by contradiction, that there exists a sequence $(z_k) \subset X \cap E_n$ such that $z_k \to x$. Again there exists m_k such that $\tilde{T}(z_k) = T(m_k)$. As $d_x\tilde{\varphi}_n = 0$, the closure of the superdifferential of a semi-concave function implies that $d_{z_k}\tilde{\varphi}_n \to 0$. We now observe that, arguing exactly as above with φ_n and $\tilde{\varphi}_n$ instead of φ and $\tilde{\varphi}$, using (4), (5), and the fact that $\varphi = \varphi_n$ and $\tilde{\varphi} = \tilde{\varphi}_n$ on C_n, one obtains

$$\varphi_n(w) < c(w, \tilde{T}(z_k)) - c(z_k, \tilde{T}(z_k)) + \tilde{\varphi}_n(z_k) \quad \forall w \in M.$$

Taking w sufficiently near to x, we can assume that we are in $\mathbb{R}^n \times N$. We now remark that, since $z_k \in E_n \subset \tilde{D}_n$, $\tilde{T}(z_k)$ vary in a compact subset of N (this follows by the construction of $\tilde{T}$). So, by hypothesis *(i)* on c, we can find a common modulus of continuity ω in a neighborhood of x for the family of uniformly semi-concave functions $z \mapsto c(z, \tilde{T}(z_k))$. Then, we get

$$\begin{aligned}\varphi_n(w) &< \frac{\partial c}{\partial x}(z_k, \tilde{T}(z_k))(w - z_k) + \omega(|w - z_k|)|w - z_k| + \tilde{\varphi}_n(z_k) \\ &= d_{z_k}\tilde{\varphi}_n(w - z_k) + \omega(|w - z_k|)|w - z_k| + \tilde{\varphi}_n(z_k).\end{aligned}$$

Letting $k \to \infty$ and recalling that $d_{z_k}\tilde{\varphi}_n \to 0$ and $\tilde{\varphi}_n(x) = \varphi_n(x) = 0$, we obtain

$$\varphi_n(w) - \varphi_n(x) \leq \omega(|w - x|)|w - x| \Rightarrow d_x\varphi_n = 0,$$

which is absurd.
Thus there exists $r > 0$ such that $B_r(x) \cap E_n$ and $X \cap E_n$ are disjoint, and (7) holds. Defining now $Y := T(A)$, by (7) we obtain

$$\begin{aligned}\nu(Y) = \mu(T^{-1}(Y)) \geq \mu(A) = \mu(E_n) + \mu(A \setminus E_n) \geq \mu(B_r(x) \cap E_n) \\ + \mu(X \cap E_n) + \mu(X \setminus E_n) = \mu(B_r(x) \cap E_n) + \mu(X) \geq \frac{1}{5}\mu(B_r(x)) + \nu(Y),\end{aligned}$$

which is absurd. □

Let now consider the special case $N = M$, with M a complete manifold. As shown in [6], this theorem applies in the following cases:

(i) $c : M \times M \to \mathbb{R}$ is defined by

$$c(x,y) := \inf_{\gamma(0)=x,\ \gamma(1)=y} \int_0^1 L(\gamma(t),\dot\gamma(t))\,dt,$$

where the infimum is taken over all the continuous piecewise C^1 curves, and the Lagrangian $L(x,v) \in C^2(TM,\mathbb{R})$ is C^2-strictly convex and uniform superlinear in v, and verifies an uniform boundedness in the fibers;

(ii) $c(x,y) = d^p(x,y)$ for any $p \in (1,+\infty)$, where $d(x,y)$ denotes a complete Riemannian distance on M.

