Generic hyperbolicity of Aubry sets on surfaces

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Abstract

Given a Tonelli Hamiltonian of class C^2 on the cotangent bundle of a compact surface, we show that there is an open dense set of potentials in the C^2 topology for which the Aubry set is hyperbolic in its energy level.

1 Introduction

Let M be a smooth compact Riemannian manifold without boundary of dimension $n \geq 2$, and $H : T^*M \to \mathbb{R}$ be a *Tonelli Hamiltonian* of class C^2. As shown by Mather [32], one can construct a compact invariant subset of T^*M which enjoys several variational properties and has the distinguished feature of being a Lipschitz graph over a part of M. This set, called the *Aubry set* associated to H and denoted by $\tilde{\mathcal{A}}(H)$, captures many important features of the Hamiltonian dynamics.

Fathi [18] established a bridge between the Aubry-Mather theory and the properties of viscosity solutions/subsolutions of the critical Hamilton-Jacobi equation associated with H, giving rise to the *weak KAM theory*. The differentials of critical (viscosity) subsolutions are uniquely determined on the projection of $\tilde{\mathcal{A}}(H)$ onto M (denoted by $\mathcal{A}(H)$), and all critical subsolutions are indeed $C^{1,1}$ on the *projected Aubry set* $\mathcal{A}(H)$. We refer the reader to Section 2.1 below for a precise definition of the Aubry set and more details about weak KAM theory.

A famous open problem concerning the structure of $\tilde{\mathcal{A}}(H)$ is the so-called "Mañé conjecture" [29] which states that, for a generic Hamiltonian, the Aubry set is either a hyperbolic equilibrium or a hyperbolic periodic orbit. In [21, 22], the second and third author obtained several results in the direction of proving the validity of the Mañé conjecture. However, all those results heavily rely on the assumption of the existence of a sufficiently smooth critical (sub-)solution. The goal of this paper is to combine some of the techniques developed in [21, 22] with tools from dynamical systems and new regularity estimates for viscosity solutions, to answer in low dimension an open problem proposed by Herman during the ICM in 1998 [25, Section 6.2, Question 2] (in the context of twist maps on $\mathbb{T}^1$, this question was posed by A. Katok, and positively solved by P. Le Calvez [27]):

Is it true that generically the Aubry set is hyperbolic?

As mentioned by Herman at the beginning of [25, Section 6], the subject of the instabilities of Hamiltonian flows and the problem of topological stability "lacks any non-trivial result". Our main theorem solves in the affirmative Herman's problem on surfaces for the C^2-topology.

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Theorem 1.1. *Let $H : T^*M \to \mathbb{R}$ be a Tonelli Hamiltonian of class C^2, and assume that $\dim M = 2$. Then there is an open dense set of potentials $\mathcal{V} \subset C^2(M)$ such that, for every $V \in \mathcal{V}$, the Aubry set associated to the Hamiltonian $H + V$ is hyperbolic in its energy level.*

The proof of Theorem 1.1 relies on the properties of Green bundles which can be associated with each orbit of the Aubry set. The basic idea is based on the following dichotomy for Green bundles: either they are always transverse, in which case one gets hyperbolicity of the Aubry set; or the Green bundles coincide along a given orbit of the Aubry set, and in this latter case, elaborating on previous works by Arnaud [4, 5], we show that the restriction to the projected Aubry set of any critical solution is C^2 along the projected orbit. This additional regularity property is not enough to apply the techniques which were introduced in [21, 22], since there the authors had to require the existence of a critical solution which is $C^{1,1}$ in a neighborhood of the projected orbit and C^2 along it. In our case, we do not have any regularity property outside the projected orbit, and critical solutions may be merely Lipschitz in any neighborhood of the projected orbit. Still, by some new refined estimates on the regularity of a critical solution near a point where the Green bundles coincide, we are able to exploit the techniques used in [21, 22] to conclude the argument and prove our theorem.

Our proof together with the shadowing lemma (see [26]) yields the following closing-type result:

Theorem 1.2. *Let $H : T^*M \to \mathbb{R}$ be a Tonelli Hamiltonian of class C^2, and assume that $\dim M = 2$. Then, for every open set $\mathcal{U} \subset T^*M$ containing $\tilde{\mathcal{A}}(H)$, and any neighborhood $\mathcal{V}$ of 0 in $C^2(M)$, there exist $\theta \in \mathcal{U}$ and $V \in \mathcal{V}$ such that the orbit with respect to the Hamiltonian $H + V$ passing through θ is periodic and hyperbolic.*

We notice that a similar statement could be deduced as a direct consequence of the results in [29, 19, 9]: more precisely, by [19, Theorem 1.5] and [9, Theorems 1 and 2] the Aubry set is upper-semicontinuous on surfaces, so [29, Theorem F] [1] implies that generically in C^∞ topology one can find a periodic orbit close to the Aubry set. However, in contrast with Theorem 1.2 above, this orbit may not be hyperbolic (even if one introduces an additional small perturbation by a potential, see [37]). Therefore, if we work in the C^2 topology, Theorem 1.1 allows us to say that the Aubry set of $H + V$ is hyperbolic, which in turn implies the hyperbolicity of the sequence of periodic orbits approaching the Aubry set (see Proposition 2.18). All in all, we get the following refinement of [29, Theorem F] in two dimensions and C^2 topology:

Theorem 1.3. *Let $H : T^*M \to \mathbb{R}$ be a Tonelli Hamiltonian of class C^2, and assume that $\dim M = 2$. Then there is a residual set of potentials $\mathcal{G} \subset C^2(M)$ such that, for every $V \in \mathcal{G}$, the Lagrangian associated with $H + V$ admits a unique minimizing measure, which is indeed a strong limit of a sequence of probability measures supported on hyperbolic periodic orbits.*

The paper is structured as follows. First, we collect several preliminary results which are fundamental for the proof of Theorem 1.1: Section 2.1 is concerned with reminders in weak KAM theory; Section 2.2 contains a result on connecting trajectories; Sections 2.3, 2.4, and 2.5 are devoted to the constructions of Green bundles, paratingent cones, and Arnaud-type results; Section 2.6 contains reminders on hyperbolicity and quasi-hyperbolicity; finally, Sections 2.7 and 2.8 contain material on semiconcave and BV functions and a lemma from harmonic analysis, which play a major role in the proof of Theorem 1.1. Section 3 is concerned with the proof of Theorem 1.1, which is split into a stability and a density part. Finally, in Section 4 we present some examples of Tonelli Hamiltonians on surfaces of positive genus whose Aubry set is a non-trivial minimal hyperbolic set.

[1] Mañé's Theorem [29, Theorem F] asserts that, given a Tonelli Hamiltonian of class C^k with $k \geq 2$, there is a residual set of potentials $\mathcal{G} \subset C^k(M)$ such that, for every $V \in \mathcal{G}$, the Lagrangian associated with $H + V$ admits a unique minimizing measure, which is indeed a strong limit of a sequence of probability measures supported on periodic orbits.

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2 Preliminary results

2.1 Reminders in Weak KAM theory

Recall that a Tonelli Hamiltonian $H : T^*M \to \mathbb{R}$ of class C^2 is a Hamiltonian satisfying the two following properties:

(H1) *Superlinear growth:* For every $K \geq 0$ there is a finite constant $C^*(K) \in \mathbb{R}$ such that

$$H(x,p) \geq K\|p\|_x + C^*(K) \qquad \forall\,(x,p) \in T^*M.$$

(H2) *Uniform convexity:* For every $(x,p) \in T^*M$, the second derivative along the fibers $\frac{\partial^2 H}{\partial p^2}(x,p)$ is positive definite.

The critical value $\mathbf{c}[H] \in \mathbb{R}$ of H may be defined as the infimum of the values $c \in \mathbb{R}$ for which there exists a smooth function $u : M \to \mathbb{R}$ satisfying

$$H(x, du(x)) \leq c \qquad \forall\, x \in M.$$

A Lipschitz function $u : M \to \mathbb{R}$ is called a *critical subsolution* (for H) if

$$H\big(x, du(x)\big) \leq \mathbf{c}[H] \qquad \text{for a.e. } x \in M. \tag{2.1}$$

It can be shown that the set of critical subsolutions, denoted by $\mathcal{SS}$, is a nonempty compact convex subset of $C^0(M;\mathbb{R})$ [18, 36]. Fathi and Siconolfi [20] proved that the set of critical subsolutions of class C^1 (denoted by $\mathcal{SS}^1$) is nonempty, and their result has been improved later by Bernard [8] who showed the existence of critical subsolutions of class $C^{1,1}$. Also, they proved that the so-called *Aubry set* can be seen as the nonempty compact subset of T^*M defined by

$$\tilde{\mathcal{A}}(H) := \bigcap_{u \in \mathcal{SS}^1} \Big\{ (x, du(x)) \,|\, x \in M \text{ s.t. } H\big(x, du(x)\big) = \mathbf{c}[H] \Big\}.$$

This set is invariant under the Hamiltonian flow, and by *Mather's graph property* it is a Lipschitz graph over the *projected Aubry set* defined as

$$\mathcal{A}(H) := \pi^*\big(\tilde{\mathcal{A}}(H)\big) \subset M,$$

where $\pi^* : T^*M \to M$ denotes the canonical projection map (see for instance [18, 36]).

The Lagrangian $L : TM \to \mathbb{R}$ associated with H by Legendre-Fenchel duality is defined by

$$L(x,v) := \max_{p \in T_x^*M} \Big\{ \langle p, v\rangle - H(x,p) \Big\} \qquad \forall\,(x,v) \in TM.$$

Thanks to (H1)-(H2), it is easy to see that L is a Tonelli Lagrangian of class C^2, that is a Lagrangian satisfying both properties of superlinear growth and uniform convexity (see [10, 18]). Critical subsolutions have important variational properties, and for instance they can be characterized as follows (see [18, 36]):

Proposition 2.1. *A function $u : M \to \mathbb{R}$ is a critical subsolution if and only if*

$$u\big(\gamma(b)\big) - u\big(\gamma(a)\big) \leq \int_a^b L\big(\gamma(s), \dot\gamma(s)\big)\, ds + \mathbf{c}[H]\,(b-a) \tag{2.2}$$

for any Lipschitz curve $\gamma : [a,b] \to M$.

The (backward) Lax-Oleinik semigroup

$$\{\mathcal{T}_t^-\}_{t\geq 0} : C^0(M;\mathbb{R}) \longrightarrow C^0(M;\mathbb{R})$$

associated with L is defined as follows: for every $t \geq 0$ and $u \in C^0(M;\mathbb{R})$, the function $\mathcal{T}_t^- u := \mathcal{T}_t^-(u)$ is given by

$$\mathcal{T}_t^- u(x) := \inf \left\{ u\big(\gamma(-t)\big) + \int_{-t}^0 L\big(\gamma(s),\dot{\gamma}(s)\big)\, ds \right\} \qquad \forall\, x \in M, \tag{2.3}$$

where the infimum is taken over all Lipschitz curves $\gamma : [-t,0] \to M$ such that $\gamma(0) = x$. The set of critical subsolutions $\mathcal{SS}$ is invariant under $\{\mathcal{T}_t^-\}_{t\geq 0}$. A critical subsolution $u : M \to \mathbb{R}$ is called a *critical solution* or a *weak KAM solution* if,

$$\mathcal{T}_t^- u = u - \mathbf{c}[H]\, t \qquad \forall\, t \geq 0. \tag{2.4}$$

Critical solutions may be characterized in several ways (see for instance [18, 36]):

Proposition 2.2. *Let $u \in C^0(M;\mathbb{R})$. The following properties are equivalent:*

(i) u is a critical solution.

(ii) $u \in \mathcal{SS}$ and, for every $x \in M$, there exists a Lipschitz curve $\gamma_x : (-\infty, 0] \to M$ with $\gamma_x(0) = x$ such that

$$u\big(\gamma_x(b)\big) - u\big(\gamma_x(a)\big) = \int_a^b L\big(\gamma_x(s),\dot{\gamma}_x(s)\big)\, ds + \mathbf{c}[H]\,(b-a) \qquad \forall\, a < b \leq 0. \tag{2.5}$$

(iii) $u \in \mathcal{SS}$ and for every smooth function $\phi : M \to \mathbb{R}$ with $\phi \leq u$ and all $x \in M$,

$$\phi(x) = u(x) \quad \Longrightarrow \quad H\big(x, d\phi(x)\big) \geq \mathbf{c}[H].$$

As shown in [35], critical solutions enjoy some regularity properties. One of them is the fact that critical solutions are semiconcave. Recall that, given an open set $\Omega \subset \mathbb{R}^n$, a function $v : \Omega \to \mathbb{R}^n$ is said to be *locally semiconcave* in Ω if, for every $x \in \Omega$, there are $C_x \geq 0$ and a ball $\mathcal{B}_x \subset \Omega$ containing x such that the function $y \mapsto v(y) - C_x|y|^2$ is concave on $\mathcal{B}_x$. A function $v : M \to \mathbb{R}$ is called *locally semiconcave* if it is locally semiconcave in charts, that is, if for every $x \in M$ there are an open neighborhood $\mathcal{V}_x$ of x and a smooth diffeomorphism $\phi_x : \mathcal{V}_x \to \phi_x(\mathcal{V}_x) \subset \mathbb{R}^n$ such that $v \circ \phi_x^{-1}$ is locally semiconcave on $\phi_x(\mathcal{V}_x) \subset \mathbb{R}^n$. Of course, if the manifold M is compact then the constant C_x can be chosen independent of the point, and we say that the function is semiconcave.

Proposition 2.3. *Any critical solution is semiconcave on M.*

Let $u : M \to \mathbb{R}$ be a critical solution for H and $x \in M$ be fixed. By assertion (ii) in Proposition 2.2 above, there exists a curve $\gamma_x : (-\infty, 0] \to M$ with $\gamma_x(0) = x$ satisfying (2.5). Since u is a critical subsolution (see Proposition 2.1), we infer that for every $T > 0$ the restriction of γ_x to $[-T, 0]$ minimizes the quantity

$$u\big(\gamma(-T)\big) + \int_{-T}^0 L\big(\gamma(s),\dot{\gamma}(s)\big)\, ds + \mathbf{c}[H]\, T,$$

among all Lipschitz curves $\gamma : [-T, 0] \to M$ such that $\gamma(0) = x$. In particular, γ_x is the projection of a Hamiltonian trajectory, and whenever u is differentiable at $\gamma_x(-T)$, by the first variation formula one gets

$$du\big(\gamma_x(-T)\big) = \frac{\partial L}{\partial v}\big(\gamma_x(-T),\dot{\gamma}_x(-T)\big).$$

We call *limiting differential* of u at $x \in M$, and we denote it by $D_x^* u$, the set of $p \in T_x^* M$ such that there is a sequence $\{x_k\}_k$ of points converging to x such that u is differentiable at x_k and $p = \lim_{k\to\infty} du(x_k)$. Note that, by the Lipschitz regularity of u, the graph of the multivalued mapping $D^* u$ is a compact subset of $T^* M$. As shown in [35], by the above discussion one can prove that there is a one-to-one correspondence between the limiting differentials and the curves satisfying (2.5):

Proposition 2.4. *Let $u : M \to \mathbb{R}$ be a critical solution and $x \in M$. For every $p \in D_x^* u$ the curve $\gamma_x : (-\infty, 0] \to M$ defined by*

$$\gamma_x(-t) := \pi^* \left(\phi_{-t}^H(x,p)\right) \qquad \forall\, t \geq 0, \tag{2.6}$$

satisfies $\gamma_x(0) = x$, (2.5), and

$$\left(\gamma_x(-t), D^*_{\gamma_x(-t)} u\right) = \left\{\phi_{-t}^H(x,p)\right\} \qquad \forall\, t \geq 0. \tag{2.7}$$

In particular u is differentiable at $\gamma_x(-t)$ for any $t > 0$. Moreover, for every curve $\gamma_x : (-\infty, 0] \to M$ satisfying $\gamma_x(0) = x$ and (2.5), there is $p \in D_x^* u$ such that (2.6) holds.

A curve of the form $\gamma_x : (-\infty, 0] \to M$ satisfying (2.5) is called a *semi-calibrated* curve. A curve defined on $\mathbb{R}$ satisfying (2.5) for any $a, b \in \mathbb{R}$ is called *calibrated*. As we said previously, the Aubry set $\tilde{\mathcal{A}}(H)$ is invariant under the Hamiltonian flow, and it is a Lipschitz graph over $\mathcal{A}(H)$. Fathi and Siconolfi [20] proved that, for every point of $\mathcal{A}(H)$, the limiting differential of a critical solution is a singleton there. In particular, since $\tilde{\mathcal{A}}(H)$ is a Lipschitz graph over $\mathcal{A}(H)$, this means that every critical solution u is differentiable on $\mathcal{A}(H)$, its differential is independent of u, and $x \mapsto du(x)$ is Lipschitz on the Aubry set. In addition, for any $(x,p) \in \tilde{\mathcal{A}}(H)$, the curve (2.6) is calibrated. All these facts are summarized in the following:

Proposition 2.5. *Let $u : M \to \mathbb{R}$ be a critical solution and $x \in \mathcal{A}(H)$. Then u is differentiable at x, $du(x)$ does not depend on u, $D_x^* u = \{du(x)\}$, and the calibrated curve $\gamma_x : \mathbb{R} \to M$ defined by*

$$\gamma_x(t) := \pi^* \left(\phi_t^H(x, du(x))\right) \qquad \forall\, t \in \mathbb{R}, \tag{2.8}$$

satisfies $\gamma_x(0) = x$,

$$u\big(\gamma_x(b)\big) - u\big(\gamma_x(a)\big) = \int_a^b L\big(\gamma_x(s), \dot\gamma_x(s)\big)\, ds + \mathbf{c}[H]\,(b-a) \qquad \forall\, a < b, \tag{2.9}$$

$\gamma_x(t) \in \mathcal{A}(H)$ for all $t \in \mathbb{R}$, and

$$\left(\gamma_x(t), D^*_{\gamma_x(t)} u\right) = \Big\{\phi_t^H(x, du(x))\Big\} \qquad \forall\, t \in \mathbb{R}. \tag{2.10}$$

Finally, the mapping $\mathcal{A}(H) \ni x \mapsto du(x)$ is Lipschitz.

We refer the reader to [21, 23] for a more detailed introduction to weak KAM theory, to the notes [36] for the proofs of the above results, and to [18] for further details.

2.2 The Dirichlet problem and the connection of trajectories

Let $H : \mathbb{R}^n \times (\mathbb{R}^n)^* \to \mathbb{R}$ be a Tonelli Hamiltonian of class C^2, $V : \mathbb{R}^n \to \mathbb{R}$ a C^2 function, and denote by H_V the Hamiltonian $H + V$. We split $\mathbb{R}^n$ as $\mathbb{R} \times \mathbb{R}^{n-1}$ and we define the $(n-1)$-dimensional disks

$$\Pi_r^\tau := \{\tau\} \times B^{n-1}(0, r) \qquad \forall\, \tau \in \mathbb{R}, \forall\, r > 0,$$

where $B^{n-1}(0,r) \subset \mathbb{R}^{n-1}$ denotes the $(n-1)$-dimensional open ball of radius r centered at the origin. Denoting by $\pi^* : \mathbb{R}^n \times (\mathbb{R}^n)^* \to \mathbb{R}^n$ the projection onto the space variable, we define the following Poincaré-type maps:

Given $\tau > 0$ small, $\tau_1, \tau_2 \in [0, \tau]$, and $(x^0, p^0) \in \Pi_{1/2}^{\tau_1} \times \mathbb{R}^n$ such that $[-2\tau, 2\tau] \ni t \mapsto \pi^*\big(\phi_t^{H_V}(x^0,p^0)\big)$ intersects $\Pi_1^{\tau_2}$ transversally, we define the maps

$$\mathcal{P}^*_{\tau_1,\tau_2}(x^0,p^0) := \phi^H_{\mathcal{T}^*_{\tau_1,\tau_2}(x^0,p^0)}(x^0,p^0), \qquad \mathcal{P}_{\tau_1,\tau_2}(x^0,p^0) := \pi^*\Big(\mathcal{P}^*_{\tau_1,\tau_2}(x^0,p^0)\Big),$$

where $\mathcal{T}^*_{\tau_1,\tau_2}(x^0,p^0) \in [-2\tau, 2\tau]$ is the first time (positive if $\tau_1 < \tau_2$, negative if $\tau_1 > \tau_2$) such that $\mathcal{P}_{\tau_1,\tau_2}(x^0,p^0) \in \Pi_1^{\tau_2}$.

As shown in [22, Lemma 5.1], the following holds (we denote by e_1 the first vector in the canonical basis of $\mathbb{R}^n = \mathbb{R} \times \mathbb{R}^{n-1}$):

Lemma 2.6. *Let $\bar{u} : B^n(0,1) \to \mathbb{R}$ be a $C^{1,1}$ function such that*

$$\frac{d}{dt}\Big(\pi^*\Big(\phi_t^H(x^0, d\bar{u}(x^0))\Big)\Big)_{|t=0} \cdot e_1 \geq \frac{1}{2} \qquad \forall\, x^0 \in \Pi_1^0.$$

Then there exists $\bar{\tau} > 0$ small such that the following properties are satisfied:

(i) For every $\tau \in (0, 5\bar{\tau}]$, the Poincaré time mapping $\mathcal{T}^{d\bar{u}}_{0,\tau} : \Pi^0_{1/2} \to \mathbb{R}$ defined by

$$\mathcal{T}^{d\bar{u}}_{0,\tau}\big(x_0\big) := \mathcal{T}^*_{0,\tau}\big(x^0, d\bar{u}(x^0)\big) \qquad \forall\, x^0 \in \Pi^0_{1/2},$$

is well-defined and is Lipschitz;

(ii) for every $\tau \in (0, 5\bar{\tau}]$, the Poincaré mapping $\mathcal{P}^{d\bar{u}}_{0,\tau} : \Pi^0_{1/2} \to \Pi_1^\tau$ defined by

$$\mathcal{P}^{d\bar{u}}_{0,\tau} := \mathcal{P}_{0,\tau}\big(x^0, d\bar{u}(x^0)\big) \qquad \forall\, x^0 \in \Pi^0_{1/2},$$

is 2-Lipschitz;

(iii) the following inclusion holds for every $\tau \in (0, 5\bar{\tau}]$:

$$\Big\{\pi^*\Big(\phi_t^H(x^0, d\bar{u}(x^0))\Big) \,|\, x^0 \in \Pi^0_{3/8},\, t \in \big[0, \mathcal{T}_{0,\tau}(x^0)\big]\Big\} \subset [0,\tau] \times B^{n-1}(0, 1/2);$$

(iv) the viscosity solution $\bar{u}_0$ to the Dirichlet problem

$$\begin{cases} H\big(z, d\bar{u}_0(z)\big) = 0 & \text{in } [0, 5\bar{\tau}] \times B^{n-1}(0,1/2), \\ \bar{u}_0 = \bar{u} & \text{on } \Pi^0_1, \end{cases}$$

is of class $C^{1,1}$.

