Generalized solutions for the Euler equations in one and two dimensions

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Abstract

In this paper we study generalized solutions (in the Brenier's sense) for the Euler equations. We prove that uniqueness holds in dimension one whenever the pressure field is smooth, while we show that in dimension two uniqueness is far from being true. In the case of the two-dimensional disc we study solutions to Euler equations where particles located at a point x go to $-x$ in a time π, and we give a quite general description of the (large) set of such solutions. As a byproduct, we can construct a new class of classical solutions to Euler equations in the disc.

Cette étude porte sur les solutions généralisées, au sens de Brenier, des équations d'Euler pour les fluides incompressibles. On démontre l'unicité en dimension un lorsque la pression est régulière. En dimension deux, on étudie le cas d'un disque dans lequel les particules en x se déplacent en $-x$ après un temps π et l'on donne une description assez générale de l'ensemble des solutions, qui est beaucoup plus étendu que prévu. Ces solutions généralisées permettent en retour de construire une nouvelle classe des solutions classiques des équations d'Euler dans le disque.

1 Introduction

The velocity field of an incompressible fluid moving inside a smooth domain $D \subset \mathbb{R}^d$ is classically represented by a time-dependent and divergence-free vector field $\boldsymbol{u}(t,x)$ which is parallel to the boundary ∂D. The Euler equations for incompressible fluids describing the evolution of such a velocity field $\boldsymbol{u}$ in terms of the pressure field p are

$$\begin{cases} \partial_t \boldsymbol{u} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} = -\nabla p & \text{in } [0,T]\times D, \\ \operatorname{div}\boldsymbol{u} = 0 & \text{in } [0,T]\times D, \\ \boldsymbol{u}\cdot n = 0 & \text{on } [0,T]\times \partial D. \end{cases} \tag{1.1}$$

If we assume that $\boldsymbol{u}$ is smooth, the trajectory of a particle initially at position x is obtained by solving

$$\begin{cases} \dot g(t,x) = \boldsymbol{u}(t, g(t,x)), \\ g(0,x) = x. \end{cases}$$

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Since $\boldsymbol{u}$ is divergence free, for each time t the map $g(t,\cdot): D \to D$ is a measure-preserving diffeomorphism of D (say $g(t,\cdot) \in \mathrm{SDiff}(D)$), which means

$$g(t,\cdot)_\# \mathscr{L}^d_{\lfloor D} = \mathscr{L}^d_{\lfloor D}$$

(here and in the sequel $f_\#\mu$ is the push-forward of a measure μ through a map f, and $\mathscr{L}^d_{\lfloor D}$ is the Lebesgue measure inside D). Writing Euler equations in terms of g, we get

$$\begin{cases} \ddot{g}(t,x) = -\nabla p\,(t, g(t,x)) & \text{in } [0,T]\times D, \\ g(0,x) = x & \text{in } D, \\ g(t,\cdot) \in \mathrm{SDiff}(D) & \text{for } t \in [0,T]. \end{cases} \tag{1.2}$$

In [2], Arnold interpreted the equation above, and therefore (1.1), as a *geodesic* equation on the space $\mathrm{SDiff}(D)$, viewed as an infinite-dimensional manifold with the metric inherited from the embedding in $L^2(D)$ and with tangent space corresponding to the divergence-free vector fields. According to this interpretation, one can look for solutions of (1.2) by minimizing

$$\int_0^T \int_D \frac{1}{2}|\dot{g}(t,x)|^2 \, d\mathscr{L}^d_{\lfloor D}(x)\, dt \tag{1.3}$$

among all paths $g(t,\cdot): [0,T] \to \mathrm{SDiff}(D)$ with $g(0,\cdot) = f$ and $g(T,\cdot) = h$ prescribed (typically, by right invariance, f is taken as the identity map $\boldsymbol{i}$). In this way, the pressure field arises as a Lagrange multiplier from the incompressibility constraint.

Shnirelman proved in [9, 10] that when $d \geq 3$ the infimum is not attained in general, and that when $d = 2$ there exists $h \in \mathrm{SDiff}(D)$ which cannot be connected to $\boldsymbol{i}$ by a path with finite action. These "negative" results motivate the study of relaxed versions of Arnold's problem.

The first relaxed version of Arnold's minimization problem was introduced by Brenier in [3]: he considered probability measures $\boldsymbol{\eta}$ in $\Omega(D)$, the space of continuous paths $\omega: [0,T] \to D$, and solved the variational problem

$$\text{minimize} \qquad \mathscr{A}_T(\boldsymbol{\eta}) := \int_{\Omega(D)} \int_0^T \frac{1}{2}|\dot{\omega}(\tau)|^2 \, d\tau \, d\boldsymbol{\eta}(\omega), \tag{1.4}$$

with the constraints

$$(e_0, e_T)_\#\boldsymbol{\eta} = (\boldsymbol{i}, h)_\#\mathscr{L}^d_{\lfloor D}, \qquad (e_t)_\#\boldsymbol{\eta} = \mathscr{L}^d_{\lfloor D} \qquad \forall\, t \in [0,T] \tag{1.5}$$

(here and in the sequel $e_t(\omega) := \omega(t)$ are the evaluation maps at time t). According to Brenier, we shall call these $\boldsymbol{\eta}$ *generalized incompressible flows* in $[0,T]$ between $\boldsymbol{i}$ and h. The existence of a minimizing $\boldsymbol{\eta}$ is a consequence of the coercivity and lower semicontinuity of the action, provided that there exists at least a generalized flow $\boldsymbol{\eta}$ with finite action (see [3]). This is the case for instance if $D = [0,1]^d$, or if D is the unit ball $B_1(0)$ (this follows from the results in [3, 5] and by [1, Theorem 3.3]).

We observe that any sufficiently regular path $g(t,\cdot): [0,1] \to \mathrm{SDiff}(D)$ induces a generalized incompressible flow $\boldsymbol{\eta} = (\Phi_g)_\#\mathscr{L}^d_{\lfloor D}$, where $\Phi_g: D \to \Omega(D)$ is given by $\Phi_g(x) = g(\cdot, x)$, but the converse is far from being true: in the case of generalized flows, particles starting from different points are allowed to cross at a later time, and particles starting from the same point are allowed to split, which is of course forbidden by classical flows. Although this crossing/splitting phenomenon could seem strange, it arises naturally if one looks for example at the hydrodynamic limit of the Euler equation. Indeed, the above model allows to describe the limits obtained by solving the Euler equations in $D \times [0, \varepsilon] \subset \mathbb{R}^{d+1}$ and, after a suitable change of variable, letting $\varepsilon \to 0$ (see for instance [6]).

In [3], a consistency result was proved: smooth solutions to (1.1) are optimal even in the larger class of the generalized incompressible flows, provided the pressure field p satisfies

$$T^2 \sup_{t \in [0,T]} \sup_{x \in D} \nabla_x^2 p(t,x) \leq \pi^2 I_d \tag{1.6}$$

(here I_d denotes the identity matrix in $\mathbb{R}^d$), and are the unique ones if the above inequality is strict.

In this paper, we will consider Problem (1.4)-(1.5) in the particular cases where $D = B_1(0)$ or D is an annulus, in dimension 1 and 2. We will be mainly be concerned with uniqueness and characterization issues, as existence always holds in these cases.

If $D = B_1(0) \subset \mathbb{R}^2$ is the unit ball, the following situation arises: an explicit solution of Euler equations is given by the transformation

$$g(t,x) = \boldsymbol{R}_t x,$$

where $\boldsymbol{R}_t : \mathbb{R}^2 \to \mathbb{R}^2$ denotes the counterclockwise rotation of an angle t. Indeed the maps $g(t,\cdot) : D \to D$ are clearly measure preserving, and moreover we have

$$\ddot{g}(t,x) = -g(t,x),$$

so that $\boldsymbol{v}(t,x) = \dot{g}(t,y)|_{y=g^{-1}(t,x)}$ is a solution to the Euler equations with the pressure field given by $p(x) = |x|^2/2$ (so that $\nabla p(x) = x$). Thus, thanks to (1.6) and by what we said above, the generalized incompressible flow induced by g is optimal if $T \leq \pi$, and is the unique one if $T < \pi$. This implies in particular that there exists a unique minimizing geodesic from $\boldsymbol{i}$ to the rotation $\boldsymbol{R}_T$ if $0 < T < \pi$. On the contrary, for $T = \pi$ more than one optimal solution exists, as both the clockwise and the counterclockwise rotation of an angle π are optimal (this shows for instance that the upper bound (1.6) is sharp). Moreover, Brenier found in [3, Section 6] an example of action-minimizing path $\boldsymbol{\eta}$ connecting $\boldsymbol{i}$ to $-\boldsymbol{i}$ in time π which is not induced by a classical solution of the Euler equations (and it cannot be simply constructed using the two opposite rotations):

$$\int_{\Omega(D)} \varphi(\omega)\, d\boldsymbol{\eta}(\omega) := \int_{D \times \mathbb{R}^d} \varphi\big(t \mapsto x\cos(t) + v\sin(t)\big)\, d\mu(x,v) \qquad \forall \varphi \in C(\Omega),$$

with μ given by (4.2). What is interestingly shown by the solution constructed by Brenier is the following: when $\boldsymbol{\eta}$ is of the form $\boldsymbol{\eta} = (\Phi_g)_\# \mathscr{L}^d_{\lfloor D}$ for a certain map g, one can always recover $g(t,\cdot)$ from $\boldsymbol{\eta}$ using the identity

$$(e_0, e_t)_\# \boldsymbol{\eta} = (\boldsymbol{i}, g(t,\cdot))_\# \mathscr{L}^d_{\lfloor D}, \qquad \forall\, t \in [0,T].$$

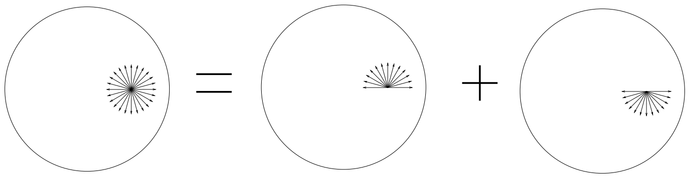

Figure 1: In Brenier's example, each particle splits uniformly in all directions. Selecting only the clockwise or the anticlockwise trajectories gives rise to two new geodesics between $\boldsymbol{i}$ and $-\boldsymbol{i}$ (see Paragraph 4.1).