Moreover, in the cases above, the following important fact holds:

Remark 2.1. *For μ_0-a.e. x, there exists an unique curve from x to $T(x)$ that minimizes the action. In fact, since $\frac{\partial c}{\partial x}(x,y)$ exists at $y = T(x)$ for μ_0-a.e. x, the fact that $\frac{\partial c}{\partial x}(x,\cdot)$ is injective on its domain of definition tells us that its velocity at time 0 is μ_0-a.e. univocally determined (see [6]).*

Let us recall the following definition, see [1, Definition 5.5.1, page 129]:

Definition 2.2 (Approximate differential). We say that $f : M \to \mathbb{R}^m$ has an approximate differential at $x \in M$ if there exists a function $h : M \to \mathbb{R}^m$ differentiable at x such that the set $\{f = h\}$ has density 1 at x with respect to the Lebesgue measure (this just means that the density is 1 in charts). In this case, the approximate value of f at x is defined as $\tilde{f}(x) = h(x)$, and the approximate differential of f at x is defined as $\tilde{d}_x f = d_x h$. It is not difficult to show that this definition makes sense. In fact, both $h(x)$, and $d_x h$ do not depend on the choice of h, provided x is a density point of the set $\{f = h\}$ for the Lebesgue measure.

We recall that many standard properties of the differential still hold for the approximate differential, such as linearity and additivity. In particular, it is simple to check that the property of being approximatively differentiable is stable by right composition with smooth maps (say C^1), and in this case the standard chain rule formula for the differentials holds. Moreover we remark that it makes sense to speak of approximate differential for maps between manifolds.
In ([6]), the following formula is proven:
in the particular case $c(x,y) = d^2(x,y)$, the optimal transport map is given by

$$T(x) = \exp_x[-\tilde\nabla_x \varphi],$$

where $\tilde\nabla_x \varphi$ denotes the approximate gradient of φ at x, which simply corresponds to the element of T_xM obtained from $\tilde{d}_x\varphi$ using the isomorphism with T_x^*M induced by the Riemannian metric.

3 The Wasserstein space W_2

Let (M, g) be a smooth complete Riemannian manifold, equipped with its geodesic distance d and its volume measure vol. We denote with $P(M)$ the set of probability measures on M. The space $P(M)$ can be endowed of the so called Wasserstein distance W_2:

$$W_2(\mu_0, \mu_1)^2 := \min_{\gamma \in \Pi(\mu_0,\mu_1)} \left\{ \int_{M\times M} d^2(x, y)\, d\gamma(x, y) \right\}.$$

The quantity W_2 will be called the *Wasserstein distance of order 2* between μ_0 and μ_1. It is well-known that it defines a metric on $P(M)$ (not necessarily finite), and so one can speak about geodesic in the metric space $(P(M), W_2)$. This space turns out, indeed, to be a length space (see for example [12], [13]). Now, whenever $W_2(\mu_0, \mu_1) < +\infty$, we know that any optimal transport plan is supported on a c-cyclical monotone set (see for example [2], [11], [13]). We denote with $P^{ac}(M)$ the subset of $P(M)$ that consists of the Borel probability measures on M that are absolutely continuous with respect to vol. Thus, if $\mu_0, \mu_1 \in P^{ac}(M)$ and $W_2(\mu_0, \mu_1) < +\infty$, we know that there exists an unique transport map between μ_0 and μ_1.

Proposition 3.1. *$P^{ac}(M)$ is a geodesically convex subset of $P(M)$. Moreover, if $\mu_0, \mu_1 \in P^{ac}(M)$ and $W_2(\mu_0, \mu_1) < +\infty$, then there is a unique Wasserstein geodesic $\{\mu_t\}_{t\in[0,1]}$ joining μ_0 to μ_1, which is given by*

$$\mu_t = (T_t)_\sharp \mu_0 := (\exp[-t\tilde{\nabla}\varphi])_\sharp \mu_0,$$

where $T(x) = \exp_x[-\tilde{\nabla}_x\varphi]$ is the unique transport map from μ_0 to μ_1 which is optimal for the cost $\frac{1}{2}d^2(x, y)$ (and so also optimal for the cost $d^2(x, y)$). Moreover:

(i) T_t is the unique optimal transport map from μ_0 to μ_t for all $t \in [0, 1]$;

(ii) T_t^{-1} the unique optimal transport map from μ_t to μ_0 for all $t \in [0, 1]$ (and, if $t \in [0, 1)$, it is locally Lipschitz);

(iii) $T \circ T_t^{-1}$ the unique optimal transport map from μ_t to μ_1 for all $t \in [0, 1]$ (and, if $t \in (0, 1]$, it is locally Lipschitz).