We now define the cylinder

$$\mathcal{C}\Big(\big(x^0,p^0\big);t;r\Big) := \Big\{\pi^*\Big(\phi_s^H(x^0,p^0)\Big) + (0,\hat{y}) \,|\, s \in [0,t],\, |\hat{y}| < r\Big\},$$

and the action

$$\begin{aligned} \mathbb{A}_V\big((x^0,p^0);\tau\big) &:= \int_0^\tau L_V\left(\pi^*\Big(\phi_t^{H_V}(x^0,p^0)\Big), \frac{d}{dt}\Big(\pi^*\Big(\phi_t^{H_V}(x^0,p^0)\Big)\Big)\right) dt \\ &= \int_0^\tau L\left(\pi^*\Big(\phi_t^{H_V}(x^0,p^0)\Big), \frac{d}{dt}\Big(\pi^*\Big(\phi_t^{H_V}(x^0,p^0)\Big)\Big)\right) \\ &\qquad\qquad -V\Big(\pi^*\Big(\phi_t^{H_V}(x^0,p^0)\Big)\Big)\, dt, \end{aligned}$$

where $\phi_t^{H_V}$ denotes the Hamiltonian flows associated to H_V. Then the following holds:

Proposition 2.7. *Let $u : B^n(0,1) \to \mathbb{R}$ be a viscosity solution of $H(x, du(x)) = \mathbf{c}[H]$ and assume that*

$$\frac{d}{dt}\left(\pi^*\left(\phi_t^H(x^0,p^0)\right)\right)_{|t=0} \cdot e_1 \geq \frac{1}{2} \qquad \forall\, x^0 \in \Pi_1^0,\, \forall\, p^0 \in D^*_{x^0}u,$$

$$\langle p_0, \dot\gamma_{0,p_0}(0)\rangle \geq -c_0 \qquad \forall\, p_0 \in D_0^*u, \qquad \textit{where} \quad \gamma_{0,p_0}(t) := \pi^*\big(\phi_t^H(0,p_0)\big) \tag{2.11}$$

for some positive constant c_0. Then, provided c_0 is sufficiently small (the smallness depending only on H), for any $\bar\tau > 0$ sufficiently small there are $\bar\delta, \bar r, \bar\epsilon \in (0,1/4)$ and $K > 0$ such that the following property holds: For any $r \in (0,\bar r), \hat\epsilon \in (0,\bar\epsilon)$, $x^0 \in \Pi_1^0, x^f \in \Pi_1^{\bar\tau}$, $p^0 \in D^*_{x^0}u$, $p^f \in D^*_{x^f}u$, and $\sigma \in \mathbb{R}$ satisfying

$$|x^0| < \bar\delta \tag{2.12}$$

and

$$\left|(x^f,p^f) - \mathcal{P}^*_{0,\bar\tau}(x^0,p^0)\right| < r\hat\epsilon, \qquad |\sigma| < r^2\hat\epsilon, \tag{2.13}$$

there exist a time $T^f > 0$ and a potential $V : \mathbb{R}^n \to \mathbb{R}$ of class C^2 such that:

(i) $\mathrm{Supp}(V) \subset \mathcal{C}\Big(\big(x^0,p^0\big); \mathcal{T}^*_{0,\bar\tau}(x^0,p^0); r\Big)$;

(ii) $\|V\|_{C^2} < K\hat\epsilon$;

(iii) $\left|T^f - \mathcal{T}^*_{0,\bar\tau}(x^0,p^0)\right| < Kr\hat\epsilon$;

(iv) $\phi^{H_V}_{T^f}\big(x^0,p^0\big) = \big(x^f,p^f\big)$;

(v) $\mathbb{A}_V\big((x^0,p^0);T^f\big) = \mathbb{A}\big((x^0,p^0);\mathcal{T}^*_{0,\bar\tau}(x^0,p^0)\big) + \big\langle du\big(\mathcal{P}_{0,\bar\tau}(x^0,p^0)\big), x^f - \mathcal{P}_{0,\bar\tau}(x^0,p^0)\big\rangle + \sigma.$

Proof of Proposition 2.7. First of all, it follows by (2.13) and Lemma 2.6(ii) that, provided $\bar\tau$ is sufficiently small (the smallness being independent of r and $\hat\epsilon$),

$$\left|\mathcal{P}^*_{\bar\tau,\bar\tau/2}\big(x^f,p^f\big) - \mathcal{P}^*_{0,\bar\tau/2}\big(x^0,p^0\big)\right| < 2r\hat\epsilon.$$

Hence, we first apply [21, Proposition 3.1] on $[0,\bar\tau/2]$ to connect (x^0,p^0) to $\mathcal{P}^*_{\bar\tau,\bar\tau/2}\big(x^f,p^f\big)$ in a time $T_1^f \sim \bar\tau/2$ with a "default" of action bounded by $Kr^2\hat\epsilon^2$. Then, thanks to (2.11), we see that assumption (A4) in [21, Proposition 4.1] is satisfied provided c_0 is small enough . Thus, if $\bar\epsilon$ is sufficiently small we can apply [21, Proposition 4.1] on $[\bar\tau/2,\bar\tau]$ to "compensate" the default of action so that (v) above holds. Moreover it is easily seen that also all the other properties are satisfied. We leave the details to the reader. □

2.3 Green bundles and reduced Green bundles

Let us endow the cotangent bundle T^*M with its standard symplectic structure ω, and denote by $V_\theta := \ker(d_\theta\pi^*)$ the vertical space in $T_\theta(T^*M)$ at any $\theta \in T^*M$ (recall that $\pi^* : T^*M \to M$ denotes the canonical projection). A subspace $E \subset T_\theta(T^*M)$ is called *Lagrangian* if it is a n-dimensional vector subspace where the symplectic bilinear form $\omega_\theta : T_\theta(T^*M) \times T_\theta(T^*M) \to \mathbb{R}$ vanishes. As an example, vertical spaces are Lagrangian. If we fix a symplectic set of local coordinates, we can identify $T_\theta(T^*M)$ with $T_xM \times T_x^*M$ and V_θ with $\{0\} \times T_x^*M$. Then, any n-dimensional vector subspace $E \subset T_\theta(T^*M)$ which is transversal to V_θ (i.e. $E \cap V_\theta = \{0\}$) can be written as the graph of some linear map $S : T_xM \to T_x^*M$, and it can be checked that E is Lagrangian if and only if S is represented by a symmetric matrix.

Given a Hamiltonian $H : T^*M \to \mathbb{R}$ of class C^2, the *Hamiltonian vector field* X_H on T^*M is defined by $\omega_\theta\big(X_H(\theta), \cdot\big) = -d_\theta H$ for any $\theta \in T^*M$. In a symplectic set of local coordinates, the Hamiltonian equations (i.e., the equations satisfied by any solution of the ODE $(\dot{x}, \dot{p}) = X_H\big((x,p)\big)$) are given by $\dot{x} = \frac{\partial H}{\partial p}, \dot{p} = -\frac{\partial H}{\partial x}$. Finally, we recall that the Hamiltonian flow ϕ_t^H of X_H preserves the symplectic form ω. In particular, the image of a Lagrangian space $E \subset T_\theta(T^*M)$ by $D_\theta \phi_t^H$ is Lagrangian in $T_{\phi_t^H(\theta}(T^*M)$. We refer the reader to [1, 11] for more details about the notions of symplectic geometry introduced above.

We recall now the construction and properties of Green bundles and reduced Green bundles along orbits of the Hamiltonian flow without conjugate points. We refer the reader to [4, 6, 14] for further details and historical accounts. For every $\theta \in T^*M$ and every $t \in \mathbb{R}$, we define the Lagrangian subspace $G_\theta^t \subset T_\theta(T^*M)$ as the pushforward of the vertical distribution at $\phi_{-t}^H(\theta)$ by ϕ_t^H, that is

$$G_\theta^t := \left(\phi_t^H\right)_* \left(V_{\phi_{-t}^H(\theta)}\right) = D_{\phi_{-t}^H(\theta)} \phi_t^H \left(V_{\phi_{-t}^H(\theta)}\right) \qquad \forall\, \theta \in T^*M. \tag{2.14}$$

The orbit of $\theta \in T^*M$ is said to be *without conjugate points* if for any $t, t' \in \mathbb{R}$,

$$t \neq t' \implies \left[D_{\phi_t^H(\theta)} \phi_{t'-t}^H \left(V_{\phi_t^H(\theta)}\right)\right] \cap V_{\phi_{t'}^H(\theta)} = \{0\}.$$

We denote by $\mathcal{D}$ the set of $\theta \in T^*M$ whose orbit has no conjugate point, and we assume that $\mathcal{D}$ is nonempty. Given $\theta \in \mathcal{D} \subset T^*M$, we fix a symplectic set of local coordinates around $\theta = (x,p)$. Then, for every $t \in \mathbb{R} \setminus \{0\}$, the Lagrangian subspace G_θ^t is transverse to the vertical subspace V_θ in $T_\theta(T^*M) \simeq T_xM \times T_x^*M$. Hence, there is a linear operator $K_\theta^t : T_xM \to T_x^*M$ such that

$$G_\theta^t = \Big\{ \big(h, K_\theta^t h\big) \in T_xM \times T_x^*M \mid h \in T_xM \Big\}.$$

Since G_θ^t is Lagrangian, the linear operator K_θ^t can be represented by a symmetric matrix in our symplectic set of local coordinates. There is a natural partial order for the Lagrangian subspaces which are transverse to the vertical, that simply corresponds to the usual order for symmetric operators. Later on, given two Lagrangian subspaces $E, E' \in T_\theta(T^*M)$ which are transverse to V_θ, we shall write $E \prec E'$ (resp. $E \preceq E'$) if the corresponding symmetric operators K, K' are such that $K' - K$ is positive definite (resp. nonnegative definite). The following property is a consequence of the uniform convexity of H in the fibers (see [4, Proposition 3.7] and [14, Proposition 1.4]):

Proposition 2.8. *Let $\theta \in \mathcal{D}$. The following properties hold:*

(i) For every $t' > t > 0$, $G_\theta^{t'} \prec G_\theta^t$.

(ii) For every $t' < t < 0$, $G_\theta^t \prec G_\theta^{t'}$.

(iii) For every $t < 0 < t'$, $G_\theta^t \prec G_\theta^{t'}$.

As a consequence, for every $\theta \in \mathcal{D}$, the sequence of Lagrangian subspaces $(0, +\infty) \ni t \mapsto G_\theta^t$ (resp. $(0, +\infty) \ni t \mapsto G_\theta^{-t}$) is decreasing (resp. increasing) and bounded from below by G_θ^{-1} (resp. bounded from above by G_θ^1). Hence, both limits as $t \to \pm\infty$ exist, which leads to the following definition:

Definition 2.9. *For every $\theta \in \mathcal{D}$, we define the positive and negative Green bundles at θ as*

$$G_\theta^+ := \lim_{t \to +\infty} G_\theta^t \quad \text{and} \quad G_\theta^- := \lim_{t \to -\infty} G_\theta^t.$$

We shall keep in mind that the positive Green bundle G^+_θ depends on the behavior of the Hamiltonian flow along the orbit of θ for large negative times, while the negative Green bundle G^-_θ depends on what happens for large positive times. By construction, we also have the following result (see [4, Corollaire 3.8 and Proposition 3.9] and [14, Proposition 1.4 (d)]):

Proposition 2.10. *Let $\theta \in \mathcal{D}$. The following properties hold:*

(i) $G^-_\theta \preceq G^+_\theta$.

(ii) $D_\theta \phi^H_t \left(G^-_\theta\right) = G^-_{\phi^H_t(\theta)}$ *and* $D_\theta \phi^H_t \left(G^+_\theta\right) = G^+_{\phi^H_t(\theta)}$ *for all* $t \in \mathbb{R}$.

Moreover, the function $\theta \mapsto G^+_\theta$ is upper-semicontinuous on $\mathcal{D}$, and $\theta \mapsto G^-_\theta$ is lower-semicontinuous on $\mathcal{D}$. Thus, if $G^+_\theta = G^-_\theta$ for some $\theta \in \mathcal{D}$ then both of them are continuous at θ.

The following result, which first appeared in [14], plays a major role in recent works by Arnaud [4, 5, 6] (see [4, Proposition 3.12], [6, Proposition 1], and [14, Proposition 1.11]):

Proposition 2.11. *Let $\theta \in \mathcal{D}$ and $\psi \in T_\theta(T^*M)$. Then the following properties hold:*

(i) $\psi \notin G^-_\theta \implies \lim_{t\to+\infty} \left\| D_\theta\big(\pi^* \circ \phi^H_t\big)(\psi)\right\| = +\infty$.

(ii) $\psi \notin G^+_\theta \implies \lim_{t\to-\infty} \left\| D_\theta\big(\pi^* \circ \phi^H_t\big)(\psi)\right\| = +\infty$.

For every $\theta = (x,p) \in T^*M$, denote by $\Sigma_\theta \subset T^*M$ the energy level

$$\Sigma_\theta := \Big\{ \theta' = (x',p') \in T^*M \,|\, H(x',p') = H(x,p) \Big\}.$$

From the previous result one easily gets the following conclusion (see [4, Exemple 2 page 17] and [14, Corollary 1.12]):

Proposition 2.12. *Let $\theta \in \mathcal{D}$ be such that $X_H(\theta) \neq 0$. Then*

$$X_H(\theta) \in G^-_\theta \cap G^+_\theta \quad \text{and} \quad G^-_\theta \cup G^+_\theta \subset T_\theta \Sigma_\theta.$$

Let $\Sigma \subset T^*M$ be a regular energy level of H, that is an energy level satisfying $\frac{\partial H}{\partial p}(x,p) \neq 0$ for every $\theta = (x,p) \in \Sigma$. By superlinear growth (H1) and uniform convexity (H2) of H, the hypersurface Σ is compact and, for every $\theta = (x,p) \in \Sigma$, the fiber $\Sigma \cap T^*_x M$ is the boundary of a uniformly convex set in $T^*_x M$. For every $\theta \in \Sigma$ we define the subspace $N_\theta \subset T_\theta \Sigma$ by

$$N_\theta := \Big\{ \psi \in T_\theta \Sigma \,|\, \big\langle D_\theta \pi^*(\psi), D_\theta \pi^*\big(X_H(\theta)\big)\big\rangle_{\pi^*(\theta)} = 0 \Big\},$$

where $\langle \cdot, \cdot \rangle$ denotes the Riemannian metric on M. By construction, we have

$$T_\theta \Sigma = N_\theta \oplus \mathbb{R} X_H(\theta) \qquad \forall\, \theta \in \Sigma.$$

For every $\theta \in \mathcal{D} \cap \Sigma$, we define the *reduced Green bundles* $\hat{G}^-_\theta$ and $\hat{G}^+_\theta$ as

$$\hat{G}^-_\theta := G^-_\theta \cap N_\theta \quad \text{and} \quad \hat{G}^+_\theta := G^+_\theta \cap N_\theta.$$

As shown in [4], the reduced Green bundles can be seen as the Green bundles associated with a specific symplectic bundle over the orbit of θ; they satisfy some of the properties of the Green bundles, in particular Proposition 2.10 (except (ii)). If M has dimension two, then, for every $\theta \in \mathcal{D} \cap \Sigma$, the reduced Green bundles $\hat{G}^+_\theta$ and $\hat{G}^-_\theta$ should be seen as lines in the plane $N_\theta \simeq T_\theta\Sigma/\mathbb{R}X_H(\theta)$. Finally we observe that, since G^+_θ depends on the behavior of the Hamiltonian flow near $\phi^H_t(\theta)$ for large negative times, its construction can be performed as soon as the orbit of $\theta \in T^*M$ has no conjugate points in negative time. In particular, this can be done for any semi-calibrated curve (see Proposition 2.4).

2.4 Paratingent cones and Green bundles

The present section is mainly inspired by ideas and techniques developed by Arnaud in [4, 5, 6, 7] to study in particular the link between Green bundles and regularity of weak KAM solutions. Before presenting Arnaud-type results, we first recall a result from [22]. Let $S \subset \mathbb{R}^k$ be a compact set which has the origin as a cluster point. The *paratingent cone* to S at 0 is the cone defined as

$$C_0(S) := \left\{ \lambda \lim_{i\to\infty} \frac{x_i - y_i}{|x_i - y_i|} \mid \lambda \in \mathbb{R}, \lim_{i\to\infty} x_i = \lim_{i\to\infty} y_i = 0,\, x_i \in S,\, y_i \in S,\, x_i \neq y_i \;\forall\, i \right\},$$

and the *paratingent space* of S at 0 is the vector space generated by $C_0(S)$:

$$\Pi_0(S) := \mathrm{Span}\, \Big\{ C_0(S) \Big\}.$$

As shown in [22], the set S is contained locally in the graph of a function from $\Pi := \Pi_0(S)$ onto its orthogonal complement $\Pi^\perp$. Let d be the dimension of Π, denote by Proj_Π the orthogonal projection onto the space Π in $\mathbb{R}^k$, and set $\mathcal{H}_S := \mathrm{Proj}_\Pi(S)$. Finally, for any $r, \nu > 0$ we define the cylinder

$$C(r,\nu) := \Big\{ (h,v) \in \Pi \times \Pi^\perp \mid |h| < r, |v| < \nu \Big\},$$

where $|\cdot|$ denotes the Euclidean norm. Also, we set $B_r := B(0,r)$. The following result holds (see [22, Lemma 3.3]).

Lemma 2.13. *There exist $r_S > 0$ and a Lipschitz function $\Psi_S : \Pi \cap \bar{B}_{r_S} \to \Pi^\perp$ such that the following properties hold:*

(i) $S \cap C(r_S, r_S) \subset \mathrm{graph}(\Psi_S)|_{B_{r_S}} := \big\{ h + \Psi_S(h) \mid h \in \Pi \cap B_{r_S} \big\}$;

(ii) $h + \Psi_S(h)$ *belongs to* $S \cap C(r_S, r_S)$ *for every* $h \in \mathcal{H}_S \cap B_{r_S}$*;*

(iii) For any $r \in (0, r_S)$*, let* $\ell(r) > 0$ *denote the Lipschitz constant of* Ψ_S *on* $\Pi \cap B_r$*. Then* $\lim_{r\downarrow 0} \ell(r) = 0$*.*

In particular $\Psi_S(0) = 0$*,* Ψ_S *is* C^1 *at* 0*, and* $\nabla\Psi_S(0) = 0$*.*

By Proposition 2.5, through each point $\theta = (x,p)$ of the Aubry set $\tilde{\mathcal{A}}(H)$ passes a calibrated curve (defined by (2.8)) which corresponds to the projection of its orbit under the Hamiltonian flow, and whose restriction to any subinterval is always minimizing the action between its endpoints. Being minimizing, such a curve has necessarily no conjugate points, hence $\theta \in \mathcal{D}$ (see any textbook on the classical theory of calculus of variations, for example [13]). We also observe that, since the Aubry set is invariant under the Hamiltonian flow,

$$D_\theta \phi_t^H \left(C_\theta \left(\tilde{\mathcal{A}}(H) \right) \right) = C_{\phi_t^H(\theta)} \left(\tilde{\mathcal{A}}(H) \right) \qquad \forall\, t \in \mathbb{R} \tag{2.15}$$

and $X_H(\theta)$ belongs to the paratingent cone to $\tilde{\mathcal{A}}(H)$ at θ, that is

$$X_H(\theta) \in C_\theta \left(\tilde{\mathcal{A}}(H) \right) \subset T_\theta \Sigma_H \qquad \forall\, \theta \in \tilde{\mathcal{A}}(H), \tag{2.16}$$

where $\Sigma_H := \big\{ H = \mathbf{c}[H] \big\}$ is a regular energy level of H. Given $\theta = (x,p) \in \tilde{\mathcal{A}}(H)$ with $X_H(\theta) \neq 0$, we define the *reduced paratingent cone* to the Aubry set as

$$\hat{C}_\theta := C_\theta \left(\tilde{\mathcal{A}}(H) \right) \cap N_\theta,$$

where N_θ has been defined in Section 2.3. If M has dimension two, $\hat{C}_\theta$ is a collection of lines in the plane N_θ. All those lines can be compared with other lines in this plane. The following proposition is a variant of Arnaud's results (compare with [4, Proposition 3.11], [4, Proposition 3.16 (3)], [6, Theorem 9]), and it follows from the Lipschitz graph property of the Aubry set.

Proposition 2.14. *Assume that* $\dim M = 2$ *and that* $\theta \in \tilde{\mathcal{A}}(H)$ *is not an equilibrium of* X_H*. Then*

$$\hat{G}_\theta^- \preceq \hat{C}_\theta \preceq \hat{G}_\theta^+.$$

In other terms, any line in $\hat{C}_\theta$ *is squeezed between* $\hat{G}_\theta^-$ *and* $\hat{G}_\theta^+$*.*

Proof of Proposition 2.14. Since $\tilde{\mathcal{A}}(H)$ is a Lipschitz graph, its paratingent cones cannot intersect the vertical bundle, hence taking $\epsilon > 0$ small enough yields

$$G_\theta^{-\epsilon} \prec C_\theta := C_\theta\left(\tilde{\mathcal{A}}(H)\right) \prec G_\theta^\epsilon. \tag{2.17}$$

To explain the meaning of the above formula notice that, for every $t \neq 0$, the Lagrangian space G_θ^t is transverse to V_θ, it does not contain $X_H(\theta)$, and it is contained in $T_\theta \Sigma_H$. Hence its intersection with N_θ is a line in the plane N_θ, and (2.17) means that the intersection of C_θ with N_θ is a collection of vector lines which are squeezed between the lines $G_\theta^{-\epsilon} \cap N_\theta$ and $G_\theta^\epsilon \cap N_\theta$. Thus, thanks to this discussion, to prove the result it is sufficient to show that no line $G_\theta^t \cap N_\theta$ with $t \in \mathbb{R} \setminus [-\epsilon, \epsilon]$ is contained in $C_\theta \cap N_\theta$. Argue by contradiction and assume that there is $\bar{t} > \epsilon$ (the other case is left to the reader) such that

$$G_\theta^{-\bar{t}} \cap N_\theta \subset C_\theta \cap N_\theta.$$

By (2.14)-(2.15), this means that $V_{\phi_{\bar{t}}^H(\theta)}$ and $D_\theta \phi_{\bar{t}}^H(C_\theta) = C_{\phi_{\bar{t}}^H(\theta)}$ do intersect, which contradicts the Lipschitz graph property of the Aubry set. □

As an application of Proposition 2.14 and Lemma 2.13, we deduce that if $\dim M = 2$ and the positive and negative Green bundles coincide for some $\theta = (x,p) \in \tilde{\mathcal{A}}(H)$ with $X_H(\theta) \neq 0$, then the Aubry set is locally contained in the graph of a Lipschitz 1-form which is C^1 at x. It will be convenient to extend the 1-form along a piece of projected orbit of the Aubry set.