In the example found by Brenier no such representation is possible (i.e. $(e_0, e_t)_\# \boldsymbol{\eta}$ is not a graph), which implies that the splitting of fluid paths starting at the same point is actually possible for optimal flows (in this case, we will say that these flows are *non-deterministic*). We moreover observe that this solution is in some sense the most isotropic: each particle starting at a point x splits uniformly in all directions and reaches the point $-x$ in time π. Due to this isotropy, it was conjectured that this solution was an extremal point in the set of minimizing geodesic [7]. However we will show that this is not the case: the decomposition of μ as the sum of its clockwise and an anticlockwise components gives rise to two new geodesics (see Figure 1 and Paragraph 4.1). The interesting property of these geodesics is that, in addition of being non-deterministic, they induce two non-trivial stationary solutions to Euler equations with a new "macroscopic" pressure field (see Paragraph 4.4). More generally, using the generalized solutions constructed in Paragraph 4.2, one can produce a new large class of stationary and non-stationary solutions to Euler equations.

The one-dimensional case is a bit particular since, if $D = [-1, 1]$, the space of measure-preserving diffeomorphisms consist of $\{\boldsymbol{i}, -\boldsymbol{i}\}$, and so the Arnold problem is trivial (there are only two continuous curves belonging to $\mathrm{SDiff}([-1, 1])$). However the relaxed problem is non-trivial, and Brenier found in [3, Section 6] an explicit example of generalized solution from $\boldsymbol{i}$ to $-\boldsymbol{i}$ in $[-1, 1]$. This generalized optimal flow is unique (see [3, Proposition 6.3]) and non-deterministic (in the sense described before). Though considering the one-dimensional case could seem peculiar, it happens to be important for the study of the multidimensional case: for instance, whenever one considers the problem from $\boldsymbol{i}$ to h with $h = (f(x_1), x_2, \ldots, x_d)$ and $D = [-1, 1]^d$, any optimal incompressible flow $\boldsymbol{\eta}$ is just a superposition of one-dimensional optimal incompressible flows from $\boldsymbol{i}$ to f in $[-1, 1]$ (see [5, Proposition 3.4]).

The aim of this paper is the following: on the one hand we will show that the uniqueness result of Brenier in dimension 1 is quite a general fact: whenever the pressure field is smooth, generalized geodesic are unique (see Section 3). On the other hand, if we move to dimension 2, the situation completely changes, and as we said before one can find a large variety of generalized geodesics. In Section 4 we describe the set of such geodesics under some additional constraints, namely rotational invariance or stationarity in time. Finally, in Paragraph 4.4 we will see that such geodesics induce classical solutions to the Euler equations with a different "macroscopic" pressure field.

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2 Preliminaries

In this section we introduce the pressure field, and we explain its relations with optimal generalized incompressible flows.

First of all, we need to relax the incompressibility constraint on $\boldsymbol{\eta}$, so that p will arise as a Lagrange multiplier.

Given a probability measure $\boldsymbol{\nu}$ on $\Omega(D)$ such that $(e_t)_\#\boldsymbol{\nu} \ll \mathscr{L}^d_{\lfloor D}$, we define its density $\rho^{\boldsymbol{\nu}}$ via the formula

$$\rho^{\boldsymbol{\nu}}(t)\mathscr{L}^d_{\lfloor D} = (e_t)_\#\boldsymbol{\nu}.$$

Definition 2.1. We say that a probability measure $\boldsymbol{\nu}$ on $\Omega(D)$ is an *almost incompressible (generalized) flow* if $\rho^{\boldsymbol{\nu}} \in C^1([0,T]\times D)$ and

$$\|\rho^{\boldsymbol{\nu}} - 1\|_{C^1([0,T]\times D)} \leq \frac{1}{2}.$$

In [4, 1], the following duality result is proved:

Theorem 2.2. *Let $\boldsymbol{\eta}$ be an optimal incompressible flow. There exists $p \in \big(C^1([0,T]\times D)\big)^*$ such that*

$$\langle p, \rho^{\boldsymbol{\nu}} - 1\rangle_{(C^1)^*,C^1} \leq \mathscr{A}_T(\boldsymbol{\nu}) - \mathscr{A}_T(\boldsymbol{\eta}) \tag{2.1}$$

for all almost incompressible flows $\boldsymbol{\nu}$ satisfying $(e_0,e_T)_\#\boldsymbol{\nu} = (\boldsymbol{i}\times h)_\#\mathscr{L}^d_{\lfloor D}$.

From the above theorem we see that, if one relaxes the incompressibility constraint, the global minimality of $\boldsymbol{\eta}$ is still preserved provided one adds to the functional the Lagrange multiplier given by p.

From (2.1) we can compute the first variation with respect to perturbations where any curve ω is replaced by its images through applications of the form $\boldsymbol{i} + \varepsilon\boldsymbol{w}$: one obtains

$$\int_{\Omega(D)}\int_0^T \left[\dot\omega(t)\cdot\frac{d}{dt}\boldsymbol{w}(t,\omega(t))\right] dt\, d\boldsymbol{\eta}(\omega) + \langle p, \operatorname{div}\boldsymbol{w}\rangle_{(C^1)^*,C^1} = 0$$

for all smooth vector fields $\boldsymbol{w}(t,x)$ vanishing near the boundary of $D\times[0,1]$. As noticed in [4, 1], the above equation uniquely identifies the pressure field p (as a distribution) up to trivial modifications, i.e. additive perturbations depending on time only. Moreover we remark that, if we define the *effective velocity* $\overline{\boldsymbol{v}}_t(x)$ by $(e_t)_\#(\dot\omega(t)\boldsymbol{\eta}) = \overline{\boldsymbol{v}}_t\mathscr{L}^d_{\lfloor D}$, and the *quadratic effective velocity* $\overline{\boldsymbol{v}\otimes\boldsymbol{v}}_t(x)$ by $(e_t)_\#(\dot\omega(t)\otimes\dot\omega(t)\boldsymbol{\eta}) = \overline{\boldsymbol{v}\otimes\boldsymbol{v}}_t\mathscr{L}^d_{\lfloor D}$, the above equation becomes

$$\partial_t\overline{\boldsymbol{v}}_t(x) + \operatorname{div}\left(\overline{\boldsymbol{v}\otimes\boldsymbol{v}}_t(x)\right) + \nabla p(t,x) = 0 \tag{2.2}$$

in the sense of distribution. The fact that in general $\overline{\boldsymbol{v}\otimes\boldsymbol{v}_t} \neq \overline{\boldsymbol{v}}_t \otimes \overline{\boldsymbol{v}}_t$ implies that $\boldsymbol{\eta}$ does not always induce a distributional solution to the Euler equations. However, as we will see in Section 4.4, in the case of $D = B_1(0) \subset \mathbb{R}^2$ the quantity $\overline{\boldsymbol{v}\otimes\boldsymbol{v}_t} - \overline{\boldsymbol{v}}_t \otimes \overline{\boldsymbol{v}}_t$ is typically a gradient, and so we can find a true distributional solution replacing the pressure with a "macroscopic" one (actually, we do not know an example in the case of $D = B_1(0)$ where $\overline{\boldsymbol{v}\otimes\boldsymbol{v}_t} - \overline{\boldsymbol{v}}_t \otimes \overline{\boldsymbol{v}}_t$ is not a gradient).

Assume now that p is smooth (indeed, this will be the case in what follows). Then we can write (2.1) as

$$\int_{\Omega(D)} \int_0^T \left(\frac{1}{2}|\dot\omega|^2 - p(t,\omega(t))\right) dt\, d\boldsymbol{\eta}(\omega) \leq \int_{\Omega(D)} \int_0^T \left(\frac{1}{2}|\dot\omega|^2 - p(t,\omega(t))\right) dt\, d\boldsymbol{\nu}(\omega).$$

From the results in [1, Section 6] (see in particular Theorems 6.8 and 6.12) one obtains that an incompressible flow $\boldsymbol{\eta}$ is optimal if and only if

$$\omega \text{ minimizes} \quad \gamma \mapsto \int_0^T \left(\frac{1}{2}|\dot\gamma|^2 - p(t,\gamma(t))\right) dt \qquad \text{for } \boldsymbol{\eta}\text{-a.e.}\omega, \tag{2.3}$$

the minimization being performed among all $\gamma \in W^{1,2}([0,T],D)$ such that $\gamma(0) = \omega(0)$ and $\gamma(T) = \omega(T)$.

From this fact one can also understand the condition (1.6) on the pressure: the Euler-Lagrange equation of the above functional is $\ddot\omega = -\nabla p(t,\omega)$ and (1.6) is the natural condition to ensure that critical points are minimizers. Moreover, if (1.6) holds with a strict inequality, then there exists a unique minimizing curve from $\omega(0) = x$ to $\omega(T) = h(x)$ for all x, and so $\boldsymbol{\eta}$ is unique.

Let us now consider the case $D = B_1(0) \subset \mathbb{R}^2$, where the pressure field is given by $p(x) = |x|^2/2$ (as proved in [3, Section 6]). The above considerations explain why there exists a unique optimal $\boldsymbol{\eta}$ from $\boldsymbol{i}$ to the rotation $\boldsymbol{R}_T$ if $0 < T < \pi$. On the other hand, as we already said in the introduction, the situation for $T = \pi$ is completely different: two classical solutions are given by

$$[0,\pi] \ni t \mapsto (x_1\cos(\pm t) + x_2\sin(\pm t), x_1\sin(\pm t) + x_2\cos(\pm t)).$$

Furthermore, one can also consider the family of minimizing curves $\omega_{x,\theta}$ connecting x to $-x$ given by

$$\omega_{x,\theta}(t) := x\cos t + \sqrt{1-|x|^2}(\cos\theta,\sin\theta)\sin t, \qquad \theta \in (0,2\pi),$$

and define

$$\boldsymbol{\eta} := \frac{1}{2\pi^2}(\omega_{x,\theta})_\#\big(\mathscr{L}^2_{\lfloor D} \times \mathscr{L}^1_{\lfloor(0,2\pi)}\big). \tag{2.4}$$

Then, as proved in [3, Section 6], $\boldsymbol{\eta}$ is a minimizer as well, and non-deterministic in between.

In Section 4 we will construct other examples of minimizers from $\boldsymbol{i}$ to $-\boldsymbol{i}$. To this aim it will be useful to introduce a different formalism.