Proof. Regarding the fact that $\mu_t \in P^{ac}(M)$ (which corresponds to say that $P^{ac}(M)$ is geodesically convex) and the Lipschitz regularity of the transport maps, they follow from the results in [6].

Since we know that the transport is unique, the proof of the rest of the proposition is quite standard. In fact, a basic representation theorem (see [13, Corollary 7.20]) states that any Wasserstein geodesic curve necessarily takes the form $\mu_t = (e_t)_\#\Pi$, where Π is a probability measure on the set Γ of minimizing geodesics $[0, 1] \to M$, and $e_t : \Gamma \to M$ is the evaluation at time t: $e_t(\gamma) := \gamma(t)$. Thus the thesis follows from Remark 2.1. □

The above result tells us that also $(P^{ac}(M), W_2)$ is a length space.

3.1 Regularity, concavity estimate and a displacement convexity result

We now consider the cost function $c(x,y) = \frac{1}{2}d^2(x,y)$. Let $\mu, \nu \in P^{ac}(M)$ with $W_2(\mu_0, \mu_1) < +\infty$, and let us denote with f and g their respective densities with respect to vol. Let

$$T(x) = \exp_x[-\tilde{\nabla}_x \varphi]$$

be the unique optimal transport map from μ to ν.
We recall that locally semiconcave functions with linear modulus admits vol-a.e. a second order Taylor expansion (see [3], [4]). Let us recall the definition of approximate hessian:

Definition 3.2 (Approximate hessian). We say that $f : M \to \mathbb{R}^m$ has a *approximate hessian* at $x \in M$ if there exists a function $h : M \to \mathbb{R}$ such that the set $\{f = h\}$ has density 1 at x with respect to the Lebesgue measure and h admits a second order Taylor expansion at x, that is there exists a self-adjoint operator $H : T_xM \to T_xM$ such that

$$h(\exp_x w) = h(x) + \langle \nabla_x h, w\rangle + \frac{1}{2}\langle Hw, w\rangle + o(\|w\|_x^2).$$

In this case the approximate hessian is defined as $\tilde{\nabla}_x^2 f := H$.

Like in the case of the approximate differential, it is not difficult to show that this definition makes sense.
Observing that $d^2(x,y)$ is locally semi-concave with linear modulus (see [6, Appendix]), we get that φ_n is locally semi-concave with linear modulus for each n. Thus we can define μ-a.e. an approximate hessian for φ (see Definition 3.2):

$$\tilde{\nabla}_x^2 \varphi := \nabla_x^2 \varphi_n \quad \text{for } x \in D_n \cap E_n,$$

where D_n was defined in the proof of Theorem 1.2, E_n denotes the full μ-measure set of points where φ_n admits a second order Taylor expansion, and $\nabla_x^2 \varphi_n$ denotes the self-adjoint operator on T_xM that appears in the Taylor expansion on φ_n at x. Let us now consider, for each set $F_n := D_n \cap E_n$, an increasing sequence of compact sets $K_m^n \subset F_n$ such that $\mu(F_n \setminus \cup_m K_m^n) = 0$. We now define the measures $\mu_m^n := \mu \llcorner K_m^n$ and $\nu_m^n := T_\sharp \mu_m^n = (\exp[-\nabla\varphi_n])_\sharp \mu_m^n$, and we renormalize them in order to obtain two probability measures:

$$\hat{\mu}_m^n := \frac{\mu_m^n}{\mu_m^n(M)} \in P_2^{ac}(M), \quad \hat{\nu}_m^n := \frac{\nu_m^n}{\nu_m^n(M)} = \frac{\nu_m^n}{\mu_m^n(M)} \in P_2^{ac}(M).$$