Corollary 2.15. *Assume that* $\dim M = 2$ *and that* $\theta = (x,p) \in \tilde{\mathcal{A}}(H)$ *with* $X_H(\theta) \neq 0$ *satisfies*

$$G_\theta^- = G_\theta^+.$$

Assume moreover that θ *is not on a periodic orbit and let* $\gamma(t) := \pi^*\big(\phi_t^H(\theta)\big)$ *for any* $t \in \mathbb{R}$*. Then, for every* $T > 0$ *there are an open neighborhood* $\mathcal{V}$ *of* $\gamma([-T,T])$ *in* M *and a function* $f : \mathcal{V} \to \mathbb{R}$ *of class* $C^{1,1}$ *which is* C^2 *along* $\gamma([-T,T])$ *such that*

$$\tilde{\mathcal{A}}(H) \cap T^*\mathcal{V} \subset Graph(df),$$

and for every $t \in [-T,T]$, $G_{\phi_t^H(\theta)}^- = G_{\phi_t^H(\theta)}^+$ *is the graph of* $D_{\gamma(t)}^2 f$ *(in a symplectic set of local coordinates in* $T^*\mathcal{V}$*).*

Proof of Corollary 2.15. By Proposition 2.12, (2.16) and Proposition 2.14, if the two Green bundles coincide, the paratingent cone $C_\theta := C_\theta\left(\tilde{\mathcal{A}}(H)\right)$ is a line which is transverse to the vertical subspace V_θ. Then, working in a symplectic set of local coordinates, by Lemma 2.13 we deduce that there are an open neighborhood $\mathcal{U}$ of x, and a Lipschitz 1-form Ψ on $\mathcal{U}$ which is C^1 at x, such that

$$\tilde{\mathcal{A}}(H) \cap T^*\mathcal{U} \subset \mathrm{Graph}(\Psi),$$

and the Lagrangian plane C_θ coincides with the graph of $d_x\Psi$. Since $\frac{\partial H}{\partial p}(\theta) \neq 0$ (because $X_H(\theta) \neq 0$ and the Aubry set is a Lipschitz graph), the set of $\theta' \in T^*M$ with $H(\theta') = \mathbf{c}[H]$ is locally (in a neighborhood of θ) a submanifold of dimension 3 of class C^2. Then up to compose Ψ with a retraction r of class at least C^1 onto the set $\{H = \mathbf{c}[H]\}$, we may assume that Ψ is a Lipschitz 1-form satisfying

$$H\big(\Psi(x)\big) = \mathbf{c}[H] \qquad \forall x \in \mathcal{U}.$$

Let $S \subset \mathcal{U}$ be a local section (that is, a smooth curve) which is transverse to γ at x. By the properties of Ψ, the map $\Phi : [-2T, 2T] \times S \to M$ defined by

$$\Phi(t, y) := \pi^* \left(\phi_t^H \big(\Psi(y) \big) \right) \qquad \forall\, t \in [-2T, 2T], \forall\, y \in S,$$

is Lipschitz, and it is C^1 along the segment $[-T, T] \times \{x\}$. Moreover, since $C_\theta = G_\theta^- = G_\theta^+$, the differential of Φ is invertible at $(t, 0)$ for every $t \in [-T, T]$. Therefore, by the Clarke Lipschitz Inverse Function Theorem (see [12, Theorem 5.1.1]), Φ admits a Lipschitz inverse $\Phi^{-1} = (\tau, \epsilon) : \mathcal{V} \to [-2T, 2T] \times S$ in a simply connected neighborhood $\mathcal{V}$ of $\gamma([-T, T])$ (remember that γ is not periodic) which is C^1 along $\gamma([-T, T])$. By construction, the 1-form α on $\mathcal{V}$ defined by

$$\alpha(x) := \phi_{\tau(x)}^H \big(\Psi(\epsilon(x)) \big) \qquad \forall\, x \in \mathcal{V},$$

is a closed Lipschitz 1-form which is C^1 along the curve $\gamma([-T, T])$. By the Poincaré lemma, we get a function satisfying the conclusions of Corollary 2.15.

We notice that an alternative way to perform the above construction is to approach Ψ by a sequence of 1-form of class C^1, to construct a sequence of functions of class C^2 by the method of characteristics (see [18]) and to get the $C^{1,1}$ function f by taking the limit. Such an approach can be found in [17].

□

2.5 Hessians and positive Green bundles

As shown by Alexandrov (see for instance [16, 39]), locally semiconcave functions are two times differentiable almost everywhere.

Theorem 2.16. *Let U be an open subset of $\mathbb{R}^n$ and $u : U \to \mathbb{R}$ be a function which is locally semiconcave on U. Then, for a.e. $x \in U$, u is differentiable at x and there exists a symmetric operator $A(x) : \mathbb{R}^n \to \mathbb{R}^n$ such that the following property is satisfied:*

$$\lim_{t \downarrow 0} \frac{u(x + tv) - u(x) - t du(x) \cdot v - \frac{t^2}{2} \langle A(x) \cdot v, v \rangle}{t^2} = 0 \qquad \forall v \in \mathbb{R}^n.$$

Moreover, $x \mapsto du(x)$ is differentiable a.e. in U (that is for a.e. $x \in U$, any section of $z \mapsto D_z^* u$ is differentiable at x), and its differential is given by $A(x)$.

We infer that if $u : M \to \mathbb{R}$ is semiconcave then, for almost every $x \in M$, u is differentiable at x, $D_x^* u$ is a singleton, du is differentiable at x and the graph of its differential is a Lagrangian subspace $D_x^2 u \subset T_{(x, du(x))}(T^*M)$. Notice that if $u : M \to \mathbb{R}$ is a critical solution, then by Proposition 2.4 regularity properties of u propagate in negative time. That is, for every $x \in M$ such that u is two times differentiable at x, the function u is two times differentiable along the semi-calibrated curve $\gamma_x : (-\infty, 0] \to M$ given by (2.6). Moreover we have

$$D_{(x, du(x))} \phi_{-t}^H \left(D_x^2 u \right) = D_{\gamma_x(-t)}^2 u \qquad \forall\, t \geq 0. \tag{2.18}$$

Recall that for every $\theta = (x, p) \in D^* u$, the Hamiltonian trajectory starting at θ at time zero has no conjugate points in negative times (see Proposition 2.4), which allows us to construct G_θ^+ at any such points. Then, proceeding as in the proof of Proposition 2.14 (replacing (2.15) by (2.18)) we obtain the following one-sided estimate (notice that, since $D_x^2 u$ is a Lagrangian subspace, the assumption on the dimension of M could be dropped, see the proof of Proposition 2.14 and [4, Proposition 3.11]):

Proposition 2.17. *Assume that* $\dim M = 2$*, let $u : M \to \mathbb{R}$ be a critical solution, and let $x \in M$ be such that $du(x)$ and $D^2 u(x)$ exist and $X_H(x, du(x)) \neq 0$. Then*

$$D_x^2 u \preceq G_{(x, du(x))}^+.$$

Later on, in the proof of Theorem 1.1, the above result together with the upper semicontinuity of the positive Green bundle will allow us to obtain a local bound from above for D^2u in a neighborhood of a given point of the projected Aubry set (see (3.13)).

2.6 Reminders on hyperbolicity

We recall here basic facts in hyperbolic dynamics, we refer the reader to the Katok-Hasselblatt monograph [26] for further details. Recall that ϕ_t^H denotes the Hamiltonian flow in T^*M. A compact ϕ_t^H-invariant set $\Lambda \subset T^*M$ is called *hyperbolic* in its energy level Σ with respect to the Hamiltonian flow if the following properties are satisfied:

(h1) At each point $\theta \in \Lambda$, the tangent space $T_\theta \Sigma$ is the direct sum of three subspaces E_θ^s, E_θ^u, and $E_\theta^0 = \mathbb{R} X_H(\theta)$.

(h2) At each point $\theta \in \Lambda$, we have $D_\theta \phi_t^H \left(E_\theta^s\right) = E^s_{\phi_t^H(\theta)}$ and $D_\theta \phi_t^H \left(E_\theta^u\right) = E^u_{\phi_t^H(\theta)}$, for any $t \in \mathbb{R}$.

(h3) There are a Riemannian metric in an open neighborhood of Λ, and constants $C \geq 1$ and $\mu > 0$ such that, for each $\theta \in \Lambda$, $\psi^s \in E_\theta^s$, and $\psi^u \in E_\theta^u$, we have

$$\begin{aligned} \left\| D_\theta \phi_t^H(\psi^s) \right\| &\leq Ce^{-\mu t} \|\psi^s\|, \\ \left\| D_\theta \phi_{-t}^H(\psi^u) \right\| &\leq Ce^{-\mu t} \|\psi^u\|, \end{aligned}$$

for all $t > 0$.

Note that, as a consequence of hyperbolicity, the splitting $T_\theta \Sigma = E_\theta^s \oplus E_\theta^u \oplus E_\theta^0$ defined for $\theta \in \Lambda$ is continuous. Let us extend it into a continuous (not necessarily invariant) splitting $T_\theta \Sigma_\theta = E_\theta^s \oplus E_\theta^u \oplus E_\theta^0$ with $E_\theta^0 = \mathbb{R} X_H(\theta)$ defined for all θ in an open neighborhood $\mathcal{V}$ of Λ. Then, for every $\theta \in \mathcal{V}$ and any $\rho \in (0,1)$, we define the family of horizontal and vertical cones $\{H_\theta^\rho\}$ and $\{V_\theta^\rho\}$ as

$$\begin{aligned} H_\theta^\rho &:= \Big\{ \xi + \eta \,|\, \xi \in E_\theta^u, \, \eta \in E_\theta^s, \, \|\eta\| \leq \rho \|\xi\| \Big\}, \\ V_\theta^\rho &:= \Big\{ \xi + \eta \,|\, \xi \in E_\theta^u, \, \eta \in E_\theta^s, \, \|\xi\| \leq \rho \|\eta\| \Big\}. \end{aligned}$$

By (h2)-(h3), for every $\theta \in \Lambda$, $\rho \in (0,1)$, and $t > 0$, we have

$$D_\theta \phi_t^H \left(H_\theta^\rho\right) \subset H^{e^{-2\mu t} C^2 \rho}_{\phi_t^H(\theta)}, \quad D_\theta \phi_{-t}^H \left(V_\theta^\rho\right) \subset V^{e^{-2\mu t} C^2 \rho}_{\phi_{-t}^H(\theta)},$$

$$\left\| D_\theta \phi_t^H(\psi) \right\| \geq \frac{e^{\mu t}}{C'(1+\rho)} \|\psi\| \qquad \forall \, \psi \in H_\theta^\rho,$$

$$\left\| D_\theta \phi_{-t}^H(\psi) \right\| \geq \frac{e^{\mu t}}{C'(1+\rho)} \|\psi\| \qquad \forall \, \psi \in V_\theta^\rho.$$

Hence, by continuity and compactness, we can find $T > 0$, $\mu' > 0$, an open neighborhood $\mathcal{V}' \subset \mathcal{V}$, and continuous disjoint cones $\mathbb{S}_\theta$, $\mathbb{U}_\theta \subset E_\theta^s \oplus E_\theta^u$ containing E_θ^s, E_θ^u respectively, such that, for every $\theta \in \mathcal{V}'$,

$$D_\theta \phi_T^H \big(\mathbb{U}_\theta\big) \subset \mathrm{Int} \left(\mathbb{U}_{\phi_T^H(\theta)} \right), \quad D_\theta \phi_{-T}^H \big(\mathbb{S}_\theta\big) \subset \mathrm{Int} \left(\mathbb{S}_{\phi_{-T}^H(\theta)} \right),$$

and

$$\begin{aligned} \left\| D_\theta \phi_T^H(\psi) \right\| &\geq e^{\mu' T} \|\psi\| \qquad \forall \, \psi \in \mathbb{U}_\theta, \\ \left\| D_\theta \phi_{-T}^H(\psi) \right\| &\geq e^{\mu' T} \|\psi\| \qquad \forall \, \psi \in \mathbb{S}_\theta. \end{aligned}$$

This shows that any ϕ_t^H-invariant compact set sufficiently close to Λ will satisfy the Alekseev cone criterion, which provides an alternative more handy characterization for hyperbolicity (see [26]). This criterion is also robust under perturbation of the dynamics, and allows us to obtain that following:

Proposition 2.18. *Let $\Lambda \subset T^*M$ be a compact ϕ_t^H-invariant set which is hyperbolic in its energy level with respect to the Hamiltonian flow. Then there exists an open neighborhood $\mathcal{V}$ of 0 in $C^2(M)$ and an open neighborhood $\mathcal{O}$ of Λ such that, for every potential $V \in \mathcal{V}$, any compact set $\Lambda' \subset \mathcal{O}$ which is ϕ_t^{H+V}-invariant is hyperbolic in its energy level with respect to ϕ_t^{H+V}.*

The above result will be useful to show the stability part (that is, openness) of Theorem 1.1. As shown in [28, 14], a way to obtain hyperbolicity is to show quasi-hyperbolicity properties. Let B be a compact metric space and $\pi : E \to B$ a vector bundle equipped with a continuous norm $|\cdot|_p$ on each fiber $\pi^{-1}(p)$. Let Ψ be a continuous $\mathbb{R}$-action $\Psi_t : \mathbb{R} \to \mathrm{Isom}(E)$ such that $\Psi_{s+t} = \Psi_s \circ \Psi_t$. We say that Ψ is *quasi-hyperbolic* if

$$\sup_{t\in\mathbb{R}} \Big\{ |\Psi_t(\xi)| \Big\} = +\infty \qquad \forall \xi \in E \setminus \{0\}.$$

The following result holds (see [14, Theorem 0.2], and compare with [5, §3] and [6, Theorem 2] where it is shown that the non-wandering assumption can indeed be dropped):

Proposition 2.19. *Assume that any point in B is non-wandering and that Ψ is quasi-hyperbolic. Then Ψ is hyperbolic.*

In the proof of Theorem 1.1, the above result allows us to obtain the hyperbolicity of the Aubry set in the case when the Green bundles are always transverse. Such an approach is nowadays classical.

2.7 Some properties of semiconcave and BV functions

2.7.1 Derivatives of semiconcave functions

Let $v : \mathbb{R}^n \to \mathbb{R}$ be a semiconcave function, i.e., v can be written as the sum of a concave function and a smooth function. Since second distributional derivatives of convex functions are nonnegative Radon measures (see [16, §6.3]), the Radon-Nikodým Theorem [2, Theorem 1.28] allows us to write D^2v as the sum of an absolutely continuous matrix-valued measure and a singular matrix-valued measure:

$$D^2 v = \nabla^2 v \, dx + D^2_{\mathcal{S}} v,$$

where $\nabla^2 v \in L^1_{loc}$ is the pointwise Hessian of v (which exists almost everywhere by Alexandrov's Theorem), and $D^2_{\mathcal{S}} v$ is a singular measure (with respect to the Lebesgue measure). Also, by semiconcavity we have that $D^2 v$ is locally bounded from above (as a measure): for any $R > 0$ there exists a constant $C_R > 0$ such that

$$\int_E \langle D^2 v \cdot e, e \rangle \leq C_R |E| \qquad \forall E \subset B_R \text{ Borel}, \forall e \in \mathbb{R}^n \text{ with } |e| = 1,$$

where $B_R := B^n(0, R)$ denotes the n-dimensional ball of radius R centered at the origin. In particular, choosing E of measure zero we get

$$\int_E \langle D^2_{\mathcal{S}} v \cdot e, e \rangle \leq 0 \qquad \forall E \subset \mathbb{R}^n \text{ Borel with } |E| = 0, \forall e \in \mathbb{R}^n \text{ with } |e| = 1. \tag{2.19}$$

Hence, since the measure $D^2_{\mathcal{S}} v$ is singular with respect to the Lebesgue measure, by the arbitrariness of E we deduce that $\langle D^2_{\mathcal{S}} v \cdot e, e \rangle$ is a negative singular measure for any vector $e \in \mathbb{R}^n$.

Since the distributional derivative of ∇v is equal to the measure $D^2 v$, by definition $\nabla v : \mathbb{R}^n \to \mathbb{R}^n$ is a function of bounded variation (see [2, §3]). Given $x' \in \mathbb{R}^{n-1}$, let us consider the function $w_{x'} : \mathbb{R} \to \mathbb{R}^n$ defined by

$$w_{x'}(s) := \nabla v(x', s) \qquad \text{for a.e. } s \in \mathbb{R}. \tag{2.20}$$

Note that, since v is differentiable almost everywhere, by Fubini's theorem the function $w_{x'}$ is defined for almost every $x' \in \mathbb{R}^{n-1}$. It is well-known that the functions $w_{x'}$ are of bounded variation on $\mathbb{R}$ for almost every $x' \in \mathbb{R}^{n-1}$ as well (see [2, Theorem 3.103] and the subsequent discussion), so their distributional derivative on $\mathbb{R}$ is a measure which can be decomposed as the sum of an absolutely continuous and a singular part:

$$Dw_{x'} = \nabla w_{x'}\, ds + D_{\mathcal{S}} w_{x'},$$

where here D is the distributional derivative on $\mathbb{R}$, $\nabla w_{x'} \in L^1_{loc}$ is the pointwise derivative of $w_{x'}$ which exists almost everywhere [2, Theorem 3.28(c)], and $D_{\mathcal{S}} w_{x'}$ is singular with respect to the one-dimensional Lebesgue measure. Also, the fundamental theorem of calculus holds between every couple of points where ∇v exists [2, Theorem 3.28]:

$$\nabla v(x', s_2) - \nabla v(x', s_1) = w_{x'}(s_2) - w_{x'}(s_1) = \int_{s_1}^{s_2} \nabla w_{x'}(s)\, ds + \int_{s_1}^{s_2} d\big(D_{\mathcal{S}} w_{x'}\big)(s), \qquad (2.21)$$

for every $s_1 < s_2$ in $\mathbb{R}$.

Let us recall that, given a vector-valued measure μ, one denotes by $|\mu|$ its total variation, which is defined as

$$|\mu|(E) := \sup\left\{ \sum_{h=0}^{\infty} |\mu(E_h)| \,:\, E_h \text{ disjoint Borel sets s.t. } E = \bigcup_{h=0}^{\infty} E_h \right\} \qquad \forall\, E \text{ Borel}.$$

It is easy to check that, with this definition, $\left|\int_E d\mu\right| \le \int_E d|\mu|$. Hence, it follows from (2.21) that

$$\big|\nabla v(x', s_2) - \nabla v(x', s_1)\big| \le \int_{s_1}^{s_2} \big|\nabla w_{x'}(s)\big|\, ds + \int_{s_1}^{s_2} d\big|D_{\mathcal{S}} w_{x'}\big|(s).$$

Finally, we recall that the derivative of $w_{x'}$ is related to $D^2 v$: if we define the family of lines $\ell_{x'} := \{(x', s) \,:\, s \in \mathbb{R}\}$, it follows from [2, Theorem 3.107] that

$$\int_E \langle D^2 v \cdot e_n, e\rangle = \int_{\mathbb{R}^{n-1}} dx' \int_{E\cap \ell_{x'}} Dw_{x'} \cdot e \qquad \forall\, E \subset \mathbb{R}^n \text{ Borel}, \forall\, e \in \mathbb{R}^n. \qquad (2.22)$$

This has the following useful consequences: since the measures

$$\langle D^2_{\mathcal{S}} v \cdot e_n, e\rangle - D_{\mathcal{S}} w_{x'} \cdot e \quad \text{and} \quad \big(\langle \nabla^2 v \cdot e_n, e\rangle - \nabla_{\mathcal{S}} w_{x'} \cdot e\big)\, dx$$

are mutually singular, we deduce that (2.22) holds with $\nabla^2 v \cdot e_n$ and $\nabla w_{x'}$ (resp., with $D^2_{\mathcal{S}} v \cdot e_n$ and $D_{\mathcal{S}} w_{x'}$) in place of $D^2 v \cdot e_n$ and $Dw_{x'}$. Hence

$$\nabla w_{x'}(s) = \nabla^2 v(x', s) \cdot e_n \qquad \text{for a.e. } (x', s) \in \mathbb{R}^n, \qquad (2.23)$$

$$\int_E \langle D^2_{\mathcal{S}} v \cdot e_n, e\rangle = \int_{\mathbb{R}^{n-1}} dx' \int_{E\cap \ell_{x'}} D_{\mathcal{S}} w_{x'} \cdot e \qquad \forall\, E \subset \mathbb{R}^n \text{ Borel}, \forall\, e \in \mathbb{R}^n. \qquad (2.24)$$

(see also [2, Theorem 3.107]). In particular, (2.19) and (2.24) imply that

$$D_{\mathcal{S}} w_{x'} \cdot e_n \quad \text{is a negative measure for a.e. } x' \in \mathbb{R}^{n-1}. \qquad (2.25)$$

2.7.2 The case of a critical solution

We now gather some extra properties when $v = u$ solves the Hamilton-Jacobi equation. Let us assume that $u : B_2 = B^n(0,2) \to \mathbb{R}$ is a semiconcave function satisfying

$$H(x, \nabla u(x)) = \mathbf{c}[H] \qquad \text{for a.e. } x \in B_2. \qquad (2.26)$$

Then $w := \nabla u$ is a function of bounded variation, and since semiconcave functions are locally Lipschitz, w is locally bounded inside B_2. Let us consider the family of bounded Borel functions $a_h : B_1 = B^n(0,1) \to \mathbb{R}$, $h \in (0,1/2)$, defined as

$$a_h(x) := \int_0^1 \frac{\partial H}{\partial x_n}\big(x + \tau h e_n, \tau \nabla u(x+he_n) + (1-\tau)\nabla u(x)\big)\, d\tau$$

and the family of bounded Borel vector fields $\xi_h : B_1 \to \mathbb{R}^n$, $h \in (0,1/2)$, given by

$$\xi_h(x) := \int_0^1 \frac{\partial H}{\partial p}\big(x + \tau h e_n, \tau \nabla u(x+he_n) + (1-\tau)\nabla u(x)\big)\, d\tau.$$

Let us recall that, since $w \in BV_{loc}(B_2)$, the following bound holds:

$$\int_{B_r} \frac{|w(x+he_n) - w(x)|}{h}\, dx \le \int_{B_{r+h}} |Dw \cdot e_n|(dx) < \infty \qquad \forall\, r \in (0,1),\, h \in (0,1/2) \tag{2.27}$$

(for smooth functions the above estimate follows from the fundamental theorem of calculus, and for the general case one argues by approximation using [2, Theorem 3.9]). Hence the measures

$$\mu_h := \frac{w(x+he_n) - w(x)}{h}\, dx$$

satisfy

$$\int_{B_1} |\mu_h|(dx) \le C,$$

which implies that, up to a subsequence, μ_h (resp. $|\mu_h|$) converge weakly* to a finite measure μ (resp. ν) as $h \to 0$. Also, there exists $\bar a : B_1 \to [0,+\infty)$ bounded such that $|a_h| \rightharpoonup^* \bar a$ in $L^\infty(B_1)$.