As we explained above, if the pressure field is smooth, any optimal $\boldsymbol{\eta}$ is concentrated on curves minimizing the action (indeed this holds under much weaker assumption on the pressure, see [1, Section 6]). In particular such curves ω solve the second order ordinary differential equations

$$\ddot{\omega} = -\nabla p(t, \omega), \tag{2.5}$$

and so they are uniquely determined by their initial position and velocity. Therefore, if we look for optimal flows $\boldsymbol{\eta}$, we can describe them just prescribing initial position and velocity of each curve: if $\Phi(\cdot, x, v)$ denotes the unique integral curve of the ODE starting from x with velocity v, we can consider probability measures μ on $D \times \mathbb{R}^d$ and define

$$\boldsymbol{\eta}_\mu := \Phi_\# \mu.$$

If we ensure that the curve $t \mapsto \Phi(t, x, v)$ belongs to D for every $t \in [0, T]$, then $\boldsymbol{\eta}_\mu$ will be a probability measure on $\Omega(D)$. Moreover, if T is chosen so that (1.6) is satisfied, then $\boldsymbol{\eta}_\mu$ is an optimal flow. Finally, it is not difficult to see that the above conditions are also necessary.

We therefore get the following:

Lemma 2.3. *Let $D \subset \mathbb{R}^d$, and denote by $\pi_D : D \times \mathbb{R}^d \to D$ the projection on the first factor. Assume that p is smooth and that* (1.6) *is satisfied, and denote by $\Phi(\cdot, x, v)$ the unique integral curve of* (2.5) *starting from x and with velocity v. Then, given a probability measures μ on $D \times \mathbb{R}^d$, the induced flow $\boldsymbol{\eta}_\mu = \Phi_\# \mu$ is a minimizer of the action (1.4) if and only if it satisfies*

$$\Phi(t, \cdot, \cdot)_\# \mu = \mathscr{L}^d_{\lfloor D} \qquad \forall\, t \in [0, T]. \tag{2.6}$$

On the other hand, any minimizer $\boldsymbol{\eta}$ is induced by a measure μ which satisfies the above condition.

Notice that condition (2.6) implies that, for μ-a.e. (x, v), the curve $[0, T] \ni t \mapsto \Phi(t, x, v)$ stays inside D. Moreover, in case (1.6) is not verified, the same lemma holds true if one adds condition (2.3) to (2.6).

The above lemma will be useful in the next sections for constructing or characterizing generalized solutions.

3 Uniqueness in 1D

As we mentioned before, existence of minimizers is always true for $D = [-1, 1]$. Moreover uniqueness holds whenever the pressure p satisfies the strict inequality $T^2 \sup_{x \in [-1,1]} p''(x) < \pi^2$. When instead of the strict inequality we have equality, uniqueness is a much harder matter (the associated differential equation may have more than one solution for prescribed starting and arrival point). A typical example is when the diffeomorphism $\boldsymbol{i}$ has to be connected to $-\boldsymbol{i}$: in this case the pressure field is $p(x) = x^2/2$, and there are infinitely many solutions of $\ddot{\gamma} = -\gamma$ in $[0, \pi]$ with $\gamma(0) = x$, $\gamma(1) = -x$, and $\gamma(t) \in [-1, 1]$ for all t.

Despite this fact, as shown by Brenier in [3, Proposition 6.3], uniqueness of geodesics holds (as we will see in the next section, the two-dimensional case is completely different).

Theorem 3.1. *If $D = [-1, 1] \subset \mathbb{R}$, Problem* (1.4)-(1.5) *for $h = -\boldsymbol{i}$ has a unique minimizer.*

Proof. By Lemma 2.3, we need to prove that the constraints (2.6) uniquely identifies a probability measure μ on the phase space $D \times \mathbb{R}$. The incompressibility constraint implies that, for any $t \in [0, \pi]$, the image measure of μ through the map $(x, v) \mapsto x \cos t + v \sin t$ is the Lebesgue measure on D. This means that the marginals of μ with respect to any one-dimensional projections $(x, v) \mapsto (x, v) \cdot (\cos t, \sin t)$ are prescribed for all $t \in [0, \pi]$. This implies that the Radon transform of μ is prescribed, so that μ is unique (see [8]). $\square$

For the reader who is not familiar with the Radon transform, we underline the possibility of getting the same result by means of the (more known) Fourier transform. Actually, since all the integrals of functions of the form $e^{i(\xi_1 x + \xi_2 v)}$ are prescribed if one knows the above projections, the Fourier transform of μ is determined.

Proposition 3.2. *Let $\psi_t : \mathbb{R}^2 \to \mathbb{R}$ denote the map $\psi_t(x, v) = x \cos t + v \sin t$. A Borel finite measure μ on $\mathbb{R}^2$ satisfies*

$$(\psi_t)_\# \mu = \mathscr{L}^1_{\lfloor [-1,1]} \qquad \forall\, t \in [0, \pi] \tag{3.1}$$

if and only if $\mu = g \cdot \mathscr{L}^2_{\lfloor B_1(0)}$, where

$$g(x, v) = \frac{1}{\pi \sqrt{1 - x^2 - v^2}}.$$

Proof. First, we show by direct computation that the measure $\mu = g \cdot \mathscr{L}^2_{\lfloor B_1(0)}$ satisfies (3.1). Since g is invariant by rotation, it is enough to prove that $(\psi_0)_\# \mu = \mathscr{L}^1_{\lfloor [-1,1]}$. To this aim, let φ be a continuous function on $\mathbb{R}$. Then

$$\begin{aligned}
\int_{B_1(0)} \varphi(x) \, \frac{dxdv}{\sqrt{1 - x^2 - v^2}} &= \int_{-1}^{1} \varphi(x) \, \frac{dx}{\pi} \int_{v=-\sqrt{1-x^2}}^{v=\sqrt{1-x^2}} \frac{dv}{\sqrt{1 - x^2}\sqrt{1 - \frac{v^2}{1-x^2}}} \\
&= \int_{-1}^{1} \varphi(x) \, \frac{dx}{\pi} \int_{y=-1}^{y=1} \frac{dy}{\sqrt{1 - y^2}} \\
&= \int_{-1}^{1} \varphi(x) \, dx.
\end{aligned}$$

As φ is arbitrary, we get $(\psi_0)_\# \mu = \mathscr{L}^1_{\lfloor [-1,1]}$. To prove that $g \cdot \mathscr{L}^2_{\lfloor B_1(0)}$ is the only possible minimizer, it suffices to observe as in Theorem 3.1 that the condition $(\psi_t)_\# \mu = \mathscr{L}^1_{\lfloor [-1,1]}$ for all $t \in [0, \pi]$ prescribes the Radon transform of μ. $\square$

We now turn to an extension of Theorem 3.1 and prove that uniqueness holds in the case of a regular pressure field. This obviously includes the quadratic pressure $p(x) = x^2/2$ that we discussed above. The idea is once again to characterize the measure μ on the couples (x, v) knowing the marginals $(e_t)_\# \boldsymbol{\eta}_\mu$ for any t, i.e. the images of μ under the applications $\mathbb{R}^2 \ni (x, v) \mapsto \Phi(t, x, v) \in \mathbb{R}$.

Theorem 3.3. *Let $D = [-1, 1]$ and suppose that the pressure p is of class C^∞. Then there exists a unique minimizer $\boldsymbol{\eta}$ to Problem* (1.4)-(1.5).

Proof. We have to show that there is a unique measure μ that satisfies the conditions in Lemma 2.3. To this aim, we will prove that the integrals with respect to μ of all the functions of the form $(x, v) \mapsto f(x)v^n$ are known. We recall that $\Phi(\cdot, x, v)$ is the solution of the ODE

$$\begin{cases} \ddot{\gamma} = -\nabla p(t, \gamma), \\ \gamma(0) = x, \, \dot{\gamma}(0) = v. \end{cases} \tag{3.2}$$

Thanks to the incompressibility condition, the integrals with respect to μ of all functions of the form $(x, v) \mapsto f(\Phi(t, x, v))$ are known. In particular, for $t = 0$, this reduces to $f(x)$. If we consider, for f smooth, the function

$$\frac{f(\Phi(t, x, v)) - f(x)}{t}$$

and we pass to the limit as $t \to 0$, we obtain the function $(x, v) \mapsto f'(x)v$. This means that the integrals of all the functions of this form are known as well (and for instance they all vanish). Since any smooth function h on $\mathbb{R}$ can be expressed as $h = f'$, we get that the integrals of all functions of the form $h(x)v$ with h smooth are known as well (and since they are all zero, we deduce in particular that the effective velocity $\overline{\boldsymbol{v}}_t = \int \dot{\omega}(t)\, d\boldsymbol{\eta}_\mu$ is identically zero). We want to go on with higher powers of v.

Let us first remark the following: for any $n \geq 0$, the n-th derivative with respect to t of $f(\Phi(t, x, v))$ is given by a sum of the form

$$(f \circ \gamma)^{(n)} = \sum_{j<n} (g_j \circ \gamma)(\dot{\gamma})^j + (f^{(n)} \circ \gamma)(\dot{\gamma})^n. \tag{3.3}$$

Such a formula can indeed be obtained by a simple induction argument, using iteratively Equation (3.2).

We now claim that the integrals of all the functions of the form $(x, v) \mapsto g(x)v^n$ are determined. This is proved inductively using Equation (3.3) at time $t = 0$ (so that γ becomes x) and noticing that the set $\{f^{(n)} : f \in C^\infty(\mathbb{R})\}$ coincides with the space of all C^∞ functions.

Thus, the integrals of $f(x)P(v)$ with respect to μ are known for any polynomial P and $f \in C^\infty(\mathbb{R})$, and the proof is completed. □

4 Weak geodesics in 2D

As shown in the last section, the regularity of the pressure field guarantees uniqueness of weak geodesics. As we will see, in two dimensions the picture is completely different.

We want to describe the set of minimizing geodesics on $[0, \pi]$ connecting the identity map $\boldsymbol{i}$ to its opposite $-\boldsymbol{i}$ on a domain $D \subset \mathbb{R}^2$ which is either the unit disc $B_1(0)$ or an annulus $A_{R_1,R_2} = \{R_1 \leq |x| \leq R_2\}$. One motivation for studying minimizers in the annulus is that, since we can decompose the disc in a disjoint union of annuli, we can use the minimizers in the annuli to construct minimizers in the disc.