We now observe observe that T is still optimal. In fact, if is not that case, we would have

$$\int_{M\times M} c(x, S(x))\, d\hat{\mu}_m^n(x) < \int_{M\times M} c(x, T(x))\, d\hat{\mu}_m^n(x)$$

for a certain S transport map from $\hat{\mu}_m^n$ to $\hat{\nu}_m^n$. This would imply that

$$\int_{M\times M} c(x, S(x))\, d\mu_m^n(x) < \int_{M\times M} c(x, T(x))\, d\mu_m^n(x),$$

and so the transport map

$$\tilde{S}(x) := \begin{cases} S(x) & \text{if } x \in K_m^n \\ T(x) & \text{if } x \in M \setminus K_m^n \end{cases}$$

would have a cost strictly less than the cost of T, which would contradict the optimality of T.
We will now apply the results of [4] to the compactly supported measures $\hat{\mu}_m^n$ and $\hat{\nu}_m^n$, in order to get information on the transport problem from μ to ν. In the sequel we will denote by $\nabla_x d_y^2$ and by $\nabla_x^2 d_y^2$ respectively the gradient and the hessian with respect to x of $d^2(x, y)$, and by $d_x \exp$ and $d(\exp_x)_v$ the two components of the differential of the map $TM \ni (x, v) \mapsto \exp_x[v] \in M$ (whenever they exist). By [4, Theorem 4.2], we get the following:

Theorem 3.3 (Jacobian identity a.e.). *There exists a subset $E \subset M$ such that $\mu(E) = 1$ and, for each $x \in E$, $Y(x) := d(\exp_x)_{-\tilde{\nabla}_x\varphi}$ and $H(x) := \frac{1}{2}\nabla_x^2 d_{T(x)}^2$ both exist and we have*

$$f(x) = g(T(x)) \det[Y(x)(H(x) - \tilde{\nabla}_x^2\varphi)] \neq 0.$$

Proof. It suffices to observe that [4, Theorem 4.2] applied to $\hat{\mu}_m^n$ and $\hat{\nu}_m^n$ gives that, for μ-a.e. $x \in K_m^n$,

$$\frac{f(x)}{\mu_m^n(M)} = \frac{g(T(x))}{\mu_m^n(M)} \det[Y(x)(H(x) - \nabla_x^2\varphi_n)] \neq 0,$$

which implies

$$f(x) = g(T(x)) \det[Y(x)(H(x) - \tilde{\nabla}_x^2\varphi)] \neq 0 \quad \text{for } \mu\text{-a.e. } x \in K_m^n.$$

Passing to the limit as $m, n \to +\infty$ we get the result. □

We can so define μ-a.e. the (weak) differential of the transport map at x as

$$d_x T := Y(x)\big(H(x) - \tilde{\nabla}_x^2\varphi\big).$$

Let us prove now that, indeed, $T(x)$ is approximately differentiable μ-a.e., and that the above differential coincides with the approximate differential of T. In order to prove this fact, let us first make a formal computation. Observe that, since the map $x \mapsto \exp_x[-\frac{1}{2}\nabla_x d_y^2] = y$ is constant, we have

$$0 = d_x(\exp_x[-\tfrac{1}{2}\nabla_x d_y^2]) = d_x \exp[-\tfrac{1}{2}\nabla_x d_y^2] - d(\exp_x)_{-\frac{1}{2}\nabla_x d_y^2}\big(\tfrac{1}{2}\nabla_x^2 d_y^2\big). \quad \forall y \in M,$$

By differentiating (in the approximate sense) the equality $T(x) = \exp[-\tilde{\nabla}_x\varphi]$ and recalling the equality $\tilde{\nabla}_x\varphi = \frac{1}{2}\nabla_x d_{T(x)}^2$, we obtain

$$\begin{aligned} \tilde{d}_x T &= d(\exp_x)_{-\tilde{\nabla}_x\varphi}\big(-\tilde{\nabla}_x^2\varphi\big) + d_x \exp[-\tilde{\nabla}_x\varphi] \\ &= d(\exp_x)_{-\tilde{\nabla}_x\varphi}\big(-\tilde{\nabla}_x^2\varphi\big) + d(\exp_x)_{-\frac{1}{2}\nabla_x d_{T(x)}^2}\big(\tfrac{1}{2}\nabla_x^2 d_{T(x)}^2\big) \\ &= d(\exp_x)_{-\tilde{\nabla}_x\varphi}\big(H(x) - \tilde{\nabla}_x^2\varphi\big), \end{aligned}$$

as wanted. In order to make the above proof rigorous, it suffices to observe that for μ-a.e. x, $T(x) \notin cut(x)$, where $cut(x)$ is defined as the set of points $z \in M$ which cannot be linked to x

by an extendable minimizing geodesic. Indeed we recall that the square of the distance fails to be semiconvex at the cut locus, that is, if $x \in cut(y)$, then

$$\inf_{0<\|v\|_x<1} \frac{d_y^2(\exp_x[v]) - 2d_y^2(x) + d_y^2(\exp_x[-v])}{|v|^2} = -\infty$$