It is easy to show that $\mu = Dw \cdot e_n = D^2 u \cdot e_n$. Furthermore, it follows from [2, Example 1.63] and (2.27) that

$$\nu(B_r) \le \liminf_{h\to 0} |\mu_h|(B_r) \le \int_{\overline{B}_r} |Dw \cdot e_n| = |\mu|(\overline{B}_r) \qquad \forall\, r \in (0,1),$$

so letting $r \nearrow 1$ we obtain $\nu(B_1) \le |\mu|(B_1)$. This information combined with the bound $|\mu| \le \nu$ (see [2, Proposition 1.62(b)]) implies that $|\mu| = \nu$, thus

$$|\mu_h| \rightharpoonup^* |\mu|. \tag{2.28}$$

We now exploit the fact that u solves the Hamilton-Jacobi equation (2.26). Since

$$\begin{aligned}
0 &= H(x+he_n, \nabla u(x+he_n)) - H(x, \nabla u(x)) \\
&= h \int_0^1 \frac{\partial H}{\partial x_n}\big(x + \tau h e_n, \tau \nabla u(x+he_n) + (1-\tau)\nabla u(x)\big)\, d\tau \\
&\quad + \bigg(\int_0^1 \frac{\partial H}{\partial p}\big(x + \tau h e_n, \tau \nabla u(x+he_n) + (1-\tau)\nabla u(x)\big)\, d\tau\bigg) \cdot \big(\nabla u(x+he_n) - \nabla u(x)\big) \\
&= h\, a_h(x) + \xi_h(x) \cdot \big(w(x+he_n) - w(x)\big),
\end{aligned}$$

we have

$$a_h + \xi_h \cdot \mu_h \equiv 0.$$

Let $\Omega \subset B_1$ be an open set and assume that there exist $h_0 > 0$ and a continuous vector field $\Xi : \Omega \to \mathbb{R}^n$ such that

$$|\Xi(x) - \xi_h(x)| \le \frac{1}{2}|\Xi(x)| \qquad \forall\, x \in \Omega,\, \forall\, h \in (0,h_0). \tag{2.29}$$

Then

$$0 = a_h + \xi_h \cdot \mu_h = a_h + (\xi_h - \Xi) \cdot \mu_h + \Xi \cdot \mu_h,$$

so that, thanks to [2, Proposition 1.62(b)] and (2.28)-(2.29), letting $h \to 0$ we obtain

$$|\Xi \cdot \mu| \leq \liminf_{h\to 0} |\Xi \cdot \mu_h| \leq \liminf_{h\to 0} \frac{1}{2}|\Xi|\,|\mu_h| + |a_h| = \frac{1}{2}|\Xi|\,|\mu| + \bar{a} \qquad \text{inside } \Omega,$$

where $|\Xi \cdot \mu|$ denotes the total-variation of the measure $\Xi \cdot \mu$ (and analogously for μ_h), and $|\Xi|$ denotes the continuous function $x \mapsto |\Xi(x)|$.

We now recall that, as observed above, the measure μ coincides with the measure $D^2u \cdot e_n$, hence

$$|\langle D^2u \cdot e_n, \Xi\rangle| \leq \frac{1}{2}|\Xi|\,|D^2u \cdot e_n| + \bar{a} \qquad \text{inside } \Omega.$$

In particular, if we restrict this inequality to the singular part of D^2u, since $\bar{a}$ is a bounded function we get

$$|\langle D^2_{\mathcal{S}}u \cdot e_n, \Xi\rangle| \leq \frac{1}{2}|\Xi|\,|D^2_{\mathcal{S}}u \cdot e_n| \qquad \text{inside } \Omega,$$

which by (2.22) can be written as a superposition of the measures $Dw_{x'}$:

$$|\Xi \cdot D_{\mathcal{S}}w_{x'}| \leq \frac{1}{2}|\Xi|\,|D_{\mathcal{S}}w_{x'}| \qquad \text{inside } \Omega, \text{ for a.e. } x'.$$

Using the polar decomposition theorem [2, Corollary 1.29], we can write $D_{\mathcal{S}}w_{x'} = \theta|D_{\mathcal{S}}w_{x'}|$, where $\theta : B_1 \to \mathbb{S}^{n-1}$ is a $|D_{\mathcal{S}}w_{x'}| \otimes dx'$-measurable function. Hence the above equation can be rewritten as

$$|\Xi \cdot \theta| \leq \frac{1}{2}|\Xi| \qquad \text{inside } \Omega,\ |D_{\mathcal{S}}w_{x'}| \otimes dx'\text{-a.e..} \tag{2.30}$$

This information is particularly useful when $n = 2$ and Ξ never vanishes: indeed, assuming for instance that $\Xi \equiv e_1$, then (2.30) implies that

$$|\theta| \leq 2|\theta \cdot e_2| \qquad \text{inside } \Omega,\ |D_{\mathcal{S}}w_{x'}| \otimes dx'\text{-a.e.,}$$

from which we get

$$|D_{\mathcal{S}}w_{x'}| \leq 2|D_{\mathcal{S}}w_{x'} \cdot e_2| \qquad \text{inside } \Omega, \text{ for a.e. } x'. \tag{2.31}$$

This means that $|D^2_{\mathcal{S}}u \cdot e_2|$ is controlled by $|\langle D^2_{\mathcal{S}}u \cdot e_2, e_2\rangle|$, or equivalently, since $D^2_{\mathcal{S}}u \cdot e_2$ is a vector-valued measure of components $\langle D^2_{\mathcal{S}}u \cdot e_2, e_1\rangle$ and $\langle D^2_{\mathcal{S}}u \cdot e_2, e_2\rangle$, the measure $|\langle D^2_{\mathcal{S}}u \cdot e_2, e_1\rangle|$ is controlled by $|\langle D^2_{\mathcal{S}}u \cdot e_2, e_2\rangle|$. Hence, the size of the pure second derivatives in the e_2 direction controls the size of the mixed second derivatives in e_1, e_2 in the region where the Hessian is singular (that is, roughly speaking, where ∇u has a jump).

2.8 A lemma from harmonic analysis

In this section we recall a classical result from harmonic analysis (see [38]), and we show its simple proof for the convenience of the reader. We denote by $|A|$ the Lebesgue measure of a set $A \subset \mathbb{R}^n$.

Lemma 2.20. *Let $f \in L^1(\mathbb{R}^n)$, and define the maximal function*

$$Mf(x) := \sup_{x\in\overline{B},\, B \text{ open ball}} \left\{ \frac{1}{|B|}\int_B |f(y)|\,dy \right\} \qquad \forall\, x \in \mathbb{R}^n.$$

There exists a dimensional constant $C_n > 0$ such that

$$\left|\left\{x \in \mathbb{R}^n : Mf(x) > \delta\right\}\right| \leq \frac{C_n}{\delta}\|f\|_{L^1(\mathbb{R}^n)} \qquad \forall\, \delta > 0.$$

Proof of Lemma 2.20. Let $K \subset \{Mf > \delta\}$ be any compact subset. By the definition of Mf, for any $x \in K$ there exists an open ball B_x such that

$$x \in \overline{B}_x, \qquad |B_x| \leq \frac{1}{\delta} \int_{B_x} |f(y)|\, dy.$$

Let ρB denote the dilation of a ball B by a factor $\rho > 0$ with respect to its center. Since $x \in \overline{B}_x \subset 2B_x$, the family of open balls $\{2B_x\}_{x \in K}$ covers K. So, by compactness we can find a finite collection of these balls which still covers K, and by Vitali's Lemma [16, §1.5.1, Theorem 1] we can select a disjoint subcollection $\{2B_{x_1}, \ldots, 2B_{x_m}\}$ such that $K \subset \cup_{j=1}^m 10B_{x_j}$. Hence

$$|K| \leq 10^n \sum_{j=1}^m |B_{x_j}| \leq \frac{10^n}{\delta} \sum_{j=1}^m \int_{B_{r_j}(x_j)} |f(y)|\, dy \leq \frac{10^n}{\delta} \|f\|_{L^1(\mathbb{R}^n)},$$

and the result follows by the arbitrariness of K. □

3 Proof of Theorem 1.1

Let $H : T^*M \to \mathbb{R}$ be a Tonelli Hamiltonian of class C^2, and denote by $L : TM \to \mathbb{R}$ its associated Lagrangian. We want to show that the set of potentials $V \in C^2(M)$ such that the Aubry set $\tilde{\mathcal{A}}(H+V)$ is hyperbolic contains an open dense set. Hence we need to prove a stability result (the openness) and a density result.

We proceed as follows: First, in Section 3.1 we show that if the Aubry set $\tilde{\mathcal{A}}(H)$ is minimal and hyperbolic, then all Aubry sets $\tilde{\mathcal{A}}(H+V)$ associated with potentials $V \in C^2(M)$ which are sufficiently small in C^2 topology are hyperbolic. Then, in Section 3.2 we show that the set of potentials $V \in C^2(M)$ such that the Aubry set of $H+V$ is minimal and hyperbolic is dense. We recall that a nonempty compact ϕ_t^H-invariant set $\Lambda \subset T^*M$ is called minimal if any orbit of ϕ_t^H contained in Λ is dense inside Λ. By Zorn's Lemma, any nonempty compact ϕ_t^H-invariant set contains a minimal subset.

3.1 The stability part

Recall that the Peierls barrier is the function $h : M \times M \to \mathbb{R}$ defined as

$$h(x,y) := \liminf_{t \to +\infty} \Big\{ h_t(x,y) + \mathbf{c}[H]t \Big\} \qquad \forall\, x, y \in M, \tag{3.1}$$

where

$$h_t(x,y) := \inf \int_0^t L\big(\gamma(s), \dot{\gamma}(s)\big)\, ds \tag{3.2}$$

and the infimum is taken over all Lipschitz curves $\gamma : [0,t] \to M$ such that $\gamma(0) = x$ and $\gamma(t) = y$ (we refer the reader to [18, 20, 36] for further details). By construction h is Lipschitz on $M \times M$ (see for instance [18, Corollary 5.3.3]) and any critical subsolution u satisfies

$$u(y) - u(x) \leq h(x,y) \qquad \forall\, x, y \in M \tag{3.3}$$

(this fact follows easily from Proposition 2.1). Moreover, it can be checked that (see [18, Proposition 5.3.8], [20, 36])

$$\mathcal{A}(H) = \Big\{ x \in M \,|\, h(x,x) = 0 \Big\}. \tag{3.4}$$

Following Mather [32], the function $\delta_M : M \times M \to \mathbb{R}$ given by

$$\delta_M(x,y) := h(x,y) + h(y,x) \qquad \forall\, x, y \in M$$

is a semi-distance (sometimes called the Mather semi-distance).

Lemma 3.1. *Assume that $\tilde{\mathcal{A}}(H)$ is minimal. Then H admits a unique weak KAM solution (up to a constant) and $\delta_M(x,y)=0$ for any $x,y\in\mathcal{A}(H)$.*

Proof of Lemma 3.1. Let $u_1,u_2:M\to\mathbb{R}$ be two weak KAM solutions. Since their differentials coincide along any orbit of the Aubry set (see Proposition 2.5) and in addition all the orbits are dense in $\mathcal{A}(H)$, there is a constant $a\in\mathbb{R}$ such that $u_1-u_2=a$ on $\mathcal{A}(H)$. By Fathi's comparison theorem (see [18, Theorem 8.5.5]), we infer that u_1 and u_2 differ by a constant on the whole M. The second assertion follows from the fact that the pointed functions $\{h(z,\cdot)\}_{z\in M}$ are weak KAM solutions (see [18, Theorem 5.3.6] or [20, Proposition 4.1]) and from the equality (using (3.4))

$$\delta_M(x,y)=\Big(h(x,y)-h(x,x)\Big)-\Big(h(y,y)-h(y,x)\Big)\qquad\forall\,x,y\in\mathcal{A}(H).$$

□

As shown in [15, Theorem C], by the uniqueness of weak KAM solutions (or equivalently the uniqueness of static classes) one obtains the upper-semicontinuity of the mapping $V\mapsto\tilde{\mathcal{A}}(H+V)$ at $V=0$ (compare with [9, corollary 5]), from which the stability of the hyperbolicity of Aubry sets follows:

Lemma 3.2. *Assume that $\tilde{\mathcal{A}}(H)$ is minimal and hyperbolic. Then there is an open neighborhood $\mathcal{V}$ of 0 in $C^2(M)$ such that, for every $V\in\mathcal{V}$, $\tilde{\mathcal{A}}(H+V)$ is hyperbolic.*

Proof of Lemma 3.2. We first show that, since H admits a unique weak KAM solution (which follows from the previous lemma), the mapping $V\in C^2(M)\mapsto\tilde{\mathcal{A}}(H+V)\subset T^*M$ is upper semicontinuous with respect to the Hausdorff topology, that is, for every open set $\mathcal{O}\in T^*M$ containing $\tilde{\mathcal{A}}(H)$ there is an open neighborhood $\mathcal{V}$ of 0 in $C^2(M)$ such that, for every $V\in C^2(M)$,

$$V\in\mathcal{V}\implies\tilde{\mathcal{A}}(H+V)\subset\mathcal{O}.$$

Without loss of generality, up to adding a constant to H we can assume that $\mathbf{c}[H]=0$.

We argue by contradiction and assume that there are an open neighborhood $\mathcal{O}$ of $\tilde{\mathcal{A}}(H)$, a sequence of potentials $\{V_k\}_k$ which tends to zero in the C^2 topology, and a sequence $\{\theta_k\}_k\subset T^*M$ satisfying $\theta_k\in\tilde{\mathcal{A}}(H+V_k)\setminus\mathcal{O}$ for all k. For every k, we pick a critical solution u_k for the Hamiltonian $H+V_k$, and we define the calibrated curves $\gamma_k(t):=\pi^*\big(\phi_t^{H+V_k}(\theta_k)\big)$. Because critical solutions (resp. calibrated curves) are uniformly Lipschitz (resp. $C^{1,1}$) [18], taking subsequences if necessary, we may assume that $\{u_k\}_k$ converge uniformly to a weak KAM solution u for H, and $\{\gamma_k\}_k$ converge in C^1 topology to a calibrated (with respect to u) curve $\gamma:\mathbb{R}\to M$ with $\gamma(0)\notin\mathcal{A}(H)$, that is,

$$u\big(\gamma(b)\big)-u\big(\gamma(a)\big)=\int_a^b L\big(\gamma(s),\dot\gamma(s)\big)\,ds=h_{b-a}(\gamma(a),\gamma(b))\qquad\forall\,a<b.\tag{3.5}$$

It can be shown that ω-limit and α-limit sets of any calibrated curves are contained in the Aubry set (see [36, Proposition 4.1]). Hence, there is a sequence $\{T_l\}_l\uparrow+\infty$ such that $\gamma(T_l)$ and $\gamma(-T_l)$ tend to $\mathcal{A}(H)$ as l tends to $+\infty$. Let us denote by d a Riemannian distance on M, and by K a Lipschitz constant for h.

Given $\eta>0$ we choose l large enough and $\alpha_l,\beta_l\in\mathcal{A}(H)$ such that

$$d\big(\gamma(-T_l),\alpha_l\big)+d\big(\gamma(T_l),\beta_l\big)<\eta.$$

Set $x:=\gamma(0)$. Then, using the definition of h (3.1), the fact that $h_{t+s}(x,y)\le h_t(x,z)+h_s(z,y)$, (3.5), and that $\delta_M(\alpha_l,\beta_l)=0$ (which follows from Lemma 3.4), we get

$$\begin{aligned}
h(x,x) &\leq h_{T_l}\big(x,\gamma(T_l)\big)+h\big(\gamma(T_l),\gamma(-T_l)\big)+h_{T_l}\big(\gamma(-T_l),x\big)\\
&= h_{T_l}\big(\gamma(-T_l),x\big)+h_{T_l}\big(x,\gamma(T_l)\big)+h\big(\beta_l,\alpha_l\big)+h\big(\gamma(T_l),\gamma(-T_l)\big)-h\big(\beta_l,\alpha_l\big)\\
&\leq h_{T_l}\big(\gamma(-T_l),x\big)+h_{T_l}\big(x,\gamma(T_l)\big)+h\big(\beta_l,\alpha_l\big)+K\big[d\big(\gamma(T_l),\beta_l\big)+d\big(\gamma(-T_l),\alpha_l\big)\big]\\
&\leq h_{T_l}\big(\gamma(-T_l),x\big)+h_{T_l}\big(x,\gamma(T_l)\big)+h\big(\beta_l,\alpha_l\big)+K\eta\\
&= h_{T_l}\big(\gamma(-T_l),x\big)+h_{T_l}\big(x,\gamma(T_l)\big)-h\big(\alpha_l,\beta_l\big)+K\eta\\
&\leq h_{T_l}\big(\gamma(-T_l),x\big)+h_{T_l}\big(x,\gamma(T_l)\big)-h\big(\gamma(-T_l),\gamma(T_l)\big)+2K\eta\\
&= \big(u(x)-u\big(\gamma(-T_l)\big)\big)+\big(u\big(\gamma(T_l)\big)-u(x)\big)-h\big(\gamma(-T_l),\gamma(T_l)\big)+2K\eta\\
&\leq u\big(\gamma(T_l)\big)-u\big(\gamma(-T_l)\big)-h\big(\gamma(-T_l),\gamma(T_l)\big)+2K\eta\leq 2K\eta,
\end{aligned}$$

where for the last inequality we used (3.3). By the arbitrariness of η this shows that $h(x,x)=0$, which implies that x belongs to $\mathcal{A}(H)$, a contradiction. This proves the upper-semicontinuity of the Aubry set, and the conclusion follows easily from Proposition 2.18. □

Thanks to Lemma 3.2, it is now sufficient to show a density result, that is, given a Tonelli Hamiltonian H of class C^2 and $\epsilon>0$, there is $V\in C^2(M)$ with $\|V\|_{C^2(M)}<\epsilon$ such that the Aubry set of $H+V$ is minimal and hyperbolic.

3.2 The density part

Let us fix a C^2 Tonelli Hamiltonian H. First of all, up to adding a small potential (in the C^2 topology) we may assume that the Aubry set $\tilde{\mathcal{A}}(H)$ is minimal, i.e., all its orbits are dense in $\tilde{\mathcal{A}}(H)$ (see [21, §5.1] where we explain how to add a potential to reduce the size of the Aubry set). We can also assume that $\tilde{\mathcal{A}}(H)$ is not an equilibrium point or a periodic orbit, as otherwise we may add an arbitrarily small potential to make it hyperbolic (see [14, Theorem D] [2] and also [30]). Thus, the critical energy level

$$\Sigma := \Big\{\theta=(x,p)\in T^*M \,|\, H(x,p)=\mathbf{c}[H]\Big\}\subset T^*M$$

satisfies the assumptions of Section 2.4. Since we work on a surface, two cases may appear. Either the positive and negative Green bundles along $\tilde{\mathcal{A}}(H)$ satisfy

$$G_\theta^-\cap G_\theta^+=\mathbb{R}X_H(\theta)\qquad\forall\,\theta\in\tilde{\mathcal{A}}(H),\tag{3.6}$$

or

$$G_{\bar\theta}^+=G_{\bar\theta}^-\quad\text{ for some }\quad\bar\theta\in\tilde{\mathcal{A}}(H).\tag{3.7}$$

In the first case (when (3.6) holds), the hyperbolicity of $\tilde{\mathcal{A}}(H)$ follows from Proposition 2.19. Indeed, consider the projection Ψ_t of the differential of the Hamiltonian flow to the bundle

$$\mathbf{N}_\theta := \Big\{\xi\in T_\theta\Sigma\,|\,\langle D_\theta\pi^*(\xi),D_\theta\pi^*(X_H(\theta))\rangle_{\pi^*(\theta)}=0\Big\},$$

that is $\Psi_t:=\Lambda\circ D\phi_t^H|_{\mathbf{N}}$ where $\Lambda: T\Sigma\to\mathbf{N}$ is the projection along the direction of the X_H:

$$\Lambda\,\xi=\xi+\beta(\xi)\,X_H(\theta)\quad\text{ with }\beta(\xi)\in\mathbb{R}\text{ such that }\Lambda\,\xi\in\mathbf{N}_\theta.$$

Since the Green bundles are always transverse, the restriction of Ψ_t to $\tilde{\mathcal{A}}(H)$ is quasi-hyperbolic (cf. [14, Corollary 2.3(d)]). Therefore, since we are assuming that $\tilde{\mathcal{A}}(H)$ is minimal, Proposition 2.19 implies that Ψ_t is a hyperbolic action and then $\tilde{\mathcal{A}}(H)$ is a hyperbolic set.

[2] Notice that Contreras and Iturriaga require the Hamiltonian to be at least of class C^3, but the proof of their Theorem D works under C^2 regularity.