As we said in Section 2 the unique pressure field in the disc is given by $p(x) = |x|^2/2$. Since the flows induced by the clockwise and the counterclockwise rotation are two classical solutions to the Euler equations also in the annulus, and (1.6) holds (with equality), we deduce that also for $D = A_{R_1,R_2}$ the (unique) pressure field is given by $p(x) = |x|^2/2$. Thus, to any measure $\mu(dx, dv)$ on the phase space $TD := D \times \mathbb{R}^2$, we can associate the measure $\boldsymbol{\eta}_\mu$ given again by

$$\boldsymbol{\eta}_\mu := \Phi_\# \mu, \qquad \Phi(\cdot, x, v) := (t \mapsto x \cos t + v \sin t),$$

and Lemma 2.3 allows us to say whether $\boldsymbol{\eta}_\mu$ is a minimizer. In the following, we will say that μ is a minimizer whenever $\boldsymbol{\eta}_\mu$ is a minimizer. Similarly, we will say that μ is incompressible whenever $\boldsymbol{\eta}_\mu$ is incompressible, i.e. whenever

$$\Phi(t, \cdot, \cdot)_\# \mu = \mathscr{L}^2_{\lfloor D}, \qquad \forall\, t \in [0, \pi]. \tag{4.1}$$

Passing from dimension 1 to dimension 2, the phase space is now of dimension 4 and we cannot hope for a uniqueness theorem like the one in the last section. Indeed, to understand why the picture now is much more complicated, let us consider the following example: as shown by Brenier [3, Section 6], the measure

$$\mu(dx, dv) = \frac{1}{2\pi\sqrt{1 - |x|^2}} \left[\mathscr{H}^1_{\lfloor\{|v| = \sqrt{1-|x|^2}\}}(dv) \right] \otimes \mathscr{L}^2_{\lfloor D}(dx) \tag{4.2}$$

induces a non-deterministic geodesic $\boldsymbol{\eta}$ from $\boldsymbol{i}$ to $-\boldsymbol{i}$ (which corresponds to the flow $\boldsymbol{\eta}$ defined in (2.4)). Observe that this solution is concentrated on the set $\{|v|^2 + |x|^2 = 1\}$.

To find new minimizers, and at the same time to give a sufficiently general description of the whole set of minimizers, we will try to reduce the dimension of the free parameters on μ by imposing some constraints.

First of all, let ϕ_t denote the Hamiltonian flow on the phase space, that is

$$\phi_t(x, v) := (x(t), v(t)) = (x \cos t + v \sin t, -x \sin t + v \cos t),$$

so that $\Phi(t, \cdot, \cdot) = \pi_D \circ \phi_t$. Since ϕ_t preserves the energy $E(x, v) := |v|^2 + |x|^2$, it is natural to try to look for measures μ which are concentrated on level sets of E (other minimizers not concentrated on a single energy level can be constructed superposing different annuli, as described at the beginning of Paragraph 4.2).

Moreover we try to look for minimizers which satisfy some additional constraints, like stationarity in time or invariance under rotations.

Definition 4.1. The measure on the phase space μ is said to be *stationary* if $\mu_t := (\phi_t)_\# \mu$ is equal to μ for all t. In terms of $\boldsymbol{\eta}$, this means that $(E_t)_\# \boldsymbol{\eta}$ does not depend on t, with $E_t(\omega) := (\omega(t), \dot\omega(t))$.

Definition 4.2. Let $\boldsymbol{R}_\theta : \mathbb{R}^2 \to \mathbb{R}^2$ denotes the counterclockwise rotation of an angle θ, and let $\bar{\boldsymbol{R}}_\theta : TD \to TD$ be defined by $\bar{\boldsymbol{R}}_\theta(x, v) = (\boldsymbol{R}_\theta x, \boldsymbol{R}_\theta v)$. We say that μ is *rotationally invariant* if $(\bar{\boldsymbol{R}}_\theta)_\# \mu = \mu$ for all $\theta > 0$.

Observe that once the constraint $\mathrm{supp}(\mu) \subset \{E(x, v) = K\}$ is imposed, we are left with 3 degrees of freedom. Since in dimension 1 (i.e. with 2 degrees of freedom) uniqueness holds, one could expect that once we impose either the stationarity or the rotational invariance of μ, then one should recover uniqueness. This is more or less true: there is still one possible choice, that on the clockwise or counterclockwise direction of the curves (see for example Paragraph 4.1). However, up to this choice, the expected uniqueness result holds (both in the case $D = B_1(0)$ and $D = A_{R_1,R_2}$):

1. Once one imposes the directions (clockwise or counterclockwise) of the particle trajectories, there is only one rotationally invariant minimizer μ that is concentrated on the (appropriate) energy level $\{E(x,v) = K\}$ (see Paragraph 4.2);
2. There is a unique stationary clockwise minimizer μ concentrated on the (appropriate) energy level $\{E(x,v) = K\}$, and in particular it is rotationally invariant (see Paragraph 4.3).

As shown by Example 4.9, rotational invariance does not imply stationarity in time (see Definition 4.1). It is an open question whether or not there is a geodesic from $\boldsymbol{i}$ to $-\boldsymbol{i}$ that is not rotationally invariant.

4.1 Clockwise/Counterclockwise decomposition of Brenier's minimizer

In this section we show that Brenier's non-deterministic geodesic $\boldsymbol{\eta}$ may be decomposed as the sum of two geodesics, one clockwise and the other counterclockwise. Let us define the two sets

$$TD^+ = \{(x,v) \,:\, x^\perp \cdot v > 0\}, \qquad TD^- = \{(x,v) \,:\, x^\perp \cdot v < 0\},$$

where $(x_1, x_2)^\perp = (x_2, -x_1)$ (i.e. $x^\perp = \boldsymbol{R}_{\pi/2} x$). Then we define the two measures

$$\mu^+ := \mu_{\lfloor TD^+}, \qquad \mu^- := \mu_{\lfloor TD^-},$$

with μ given by (4.2).

Lemma 4.3. *The measures μ^+ and μ^- are stationary.*

Proof. Since μ is stationary (see [3, Section 6]) we get the following identities:

$$\mu = \mu^+ + \mu^- = \mu_t^+ + \mu_t^- \qquad \forall\, t \in [0, \pi], \tag{4.3}$$

where $\mu_t^\pm := (\phi_t)_\# \mu^\pm$. This implies that the supports of the two measures μ_t^+ and μ_t^- are contained for all times in the support of μ.

We now observe that the conditions $x^\perp \cdot v > 0$ and $x^\perp \cdot v < 0$ are stationary in time, as $\frac{d}{dt}(x^\perp(t) \cdot v(t)) = 0$. Therefore, we necessarily have

$$\operatorname{supp}(\mu_t^+) \subset \operatorname{supp}(\mu^+), \qquad \operatorname{supp}(\mu_t^-) \subset \operatorname{supp}(\mu^-). \tag{4.4}$$

Combining (4.3) and (4.4), we easily get

$$\mu_t^+ = \mu^+, \qquad \mu_t^- = \mu^- \qquad \forall\, t \in [0, \pi].$$

$\square$

Proposition 4.4. *The measures $\boldsymbol{\eta}_{\mu^+}$ and $\boldsymbol{\eta}_{\mu^-}$ are two weak geodesics from $\boldsymbol{i}$ to $-\boldsymbol{i}$.*

Proof. It is enough to prove that the measures $\boldsymbol{\eta}_{\mu^+}$ and $\boldsymbol{\eta}_{\mu^-}$ are incompressible. Since μ^+ and μ^- are stationary, $(e_t)_\#\boldsymbol{\eta}_{\mu^+}$ and $(e_t)_\#\boldsymbol{\eta}_{\mu^-}$ do not depend on t. The incompressibility then comes from the fact that $(e_0)_\#\boldsymbol{\eta}_{\mu^+} = (e_0)_\#\boldsymbol{\eta}_{\mu^-} = \mathscr{L}^d_{\lfloor D}$. □

4.2 Rotationally invariant geodesics on an annulus

In this subsection we are concerned with optimal rotationally invariant measures concentrated on the set $TD_K := \{(x,v) \in TD \,:\, E(x,v) = K\}$. We consider the case of D being an annulus A_{R_1,R_2}, with the aim of proving existence of minimizers on such domain (the disc corresponds to the case $R_1 = 0$). In this way, our existence results can also be used to build new minimizers on the disc: one performs a partition of the disc into annuli and then uses one such minimizer in each of them. This produces a whole class of minimizers to Problem (1.4)-(1.5) in the disc which was not known before. Moreover, notice that we will build minimizers in the annulus with radii R_1 and R_2 which are concentrated on $TD_{R_1^2+R_2^2}$. By superposing them, one can construct minimizers on the disc where velocities at point x have a modulus which is neither $|x|$ (as in the deterministic rotational solution), nor $\sqrt{1-|x|^2}$ (as in Brenier's non-deterministic minimizer). We also remark that one can recover the non-deterministic minimizer of Brenier considering $R_1 = 0$ and $R_2 = 1$, while the deterministic solutions correspond to the limit $|R_2 - R_1| \to 0$, where one superposes infinitely many annuli, each of them corresponding to a single circle. Anyway, even besides this superposition procedure, the understanding of the minimizers on annuli has brought many interesting consequences to the case of the disc as well.

Without loss of generality, we can assume $R_2 = 1$. Take $0 \leq R < 1$, and let $D = A_{R,1}$. We consider the set TD_{1+R^2} and notice that level sets of the energy are invariant under the Hamiltonian flow ϕ_t. However, given a point x, not all initial speed with modulus $\sqrt{1+R^2-|x|^2}$ are such that the trajectory $(x(t), v(t))$ stays in the annulus (see also Figure 2). In all that follows, we will only consider measures concentrated on TD_{1+R^2}.

Since we consider rotationally invariant measures μ, we can characterize them by identifying their behaviour on a single ray of the disc. Moreover, as their x marginal is the Lebesgue measure (thanks to the incompressibility condition for $t = 0$), they can be written in the form

$$\mu = \big((\boldsymbol{R}_\theta)_\#\mu_r(dv)\big) \otimes r\,dr\,d\theta, \tag{4.5}$$

where μ_r are measures on the set of possible velocities corresponding to the points x with $|x| = r$.

Since the variable θ does not play any role here, and for any x the velocities are actually concentrated on a one-dimensional set (thanks to the constraint $(x,v) \in TD_{1+R^2}$), we may actually reduce the total number of variables from 4 to 2. Hence, we consider the following projection from the 4-dimensional space TD to the 2-dimensional space $\mathcal{P} := [-\frac{1-R^2}{2}, \frac{1-R^2}{2}] \times \mathbb{R}$ given by

$$\pi_{\mathcal{P}} : (x,v) \mapsto (a,b) := (|x|^2 - \frac{1+R^2}{2}, v\cdot x).$$

We remark that this projection will turn out to have a very interesting behaviour with respect to the flow ϕ_t (see Lemma 4.11 and Figure 3).