(see [4, Proposition 2.5]). Fix now $x \in F_n$. Since we know that $\frac{1}{2}d^2(z, T(x)) \geq \varphi_n(z) - \psi(T(x))$ with equality for $z = x$, we obtain a bound from below of the Hessian of $d^2_{T(x)}$ at x in term of the Hessian of φ_n at x (see the proof of [4, Proposition 4.1(a)]). Thus, since each φ_n admits vol-a.e. a second order Taylor expansion, we obtain that, for μ-a.e. x,

$$x \notin cut(T(x)), \quad \text{or equivalently} \quad T(x) \notin cut(x).$$

This implies that all the computations we made above in order to prove the formula for $\tilde{d}_x T$ are correct. Indeed the exponential map $(x, v) \mapsto \exp_x[v]$ is smooth if $\exp_x[v] \notin cut(x)$, the function d_y^2 is smooth around any $x \notin cut(y)$ (see [4, Paragraph 2]), and $\tilde{\nabla}_x\varphi$ is approximatively differentiable μ-a.e. Thus, recalling that, once we consider the right composition of an approximatively differentiable map with a smooth map, the standard chain rule holds (see the remarks after Definition 2.2), we have proved the following regularity result for the transport map:

Proposition 3.4 (Approximate differentiability of the transport map). *The transport map is approximatively differentiable for μ-a.e. x and its approximate differential is given by the formula*

$$\tilde{d}_x T = Y(x)\big(H(x) - \tilde{\nabla}_x^2\varphi\big),$$

where Y and H are defined in Theorem 3.3.

For proving our displacement convexity result, it will be useful the following change of variables formula.

Proposition 3.5 (Change of variables for optimal maps). *If $A : [0 + \infty) \to \mathbb{R}$ is a Borel function such that $A(0) = 0$, then*

$$\int_M A(g(y))\, d\,\mathrm{vol}(y) = \int_E A\left(\frac{f(x)}{J(x)}\right) J(x)\, d\,\mathrm{vol}(x),$$

where $J(x) := \det[Y(x)(H(x) - \tilde{\nabla}_x^2\varphi)] = \det[\tilde{d}_x T]$ (either both integrals are undefined or both take the same value in $\overline{\mathbb{R}}$).

The proof follows by the Jacobian identity proved in Theorem 3.3 exactly as in [4, Corollary 4.7].

Let us now define for $t \in [0, 1]$ the measure $\mu_t := (T_t)_\sharp\mu$, where

$$T_t(x) = \exp_x[-t\tilde{\nabla}_x\varphi].$$

By the results in [6] and Proposition 3.1, we know that T_t coincides with the unique optimal map pushing μ forward to μ_t, and that μ_t is absolutely continuous with respect to vol for any

$t \in [0,1]$.
Given $x, y \in M$, following [4], we define for $t \in [0,1]$

$$Z_t(x,y) := \{z \in M \mid d(x,z) = td(x,y) \text{ and } d(z,y) = (1-t)d(x,y)\}.$$

If now N is a subset of M, we set

$$Z_t(x,N) := \cup_{y \in N} Z_t(x,y).$$

Letting $B_r(y) \subset M$ denote the open ball of radius $r > 0$ centered at $y \in M$, for $t \in (0,1]$ we define

$$v_t(x,y) := \lim_{r \to 0} \frac{\operatorname{vol}(Z_t(x, B_r(y)))}{\operatorname{vol}(B_{tr}(y))} > 0$$

(the above limit always exists, thought it will be infinite when x and y are conjugate points, see [4]). Arguing as in the proof of Theorem 3.3, by [4, Lemma 6.1] we get the following:

Theorem 3.6 (Jacobian inequality). *Let E be the set of full μ-measure given by Theorem 3.3. Then, for each $x \in E$, $Y_t(x) := d(\exp_x)_{-t\tilde{\nabla}_x\varphi}$ and $H_t(x) := \frac{1}{2}\nabla_x^2 d^2_{T_t(x)}$ both exist for all $t \in [0,1]$ and the Jacobian determinant*

$$J_t(x) := \det[Y_t(x)(H_t(x) - t\tilde{\nabla}_x^2\varphi)] \tag{8}$$

satisfies

$$J_t^{\frac{1}{n}}(x) \geq (1-t)\,[v_{1-t}(T(x),x)]^{\frac{1}{n}} + t\,[v_t(x,T(x))]^{\frac{1}{n}}\, J_1^{\frac{1}{n}}(x).$$

We now consider as source measure $\mu_0 = \rho_0\, d\operatorname{vol}(x) \in P^{ac}(M)$ and as target measure $\mu_1 = \rho_1\, d\operatorname{vol}(x) \in P^{ac}(M)$, and we assume as before that $W_2(\mu_0,\mu_1) < +\infty$. By Proposition 3.1 we have

$$\mu_t = (T_t)_\sharp[\rho_0\, d\operatorname{vol}] = \rho_t\, d\operatorname{vol} \in P_2^{ac}(M),$$

for a certain $\rho_t \in L^1(M, d\operatorname{vol})$.
We now want to consider the behavior of the functional

$$U(\rho) := \int_M A(\rho(x))\, d\operatorname{vol}(x)$$

along the path $t \mapsto \rho_t$. In Euclidean spaces, this path is called *displacement interpolation* and the functional U is said to be *displacement convex* if

$$[0,1] \ni t \mapsto U(\rho_t) \quad \text{is convex for every } \rho_0, \rho_1.$$

A sufficient condition for the displacement convexity of U in $\mathbb{R}^n$ is that $A : [0,+\infty) \to \mathbb{R} \cup \{+\infty\}$ satisfy

$$(0,+\infty) \in s \mapsto s^n A(s^{-n}) \text{ is convex and non-increasing, with } A(0) = 0 \tag{9}$$

(see [7], [9]). Typical examples include the entropy $A(\rho) = \rho \log \rho$ and the L^q-norm $A(\rho) = \frac{1}{q-1}\rho^q$ for $q \geq \frac{n-1}{n}$.

By all the results collected above, arguing as in the proof of [4, Theorem 6.2], we can prove that the displacement convexity of U is still true on Ricci non-negative manifolds under the assumption (9).

Theorem 3.7 (Displacement convexity on Ricci non-negative manifolds). *If* $\mathrm{Ric} \geq 0$ *and A satisfies (9), then U is displacement convex.*

Proof. As we remarked above, T_t is the optimal transport map from μ_0 to μ_t. So, by Theorem 3.3 and Proposition 3.5, we get

$$U(\rho_t) = \int_M A(\rho_t(x))\, d\,\mathrm{vol}(x) = \int_{E_t} A\left(\frac{\rho_0(x)}{\left(J_t^{\frac{1}{n}}(x)\right)^n}\right)\left(J_t^{\frac{1}{n}}(x)\right)^n d\,\mathrm{vol}(x), \tag{10}$$

where E_t is the set of full μ_0-measure given by Theorem 3.3 and $J_t(x) \neq 0$ is defined in (8). Since $\mathrm{Ric} \geq 0$, we know that $v_t(x,y) \geq 1$ for every $x, y \in M$ (see [4, Corollary 2.2]). Thus, for fixed $x \in E_1$, Theorem 3.6 yields the concavity of the map

$$[0,1] \ni t \mapsto J_t^{\frac{1}{n}}(x).$$

Composing this function with the convex non-increasing function $s \mapsto s^n A(s^{-n})$ we get the convexity of the integrand in (10). The only problem in order to conclude the displacement convexity of U is that the domain of integration appears to depend on t. But, since by Theorem 3.3 E_t is a set of full μ_0-measure for any $t \in [0,1]$, we obtain that, for fixed $t, t', s \in [0,1]$,

$$U(\rho_{(1-s)t+st'}) \leq (1-s)U(\rho_t) + sU(\rho_{t'}),$$

simply by computing each of the three integrals above on the full measure set $E_t \cap E_{t'} \cap E_{(1-s)t+st'}$. □

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