In the second case (when (3.7) holds), the results in Section 2.4 show that critical solutions restricted to the Aubry sets are C^2 at $x = \pi^*(\theta)$. As we will show below, this property allows us to implement the techniques developed in [21, 22] to close the orbit of $\bar{\theta}$ into a periodic orbit. However, the construction of a critical subsolution for the new Hamiltonian (which is unavoidable to close the orbit into an genuine Aubry set) becomes much more difficult than in [21, 22] because of the lack of regularity of critical solutions in a neighborhood of the orbit passing through x (in [21, 22], the authors had to assume extra regularity on a critical solution to make their argument work). Still, thanks to the preparatory results on semiconcave and BV functions given in Section 2.7, we will be able to perform such a construction and make the whole proof work. So, the goal of the next section is to prove the following result, from which Theorem 1.1 follows.

Proposition 3.3. *Let $H : T^*M \to \mathbb{R}$ be a Hamiltonian of class C^2, and assume that $\dim M = 2$ and that $\tilde{\mathcal{A}}(H)$ is minimal. Let $\mathcal{V}$ be a neighborhood of 0 in $C^2(M)$ and $\bar{\theta} \in \tilde{\mathcal{A}}(H)$ with $X_H(\bar{\theta}) \neq 0$ be such that $G^+_{\bar{\theta}} = G^-_{\bar{\theta}}$. Then there exists $V \in \mathcal{V}$ such that the Aubry set associated to the Hamiltonian $H + V$ is a hyperbolic periodic orbit (in its energy level).*

3.3 Proof of Proposition 3.3

From now on, we assume that the Aubry set $\tilde{\mathcal{A}}(H)$ is a minimal set which is neither an equilibrium point nor a periodic orbit. Without loss of generality, up to adding a constant to H (which does not change the dynamics), we can assume that $\mathbf{c}[H] = 0$. Let L denote the Lagrangian associated to H. Given $\epsilon > 0$, our goal is to find a potential $V : M \to \mathbb{R}$ of class C^2 with $\|V\|_{C^2} < \epsilon$, together with a Lipschitz function $v_V : M \to \mathbb{R}$, and a C^1 curve $\gamma : [0, T'] \to M$ with $\gamma(0) = \gamma(T')$, such that the following properties are satisfied:

(P1) $H_V\big(x, dv_V(x)\big) \leq 0$ for a.e. $x \in M$.

(P2) $\int_0^{T'} L_V(\gamma(t), \dot{\gamma}(t))\, dt = 0$.

Indeed, as explained in [21, §5.1] (see also [23]), if we are able to do this then (P1) implies that $c[H_V] \leq 0$ (see Subsection 2.1), while (P2) together with (2.2) yields $c[H_V] \geq 0$. Therefore, by (3.4) and the definition of the Peierls barrier h (3.1), the closed curve $\Gamma := \gamma([0, T'])$ is contained in the projected Aubry set of H_V. Now, if $W : M \to \mathbb{R}$ is any smooth function such that $W = 0$ on Γ, $W > 0$ outside Γ, and $\|W\|_{C^2} < \epsilon - \|V\|_{C^2}$, then the function v is a critical subsolution of $H_{V-W} = H + V - W$ which is strict outside Γ, and we have $\int_0^T L_{V-W}(\gamma(t), \dot{\gamma}(t))\, dt = 0$. By (3.4), this implies that the projected Aubry set of H_{V-W} coincides with the periodic curve $t \mapsto \gamma(t)$. Moreover, as shown in [14, Theorem D], we can add a potential, small in the C^2 topology, which preserves the periodic orbit and makes it a hyperbolic Aubry set. Hence, we are left with finding V, v_V, and γ such that (P1) and (P2) hold.

Fix $\epsilon > 0$, and let $\bar{\theta} = (\bar{x}, \bar{p}) \in \tilde{\mathcal{A}}(H)$ be as in the statement of Proposition 3.3. Let us denote by $\bar{\theta}(\cdot) = (\bar{\gamma}(\cdot), \bar{p}(\cdot))$ the orbit of $\bar{\theta}$ by the Hamiltonian flow, and by $\bar{\Pi} \subset M$ a local section (that is, a smooth curve) which is transverse to $\bar{\gamma}$ at $t = 0$. Let $u : M \to \mathbb{R}$ be a critical solution for H. Recall that u is differentiable on the projected Aubry set $\mathcal{A}(H)$, and that the restriction of du to $\mathcal{A}(H)$ is Lipschitz (see Proposition 2.5).

The following lemma will be needed to apply Proposition 2.7.

Lemma 3.4. *Let $c_0 > 0$ be as in Proposition 2.7. There exists $\bar{t} > 0$ such that, on any time interval of the form $[t_0, t_0 + \bar{t}]$ there is a time $t' \in [t_0, t_0 + \bar{t}]$such that*

$$\frac{d}{dt}\Big\{u\Big(\phi^H_t(\bar{x}, \bar{p})\Big)\Big\}_{|t=t'} = \big\langle du\big(\bar{\gamma}(t')\big), \dot{\bar{\gamma}}\big(t'\big)\big\rangle \geq -c_0. \tag{3.8}$$

Proof of Lemma 3.4. If not

$$u\big(\bar\gamma(t_0+\bar t)\big) - u\big(\bar\gamma(t_0)\big) = \int_{t_0}^{t_0+\bar t} \big\langle du\big(\bar\gamma(s)\big), \dot{\bar\gamma}(s)\big\rangle\, ds \le -\int_{t_0}^{t_0+\bar t} c_0 = -c_0\bar t.$$

Since u is bounded, this is impossible if $\bar t$ is sufficiently large. □

Up to replacing H by $4H/\bar t$, we can assume that the constant $\bar t$ appearing in the previous lemma satisfies [3]

$$\bar t = 1/4. \tag{3.9}$$

Let us take $T > 0$ to be fixed. Since $\bar\gamma$ can never intersect itself, there exist an open neighborhood $\mathcal{U}$ of $\bar\gamma([0,T])$ in M, and a C^2 diffeomorphism $\Phi : \mathcal{U} \to \mathcal{U}' := \Phi(\mathcal{U}) \subset \mathbb{R}^2$, such that, in the new system of coordinates, the curve $\Phi\left(\bar\gamma_{|[0,T]}\right)$ is a straight segment. Hence, using still $\bar\gamma$ instead of $\Phi(\bar\gamma)$ to denote this curve (by a slight abuse of notation), we can assume that

(π_1) $\bar\gamma(t) = (te_1, 0)$ for any $t \in [-1, T]$;

(π_2) $[-1, T] \times [-\rho, \rho] \subset \mathcal{U}'$.

(Here and in the sequel, (e_1, e_2) denotes the canonical basis in $\mathbb{R}^2$.) Also, in this new set of coordinates, we can see H as a Hamiltonian on $T^*\mathcal{U}' \subset T^*\mathbb{R}^2 = \mathbb{R}^2 \times (\mathbb{R}^2)^*$, and the critical solution u as a semiconcave function on $\mathbb{R}^2$. We set

$$\Pi_r^t := \Big\{(te_1, y)\,|\, y \in [-r, r]\Big\} \qquad \forall\, t, r \in \mathbb{R}.$$

The intersection of the Aubry set (resp. projected Aubry set) with $T^*\mathcal{U}$ (resp. with $\mathcal{U}$) is transported by Φ. Let us denote by $\tilde{\mathcal{A}}$ and $\mathcal{A}$ their respective images in $T^*\mathcal{U}'$ and $\mathcal{U}'$. The Green bundles $G^+_{\phi_t^H(\bar\theta)} = G^-_{\phi_t^H(\bar\theta)}$ for $t \in [-1, T]$, and G^+_θ with $\theta \in T^*\mathcal{U}$, are also transported by Φ. We denote them respectively by G_t and G^+_θ in $T(T^*\mathcal{U}')$. We now apply (3.7) and Corollary 2.15 to deduce that, up to reduce the size of ρ and $\mathcal{U}'$, there is a function $f : \mathcal{U}' \to \mathbb{R}$ of class $C^{1,1}$ such that the 1-form $\Psi := df$ on $\mathcal{U}'$ satisfies the following properties:

(π_3) Ψ is C^1 along $\bar\gamma([-1, T])$;

(π_4) $\tilde{\mathcal{A}} \cap T^*\mathcal{U}' \subset \mathrm{Graph}(\Psi)$;

(π_5) for every $t \in [-1, T], G_t = \mathrm{Graph}\big(L_t := d_{\bar\gamma(t)}\Psi\big) \subset \mathbb{R}^2 \times (\mathbb{R}^2)^*$.

3.3.1 Some preliminary regularity estimates on u

Let us recall that u is semiconcave (see Proposition 2.3), so the discussion in Section 2.7 (see in particular Section 2.7.2) applies. Also, since $\bar\gamma([0,T]) = \{te_1\}_{t\in[0,T]}$ (see (π_1)) and $te_1 \in \mathcal{A}$ (hence u is differentiable at te_1, see Proposition 2.5), by upper-semicontinuity of the limiting differential of semiconcave functions there is a modulus of continuity $\omega : \mathbb{R}^+ \to \mathbb{R}^+$ (that is, ω is nondecreasing with $\lim_{r\downarrow 0}\omega(r) = 0$), possibly depending on T, such that

$$\Big|(x,p) - (te_1, \nabla u(te_1))\Big| \le \omega(r) \qquad \forall\, x \in \Pi_r^t,\ p \in D_x^* u(x),\ t \in [0,T],\ r \in (0,\rho) \tag{3.10}$$

and (since $\frac{\partial H}{\partial p}\big(te_1, \nabla u(te_1)\big) = e_1$, see ($\pi_1$))

$$\left|\frac{\partial H}{\partial p}(x,p) - e_1\right| \le \omega(r) \qquad \forall\, x \in \Pi_r^t,\ p \in D_x^* u(x),\ t \in [0,T],\ r \in (0,\rho). \tag{3.11}$$

[3]Notice that the flow of the Hamiltonian $\bar H(x,p) := 4H(x,p)/\bar t$ is just a reparameterization of the flow of H, and u is still a solution of $\bar H(x, du) = 0$. The advantage of choosing $\bar t = 1/4$ is that later we will be able to connect trajectories over time intervals of length 1 instead of $\bar t$.

As in (2.20), for a.e. $t \in [0,T]$ we define the function $w_t : [-\rho,\rho] \to \mathbb{R}^n$ by

$$w_t(s) := \nabla u(t,s) \qquad \text{for a.e. } s \in [-\rho,\rho],$$

and we recall the following decomposition for Dw_t (see Section 2.7):

$$Dw_t = \nabla w_t \, ds + D_{\mathcal{S}} w_t,$$

where $\nabla w_t \, ds$ is absolutely continuous and $D_{\mathcal{S}} w_t$ is singular with respect to ds.

We notice that (3.11) implies that (2.29) holds with $\Xi(x) \equiv e_1$, hence it follows from (2.31) that

$$|D_{\mathcal{S}} w_t| \le 2|D_{\mathcal{S}} w_t \cdot e_2| \qquad \text{inside } \mathcal{U}', \text{ for a.e. } t \in [0,T]. \tag{3.12}$$

Also, Proposition 2.17 combined with the upper semicontinuity of the positive Green bundle provides an upper bound on $D^2 u$ in a neighborhood of a the curve $\bar\gamma([0,T])$. More precisely, we recall that $\langle D^2_{\mathcal{S}} v \cdot e, e\rangle$ is a nonpositive measure for any vector $e \in \mathbb{R}^n$ (see Section 2.7). Also, by (π_5) and Propositions 2.10 and 2.17 we deduce that there exists a modulus of continuity $\omega' : \mathbb{R}^+ \to \mathbb{R}^+$, possibly depending on T, such that, for a.e. $t \in [0,T]$,

$$\nabla^2 u(x) \le L_t + \omega'(r)\mathrm{Id} \qquad \text{for a.e. } x \in \Pi^t_r. \tag{3.13}$$

(Recall that $\nabla^2 u$ denotes the pointwise Hessian of u, which exists almost everywhere.) We denote by $\mathscr{O}$ the orbit of $\bar\gamma$ in $\mathcal{U}'$, that is $\mathscr{O} := \bar\gamma(\mathbb{R}) \cap \mathcal{U}'$.

In the next lemma we use (3.7) to show that, for a.e. t, Dw_t is close in total variation to a constant matrix. From now on, we always denote a modulus of continuity by ω and a positive constant by C, their values might change from line to line but otherwise they depend only on T and the data (i.e., H, u, etc.).

Lemma 3.5. *Let Ψ be as in (π_3)-(π_5). There exist a modulus of continuity $\omega : \mathbb{R}^+ \to \mathbb{R}^+$ and a constant $C > 0$ such that the following properties hold for any $r \in (0,\rho]$:*

(i) For a.e. $t \in [0,T]$, for every $y_1 = (t,\ell_1), y_2 = (t,\ell_2) \in \mathscr{O} \cap \Pi^t_r$ with $\ell_2 > \ell_1$,

$$\int_{\ell_1}^{\ell_2} |\nabla w_t(s) - L_t \cdot e_2| \, ds + \int_{\ell_1}^{\ell_2} d|D_{\mathcal{S}} w_t|(s) \le \omega(r)\, |\ell_2 - \ell_1|.$$

(ii) For every $y_1, y_2 \in \mathscr{O} \cap \Pi^T_\rho$ there exists a family of matrices $\{M_{-t}\}_{t\in[0,T]}$, with

$$\big|M_{-t}\big| + \big|(M_{-t})^{-1}\big| \le C,$$

such that the following holds for any constant $N \ge 1$: for every $z, z' \in \Pi^T_\rho \cap [y_1,y_2]$ such that u is differentiable at z, z' and $|z'-z| \ge \frac{|y_1-y_2|}{N}$, we have

$$\Big|\pi^*\big(\phi^H_{-t}(z,\nabla u(z))\big) - \pi^*\big(\phi^H_{-t}(z',\nabla u(z'))\big) - M_{-t}(z-z')\Big| \le N\,\omega(r)\,|z-z'|.$$

Proof of Lemma 3.5. We begin by observing that $|y_2 - y_1| = |\ell_2 - \ell_1|$. Since the graph of ∇u restricted to $\bar\gamma([0,T]) = \{te_1\}_{t\in[0,T]}$ is contained inside the graph of Ψ and the latter is C^1 there (see (π_1) and (π_3)), for a.e. $t \in [0,T]$ we get

$$\begin{aligned}
\Big|w_t(\ell_2) - w_t(\ell_1) - L_t \cdot (y_2 - y_1)\Big| &= \Big|\Psi(y_2) - \Psi(y_1) - d_{\bar\gamma(t)}\Psi \cdot (y_2 - y_1)\Big| \\
&= \left|\int_0^1 d_{y_1+s(y_2-y_1)}\Psi \cdot (y_2-y_1)\, ds - d_{\bar\gamma(t)}\Psi \cdot (y_2-y_1)\right| \\
&\le \int_0^1 \big|d_{y_1+s(y_2-y_1)}\Psi - d_{\bar\gamma(t)}\Psi\big|\, |y_2-y_1|\, ds \qquad (3.14)\\
&\le \omega(r)\,|\ell_2-\ell_1|,
\end{aligned}$$

for some modulus of continuity $\omega : \mathbb{R}^+ \to \mathbb{R}^+$. So, rewriting the above expression using the fundamental theorem of calculus (see (2.21)), for a.e. $t \in [0,T]$ we have (observe that $\frac{y_2-y_1}{|y_2-y_1|} = e_2$)

$$\left| \int_{\ell_1}^{\ell_2} \big[\nabla w_t(s) - L_t \cdot e_2\big]\, ds + \int_{\ell_1}^{\ell_2} d[D_{\mathcal{S}} w_t](s) \right| \leq \omega(r)\, |\ell_2 - \ell_1|,$$

which implies in particular that

$$\left| \int_{\ell_1}^{\ell_2} \big[\nabla w_t(s) \cdot e_2 - \langle L_t \cdot e_2, e_2\rangle\big]\, ds + \int_{\ell_1}^{\ell_2} d[D_{\mathcal{S}} w_t \cdot e_2](s) \right| \leq \omega(r)\, |\ell_2 - \ell_1|.$$

This estimate combined with (2.23), (3.13), and (2.25), gives

$$\int_{\ell_1}^{\ell_2} |\nabla w_t(s) \cdot e_2 - \langle L_t \cdot e_2, e_2\rangle|\, ds + \int_{\ell_1}^{\ell_2} d|D_{\mathcal{S}} w_t \cdot e_2|(s) \leq \omega(r)\, |\ell_2 - \ell_1| \qquad \text{for a.e. } t \in [0,T],$$

which shows that $Dw_t \cdot e_2$ is L^1-close to $\langle L_t \cdot e_2, e_2\rangle$.

We now need to control $Dw_t \cdot e_1$. For this, we first apply (3.12) to obtain that the singular part of Dw_t is controlled by $D_{\mathcal{S}} w_t \cdot e_2$: indeed (3.12) and the bound above imply

$$\int_{\ell_1}^{\ell_2} d|D_{\mathcal{S}} w_t|(s) \quad \leq \quad 2\int_{\ell_1}^{\ell_2} d|D_{\mathcal{S}} w_t \cdot e_2|(s) \leq 2\,\omega(r)\, |\ell_2 - \ell_1| \qquad \text{for a.e. } t \in [0,T].$$

Hence it suffices to control only the absolutely continuous part of Dw_t.

Recall that, thanks to (2.23), for a.e. $t \in [0,T]$ we have

$$\nabla w_t(s) = \nabla^2 u(t,s) \cdot e_2 \qquad \text{for a.e. } s \in [-r,r],$$

where $\nabla^2 u$ is the Hessian of u, which exists at almost every point. Hence it suffices to prove the closeness of ∇w_t to $L_t \cdot e_2$ only at points where u is twice differentiable.

For every $x_\ell := (t,\ell) \in \Pi_r^t$ where u is twice differentiable, consider the curve

$$\big(x_\ell(\tau), p_\ell(\tau)\big) := \phi_\tau^H(x_\ell, \nabla u(x_\ell)).$$

It follows from (3.10) and (π_1) that

$$\big|x_\ell(-\tau) - (t-\tau)e_1\big| \leq \omega(r) \qquad \forall\, t \in [0,T],\ \tau \in [0,1].$$

Also, since the trajectories do not cross backward in time, u is differentiable along them, and $p_\ell(-\tau) = \nabla u(x_\ell(-\tau))$ (see Proposition 2.4), we have (here we use $\dot{x}_s(\tau)$ to denote the derivative with respect to τ)

$$\frac{d}{d\tau}\big[p_\ell(-\tau)\big] = \frac{d}{d\tau}\big[\nabla u(x_\ell(-\tau))\big] = -\nabla^2 u(x_\ell(-\tau)) \cdot \dot{x}_\ell(-\tau) \qquad \forall\, \tau \in [0,1], \tag{3.15}$$

Since p_ℓ is uniformly bounded and solves the Hamiltonian system, also $\frac{d}{d\tau}\big[p_\ell(-\tau)\big]$ is uniformly bounded, hence we have

$$\Big|\big\langle \nabla^2 u(x_\ell(-\tau)) \cdot e, \dot{x}_\ell(-\tau)\big\rangle - \big\langle \nabla^2 u(x_\ell(0)) \cdot e, \dot{x}_\ell(0)\big\rangle\Big| \leq C\tau\, |e| \qquad \forall\, e \in \mathbb{R}^2,\ \forall\, \tau \in [0,1]. \tag{3.16}$$

To simplify the notation, set $x_s := x_{\ell_1+s(\ell_2-\ell_1)}$. Then, it follows from (3.15)-(3.16) and the smoothness in τ of the curves $\tau \mapsto x_s(\tau)$ that, for every $\tau \in [0,1]$,

$$\begin{aligned}
&\int_0^1 |\nabla w_t(\ell_1 + s(\ell_2-\ell_1)) \cdot \dot x_s(0) - \langle L_t \cdot e_2, \dot x_s(0)\rangle|\, ds \\
=\ & \int_0^1 \big|\langle \nabla^2 u(x_s(0)) \cdot e_2, \dot x_s(0)\rangle - \langle L_t \cdot e_2, \dot x_s(0)\rangle\big|\, ds \\
\le\ & \int_0^1 \left|\frac{1}{\tau}\int_{-\tau}^0 \langle \nabla^2 u(x_s(\sigma)) \cdot \dot x_s(\sigma), e_2\rangle\, d\sigma - \langle L_t \cdot e_2, \dot x_s(0)\rangle\right| ds \\
&+ \int_0^1 \frac{1}{\tau}\int_{-\tau}^0 \Big|\langle \nabla^2 u(x_s(\sigma)) \cdot e_2, \dot x_s(\sigma)\rangle - \langle \nabla^2 u(x_s(0)) \cdot e_2, \dot x_s(0)\rangle\Big|\, d\sigma\, ds \\
\le\ & \int_0^1 \left|\left\langle \frac{\nabla u(x_s(0)) - \nabla u(x_s(-\tau))}{\tau}, e_2\right\rangle - \langle L_t \cdot e_2, \dot x_s(0)\rangle\right| ds + C\tau. \qquad (3.17)
\end{aligned}$$

By (π_1) and (π_5) (note that ∇u varies smoothly along $\bar\gamma([0,T])$, since it solves the Hamiltonian system) we have

$$\left|\frac{\nabla u(te_1)\cdot e_2 - \nabla u((t-\tau)e_1)\cdot e_2}{\tau} - \langle L_t \cdot e_1, e_2\rangle\right| \le C\tau \qquad \forall\, \tau \in [0,1].$$

Hence, by (3.10), (3.11), and (3.17), for every $\tau \in [0,1]$ we get

$$\int_0^1 \big|\nabla w_t(\ell_1 + s(\ell_2-\ell_1)) \cdot \dot x_s(0) - \langle L_t \cdot e_2, \dot x_s(0)\rangle\big|\, ds \le C\tau + \frac{\omega(r)}{\tau}.$$

Thus, choosing $\tau := \sqrt{\omega(r)}$ and using that $|\dot x_s(0) - e_1| \le \omega(r)$ and that L_t is bounded (since u is universally $C^{1,1}$ on the Aubry set), we get

$$\begin{aligned}
\frac{1}{|\ell_2-\ell_1|}\int_{\ell_1}^{\ell_2} |\nabla w_t(s)\cdot e_1 - \langle L_t \cdot e_2, e_1\rangle\big|\, ds &= \int_0^1 \big|\nabla w_t(\ell_1 + s(\ell_2-\ell_1)) \cdot e_1 - \langle L_t \cdot e_2, e_1\rangle\big|\, ds \\
&\le C\sqrt{\omega(r)},
\end{aligned}$$

concluding the proof of (i).