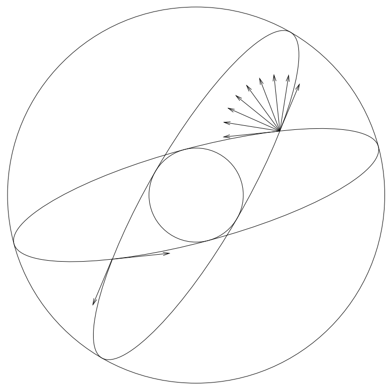

Figure 2: Given a point x in the annulus with inner radius R and outer radius 1, only some initial velocities (with modulus $\sqrt{1+R^2-|x|^2}$ because of the energy constraint) correspond to trajectories that remain in the annulus. Observe that the two extremal trajectories are tangent to the inner and outer circles.

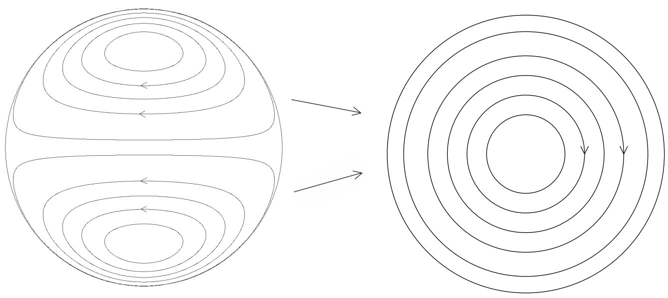

Figure 3: A point (x, v) is represented on the left disc by the coordinates $(v_1, v_2) := (v \cdot \frac{x}{|x|}, -v \cdot \frac{x^\perp}{|x|})$. On the other hand, thanks to the energy constraint, the coordinates (v_1, v_2) prescribe (x, v) up to the direction of x. Therefore, by the rotational invariance, the dynamics $t \mapsto (x(t), v(t))$ is completely described by the corresponding trajectories in the coordinates (v_1, v_2). As illustrated on this figure, the flow induced by ϕ_t in the space (v_1, v_2) is better understood in the space (a, b), where it just consists of a rotation (see Lemma 4.11).

Let us finally define the map $\mathcal{S}(x,v) := \Big(x, 2(v\cdot x)\frac{x}{|x|^2} - v\Big)$, which correspond to a reflection of v with respect to the axis parallel to x, so that $\pi_{\mathcal{P}}(x,v) = \pi_{\mathcal{P}}(\mathcal{S}(x,v))$. Notice that, if $\pi_{\mathcal{P}}(x,v) = (a,b)$, then $\pi_{\mathcal{P}}^{-1}(a,b) = \{(x,v), \mathcal{S}(x,v)\}$. We are now able to state the main results of this section and illustrate it through some comments and examples.

Proposition 4.5. *If μ is incompressible, optimal and concentrated on TD_{1+R^2}, we have*

$$(\pi_{\mathcal{P}})_{\#}\mu = \frac{1/(2\pi)}{\sqrt{(\frac{1-R^2}{2})^2 - a^2 - b^2}} \cdot \mathscr{L}^2_{\lfloor B_{\frac{1-R^2}{2}}}$$

Corollary 4.6. *If μ is a rotationnaly invariant minimizer concentrated on TD_{1+R^2}, then $\mu + \mathcal{S}_{\#}\mu$ is uniquely determined. Hence, there is a unique rotationnaly invariant clockwise minimizer μ concentrated on TD_{1+R^2}.*

To treat specific examples, it is convenient to express the measures μ_r in terms of the angles $\alpha \in [-\pi,\pi]$ defining the vectors v. More precisely, if we fix a point x with $|x| = r$, we can associate to any vector v the angle α between the directions of x and v. This correspondence is one-to-one if restricted to the vectors $v \in \mathbb{R}^2$ with $|v| = \sqrt{1+R^2-r^2}$. Since

$$re_1 \cdot v = r\sqrt{1+R^2-r^2}\cos\alpha, \tag{4.6}$$

the condition on μ_r induced by Proposition 4.5 reads in terms of the angles α as a condition on the image measure under the map $\alpha \mapsto |\alpha|$ (corresponding to the identification of the two vectors v and $2(v\cdot x)\frac{x}{|x|^2} - v$). This measure can be explicitely computed using (4.5), (4.6) and Proposition 4.5, and is given by

$$\frac{H_a \sin\alpha}{\pi\sqrt{H_a^2(\sin\alpha)^2 - R^2}} \cdot \mathscr{L}^1_{\lfloor I_a},$$

where

$$H_a = \sqrt{\left(\frac{1+R^2}{2}\right)^2 - a^2}, \qquad I_r = \{\alpha \in [0,\pi] \,:\, \sin\alpha \geq R/H_a\}, \qquad a = r^2 - \frac{1+R^2}{2}.$$

The interval I_r is of the form $[\frac{\pi}{2} - \alpha(r), \frac{\pi}{2} + \alpha(r)]$, where $\frac{\pi}{2} - \alpha(r)$ and $\frac{\pi}{2} + \alpha(r)$ are the two angles corresponding to the extremal trajectories remaining inside the annulus (see Figure 2). We can notice that in the case of the annulus $\alpha(\cdot)$ is strictly concave with $\alpha(R) = \alpha(1) = 0$ (see Figure 4), and the angles near the boundary of the interval I_r are more charged than the interior ones. On the other hand, in the case of the disc (that is $R = 0$) we get the constant density on $I_r = [0,\pi]$ for all $r \in [0,1]$.

Example 4.7. We notice that if we take the radius of the inner circle as being $R = 0$ and μ_r giving symmetrically the same mass to the two intervals $[0,\pi]$ and $[-\pi,0]$, we obtain Brenier's geodesic (4.2). On the other hand, the two possible measures concentrated respectively on $[0,\pi]$ and $[-\pi,0]$, correspond to the minimizers μ^+ and μ^- described in Section 4.1.

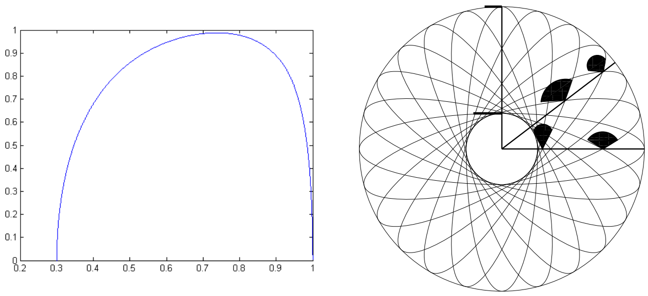

Figure 4: The figure on the left represents the function $r \mapsto \alpha(r)$ in the case of an annulus with inner radius 0.3 and outer radius 1. On the right, given a point of intersection of two ellipses, the tangents to the ellipses delimitate the interval of admissible velocities.

Example 4.8. There is a unique possible μ_r that is concentrated on the angles $[0, \pi]$. In other words, there is a unique clockwise rotationally invariant minimizer. It is to be noted that the macroscopic velocity is not zero in that case, and that the velocity field is stationary.

Example 4.9. If we design μ_r as being concentrated on the angles $[0, \frac{\pi}{2}] \cup [-\pi, -\frac{\pi}{2}]$, we obtain an example of a non-stationary minimizer. Indeed, since the scalar products $x \cdot v$ change their signs after a time $t = \pi/2$, while $x^\perp \cdot v$ is preserved in time, the angles corresponding to the velocities at time $t = \pi/2$ belong to the intervals $[-\frac{\pi}{2}, 0] \cup [\frac{\pi}{2}, \pi]$. In particular the velocity field is not stationary. The effective velocity is zero at time $t = 0$ but this condition is not preserved along time. For instance, at time $t = \pi/4$ all the points in the annulus $\{|x|^2 \geq 1/2\}$ have clockwise velocities, while all points in the disc $\{|x|^2 \leq 1/2\}$ have counterclockwise velocities (see Figure 5).

Example 4.10. If μ_r is concentrated on the angles $[0, \frac{3\pi}{4}] \cup [-\pi, -\frac{3\pi}{4}]$, we obtain an example of minimizer which is non-stationary and whose effective velocity never vanishes.

A more complicated example of generalised solution can be constructed as follows: given a (Borel) partition of $[R, 1] = A_1 \cup A_2$, we can for instance take μ_r being concentrated on the angles $[0, \frac{3\pi}{4}] \cup [-\pi, -\frac{3\pi}{4}]$ for $r \in A_1$, and concentrated on the angles $[\frac{\pi}{2}, \pi] \cup [-\frac{\pi}{2}, 0]$ for $r \in A_2$.

4.2.1 Proof of Proposition 4.5

It is simple to check that ϕ_t gives rise to a unique well-defined flow s_t on the space of couples $(a, b) = (|x|^2 - \frac{1+R^2}{2}, v \cdot x)$, such that

$$\pi_{\mathcal{P}} \circ \phi_t(x, v) = s_t \circ \pi_{\mathcal{P}}(x, v). \tag{4.7}$$

Lemma 4.11. *The unique flow s_t satisfying Equation (4.7) is given by $s_t(a, b) = \boldsymbol{R}_{2t}(a, b)$.*

Proof. By a direct computation one checks that, if $a = |x|^2 - \frac{1+R^2}{2}$ and $b = v \cdot x$, then

$$a(t) := |x(t)|^2 - \frac{1 + R^2}{2}, \qquad b(t) := v(t) \cdot x(t)$$

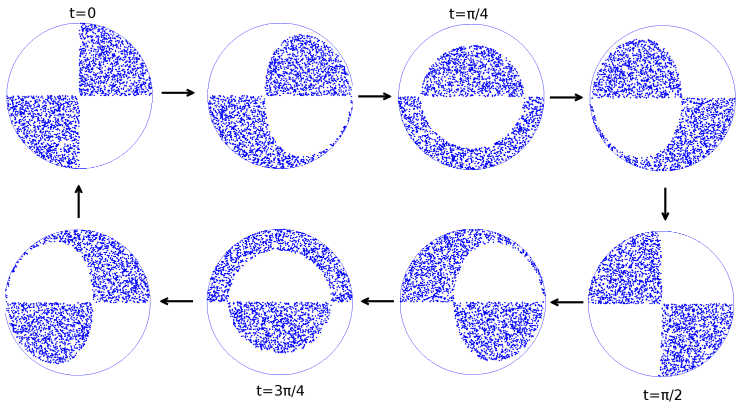

Figure 5: We illustrate the example 4.9 by representing the image of the measures μ_t through the map $(x, v) \mapsto (v_1, v_2)$ introduced in Figure 3.

are given by

$$a(t) = \cos(2t)a + \sin(2t)b, \qquad b(t) = -\sin(2t)a + \cos(2t)b$$

(recall that $x(t) = x\cos t + v\sin t$ and $v(t) = -x\sin t + v\cos t$). □

Lemma 4.12. *If μ is incompressible, then*

$$(p_1)_\#(\pi_\mathcal{P})_\#\left[(\phi_t)_\#\mu\right] = 2\pi\, dt_{\lfloor[-\frac{1-R^2}{2},\frac{1-R^2}{2}]},$$

where $p_1(a,b) := a$.