Let us now prove the second assertion. To simplify the notation, for a.e. $\bar t \in [T-1,T]$ we define the functions [4]

$$\psi_{-t}^{\bar t}(s) := \pi^*\big(\phi_{-t}^H(z, \nabla u(z))\big) = \pi^*\big(\phi_{-t}^H\big((\bar t, s), w_{\bar t}(s)\big)\big), \qquad \text{for a.e. } z = (\bar t, s) \in \Pi_r^{\bar t},\ \forall\, t \in [0,T].$$

By the chain-rule formula for BV functions [2, Theorem 3.96], the following hold: if we decompose the distributional derivative $D\psi_{-t}^{\bar t}$ into its absolutely continuous part $\nabla\psi_{-t}^{\bar t}$ and its singular part $D_{\mathcal S}\psi_{-t}^{\bar t}$, we have

$$\nabla\psi_{-t}^{\bar t}(s) = d\pi^*\big(\phi_{-t}^H\big((\bar t, s), w_{\bar t}(s)\big)\big)\cdot\Big(\partial_{x_2}\phi_{-t}^H\big((\bar t, s), w_{\bar t}(s)\big) + \partial_p\phi_{-t}^H\big((\bar t, s), w_{\bar t}(s)\big)\cdot\nabla w_{\bar t}(s)\Big)$$

for a.e. $s \in [-r,r]$, and

$$\big|D_{\mathcal S}\psi_{-t}^{\bar t}\big| \le C\big|D_{\mathcal S} w_{\bar t}\big|.$$

Given $z, z' \in \Pi_r^{\bar t}$, let us denote by $\int_z^{z'} d\mu$ the integral of a measure μ over the segment joining z to z'. Then, by (i) and (3.10), for every $\tau \in [0,T]$ we have

$$\begin{aligned}
\int_{y_1}^{y_2} d|D\psi_{-t}^{\bar t} - M_{-t}^{\bar t}\cdot e_2| &= \int_{y_1}^{y_2} \big|\nabla\psi_{-t}^{\bar t}(s) - M_{-t}^{\bar t}\cdot e_2\big|\, ds + \int_{y_1}^{y_2} d|D_{\mathcal S}\psi_{-t}^{\bar t}|(s) \\
&\le \omega(r)|y_2 - y_1|, \qquad (3.18)
\end{aligned}$$

[4] Notice that, since u is differentiable a.e., for a.e. $t \in [0,T]$ we have that u is differentiable at a.e. $z \in [y_1, y_2]$.

where

$$M_{-t}^{\bar t} := d\pi^*\big(\phi_{-t}^H(\bar t e_1, e_1)\big)\cdot\Big(\partial_x\phi_{-t}^H(\bar t e_1, e_1) + \partial_p\phi_{-t}^H(\bar t e_1, e_1)L_{\bar t}\Big),$$

and we used that $|L_{\bar t}|$ is universally bounded (because u is universally $C^{1,1}$ on the Aubry set) to estimate

$$\begin{aligned}\Big|\partial_p\phi_{-t}^H\big((\bar t, s), w_{\bar t}(s)\big)\cdot\nabla w_{\bar t}(s) - \partial_p\phi_{-t}^H(\bar t e_1, e_1)L_{\bar t}\cdot e_2\Big| \\ \le \Big|\partial_p\phi_{-t}^H\big((\bar t, s), w_{\bar t}(s)\big)\cdot\big(\nabla w_{\bar t}(s) - L_{\bar t}\cdot e_2\big)\Big| \\ + \Big|\Big(\partial_p\phi_{-t}^H\big((\bar t, s), w_{\bar t}(s)\big) - \partial_p\phi_{-t}^H(\bar t e_1, e_1)\Big)L_{\bar t}\cdot e_2\Big|\end{aligned}$$

The boundedness of $|L_{\bar t}|$ implies also that the norm $M_{-t}^{\bar t}$ is bounded on $[0,T]$ by a constant depending only on T. Also, since u is semiconcave, a simple Gronwall argument shows that the backward flow $t\mapsto\psi_{-t}(z)$ is not "too much contractive": there exists a universal constant $C>0$ such that

$$|\psi_{-t}^{\bar t}(z) - \psi_{-t}^{\bar t}(z')| \ge e^{-Ct}|z-z'| \qquad \forall\, t\in[0,T],\ \forall z, z'\in\Pi_r^{\bar t}. \tag{3.19}$$

Before proving the validity of the above estimate, we first show how we use it to conclude the proof.

From (3.19) we deduce that $|(M_{-t}^{\bar t})^{-1}|\le e^{CT}$ and that the trajectories cannot cross backward in time. Also, from (3.18) and the assumption $|z'-z|\ge\frac{|y_2-y_1|}{N}$ we deduce that

$$\begin{aligned}\big|\psi_{-t}^{\bar t}(z) - \psi_{-t}^{\bar t}(z') - M_{-t}^{\bar t}(z-z')\big| &\le \int_{z'}^{z} d|D\psi_{-t}^{\bar t} - M_{-t}^{\bar t}\cdot e_2| \le \int_{y_1}^{y_2} d|D\psi_{-t}^{\bar t} - M_{-t}^{\bar t}\cdot e_2| \\ &\le \omega(r)\,|y_1-y_2| \le N\omega(r)\,|z'-z|\end{aligned}$$

for a.e. $t\in[0,T]$ and a.e. $z,z'\in\Pi_r^t$. By a simple approximation argument, the above estimate extends to $\bar t = T$ and every $z,z'\in\Pi_r^T$ such that u is differentiable at z,z', which proves Lemma 3.5 with $M_{-t} := M_{-t}^T$.

To finish the proof, we need to show the validity of (3.19), and we notice that (by triangle inequality in z and because $t\mapsto\psi_{-t}^{\bar t}$ enjoys the semigroup property) it is sufficient to prove the result for z,z' close to each other and for small times. By semiconcavity and compactness, there is a constant $K>0$ such that for every $(\bar x,\bar p)\in T^*\mathcal{U}'$ with $H(\bar x,\bar p)\le 0$ and every $x,x'\in\mathcal{U}'$ and $p\in\operatorname{conv}(D_x^*u)$, $p'\in\operatorname{conv}(D_{x'}^*u)$, we have (see [10])

$$\Big\langle\frac{\partial^2 H}{\partial p^2}\big(\bar x,\bar p\big)(p-p'), x-x'\Big\rangle \le K\big|x-x'\big|^2. \tag{3.20}$$

In particular, the above inequality holds for any $p=\nabla u(x)$, $p'=\nabla u(x')$ with u differentiable at x,x'. For any $z,z'\in\Pi_r^{\bar t}$ close enough and $t>0$ small, there is $(\bar x,\bar p)\in T^*\mathcal{U}'$ with $H(\bar x,\bar p)\le 0$

such that

$$\begin{aligned}
& \left\langle \frac{d}{dt}\left(\psi_{-t}^{\bar t}(z)-\psi_{-t}^{\bar t}(z')\right), \psi_{-t}^{\bar t}(z)-\psi_{-t}^{\bar t}(z')\right\rangle \\
= & \left\langle -\frac{\partial H}{\partial p}\left(\phi_{-t}^{H}(z,\nabla u(z))\right)+\frac{\partial H}{\partial p}\left(\phi_{-t}^{H}(z',\nabla u(z'))\right), \psi_{-t}^{\bar t}(z)-\psi_{-t}^{\bar t}(z')\right\rangle \\
= & \left\langle -\frac{\partial H}{\partial p}\left(\psi_{-t}^{\bar t}(z),\nabla u\big(\psi_{-t}^{\bar t}(z)\big)\right)+\frac{\partial H}{\partial p}\left(\psi_{-t}^{\bar t}(z'),\nabla u\big(\psi_{-t}^{\bar t}(z')\big)\right), \psi_{-t}^{\bar t}(z)-\psi_{-t}^{\bar t}(z')\right\rangle \\
= & \left\langle -\frac{\partial H}{\partial p}\left(\psi_{-t}^{\bar t}(z),\nabla u\big(\psi_{-t}^{\bar t}(z)\big)\right)+\frac{\partial H}{\partial p}\left(\psi_{-t}^{\bar t}(z),\nabla u\big(\psi_{-t}^{\bar t}(z')\big)\right), \psi_{-t}^{\bar t}(z)-\psi_{-t}^{\bar t}(z')\right\rangle \\
& +\left\langle \frac{\partial H}{\partial p}\left(\psi_{-t}^{\bar t}(z'),\nabla u\big(\psi_{-t}^{\bar t}(z')\big)\right)-\frac{\partial H}{\partial p}\left(\psi_{-t}^{\bar t}(z),\nabla u\big(\psi_{-t}^{\bar t}(z')\big)\right), \psi_{-t}^{\bar t}(z)-\psi_{-t}^{\bar t}(z')\right\rangle \\
= & \left\langle \frac{\partial^2 H}{\partial p^2}(\bar x,\bar p)\left(\nabla u\big(\psi_{-t}^{\bar t}(z')\big)-\nabla u\big(\psi_{-t}^{\bar t}(z)\big)\right), \psi_{-t}^{\bar t}(z)-\psi_{-t}^{\bar t}(z')\right\rangle \\
& +\left\langle \frac{\partial H}{\partial p}\left(\psi_{-t}^{\bar t}(z'),\nabla u\big(\psi_{-t}^{\bar t}(z')\big)\right)-\frac{\partial H}{\partial p}\left(\psi_{-t}^{\bar t}(z),\nabla u\big(\psi_{-t}^{\bar t}(z')\big)\right), \psi_{-t}^{\bar t}(z)-\psi_{-t}^{\bar t}(z')\right\rangle .
\end{aligned}$$

By (3.20) and C^2 regularity of H, we infer that there is some universal constant $K'>0$ such that

$$\left\langle \frac{d}{dt}\left(\psi_{-t}^{\bar t}(z)-\psi_{-t}^{\bar t}(z')\right), \psi_{-t}^{\bar t}(z)-\psi_{-t}^{\bar t}(z')\right\rangle \geq -K'\left|\psi_{-t}^{\bar t}(z)-\psi_{-t}^{\bar t}(z')\right|^2.$$

We conclude easily by Gronwall's lemma. □

The following bound will be crucial to estimate the action.

Lemma 3.6. *There exist a modulus of continuity $\omega:\mathbb{R}^+\to\mathbb{R}^+$ and a constant $K'>0$ such that the following holds: Let $y_1,y_2\in\mathscr{O}\cap\Pi_\rho^t$ for some $t\in[0,T]$. Then, for every $r\in(0,\rho]$, and for every $z_1,z_2\in\Pi_r^t\cap[y_1,y_2]$ such that u is differentiable at z_1,z_2 and $\frac{|y_1-y_2|}{10N}\leq|z_1-z_2|\leq\frac{|y_1-y_2|}{N}$,*

(i)

$$\big|u(z_2)-u(z_1)-\langle\nabla u(z_1),z_2-z_1\rangle\big|\leq K'\frac{|y_1-y_2|^2}{N};$$

(ii)

$$\big|\nabla u(z_2)-\nabla u(z_1)\big|\leq K'\Big(\omega(r)+\frac{1}{N}\Big)|y_1-y_2|.$$

Proof of Lemma 3.6. Since u is semiconcave, there exists a universal constant C such that $v:=u-C|x|^2$ is concave. Since

$$\Big|\big(u(z_2)-u(z_1)-\langle\nabla u(z_1),z_2-z_1\rangle\big)-\big(v(z_2)-v(z_1)-\langle\nabla v(z_1),z_2-z_1\rangle\big)\Big|\leq C|z_1-z_2|^2\leq C\frac{|y_1-y_2|^2}{N^2},$$

it suffices to prove the result (i) with v in place of u.

By concavity of v, since z_2-z_1 is parallel to y_2-y_1, and $|z_1-z_2|\geq\frac{|y_1-y_2|}{10N}$, we get

$$\begin{aligned}
0 & \geq v(z_2)-v(z_1)-\langle\nabla v(z_1),z_2-z_1\rangle \\
& \geq \langle\nabla v(z_2)-\nabla v(z_1),z_2-z_1\rangle \\
& \geq \langle\nabla v(y_2)-\nabla v(y_1),z_2-z_1\rangle \\
& \geq \frac{1}{10N}\langle\nabla v(y_2)-\nabla v(y_1),y_2-y_1\rangle \\
& \geq -C\,\frac{|y_1-y_2|^2}{N},
\end{aligned}$$

where for the last estimate we used that u (and hence v) is $C^{1,1}$ with a universal bound on the Aubry set. This proves (i).

For (ii), we recall that $\int_z^{z'} d\mu$ denotes the integral of a measure μ over the segment joining z to z'. Hence, using the same notation as before, we apply Lemma 3.5(i) and use that $|L_t|$ is universally bounded (because of the $C^{1,1}$ regularity of u on the Aubry set) to get, for a.e. $t \in [0, T]$,

$$\begin{aligned}
\big|\nabla u(z_2) - \nabla u(z_1)\big| &\leq \int_{z_1}^{z_2} d|Dw_t| \\
&\leq \int_{z_1}^{z_2} d|Dw_t - L_t \cdot e_2| + |L_t|\,|z_1 - z_2| \\
&\leq \int_{y_1}^{y_2} d|Dw_t - L_t \cdot e_2| + C\,|z_1 - z_2| \\
&\leq \omega(r)\,|y_1 - y_2| + C\,\frac{|y_1 - y_2|}{N}.
\end{aligned}$$

By approximation, this estimate extends to every $t \in [0, T]$. □

3.3.2 The connection

Given $y_1, y_2 \in \mathbb{R}$, we set

$$I^{1/3}\big(y_1; y_2\big) := \Big\{y \in \mathbb{R} \,|\, \mathrm{dist}\big(y, [y_1, y_2]\big) < \big|y_1 - y_2\big|/3\Big\}.$$

Lemma 5.2 in [22] (see also [3, Remarque 6.3.3]) applied with $n = 1$ yields the following result:

Lemma 3.7. *Let $\hat r > 0$ and Y be a finite set in $\mathbb{R}$ such that $B(0, \hat r/12) \cap Y$ contains at least two points. Then there are $y_1 \neq y_2 \in Y$ such that the interval $I^{1/3}(y_1; y_2)$ is included in $B(0, \hat r)$ and does not intersect $Y \setminus \{y_1, y_2\}$.*

Given $\hat r \in (0, \rho)$ small enough ($\hat r$ much smaller than ρ and ϵ), let $T_{\hat r} \gg T$ be the first time such that $\bar\gamma(T_{\hat r}) \in \Pi^T_{\hat r/12}$, and define the set

$$W := \Big\{w_0 := \bar x, w_1 := \bar\gamma(t_1), \ldots, w_J := \bar\gamma(T_{\hat r})\Big\} \subset \mathcal{A} \tag{3.21}$$

obtained by intersecting the curve

$$[T, T_{\hat r}] \ni t \mapsto \bar\gamma(t)$$

with $\Pi^T_{\hat r}$. We apply Lemma 3.7 with $Y = W \subset \Pi^T_{\hat r}$ to find two points

$$\hat y_1 = w_j \quad \text{and} \quad \hat y_2 = w_l \quad \text{with } j > l \tag{3.22}$$

which satisfy the properties described in the statement of the lemma. Set

$$N := \left\lfloor \frac{1}{\epsilon} \right\rfloor + 1, \qquad \eta := 2N + 1,$$

and consider a sequence of points $\hat z_1, \ldots, \hat z_\eta$ in the segment $[\hat y_1, \hat y_2] \subset \Pi^T_\rho$ which satisfy [5]

$$\hat z_1 := \hat y_1, \quad \hat z_\eta := \hat y_2,$$

[5]Since u is differentiable a.e., by Fubini's Theorem, for a.e. $T \in (0, \infty)$ we can find points $\hat z_1, \ldots, \hat z_\eta$ such that (3.23) and (3.24) hold. Notice that we do not yet fix the points $\hat z_i$, since later we will need to impose that they satisfy some additional conditions, see in particular (π_9) below.

$$u \text{ is differentiable at } \hat{z}_i \quad \forall\, i = 1, \ldots, \eta, \tag{3.23}$$

and

$$\hat{z}_i \in \Big[\hat{y}_1 + \frac{i-4/3}{2N}\big(\hat{y}_2 - \hat{y}_1\big), \hat{y}_1 + \frac{i-2/3}{2N}\big(\hat{y}_2 - \hat{y}_1\big)\Big] \qquad \forall\, i = 2, \ldots, \eta - 1. \tag{3.24}$$

Notice that

(π_6) $\frac{|\hat{y}_2 - \hat{y}_1|}{6N} \leq \big|\hat{z}_{i+1} - \hat{z}_i\big| \leq \frac{5|\hat{y}_2 - \hat{y}_1|}{6N} \quad \forall\, i = 1, \ldots, \eta - 1.$

We now fix T to be an arbitrary time in $[\eta+1, \eta+2]$ chosen so that (3.23) and (3.24) hold. Applying Lemma 3.4 (recall also (3.9)), for any $i = 1, \ldots, \eta - 1$ we can find a time $t_i \in [i - 1 - 1/8, i - 1 + 1/8]$ such that

$$\bar{\gamma}(t_i) \text{ satisfies (3.8)} \qquad \forall\, i = 1, \ldots, \eta - 1. \tag{3.25}$$

Given $z \in [\hat{y}_1, \hat{y}_2]$ such that u is differentiable at z, we can consider the calibrated curve $\gamma_z : (-\infty, 0] \to M$ as in (2.6). Notice that, since $\hat{y}_1$ and $\hat{y}_2$ belong to the projected Aubry set, u is differentiable at $\hat{y}_1$ and $\hat{y}_2$ and those curves are transverse to Π_ρ^T (remember (3.10)-(3.11)), the curves $\gamma_{\hat{y}_1}$ and $\gamma_{\hat{y}_2}$ are unique and disjoint and moreover the curves γ_z cannot intersect $\gamma_{\hat{y}_1}$ and $\gamma_{\hat{y}_2}$ (see Propositions 2.4 and 2.5). Hence, provided $\hat{r}$ is sufficiently small, for any $z \in [\hat{y}_1, \hat{y}_2]$ where u is differentiable there exist $T_z \in [T-1, T+1]$ such that

$$\gamma_z(-T_z) \in \Pi_\rho^0, \quad \gamma_z([-T_z, 0]) \subset [-1, T] \times [-\rho, \rho].$$

Recalling (3.23), we now define the following points for all $i = 1, \ldots, \eta - 1$ (see Figure 1), where $\bar{\tau} \in (0, 1/10)$ is the same as in Proposition 2.7:

$$z_{i,-} := \gamma_{\hat{z}_i}([-T_{\hat{z}_i}, 0]) \cap \Pi_\rho^{t_i}, \qquad z_{i,+} := \gamma_{\hat{z}_i}([-T_{\hat{z}_i}, 0]) \cap \Pi_\rho^{t_i + \bar{\tau}},$$

$$z'_{i,-} := \gamma_{\hat{z}_{i+1}}([-T_{\hat{z}_{i+1}}, 0]) \cap \Pi_\rho^{t_i}, \qquad z'_{i,+} := \gamma_{\hat{z}_{i+1}}([-T_{\hat{z}_{i+1}}, 0]) \cap \Pi_\rho^{t_i + \bar{\tau}}.$$

Also, we set

$$y_1^t := \gamma_{\hat{y}_1}([-T_{y_1}, 0]) \cap \Pi_\rho^t, \qquad y_2^t := \gamma_{\hat{y}_2}([-T_{y_2}, 0]) \cap \Pi_\rho^t. \tag{3.26}$$

By Lemma 3.5(ii), provided $\hat{r}$ is sufficiently small, (π_6) yields

(π_7) $\frac{|y_1^{t_i+\bar{\tau}} - y_2^{t_i+\bar{\tau}}|}{7N} \leq |z_{i,+} - z'_{i,+}| \leq \frac{|y_1^{t_i+\bar{\tau}} - y_2^{t_i+\bar{\tau}}|}{N} \leq \frac{C}{N}\hat{r} \quad \forall\, i = 1, \ldots, \eta - 1.$

Also, using again Lemma 3.5(ii), it follows from the construction of $\hat{y}_1$ and $\hat{y}_2$ (see Lemma 3.7) that, for any $t \in [0, T]$, all points of $\bar{\gamma}([0, T_{\hat{r}}])$ on Π_ρ^t are at distance at least $\frac{|y_1^t - y_2^t|}{4}$ from $\{y_1^t, y_2^t\}$ (besides the points $\{y_1^t, y_2^t\}$ themselves), that is

(π_8) $\operatorname{dist}\Big(\Big(\big(\bar{\gamma}([0, T_{\hat{r}}]) \cap \Pi_\rho^t\big) \setminus \{y_1^t, y_2^t\}\Big), \{y_1^t, y_2^t\}\Big) \geq \frac{|y_1^t - y_2^t|}{4} \quad$ for all $t \in [0, T]$.

Since u is differentiable at $z_{i,-}, z'_{i,-}, z_{i,+}, z'_{i,+}$ (see (3.23) and Proposition 2.4), by Lemma 3.6(ii) and (π_7) it follows that (provided $\hat{r}$ is small enough)

$$\big|\nabla u(z_{i,+}) - \nabla u(z'_{i,+})\big| \leq \frac{2K'}{N}\,\big|y_1^{t_i+\bar{\tau}} - y_2^{t_i+\bar{\tau}}\big|.$$

Hence, since $\mathcal{P}^*_{0,\bar{\tau}}\big(z_{i,-}, \nabla u(z_{i,-})\big) = \big(z_{i,+}, \nabla u(z_{i,+})\big)$ and thanks to (3.25), if $\hat{r}$ is sufficiently small (so that $z_{i,-}$ is close to $\bar{\gamma}(t_i)$) we can apply Proposition 2.7 with $x^0 = z_{i,-}$, $x^f = z'_{i,+}$, $p^0 = \nabla u(z_{i,-})$, $p^f = \nabla u(z'_{i,+})$, $r = |y_1^{t_i+\bar{\tau}} - y_2^{t_i+\bar{\tau}}|$, and $\hat{\epsilon} = 2(1+K')/N$ (with K' as in Lemma 3.6), to find a potential V_i which permits to connect $z_{i,-}$ to $z'_{i,+}$ on a time interval $[\hat{t}_i, \hat{t}_i + T_i^f]$.

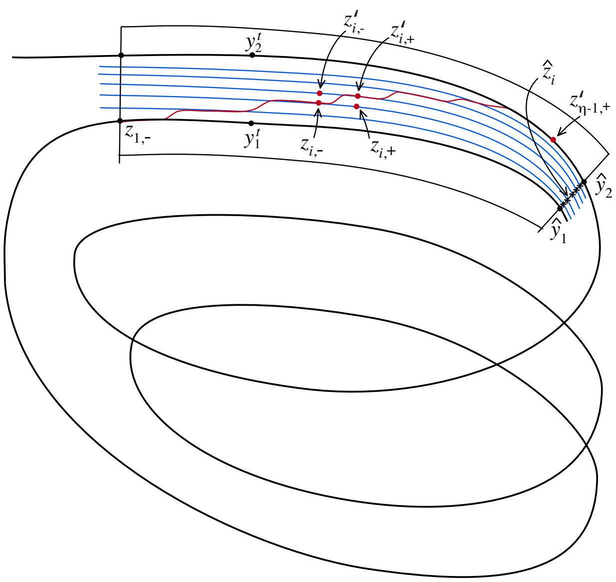

Figure 1: The points $z_{i,-}$ and $z_{i,+}$ (resp. $z'_{i,-}$ and $z'_{i,+}$) are obtained from z_i (resp. z_{i+1}) by intersecting the trajectory passing through $\hat{z}_i$ (resp. $\hat{z}_{i+1}$) with the segments $\Pi_\rho^{t_i}$ and $\Pi_\rho^{t_i+\bar{\tau}}$. Analogously, the points y_1^t (resp. y_2^t) defined in (3.26) are obtained by intersecting the trajectory passing through $\hat{y}_1$ (resp. $\hat{y}_2$) with the segment Π_ρ^t. Our goal is to connect $z_{i,-}$ to $z'_{i,+}$ with a control on the action, in order to obtain a closed curve which satisfies (P2).

Notice that the constant $\sigma = \sigma_i$ appearing in the value of the action is an arbitrary number less than $\hat{\epsilon} r^2 = 2(1+K')\frac{|y_1^{t_i+\bar{\tau}} - y_2^{t_i+\bar{\tau}}|^2}{N}$.

We now construct a curve $\gamma : [0, T'] \to M$ by concatenating $\gamma_1 : [0, T_1] \to M$ with $\gamma_2 : [T_1, T'] \to M$, where

$$\gamma_2(t) := \pi^* \left(\phi_{t-T_1}^H \left(\hat{y}_2, \nabla u(\hat{y}_2)\right)\right) \qquad \text{connects } z'_{\eta-1,+} \text{ to } z_{1,-},$$

while γ_1 is obtained as a concatenation of $2\eta - 1$ pieces: defining by $V := \sum_i V_i$ (notice that the support of the V_i's are all disjoint, so the C^2 norm of V is bounded by $\max_i \|V_i\|_{C^2}$), for every $i = 1, \ldots, \eta - 1$ we use the flow $(t, z) \mapsto \pi^* \left(\phi_t^{H+V}(z, \nabla u(z))\right)$ to connect $z_{i,-}$ to $z'_{i,+}$ on a time interval $[\hat{t}_i, \hat{t}_i + T_i^f]$, while on $[\hat{t}_i + T_i^f, \hat{t}_{i+1}]$ ($i = 1, \ldots, \eta - 1$) we just use the original flow $(t, z) \mapsto \pi^* \left(\phi_t^H(z, \nabla u(z))\right)$ to send $z'_{i,+}$ onto $z_{i+1,-}$. (See [21, Subsection 5.3] for more detail.)

In this way we obtain a closed curve $\gamma : [0, T'] \to M$ (see Figure 1) whose action is given by the following formula (see [21, Section 5.4]):

$$\int_0^{T'} L_V(\gamma(t), \dot{\gamma}(t))\, dt = \sum_{i=1}^{\eta-1} \Big[\big\langle \nabla u(z_{i,+}), z'_{i,+} - z_{i,+} \big\rangle - \Big(u(z'_{i,+}) - u(z_{i,+}) \Big) \Big] + \sigma_i.$$

Thanks to (π_7) and Lemma 3.6 we deduce that

$$\Big| \big\langle \nabla u(z_{i,+}), z'_{i,+} - z_{i,+} \big\rangle - \Big(u(z'_{i,+}) - u(z_{i,+}) \Big) \Big| \leq K' \frac{|y_1^{t_i+\bar{\tau}} - y_2^{t_i+\bar{\tau}}|^2}{N}.$$

Hence, since σ_i can be any arbitrary number less than $2(1+K')\frac{|y_1^{t_i+\bar{\tau}} - y_2^{t_i+\bar{\tau}}|^2}{N}$, we can choose

$$\sigma_i := \Big(u(z'_{i,+}) - u(z_{i,+}) \Big) - \big\langle \nabla u(z_{i,+}), z'_{i,+} - z_{i,+} \big\rangle$$

to enforce

$$\int_0^{T'} L_V(\gamma(t), \dot\gamma(t))\, dt = 0,$$

as desired. This concludes the proof of (P2).

3.3.3 A "good" critical subsolution for H

To prove (P1), we first need to construct a $C^{1,1}$ critical subsolution v which is "C^2 in average". Recall that, for every $t > 0$, the function $h_t : M \times M \to \mathbb{R}$ is defined by

$$h_t(x, y) := \inf \int_0^t L\big(\gamma(s), \dot\gamma(s)\big)\, ds,$$

where the infimum is taken over all Lipschitz curves $\gamma : [0, t] \to M$ such that $\gamma(0) = x$ and $\gamma(t) = y$.

Lemma 3.8. *Let Ψ be as in (π_3)-(π_5), and let $y_1^t = (t, \ell_1^t), y_2^t = (t, \ell_2^t) \in \mathscr{O} \cap \Pi_\rho^t$ be as in* (3.26). *There exists $s_0 > 0$ small but universal such that the critical subsolution $v : M \to \mathbb{R}$ defined by*

$$v(x) := \mathcal{T}_{s_0}^+ u(x) = \sup_{y \in M} \Big\{ u(y) - h_{s_0}(x, y) \Big\} \qquad \forall\, x \in M,$$

is universally $C^{1,1}$ and, for a.e. $t \in [0, T]$, it satisfies [6]

$$\int_0^1 \big|\langle \nabla^2 v\big(y_1^t + s(y_2^t - y_1^t)\big) \cdot e_2, e_2\rangle - \langle L_t \cdot e_2, e_2\rangle\big|\, ds \le \omega\big(|\ell_1^t| + |\ell_2^t|\big)$$

for some universal modulus of continuity $\omega : \mathbb{R}^+ \to \mathbb{R}^+$.

Proof of Lemma 3.8. The fact that v is a critical subsolution is standard, see for instance [8].

By semiconcavity, there is a bounded family of $C^2(\mathcal{U}', \mathbb{R})$ such that

$$u = \inf_{f \in \mathcal{F}} \{f\}.$$

Moreover, thanks to the estimate (3.13) on $D^2 u$ provided by the Green bundles, we may assume that

$$\nabla^2 f(x) \le L_t + \omega'(r)\mathrm{Id} \qquad \forall\, x \in \Pi_r^t,\ r \in (0, \rho],\ t \in [0, T],\ f \in \mathcal{F}. \tag{3.27}$$

Then, for every $s_0 > 0$ we have

$$v = \inf_{f \in \mathcal{F}} \{T_{s_0}^+ f\}.$$

By [8] it is known that, for $s_0 > 0$ small enough, $v \in C^{1,1}$, $v = u$, and $\nabla v = \nabla u$ on the projected Aubry set. Since $\nabla v = \nabla u$ on $\mathscr{O}$, by (3.14) we get

$$\nabla v(y_2^t) = \nabla v(y_1^t) + L_t \cdot (y_2^t - y_1^t) + \omega\big(|\ell_1^t| + |\ell_2^t|\big)\, |y_1^t - y_2^t| \qquad \forall\, t \in [0, T].$$

Since $y_2^t - y_1^t$ is parallel to e_2, rewriting the above expression using the fundamental theorem of calculus (recall that $v \in C^{1,1}$) we have

$$\left| \int_0^1 \langle \nabla^2 v\big(y_1^t + s(y_2^t - y_1^t)\big) \cdot e_2, e_2\rangle\, ds - \langle L_t \cdot e_2, e_2\rangle \right| \le \omega\big(|\ell_1^t| + |\ell_2^t|\big) \qquad \text{for a.e. } t \in [0, T], \tag{3.28}$$

where $\nabla^2 v$ is the pointwise Hessian of v.

[6] Notice that, being $C^{1,1}$, v is twice differentiable a.e.

By [8, Lemma 3] there is $s_0 > 0$ such that, for every $s \in [0, s_0]$, $\mathcal{T}_s^+(\mathcal{F})$ is a bounded set in $C^2(M, \mathbb{R})$. Since the Hessian of a C^2 function f is transported by the linearized Hamiltonian flow along the calibrating trajectories (see for instance the discussion in [21] after Lemma 5.3), and since all the trajectories are close (as a function of r) to the trajectory passing through te_1 (see (3.11)), we deduce from (3.27) that, for all $r \in (0, \rho]$,

$$\nabla^2 v \le L_t + \omega'(r)\mathrm{Id} \qquad \text{a.e. on } \Pi_r^t, \text{ for a.e. } t \in [0, T].$$

Hence, combining this bound with (3.28) we easily get

$$\int_0^1 \left|\left\langle \left[\nabla^2 v\big(y_1^t + s(y_2^t - y_1^t)\big) - L_t\right] \cdot e_2, e_2\right\rangle\right| ds \le \omega\big(|\ell_1^t| + |\ell_2^t|\big) + \omega'\big(|\ell_1^t| + |\ell_2^t|\big),$$

as desired. □

Combining Lemmas 3.8 and 2.20, we can also prove that there are many points where v is "C^2 in average".

Lemma 3.9. *With the same notation as in Lemma 3.8, let $y_1^t, y_2^t \in \mathscr{O} \cap \Pi_r^t$, $r := |\ell_1^t| + |\ell_2^t| \in (0, \rho]$. Then, for a.e. $t \in [0, T]$ there exists a set $A_t \subset [y_1^t, y_2^t]$ such that*

$$|A_t| \ge \big(1 - \sqrt{\omega(r)}\big)|y_1^t - y_2^t|$$

and

$$\frac{1}{R}\int_0^R \left|\langle \nabla^2 v(z + se_2) \cdot e_2, e_2\rangle - \langle L_t \cdot e_2, e_2\rangle\right| ds \le C_1\sqrt{\omega(r)}$$

for all $z \in A_t$ and $R \in \big[-|y_1^t - z|, |y_2^t - z|\big]$.

Proof of Lemma 3.9. We simply apply Lemma 2.20 to the one dimensional function

$$f(s) := \left|\langle \nabla^2 v(z + se_2) \cdot e_2, e_2\rangle - \langle L_t \cdot e_2, e_2\rangle\right| \chi_{[-|y_1 - z|, |y_2 - z|]}(s)$$

with $\delta = C_1\sqrt{\omega(r)}$, and use Lemma 3.8. □

From the results above we see that, provided r is sufficiently small, we can shift the system of coordinates in the variable t by an arbitrary small amount (sat, $t \mapsto t + \tau_0$ with $|\tau_0|$ arbitrarily small) to ensure that the above lemmas all apply with $t = t_i$ for all $i = 1, \dots, \eta - 1$, and then the points $\hat{z}_1, \dots, \hat{z}_\eta$ can be chosen so such that

(π_9) $z_{i,-}, z'_{i,-} \in A_{t_i}$ for all $i = 1, \dots, \eta - 1$.

3.3.4 Construction of a global critical subsolution

As before, we will denote by $\omega : \mathbb{R}^+ \to \mathbb{R}^+$ a modulus of continuity which may change from line to line.

Our goal is to construct a critical subsolution $v_V : M \to \mathbb{R}$ satisfying (P1). We proceed as follows: first, for any $i = 1, \dots, t_i$ we define u_0^i and u_V^i as the $C^{1,1}$ solutions of the Dirichlet problems

$$\begin{cases} H\big(z, \nabla u_0^i(z)\big) = 0 & \text{in } [t_i, t_i + 3\bar{\tau}] \times \Pi_\rho^{t_i}, \\ u_0^i = v & \text{on } \Pi_\rho^{t_i}, \end{cases}$$

$$\begin{cases} H_V\big(z, \nabla u_V^i(z)\big) = 0 & \text{in } [t_i, t_i + 3\bar{\tau}] \times \Pi_\rho^{t_i}, \\ u_V^i = v & \text{on } \Pi_\rho^{t_i}, \end{cases}$$

where v is as in Lemma 3.8 (see Lemma 2.6(iv)). Let $\gamma : [0, T'] \to M$ be the closed trajectory constructed in Section 3.3.2, and define $\Gamma_i := \gamma([\hat{t}_i + T_i^f, \hat{t}_{i+1}])$ to be the piece of curve which connects $z'_{i,+}$ to $z_{i+1,-}$ (see Figure 2). In complete analogy with [21, Section 5.5, Property ($\pi 3$)], we have

$$u_0^i = u_V^i, \quad \nabla u_0^i = \nabla u_V^i \qquad \text{on } \Gamma_i \cap \mathcal{C}_i, \text{ where } \quad \mathcal{C}_i := \bigcup_{t \in [t_i, t_i+3\bar{\tau}]} [y_1^t, y_2^t]. \tag{3.29}$$

Also, because y_1^t and y_2^t are in the projected Aubry set and V is supported in the interior of $\mathcal{C}_i$, we get that the values of u_0^i, u_V^i, v are all transported along the curves, and the same happens to their gradients (see [21, Lemma 5.3] and the discussion immediately after it), so in analogy with [21, Section 5.5, Property ($\pi 7$)] we get

$$u_0^i = u_V^i = v, \quad \nabla u_0^i = \nabla u_V^i = \nabla v \qquad \text{on } \partial_{lat}\mathcal{C}_i, \text{ where } \quad \partial_{lat}\mathcal{C}_i := \bigcup_{t \in [t_i, t_i+3\bar{\tau}]} \{y_1^t, y_2^t\}. \tag{3.30}$$

We claim that

$$|u_0^i(x) - u_V^i(x)| \leq \omega(\hat{r} + \epsilon)\,\mathrm{dist}(x, \Gamma_i)^2 \qquad \forall\, x \in \mathcal{C}_i. \tag{3.31}$$

Indeed, since by (π_1), (π_9), and Lemmas 3.8 and 3.9

$$\frac{1}{R}\int_0^R \big|\langle \nabla^2 v(z + s e_2)\cdot e_2, e_2\rangle - \langle L_{t_i}\cdot e_2, e_2\rangle\big|\, ds \leq C_1\sqrt{\omega\big(|\ell_1^{t_i}| + |\ell_2^{t_i}|\big)}$$

for $z = z_{i,-}, z'_{i,-}$ and $R \in \big[-|y_1^{t_i} - z|, |y_2^{t_i} - z|\big]$, and the flow of the vector field $\frac{\partial H}{\partial p}(x, \nabla v(x))$ is bi-Lipschitz (since v is $C^{1,1}$), we deduce that (recall that the Hessian of a solution is propagated along the linearized flow)

$$\frac{1}{R}\int_0^R \big|\langle \nabla^2 u_0^i(z_i^t + s e_2)\cdot e_2, e_2\rangle - \langle L_t\cdot e_2, e_2\rangle\big|\, ds \leq C\sqrt{\omega\big(|\ell_1^{t_i}| + |\ell_2^{t_i}|\big)}$$

for all $R \in \big[-|y_1^t - z|, |y_2^t - z|\big]$ and $t \in [t_i + 2\bar{\tau}, t_i + 3\bar{\tau}]$, where

$$z_i^t := \gamma_{\hat{z}_{i+1}}([-T_{\hat{z}_{i+1}}, 0]) \cap \Pi_\rho^t.$$

Also, because $\|V\|_{C^2} \leq \epsilon$, the linearized flows of H and H_V are close in terms of ϵ (see [21, Proof of Lemma 5.5]), hence

$$\frac{1}{R}\int_0^R \big|\langle \nabla^2 u_V^i(z_i^t + s e_2)\cdot e_2, e_2\rangle - \langle L_t\cdot e_2, e_2\rangle\big|\, ds \leq C\sqrt{\omega\big(|\ell_1^{t_i}| + |\ell_2^{t_i}|\big)} + \omega(\epsilon)$$

for all $R \in \big[-|y_1^t - z|, |y_2^t - z|\big]$ and $t \in [t_i + 2\bar{\tau}, t_i + 3\bar{\tau}]$. Since $|\ell_1^{t_i}| + |\ell_2^{t_i}| \leq C\hat{r}$ (because $\hat{y}_1, \hat{y}_2 \in \Pi_{\hat{r}}^T$ and the flow is Lipschitz on the Aubry set), this estimate combined with (3.29) and a simple Taylor expansion proves (3.31).

We consider now for every i a smooth nonincreasing function $\Theta_i : \mathbb{R} \to [0, 1]$ such that

$$\begin{cases} \Theta_i(\lambda) = 1 & \text{if } \lambda \in [t_i, t_i + 3\bar{\tau}/2], \\ \Theta_i(\lambda) = 0 & \text{if } \lambda \in [t_i + 5\bar{\tau}/2, t_i + 3\bar{\tau}], \end{cases}$$

and we set

$$\Theta_i(z) := \Theta_i(z_1) \qquad \forall z = (z_1, z_2) \in \mathbb{R}^2.$$

Then we define $\tilde{u}_i$ as

$$\tilde{u}_i(z) := \begin{cases} \Theta_i(z) u_V^i(z) + \big(1 - \Theta_i(z)\big) v(z) & \text{for } z \in \mathcal{C}_i, \\ v(z) & \text{for } z \in \mathcal{C}_i' \setminus \mathcal{C}_i, \end{cases}$$

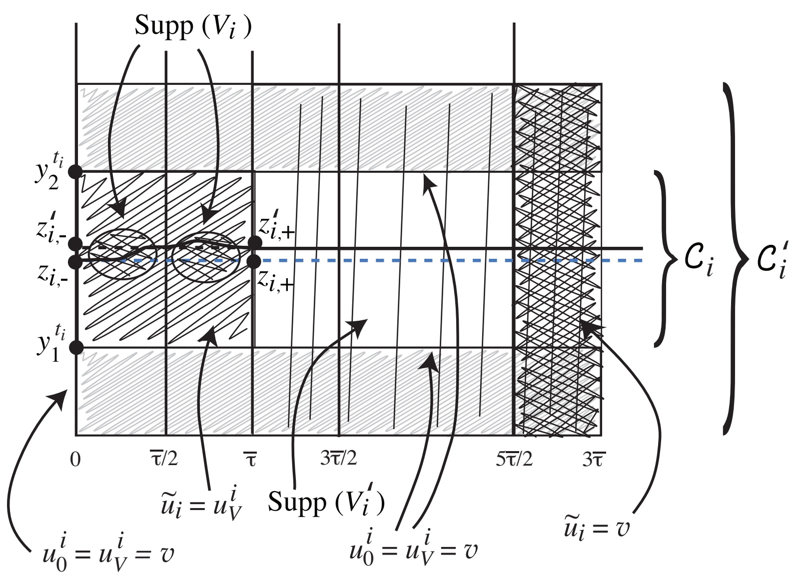

Figure 2: The curve Γ_i corresponds to the horizontal segment starting from the point $z'_{i,+}$ and going to the right. The function $\tilde{u}_i$ is obtained by interpolating (using a cut-off function) between u_V^i (the viscosity solution for H_V) and v (the critical subsolution for H constructed in Section 3.3.3) inside the "cylinder" $\mathcal{C}'_i$. Then, by adding a new potential V'_i, small in C^2 topology and supported inside $\mathcal{C}'_i \cap \{z = (z_1, \hat{z}) \mid z_1 \in [\bar{\tau}, 3\bar{\tau}]\}$, we can ensure that $H_{V_i+V'_i}(z, \nabla \tilde{u}_i(z)) \le 0$. Since the cylinders $\mathcal{C}'_i$ are disjoint, we can repeat this construction for $i = 1, \dots, \eta - 1$ to find a global critical subsolution $\tilde{u}$ and a potential $\tilde{V}$ so that (P1) and (P2) hold.

where

$$\mathcal{C}'_i := \bigcup_{t \in [t_i, t_i + 3\bar{\tau}]} [y_1^{t,'}, y_2^{t,'}], \qquad y_1^{t,'} := y_1^t - \frac{y_2^t - y_1^t}{4}, \quad y_2^{t,'} := y_2^t + \frac{y_2^t - y_1^t}{4}$$

(see Figure 2).

Thanks to (3.30), $\tilde{u}_i$ is of class $C^{1,1}$ inside $\mathcal{C}'_i$. Moreover, for every $z \in \mathcal{C}_i$ we have

$$\nabla \tilde{u}_i(z) = \big(u_V^i(z) - v(z)\big)\nabla\Theta_i(z) + \Theta_i(z)\nabla u_V^i(z) + \big(1 - \Theta_i(z)\big)\nabla v(z).$$

Set

$$P_i(z) := \Theta_i(z)\nabla u_V^i(z) + \big(1 - \Theta_i(z)\big)\nabla v(z) \qquad \forall\, z \in \mathcal{C}_i.$$

By convexity of H in the p variable we get

$$H_V\left(z, P_i(z)\right) \le 0 \qquad \forall\, z \in \mathcal{C}_i.$$

Moreover, since v is a subsolution for H,[7]

$$H_V\left(z, \nabla \tilde{u}_i(z)\right) \le 0 \qquad \forall\, z \in \mathcal{C}'_i \setminus \mathcal{C}_i, \qquad u_0^i(z) - v(z) \ge 0 \qquad \forall\, z\, \in \mathcal{C}_i.$$

[7]The latter inequality comes from the fact that, by the construction of u_0^i, for every $z \in \mathcal{C}_i$, there are $w \in \Pi_\rho^{t_i}$, $\tau > 0$, and a curve $\gamma : [0, \tau] \to \mathcal{C}_i$, such that $\gamma(0) = w$, $\gamma(\tau) = z$, and

$$u_0^i(z) = v(w) + \int_0^\tau L\left(\gamma(t), \dot{\gamma}(t)\right)\, dt.$$

Also, since v is subsolution for H we have

$$v(z) \le v(w) + \int_0^\tau L\left(\gamma(t), \dot{\gamma}(t)\right)\, dt,$$

which proves that $v \le u_0^i$.