Proof. Let us define $\pi_r(x) = |x|$ and $\pi_{r^2}(x) = |x|^2$. As μ is incompressible

$$(\pi_r)_\#\left[(\phi_t)_\#\mu\right] = 2\pi r\, dr_{\lfloor[R,1]},$$

that is

$$(\pi_{r^2})_\#\left[(\phi_t)_\#\mu\right] = 2\pi\, dt_{\lfloor[R^2,1]}.$$

Since $p_1 \circ \pi_\mathcal{P}(x) = |x|^2 - \frac{1+R^2}{2}$, the result follows. □

We are ready to prove Proposition 4.5:

Proof. (of Proposition 4.5) Recalling that $\pi_\mathcal{P} \circ \phi_t = s_t \circ \pi_\mathcal{P}$, by Lemma 4.12 if μ is an incompressible minimizer then

$$(p_1)_\#(s_t)_\#\left[(\pi_\mathcal{P})_\#\mu\right] = 2\pi\, dt_{\lfloor[-\frac{1-R^2}{2},\frac{1-R^2}{2}]}.$$

Since $p_1 \circ s_t = \psi_{2t}$, the result follows by Proposition 3.2. □

4.3 Uniqueness of clockwise stationary minimizers

The goal of this section is to prove uniqueness of clockwise stationary minimizers on an annulus (or a disc, if $R = 0$) concentrated on TD_{1+R^2}, and at the same time we will show that they are rotationally invariant. Thanks to Corollary 4.6, these two facts are actually equivalent: if one has uniqueness, then the unique minimizer is rotationally invariant (since we already know an example of minimizer which is clockwise, stationary, and rotationally invariant); if we prove rotationally invariance then we get uniqueness from the same corollary.

The idea of the proof is the following.

We fix a minimizer μ. Let us denote by $\mathcal{R}(\theta) \subset D$ the ray forming an angle θ with the axis $\{x_2 = 0\}$, and by $\mathcal{V}(\theta) \subset TD$ the set of pairs (x, v) with $x \in \mathcal{R}(\theta)$ and v an admissible velocity for x. First, in Lemma 4.14, we notice that the velocity v is never parallel to the position x, so that trajectories consist of non-degenerate ellipses. This fact implies that given two rays $\mathcal{R}(\theta_1)$ and $\mathcal{R}(\theta_2)$ every particle on the first one will reach the other one at some time, and this allows to define a family of one-to-one applications $T_{\theta_1,\theta_2}(x, v)$ that map $\mathcal{V}(\theta_1)$ onto $\mathcal{V}(\theta_2)$. We then disintegrate μ along rays, getting $\mu = \mu_\theta \otimes d\theta$ for a family of measures μ_θ on $\mathcal{V}(\theta)$. Using the stationarity assumption we want to find relations between μ_{θ_1} and μ_{θ_2}. Yet, in general, μ_{θ_2} is different from $(T_{\theta_1,\theta_2})_\#\mu_{\theta_1}$, since there is a Jacobian factor $g(x, v)$ to take into account (see Lemma 4.15), and what we actually have is $g \cdot \mu_{\theta_2} = (T_{\theta_1,\theta_2})_\#(g \cdot \mu_{\theta_1})$ (see Lemma 4.16). From this fact we will deduce $h \cdot \mu_{\theta_2} = (T_{\theta_1,\theta_2})_\#(h \cdot \mu_{\theta_1})$, with $h(x, v) = \frac{1}{|x|^2}$. We then consider the image measure $m_\theta := (\pi_{\mathcal{P}})_\#(h \cdot \mu_\theta)$, in the same spirit as in the previous paragraph (but, since we have not yet proven rotationally invariance, now we have to look at each ray separately). To define the analogous of the maps $p_1 \circ s_t$, we introduce some applications S_{θ_1,θ_2} such that $S_{\theta_1,\theta_2} \circ \mathcal{P} = p_1 \circ \mathcal{P} \circ T_{\theta_1,\theta_2}$. We will prove that the images of a measure through this family of maps are sufficient to prescribe such a measure. Moreover the measures $(S_{\theta_1,\theta_2})_\# m_{\theta_1}$ are always equal to a given measure λ, independent of μ, θ_1 and θ_2 (see Proposition 4.17). This proves that m_θ does not actually depend on θ and μ is rotationally invariant (see Theorem 4.20).

Before proceeding to the proof, we need to introduce some notations. We will denote by $\mathcal{V}(\theta)_\varepsilon$ the set $\cup_{|t|\le\varepsilon}\{\phi_t(x, v) \,|\, (x, v) \in \mathcal{V}(\theta)\}$. Moreover we set $\mathcal{V} = \cup_\theta \mathcal{V}(\theta) = TD$, and we define

$$g(x, v) := \frac{x^\perp \cdot v}{|x|^2}, \qquad h(x, v) := \frac{1}{|x|^2}.$$

Notice that g is non-negative μ-a.e., as we are considering clockwise minimizers.

The two following lemmas are immediate if D is an annulus.

Lemma 4.13. *For a.e. $\theta \in [0, 2\pi]$, the measure $g \cdot \mu_\theta$ is finite and $h \cdot \mu_\theta$ is σ-finite.*

Proof. Let us first prove that $g \cdot \mu_\theta$ is finite. As the set of admissible velocities is bounded, it is enough to estimate $\int_{\mathcal{V}(\theta)} \frac{1}{|x|}\mu_\theta(dx)$. Since the x marginal of μ is the Lebesgue measure, we get

$$\int_0^{2\pi} \int_{\mathcal{V}(\theta)} \frac{1}{|x|}\, d\mu_\theta(x)\, d\theta = \int_D \frac{1}{|x|}\, dx < +\infty,$$

and the result follows. The fact that $h \cdot \mu_\theta$ is σ-finite can be proved in the same way using that $\int_{D\cap\{|x|>\varepsilon\}} \frac{1}{|x|^2}\,dx$ if finite for any $\varepsilon > 0$. $\square$

Lemma 4.14. *We have* $\mu(\{(x,v) : x^\perp \cdot v = 0\}) = 0$

Proof. Notice that since we are in dimension two, if $x^\perp \cdot v = 0$, then x and v are parallel. Hence the trajectory stays on a straight line joining x and $-x$ and it passes through 0. If D is an annulus this is obviously not allowed, and the the thesis is proven. In the case of the disc, we need to show that the set A of pairs (x,v) such that the corresponding trajectory passes through the origin is negligible. We can do it by considering a small ball $B_\varepsilon(0)$ and the indicator function $f(t,x,v) := \chi_{B_\varepsilon(0)\times\mathbb{R}^2}(\phi_t(x,v))$. We have

$$\pi^2\varepsilon^2 = \int_0^\pi \left(\int f\,d\mu\right)dt = \int \left(\int_0^\pi f\,dt\right)d\mu \geq \int_A \left(\int_0^\pi f\,dt\right)d\mu \geq \mu(A)\varepsilon.$$

The first equality is justified by the fact that, by stationarity, for any $t \in [0,\pi]$ the integral with respect to μ gives the area of the ball $B_\varepsilon(0)$, while the last inequality arises from the fact that any trajectory passing through the origin stays in the ball $B_\varepsilon(0)$ at least a time ε (recall that the velocity is bounded by 1). Dividing by ε in the above inequality and letting $\varepsilon \to 0$ gives $\mu(A) = 0$. $\square$

Thanks to the previous lemma, we can introduce a negligible set N_0 such that, if $\theta \notin N_0$, then μ_θ gives no mass to the set of velocites parallel to the radius $\mathcal{R}(\theta)$.

Lemma 4.15. *Let μ be any finite measure on $\mathcal{V}$. Then, for any continuous function $f : \mathcal{V} \to \mathbb{R}$ supported in a set $\{(x,v) : |x| \geq c > 0\}$ (in the annulus case this assumption is obviously not necessary), we have*

$$\lim_{\varepsilon\to 0} \frac{1}{2\varepsilon} \int_{\mathcal{V}(\theta)_\varepsilon} f\,d\mu = \int_{\mathcal{V}(\theta)} f(x,v)\frac{x^\perp \cdot v}{|x|^2}\,d\mu_\theta$$

for almost every $\theta \in [0,2\pi]$.

Proof. We remark that the measure μ has compact support, and the map $\theta \mapsto \mu_\theta$ is a measurable map with values in the space of probability measures endowed with the weak-$*$ topology (i.e., in the duality with continuous functions). Since the space of continuous function is separable, it is simple to prove that almost every $\theta \in [0,2\pi]$ is a Lebesgue point, that is there exists a set of zero measure $N \subset [0,2\pi]$ such that, if $\theta_0 \in [0,2\pi] \setminus N$, then

$$\frac{1}{2\varepsilon}\int_{\theta_0-\varepsilon}^{\theta_0+\varepsilon}\left(\int f\,d\mu_\theta\right)d\theta \to \int f\,d\mu_{\theta_0}.$$

We will prove the thesis for $\theta_0 \in [0,2\pi] \setminus N$.

Let us set for simplicity $\theta_0 = 0$. First, we need to express in the variables $(\theta,|x|,v)$ the condition of belonging to the set $\mathcal{V}(0)_\varepsilon$. For fixed $x_0 = |x_0|e_1$ and v_0, we want to estimate the measure of the set $\Theta_\varepsilon(x_0,v_0)=\{\theta:(\theta,|x_0|,v_0)\in\mathcal{V}(0)_\varepsilon\}$. A point $(\boldsymbol{R}_\theta x_0,v_0)$ belongs to $\mathcal{V}(0)_\varepsilon$ if and only if it can be written as

$$\boldsymbol{R}_\theta x_0=x\cos t+v\sin t,\qquad v_0=-x\cos t+v\sin t,$$

for some $(x,v)\in\mathcal{V}(\theta)$, $t\in[-\varepsilon,\varepsilon]$. By taking the scalar product with $x^\perp$ in the first equality we get

$$|x_0||x|\sin\theta=x^\perp\cdot v\sin t.$$

Notice that, since $|t|\le\varepsilon$, x and v are respectively close to x_0 and v_0. It is therefore not difficult to see that the set of admissible θ is a closed interval of the form

$$\Theta_\varepsilon(x_0,v_0)=\big[-\varepsilon\frac{x_0^\perp\cdot v_0}{|x_0|^2}+o(\varepsilon),\varepsilon\frac{x_0^\perp\cdot v_0}{|x_0|^2}+o(\varepsilon)\big],$$

where we used $x\approx x_0$, $v\approx v_0$, and $\sin(s)=s+o(s)$. This implies in particular

$$\mathscr{L}^1\left(\Theta_\varepsilon(x_0,v_0)\right)=2\varepsilon g(x_0,v_0)+o(\varepsilon).$$

Let us now suppose that $\theta\mapsto\mu_\theta$ is constant and that the function f does not depend on the variable θ. In this case we have to evaluate the limit of the integral

$$\frac{1}{2\varepsilon}\int\int f(x,v)\chi_{\Theta_\varepsilon(x,v)}(\theta)d\mu_0\,d\theta=\int f(x,v)\frac{2(\varepsilon g(x,v)+o(\varepsilon))}{2\varepsilon}d\mu_0,$$

and the result is evident. In the general case (that is f depends also on θ and the application $\theta\mapsto\mu_\theta$ is not constant), the result is the same: to evaluate the same integral we simply use the fact that f is uniformly continuous and that 0 is a Lebesgue point for the map $\theta\mapsto\mu_\theta$. $\square$

Now, by means of Lemma 4.15, we prove the following:

Lemma 4.16. *For $\theta_1,\theta_2\in[0,2\pi]$, let $T_{\theta_1,\theta_2}:\mathcal{V}(\theta_1)\to\mathcal{V}(\theta_2)$ denote the application given by*

$$T_{\theta_1,\theta_2}(x,v)=\phi_t(x,v),\qquad \text{with } t=t(\theta_1,\theta_2) \text{ such that } \phi_t(x,v)\in\mathcal{V}(\theta_2).$$

Then there exists a negligible set $N\subset[0,2\pi]$ such that, if $\bar\theta_1,\bar\theta_2\notin N$, then $(T_{\bar\theta_1,\bar\theta_2})_\#(g\cdot\mu_{\bar\theta_1})=g\cdot\mu_{\bar\theta_2}$, $(T_{\bar\theta_1,\bar\theta_2})_\#(h\cdot\mu_{\bar\theta_1})=h\cdot\mu_{\bar\theta_2}$.

Proof. Let us fix a dense and countable subset $\mathcal{D}$ in the set of continuous function $f:\mathcal{V}\to\mathbb{R}$ vanishing in a neighborhood of $\{x=0\}$. For each $f\in\mathcal{D}$, we have a negligible set N_f given by Lemma 4.15. Take $N_1=\bigcup_{f\in\mathcal{D}}N_f$, which is still negligible, and set $N=N_0\cup N_1$, where N_0 is the negligible set defined accordingly to Lemma 4.14. Then, take $\bar\theta_1,\bar\theta_2\notin N$.

Fix $\varepsilon,\delta>0$, take $f\in\mathcal{D}$, and fix a partition of $\mathcal{V}(\theta_1)$ into disjoint measurable sets A_i such that for each i there exists $(x_i,v_i)\in\mathcal{V}(\theta_1)$ with $A_i\subset B_\delta((x_i,v_i))$. Let us denote by $(A_i)_\varepsilon$ the subset of $\mathcal{V}(\theta_1)_\varepsilon$ given by the points $\phi_t(x,v)$ for $|t|\le\varepsilon$ and $(x,v)\in A_i$. These sets $(A_i)_\varepsilon$ give a partition of $\mathcal{V}(\theta_1)_\varepsilon$. Let t_i be the time in $[0,2\pi]$ such that $\phi_{t_i}(x_i,v_i)\in\mathcal{V}(\theta_2)$ (thanks to Lemma 4.14 we can assume that x_i and v_i are not parallel, and so this time exists and is unique). Set

$B_i := \phi_{t_i}(A_i)$, and notice that $(B_i)_\varepsilon = \phi_{t_i}((A_i)_\varepsilon)$ (where $(B_i)_\varepsilon$ denotes the set of points $\phi_t(x, v)$ for $|t| \leq \varepsilon$ and $(x, v) \in B_i$). Notice also that both the sets B_i and the sets $(B_i)_\varepsilon$ are disjoint.

Since μ is stationary, we get

$$\int_{(B_i)_\varepsilon} f \, d\mu = \int_{(A_i)_\varepsilon} f \circ \phi_{t_i} \, d\mu,$$

and summing up over i we have

$$\int_{\bigcup_i (B_i)_\varepsilon} f \, d\mu = \int_{\mathcal{V}(\theta_1)_\varepsilon} f \circ \phi_{t(x,v)} \, d\mu,$$

where $t(x, v) := t_i$ if $(x, v) \in (A_i)_\varepsilon$.

Now we let the partition get finer and finer, i.e. $\delta \to 0$. For all $(x, v) \in \mathcal{V}(0)$ we have $\phi_{t(x,v)}(x, v) \to \tilde{T}(x, v)$, where the map $\tilde{T} : \mathcal{V}(\theta_1)_\varepsilon \to \mathcal{V}(\theta_2)_\varepsilon$ is the extension of T_{θ_1,θ_2} defined by $\tilde{T}(\phi_t(x, v)) := \phi_t(T_{\theta_1,\theta_2}(x, v))$. Moreover, the set $\bigcup_i (B_i)_\varepsilon$ converges to $\mathcal{V}(\theta_2)_\varepsilon$ (in the sense that the corresponding indicator functions converge pointwisely, up to the boundary of $\mathcal{V}(\theta_2)_\varepsilon$). This means that, for any ε such that $\mu\left(\partial(\mathcal{V}(\theta_2)_\varepsilon)\right) = 0$ (i.e. for all but a countable quantity of ε), we get at the limit as $\delta \to 0$

$$\int_{\mathcal{V}(\theta_2)_\varepsilon} f \, d\mu = \int_{\mathcal{V}(\theta_1)_\varepsilon} f \circ \tilde{T} \, d\mu.$$

By Lemmas 4.13 and 4.15, letting $\varepsilon \to 0$ and recalling that T_{θ_1,θ_2} is the restriction of $\tilde{T}$ to $\mathcal{V}(\theta_1)$, we obtain

$$\int_{\mathcal{V}(\theta_2)} f \, g \, d\mu_{\theta_2} = \int_{\mathcal{V}(\theta_1)} f \circ T_{\theta_1,\theta_2} \, g \, d\mu_{\theta_1},$$

and thanks to the density of $\mathcal{D}$ the above equality implies $(T_{\theta_1,\theta_2})_\#(g \cdot \mu_{\theta_1}) = g \cdot \mu_{\theta_2}$.

To replace g with h, just notice that $x^\perp \cdot v$ is invariant under the flow, so that, if we define $\tilde{f}(x, v) := f(x, v)(x^\perp \cdot v)$ for a continuous function f, we get

$$\int_{\mathcal{V}(\theta_2)} \frac{\tilde{f}}{|x|^2} \, d\mu_{\bar{\theta}} = \int_{\mathcal{V}(\theta_1)} \frac{\tilde{f} \circ T}{|x|^2} \, d\mu_0.$$

Since by Lemma 4.14 the set $\{(x, v) : x^\perp \cdot v = 0\})$ is μ-negligible, and by Lemma 4.13 the measures $h \cdot \mu_\theta$ are σ-finite for a.e. θ, the result follows easily by the arbitrariness of $\tilde{f}$. $\square$

Combining the previous lemmas, we easily obtain the following:

Proposition 4.17. *Let us decompose μ into $\mu_\theta \otimes d\theta$ and let m_θ be the image of the measure $h \cdot \mu_\theta$ through the map $\mathcal{P} : (x, v) \mapsto (a, b)$, with $a = |x|^2 - \frac{1+R^2}{2}$, $b = x \cdot v$ ($\mathcal{P}$ is one-to-one thanks to the assumptions on μ to be clockwise and concentrated on TD_{1+R^2}). Define the maps $S_{\theta_0,\theta}$ by $S_{\theta_0,\theta} \circ \mathcal{P} := p_1 \circ \mathcal{P} \circ T_{\theta_0,\theta}$. Then, if $\theta_0, \theta \notin N$, we have*

$$(S_{\theta_0,\theta})_\# m_{\theta_0} = \lambda, \qquad \text{with} \quad \lambda(da) = \frac{1}{a + \frac{1+R^2}{2}} da \quad \text{on } [-(1 - R^2)/2, (1 - R^2)/2]. \tag{4.8}$$

Proof. We have proved that, for $\theta_0, \theta \notin N$,

$$h \cdot \mu_\theta = (T_{\theta_0,\theta})_\#(h \cdot \mu_{\theta_0}).$$

We notice that the measures $h \cdot \mu_\theta$ are not known a priori (they may depend on the particular choice of the solution μ), but their projections on the x variable are known (since the projection on the x variable of μ is $\mathscr{L}^2_{\lfloor D}$ and h depends only on x). Rewriting everything in terms of the variable (a, b), and projecting the measures m_θ on the a variable through p_1, we immediately get

$$(p_1)_\# m_\theta = \frac{1}{a + \frac{1+R^2}{2}} \cdot \mathscr{L}^1_{\lfloor[-(1-R^2)/2,(1-R^2)/2]} = \lambda. \quad \square$$

The goal now is to prove that the above condition on the images of a measure through the maps $S_{\theta_0,\theta}$ suffices to identify it:

Lemma 4.18. *Satisfying condition* (4.8) *for a.e.* $\theta \in [0, 2\pi]$ *uniquely prescribes* m_{θ_0}.

Proof. We first remark that $S_{\theta_0,\theta}$ can be explicitly written as

$$S_{0,\theta}(a, b) \mapsto \gamma(\theta, a, b) := a \cos t(\theta, a, b) + b \sin t(\theta, a, b) \tag{4.9}$$

with

$$t(\theta, a, b) := 2 \arctan \left(\frac{c \sin(\theta - \theta_0)}{-b \sin(\theta - \theta_0) + d \cos(\theta - \theta_0)} \right)$$

where $c = a + \frac{1+R^2}{2}$, $d = \sqrt{\left(\frac{1+R^2}{2}\right)^2 - a^2 - b^2}$.

It is useful to notice that the term d is strictly positive m_{θ_0}-a.e. Indeed, in the case of an annulus this is true since $R > 0$ and m_{θ_0} is concentrated on the ball $\sqrt{a^2 + b^2} \le (1 - R^2)/2 < (1 + R^2)/2$. In the case of the disc, if $\sqrt{a^2 + b^2} = 1/2$, then $|b| = |x||v|$, which implies that x and v are parallel; however, we already saw in Lemma 4.14 that this only happens on a μ-negligible set.

Let us set $s = \tan(\theta - \theta_0)$ so that t becomes $t(s, a, b) = 2 \arctan(-cs/(bs + d))$. We observe that the integrals with respect to m_{θ_0} of all the functions of the form $(a, b) \mapsto f(\gamma(s, a, b))$ are known and, by passing to the limit in the incremental ratios, the integrals with respect to m_{θ_0} of all the functions

$$(a, b) \mapsto \frac{d^n}{ds^n} f(\gamma(s, a, b))|_{s=0}$$

are known as well.

Since in the case of the disc the measure m_{θ_0} is only σ-finite, we will stick at the beginning to functions f which vanish near the origin $x = 0$ (corresponding to $a = -(1 - R^2)/2$), so that for s small also the composition $f(\gamma(s, a, b))$ is zero near the origin (as $\gamma(s, a, b)$ is close to a). In the end this class of functions will be sufficient to identify the measure, since m_0 does not give mass to the point $(-(1 - R^2)/2, 0)$.

Notice that $\gamma(0, a, b) = a$, so that the integrals of all the functions of a are known. By taking the first derivative in s at $s = 0$, we also know the integrals of all the functions of the form

$f'(a)\dot\gamma(0,a,b)$. Set $w=\dot\gamma(0,a,b)$. Since the correspondence $(a,b)\leftrightarrow(a,w)$ is one to one, it is sufficient to prove that the integrals of all functions of the form $f(a)w^n$ are known.

To this aim, it suffices to prove that $\gamma^{(n)}(0)$ is a sum of terms which include functions of a and powers of w, up to the exponent n at most. This will be done in Lemma 4.19 below.

To conclude, one notices that:

- We know the integrals of all the functions of the form $\frac{d^n}{ds^n}f(\gamma(s,a,b))|_{s=0}$.
- By Faa di Bruno's formula and by Lemma 4.19, a function of this kind is of the form $f^{(n)}(a)\dot\gamma(0)^n+\sum_{i<n}g_i(a)f^{(i)}(a)\dot\gamma(0)^n+\sum_{i,j\,:\,j<n}g_{i,j}(a)f^{(i)}(a)\dot\gamma(0)^j$.
- The last term of this sum is composed by functions whose integrals are known by recurrence.
- If we set $h=f^{(n)}+\sum_{i<n}g_if^{(i)}$, we conclude that the integrals of all the functions of the form $h(a)\dot\gamma(0)^n$ are known.
- The function h is a completely arbitrary function among those who vanish near $a=-(1-R^2)/2$, since one can always solve the linear differential equation $h=f^{(n)}+\sum_{i<n}g_if^{(i)}$ in the unknown f, imposing vanishing boundary conditions at such a point: if h vanishes on a neighborhood of that point, f will vanish too.
- Hence, all the polynomial functions in a and w belong to the space of the functions whose integrals are known. By density of polynomials in the space of all continuous functions of a and w (i.e. in the space of all continuous functions of a and b, thanks to the one-to-one correspondence $(a,b)\leftrightarrow(a,w)$), we get that the measure m_0 is prescribed.

□

Lemma 4.19. *With the notations of Lemma 4.18, $\gamma^{(n)}(0)$ is a sum of terms which include functions of a and powers of w, up to the exponent n at most.*

Proof. To prove such a structure result on $\gamma^{(n)}$ we use Faa di Bruno's formula: write $\gamma(s)=g\circ t(s)$, where $t(s)=t(s,a,b)$ was defined in Lemma 4.18, to get

$$\gamma^{(n)}(s)=\sum_{m_1+2m_2+\ldots+km_k=n}C_{m_1,\ldots,m_k}g^{(m_1+\cdots+m_k)}\prod_{j=1}^{k}[t^{(j)}]^{m_j}(s)$$

We have

$$\dot t(s)=\frac{-2bd}{P(s)},\qquad P(s)=c^2s^2+(bs+d)^2.$$

From the relation $P(s)\dot t(s)=$ constant, taking into account that P is a quadratic polynomial we get

$$C_n\ddot P(s)t^{(n)}(s)+C_{n+1}\dot P(s)t^{(n+1)}(s)+Pt^{(n+2)}(s)=0.$$

Computing everything at $s = 0$, since $P(0) = d^2$, $\dot{P}(0) = 2bd$, and $\ddot{P}(0) = 2(c^2 + b^2)$, we obtain

$$\dot{t}(0) = -2\frac{c}{d}, \qquad \ddot{t}(0) = 4\frac{bc}{d^2}, \qquad t^{(n+2)}(0) = -2C_{n+1}\frac{b}{d}t^{(n+1)}(0) - 2C_n\frac{c^2+b^2}{d^2}t^{(n)}(0).$$

Moreover we have $\dot{\gamma}(0) = b\dot{t}(0) = -2\frac{b}{d}c$, with c function of a. This means that we can write

$$t^{n+2}(0) = f_1(a)\dot{\gamma}(0)t^{(n+1)}(0) + f_2(a)\dot{\gamma}(0)^2 t^{(n)}(0) + \frac{f_3(a)}{d^2}t^{(n)}(0)$$

(recall that $d > 0$).

We now observe that $b^2 + d^2 = \frac{1}{4}(1+R^2)^2 - a^2$ is a function of a. Hence, we can rewrite the last term in the right hand side as $f_4(a)t^{(n)}(0)(b^2+d^2)/d^2$. Recalling that $\dot{\gamma}(0) = b\dot{t}(0) = -2\frac{b}{d}c$, and c is a function of a, we obtain

$$f_4(a)t^{(n)}(0)\frac{b^2+d^2}{d^2} = f_5(a)t^{(n)}(0)\dot{\gamma}(0)^2 + f_4(a)t^{(n)}(0)$$

Collecting all together, and recalling that $\dot{t}(0) = \dot{\gamma}(0)/b$ and $\ddot{t}(0) = f_6(a)\dot{\gamma}(0)^2/b$, we get by induction

$$t^{(n)}(0) = \frac{1}{b}\sum_{i\,:\,n-2i\geq 1} \tilde{f}_i(a)\dot{\gamma}(0)^{n-2i}.$$

We now put everything inside Faa di Bruno's formula. All the terms $g^{(m_1+\cdots+m_k)}$ are either $\pm a$ or $\pm b$: they are $\pm a$ if $m_1 + \dots m_k$ is even, $\pm b$ if it is odd. Moreover in the product we have a factor b^{-1} to the power $m_1 + \cdots + m_k$. Consequently we obtain a sum where the terms are of the form

$$\frac{a}{b^{2k}}f(a)\dot{\gamma}(0)^{n-2h} \qquad \text{or} \qquad \frac{b}{b^{2k+1}}\tilde{f}(a)\dot{\gamma}(0)^{n-2h} = \frac{1}{b^{2k}}\tilde{f}(a)\dot{\gamma}(0)^{n-2h},$$

and the exponent of $\dot{\gamma}(0)$ is always strictly larger than the exponent of b. To get rid of b^{2k}, we use again the fact that $b^2 + d^2$ is a function of a: if we multiply $b^{-2k}f(a)\dot{\gamma}(0)^{n-2h}$ by b^2+d^2, and we consider that $\dot{\gamma}(0) = -2\frac{b}{d}c$ and c is a function of a, we get

$$\frac{1}{b^{2k}}f(a)\dot{\gamma}(0)^{n-2h}(b^2+d^2) = \frac{1}{b^{2(k-1)}}f(a)\dot{\gamma}(0)^{n-2h} + \frac{1}{b^{2(k-1)}}f(a)\dot{\gamma}(0)^{n-2(h+1)}.$$

Iterating this last procedure k times, we finally get the desired result. □

The following conclusion holds:

Theorem 4.20. *If $D \subset \mathbb{R}^2$ is either the disc or the annulus, then there exists only one stationary clockwise minimizer, and it is rotationally invariant.*

Proof. Take a minimizer $\mu = \mu_\theta \otimes d\theta$ and define m_θ as in Proposition 4.17. We have proved that, for $\theta_0 \notin N$, the images of m_{θ_0} through the family of maps $S_{\theta_0,\theta}$ are always λ for a.e. θ. This prescribes m_{θ_0}, and hence μ_{θ_0} (here the clockwise assumption is essential). Therefore, for $\theta_0 \notin N$, all the measures m_{θ_0} are equal, and the corresponding μ_{θ_0} are obtained one from another by applying the suitable rotation $\bar{R}_\theta$, that is $\mu_{\theta_2} = (\bar{R}_{\theta_2-\theta_1})_\# \mu_{\theta_1}$. Hence μ is rotationally invariant, and we conclude applying the uniqueness result from Corollary 4.6. □

4.4 Microscopic versus macroscopic pressure

It is interesting to observe that the minimizers we constructed in the previous paragraphs induce classical solutions to the Euler equations with a new "macroscopic" pressure which differs from the microscopic one $p(x) = |x|^2/2$.

For example consider the minimizers constructed in Paragraph 4.1. They provide an example of generalized solutions with non-zero effective velocity: we have

$$\overline{\boldsymbol{v}}_t^{\pm}(x) := \int v\mu_t^{\pm}(x, dv) = c\sqrt{1-|x|^2}\,\frac{x^{\perp}}{|x|},$$

for a certain constant $c > 0$, and $\overline{\boldsymbol{v}}_t^{\pm}(x)$ are stationary solutions to the Euler equations with the new "macroscopic" pressure given by

$$\bar{p}(x) := \frac{1}{2}c^2\int_0^{|x|^2}\sqrt{\frac{1}{s}-1}\,ds = \frac{1}{2}c^2\Big(|x|\sqrt{1-|x|^2} + \arctan\frac{\sqrt{1-|x|^2}}{|x|}\Big).$$

Indeed

$$\nabla\bar{p}(x) = c^2\frac{x}{|x|}\sqrt{1-|x|^2} = -\mathrm{div}(\overline{\boldsymbol{v}}_t^{\pm}\otimes\overline{\boldsymbol{v}}_t^{\pm})(x),$$

and therefore $\overline{\boldsymbol{v}}_t$ satisfies

$$\mathrm{div}(\overline{\boldsymbol{v}}_t^{\pm}\otimes\overline{\boldsymbol{v}}_t^{\pm})(x) + \nabla\bar{p}(x) = 0.$$

More interestingly, the minimizers provided in Examples 4.9 and 4.10 induce non-stationary generalized solutions to the Euler equations.

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