Also, since $\nabla\Theta_i$ points in the direction of $-e_1$, by (3.11) we get

$$\langle \frac{\partial H}{\partial p}\left(z, P_i(z)\right), \nabla\Theta_i(z)\rangle \leq -\frac{1}{2}|\nabla\Theta_i(z)| \qquad \forall\, z \in \mathcal{C}_i.$$

Then, using that $u_0^i \geq v$ and Taylor's formula, we obtain

$$\begin{aligned} H_V\left(z, \nabla\tilde{u}_i(z)\right) &\leq H_V\left(z, P_i(z)\right) + \left(u_V^i(z) - v(z)\right)\langle \frac{\partial H}{\partial p}\left(z, P_i(z)\right), \nabla\Theta_i(z)\rangle \\ &\quad + K\left|\nabla\Theta_i(z)\right|^2 \left|u_V^i(z) - v(z)\right|^2 \\ &\leq \left(u_V^i(z) - u_0^i(z)\right)\langle \frac{\partial H}{\partial p}\left(z, P_i(z)\right), \nabla\Theta_i(z)\rangle - \frac{1}{2}\left|u_0^i(z) - v(z)\right| \left|\nabla\Theta_i(z)\right| \\ &\quad + 2K\left|\nabla\Theta_i(z)\right|^2 \left|u_V^i(z) - u_0^i(z)\right|^2 + 2K\left|\nabla\Theta_i(z)\right|^2 \left|u_0^i(z) - v(z)\right|^2 \\ &\leq C\left|\nabla\Theta_i(z)\right| \left|u_V^i(z) - u_0^i(z)\right|, \end{aligned}$$

where we used that $|u_V^i - u_0^i|$ and $|u_0^i - v|$ are small to absorb the quadratic terms into the linear ones. Since the last term in the above equation is of order $\omega(\hat{r}+\epsilon)\,\mathrm{dist}(z,\Gamma_i)^2$ (by (3.31)) and vanishes both outside $\mathcal{C}_i$ and outside the support of Θ, we deduce that

$$H_V\left(z, \nabla\tilde{u}_i(z)\right) \leq 0$$

inside $\mathcal{C}_i' \cap \left([t_i, t_i + 3\bar{\tau}/2] \cup [t_i + 5\bar{\tau}/2, t_i + 3\bar{\tau}]\right)$, and (using (π_8))

$$H_V\left(z, \nabla\tilde{u}_i(z)\right) \leq \omega(\hat{r}+\epsilon)\, \min\left\{\mathrm{dist}(z,\Gamma_i)^2, \mathrm{dist}(z, \partial_{lat}\mathcal{C}_i')^2\right\}$$

inside $\mathcal{C}_i' \cap [t_i + 3\bar{\tau}/2, t_i + 5\bar{\tau}/2]$, where

$$\partial_{lat}\mathcal{C}_i' := \bigcup_{t \in [t_i + 2\bar{\tau}, t_i + 3\bar{\tau}]} \left\{y_1^{t,'}, y_2^{t,'}\right\}.$$

By choosing $\hat{r}$ and ϵ sufficiently small, it is easy to see that we can add a potential $V_i' \leq 0$, small in C^2 topology, which vanishes on Γ and supported inside $\mathcal{C}_i'$, so that

$$H_{V+V_i'}\left(z, \nabla\tilde{u}_i(z)\right) \leq 0 \quad \text{in } \mathcal{C}_i'$$

and property (P2) is preserved. Then the function $\tilde{u}$ obtained by gluing together the functions $\tilde{u}_i$ with v is a global critical subsolution for $H_{\tilde{V}}$ with $\tilde{V} := V + \sum_i V_i'$ (notice that the support of the V_i''s are all disjoint, so the C^2 norm of $\sum_i V_i'$ is bounded by $\max_i \|V_i'\|_{C^2}$), yielding (P1) and concluding the proof of Proposition 3.3.

4 Examples

Recall that a minimal set of a Lipschitz vector field on a surface is called exceptional if it is neither a fixed point, nor a closed trajectory, nor the whole surface (see [33]). By the Poincaré-Bendixon Theorem, exceptional minimal sets do not exist on the two-dimensional sphere. The purpose of this section is to construct Tonelli Hamiltonians with exceptional minimal Aubry sets on orientable surfaces with positive genus. Such a counter-example in the setting of twist maps was given by Goroff [24].

4.1 Preliminaries on the Mather functions

Let M be a smooth compact Riemannian manifold without boundary of dimension $n \geq 2$ and $H : T^*M \to \mathbb{R}$ a Tonelli Hamiltonian of class C^2. Denote by $L : TM \to \mathbb{R}$ the Tonelli

Lagrangian of class C^2 associated with H by Legendre-Fenchel duality (see Section 2.1). The flow ϕ_t^L of L is conjugated with the Hamiltonian flow through the Legendre transform $\mathcal{L}$: $T^*M \to TM$ defined by $\mathcal{L}(x,p) := \Big(x, \frac{\partial H}{\partial p}(x,p)\Big)$, that is

$$\phi_t^L = \mathcal{L} \circ \phi_t^H \circ \mathcal{L}^{-1}.$$

Denote by $\mathcal{M}(L)$ the set of probability measures on TM which are invariant under the Lagrangian flow. Recall that the *homology* $\rho(\mu) \in H_1(M,\mathbb{R}) = H^1(M,\mathbb{R})^*$ of a measure $\mu \in \mathcal{M}(L)$ is determined by

$$\langle \rho(\mu), [\omega] \rangle = \int_{TM} \omega_x(v)\, d\mu(x,v),$$

where ω is any closed 1-form on M and $[\omega] \in H^1(M,\mathbb{R})$ is its cohomology class. The *action* of μ with respect to L is defined as

$$\mathbb{A}_L(\mu) := \int_{TM} L\, d\mu.$$

The Mañé critical value of L and H can be recovered as

$$\mathbf{c}[H] = \mathbf{c}[L] := -\min\Big\{ \mathbb{A}_L(\mu) \,|\, \mu \in \mathcal{M}(L) \Big\}.$$

The *Mather α and β functions* associated with L (or equivalently with H),

$$\alpha_L \,:\, H^1(M,\mathbb{R}) \longrightarrow \mathbb{R} \qquad \beta_L \,:\, H_1(M,\mathbb{R}) \longrightarrow \mathbb{R},$$

are defined as

$$\beta_L(h) := \min \Big\{ \mathbb{A}_L(\mu) \,|\, \mu \in \mathcal{M}(L),\, \rho(\mu) = h \Big\} \qquad \forall\, h \in H_1(M,\mathbb{R})$$

and

$$\alpha_L\big([\omega]\big) := \mathbf{c}[L - \omega] \qquad \forall\, [\omega] \in H^1(M,\mathbb{R}).$$

They are convex functions with superlinear growth which are conjugate (see [31, Theorem 1]), that is

$$\alpha_L(c) = \max\Big\{ \langle h, c\rangle - \beta_L(h) \,|\, h \in H_1(M,\mathbb{R}) \Big\} \qquad \forall\, c \in H^1(M,\mathbb{R})$$

and

$$\beta_L(h) = \max\Big\{ \langle h, c\rangle - \alpha_L(c) \,|\, c \in H^1(M,\mathbb{R}) \Big\} \qquad \forall\, h \in H_1(M,\mathbb{R}).$$

Let us now introduce some definitions and notation. We call *flat* of β_L any non-trivial maximal convex domain in $H_1(M,\mathbb{R})$ on which β_L is an affine function. Moreover we say that a flat is *radial* if it is contained in a set of the form $\langle h \rangle = \{th \mid t \in \mathbb{R}\}$ with $h \in H_1(M,\mathbb{R})$. By conjugation, any flat F of β_L is associated with a non-differentiability point of α_L: more precisely, if $c \in H^1(M,\mathbb{R})$ satisfies

$$\alpha_L(c) = \langle h, c\rangle - \beta_L(h) \qquad \forall\, h \in F,$$

then all affine functions $c' \mapsto \langle h, c'\rangle - \beta_L(h)$ with slope $h \in F$ are supporting functions for α at c. Given $h \in H_1(M,\mathbb{R})$ and $c = [\omega] \in H^1(M,\mathbb{R})$, let

$$\mathcal{M}_h(L) : = \arg\min \Big\{ \mathbb{A}_L(\mu) \;\Big|\; \mu \in \mathcal{M}(L),\, \rho(\mu) = h \Big\},$$

$$\mathcal{M}^c(L) = \mathcal{M}^\omega(L) : = \arg\min \Big\{ \mathbb{A}_{L-\omega}(\mu) \;\Big|\; \mu \in \mathcal{M}(L) \Big\}.$$

Note that, by the above properties, for every $c \in H^1(M,\mathbb{R})$ we have

$$\rho\big(\mathcal{M}^c(L)\big) = \Big\{ h \in H_1(M,R) \,|\, \alpha_L(c) + \beta_L(h) = \langle h, c\rangle \Big\}. \tag{4.1}$$

Finally, we recall that a homology class is *rational* if there is $t \in \mathbb{R}$ such that $th \in H_1(M,\mathbb{Z})$.

4.2 Exceptional minimal hyperbolic Aubry sets on the 2-torus

Let M be a torus of dimension 2 and fix P a point in M. The open manifold $M \setminus \{P\}$ can be equipped with a hyperbolic metric of curvature -1 (we refer the reader to [34] and references therein for further details in hyperbolic geometry). Let us fix a simple close curve χ with length $\ell > 0$ which bounds a small open disc D containing P, and another simple closed curve χ' with length $\ell' \in (0, \ell)$ which is contained in D and which bounds a small open disc D' containing P. We can choose χ' so small that $d(\chi, \chi') > \ell$, where d denotes the distance with respect to the hyperbolic metric. We now change the hyperbolic metric on $D' \setminus \{P\}$ into a smooth metric on D' which coincides with the former metric on the boundary of D'. In this way we obtain a smooth metric g on M. We will be concerned with the geodesic Lagrangian $L : TM \to \mathbb{R}$ defined as

$$L(x, v) := \tfrac{1}{2} \|v\|_x^2 \qquad \forall\, (x, v) \in TM, \tag{4.2}$$

and we denote by H the associated Hamiltonian. We notice that, for every $c = [\omega] \in H^1(M, \mathbb{R})$, $\mathbf{c}[L - \omega] \geq 0$ and $\mathbf{c}[L - \omega] = 0 \Leftrightarrow [\omega] = 0$. For every $c = [\omega] \in H^1(M, \mathbb{R})$ we denote respectively by $\tilde{\mathcal{A}}(c)$ and $\mathcal{A}(c)$ the Aubry set and projected Aubry set of the Hamiltonian associated with the Lagrangian $L - \omega$, that is, of the Hamiltonian given by $H(x, p) = \frac{1}{2}\|p + \omega_x\|^2$, where $\| \cdot \|$ denotes the cometric on T^*M. In the sequel, by abuse of notation, we will look at the Aubry set as a subset of TM via the identification between TM and T^*M given by the Legendre transform.

Lemma 4.1. *For every closed form ω with $c = [\omega] \neq 0$, we have*

$$\mathcal{A}(c) \cap D' = \emptyset.$$

Proof of Lemma 4.1. Let us first show that $\mathcal{A}(c)$ cannot be included in D. Argue by contradiction and pick a positively recurrent point $\theta = (x, p)$ of the Aubry set $\tilde{\mathcal{A}}(c)$. Let $\gamma(t) = \pi^*\left(\phi_t^H(x, p)\right)$ for $t \in \mathbb{R}$. Then there exists a sequence $t_k \to +\infty$ such that $\theta = \lim_{k\to\infty} \phi_{t_k}^H(\theta)$. Since θ belongs to the Aubry set, the curve γ is calibrated, that is, for every critical solution $u : M \to \mathbb{R}$ we have

$$0 = \lim_{k\to\infty} u\big(\gamma(t_k)\big) - u(x) = \lim_{k\to\infty} \int_0^{t_k} \left[L\big(\gamma(t), \dot{\gamma}(t)\big) - \omega_{\gamma(t)}\big(\dot{\gamma}(t)\big)\right] dt + \mathbf{c}[L - \omega]\, t_k.$$

Let $f : D \to \mathbb{R}$ be a smooth function such that $df = \omega$ on D. Since $\gamma([0, \infty)) \subset \mathcal{A}(c) \subset D$ by assumption, we have

$$\int_0^{t_k} \omega_{\gamma(t)}\big(\dot{\gamma}(t)\big)\, dt = f\big(\gamma(t_k)\big) - f\big(\gamma(0)\big).$$

Hence, since $L \geq 0$, combining the two estimates above and letting $k \to \infty$ we obtain $\mathbf{c}[L-\omega] \leq 0$. Recalling that $\mathbf{c}[L - \omega] \geq 0$ and $\mathbf{c}[L - \omega] = 0 \Leftrightarrow [\omega] = 0$, we infer that $\mathbf{c}[L - \omega] = 0$ which means that ω is exact, a contradiction.

Assume now that $\mathcal{A}(c)$ intersects both D' and $M \setminus D$. Then there are a calibrated curve $\gamma : \mathbb{R} \to \mathcal{A}(c)$ and T_1 such that $\gamma(0) \in \partial D'$ and $\gamma(T_1) \in \partial D$. The α-limit and ω-limit sets of γ contain positively recurrent points, so (by the previous argument) they cannot be contained in D. Therefore we may assume that $T_1 < 0$ and that there is $T_2 > 0$ such that $\gamma(T_2) \in \partial D$ and $\gamma((T_1, T_2)) \subset D$. Let $\bar{\chi} : [T_1, T_2] \to \partial D$ be a smooth constant-speed curve corresponding to piece of the curve χ joining $\gamma(T_1)$ to $\gamma(T_2)$ with constant speed. Since $d(\chi, \chi') > \ell$ and while the length of χ is ℓ

$$\int_{T_1}^{T_2} \left\|\dot{\gamma}(t)\right\|_{\gamma(t)}^2 dt > \frac{\ell^2}{T_2 - T_1} \geq \int_{T_1}^{T_2} \left\|\dot{\bar{\chi}}(t)\right\|_{\bar{\chi}(t)}^2 dt,$$

which shows that (since both curves are contained in $\bar{D}$ the integral of ω along them just depends on their end-points)

$$\int_{T_1}^{T_2} \Big[\big\|\dot\gamma(t)\big\|^2_{\gamma(t)} - \omega_{\gamma(t)}\big(\dot\gamma(t)\big)\Big]\,dt > \frac{\ell^2}{T_2 - T_1} \geq \int_{T_1}^{T_2} \Big[\big\|\dot{\bar\chi}(t)\big\|^2_{\bar\chi(t)} - \omega_{\bar\chi(t)}\big(\dot{\bar\chi}(t)\big)\Big]\,dt.$$

This contradicts the minimality of γ (see (2.2) and (2.9)), proving the result. □

Lemma 4.2. *The function β_L has no flat.*

Proof of Lemma 4.2. By homogeneity of L, the function β_L is quadratic in the radial direction, that is $\beta(th) = t^2\beta(h)$ for any $h \in h_1(M,\mathbb{R})$ and $t \geq 0$. Thus it suffices to show that any flat of β_L has to be radial. Argue by contradiction and suppose that there is a flat $F \subset H_1(M,\mathbb{R})$ which is not radial. Let ρ_1, ρ_2 be two extremal points in F which are linearly independent and let $\mu_i \in \mathcal{M}_{\rho_i}(L)$, $i = 1, 2$. Then there is a cohomology class $c = [\omega]$ such that

$$F = \rho(\mathcal{M}^\omega(L)) = \Big\{\, h \in H_1(M,\mathbb{R}) \mid \alpha_L(c) + \beta_L(h) = \langle h, c\rangle \,\Big\}.$$

Since the ergodic components of μ_1 and μ_2 are also in $\mathcal{M}^\omega(L)$, their homologies are also in F. Since ρ_1, ρ_2 are extremal points of F and ρ is linear, the homologies of the ergodic components of μ_1 and μ_2 are respectively ρ_1 and ρ_2. In conclusion, we can assume that μ_1, μ_2 are ergodic. We need to show that the projection of the orbits in the support of μ_1 and μ_2 intersect. Since $\mu_1, \mu_2 \in M^\omega(L)$ are minimizing measures for the Lagrangian $L - \omega$, the intersection will contradict the Mather's graph property, proving the result.

In the 2-torus M, any two integral homology classes in $H_1(M,\mathbb{Z})$ which are linearly independent intersect. Let $I : H_1(M,\mathbb{Z}) \times H_1(M,\mathbb{Z}) \to \mathbb{R}$ be the intersection form which extends by bilinearity to real homologies. Then, if r_2 is not a multiple of r_1 in $H_1(M,\mathbb{R})$, we have $I[r_1, r_2] \neq 0$.

We denote by $\pi : TM \to M$ the canonical projection. For each $i = 1, 2$, let $(x_i, v_i) \in TM$ be a generic point for μ_i, Σ_i a small transversal segment to v_i in M containing x_i, T a large return time to Σ_i of the projected flow of (x_i, v_i) so that $\pi\big(\phi^L_T(x_i, v_i)\big) \in \Sigma_i$, and $\Gamma_i(T)$ a small segment in Σ_i joining x_i to $x_i(T) = \pi\big(\phi^L_T(x_i, v_i)\big)$. For each $i = 1, 2$, we define $C_i(T)$ to be the closed curve $C_i(T) := \pi\big(\phi^L_{[0,T]}(x_i, v_i)\big) * \Gamma_i(T)$ obtained by concatenating $\pi\big(\phi^L_{[0,T]}(x_i, v_i)\big)$ with $\Gamma_i(T)$. Note that, without loss of generality, we may assume that $\Sigma_1 \cap \Sigma_2 = \emptyset$. Choose now two sequences of return times $\{T^1_k\}_k, \{T^2_l\}_l$ such that $\lim_{k\to\infty} T^1_k = \lim_{l\to\infty} T^2_l = +\infty$ and $\lim_{k\to\infty} \operatorname{diam}\Gamma_i(T^1_k) = \lim_{l\to\infty} \operatorname{diam}\Gamma_i(T^1_l) = 0$. Since the points (x_i, v_i) are generic points for μ_i, Birkhoff's Theorem ensures that

$$\lim_{k\to\infty} \frac{1}{T^1_k}[C_1(T^1_k)] = \rho_1 \quad \text{and} \quad \lim_{k\to\infty} \frac{1}{T^2_l}[C_2(T^2_l)] = \rho_2 \quad \text{in } H_1(M,\mathbb{R}).$$

Then by bilinearity of the intersection form, we have

$$0 \neq I[\rho_1, \rho_2] = \lim_{k,l\to\infty} \frac{1}{T^1_k\, T^2_l} I\big[C_1(T^1_k), C_2(T^2_l)\big].$$

In order to obtain the contradiction we have to show that there is at least one intersection in $I\big[C_1(T^1_k), C_2(T^2_l)\big]$ which is not due to the small closing segments $\Gamma_1(T^i_k), \Gamma_2(T^i_l)$.

Note that if μ_1 (resp. μ_2) is supported on a periodic orbit then we can take as T^1_k (resp. T^2_l) a multiple of the period and there is no joining segment $\Gamma_1(T^1_k)$ (resp. $\Gamma_2(T^2_l)$). This proves that the intersection occurs when both μ_1, μ_2 are supported on periodic orbits, giving the desired contradiction.

For the general case, let ψ be the induced Hamiltonian flow on the projected Aubry set $\mathcal{A}(c)$ in M, that is

$$\psi_t(x) := \pi^*\big(\phi^H_t(x, du(x))\big),$$

where $u : M \to \mathbb{R}$ is a critical solution (see Proposition 2.5). We fix $\tau > 0$ small enough so that $\psi_{(0,\tau]}\big(\mathcal{A}(c) \cap \Sigma_1\big) \cap \Sigma_1 = \emptyset$, and define $B_1(T_k^1) := \psi_{[0,\tau]}(\mathcal{A}(c) \cap \Gamma_1(T_k^1))$. Let $\chi_{B_1(T_k^1)}$ be the characteristic function of $B_1(T_k^1)$. Since $\chi_{B_1(T_k^1)} \leq 1$ and the part of $C_2(T_l^2)$ which may intersect $\Gamma_1(T_k^1)$ is contained in $\mathcal{A}(c)$ (recall that, by construction, $\Sigma_1 \cap \Sigma_2 = \emptyset$), we have

$$\#\big[C_2(T_l^2) \cap \Gamma_1(T_k^1)\big] \leq \frac{1}{\tau} \int_0^{T_l^2} (\chi_{B_1(T_k^1)} \circ \pi)(\varphi_s^L(x_2, v_2))\, ds \leq \frac{T_l^2}{\tau} \qquad \forall\, k, l.$$

Therefore

$$\limsup_{l \to \infty} \frac{1}{T_l^2} \#\big[C_2(T_l^2) \cap \Gamma_1(T_k^1)\big] \leq \frac{1}{\tau} \qquad \forall\, k,$$

which implies

$$\lim_{k,l \to \infty} \frac{1}{T_k^1\, T_l^2} \left| I\left[\Gamma_1(T_k^1), C_2(T_l^2)\right] \right| \leq \lim_{k \to \infty} \frac{1}{T_k^1}\, \frac{1}{\tau} = 0.$$

Similarly

$$\lim_{k,l \to \infty} \frac{1}{T_k^1\, T_l^2} \left| I\left[\Gamma_2(T_k^2), C_1(T_l^1)\right] \right| = 0,$$

which proves that projection of the orbits in the support of μ_1 and μ_2 intersect, a contradiction. □

Let $h \in H_1(M, \mathbb{R})$ be an irrational homology class, and let $c = [\omega] \in H^1(M, \mathbb{R})$ be a cohomology class such that $\alpha_L(c) + \beta_L(h) = \langle c, h \rangle$. Since β_L has no flat, the set $\rho(\mathcal{M}^\omega(L))$ is a singleton (see (4.1)). Let Λ be a minimal set in $\tilde{\mathcal{A}}(c)$, and $U \in C^\infty(M, \mathbb{R})$ a C^2-small smooth non-negative function on M such that $U^{-1}(\{0\}) = \pi^*(\Lambda)$. Then the Aubry set for the Lagrangian $L - \omega + U$ is the minimal set Λ. Moreover Λ is not a closed orbit because (the image through the Legendre transform of) any ergodic measure in Λ is in $\mathcal{M}^\omega(L)$, thus has homology h, which is irrational.

The Euler-Lagrange flow of L is the geodesic flow of the metric g, which is uniformly hyperbolic (outside TD'). Then the Lagrangian $L - \omega$ has the same flow as L. Moreover since the projected Aubry set does not cross D' (Lemma 4.1) and U is C^2-small, the invariant set Λ remains hyperbolic with respect to $L - \omega + U$. Hence Λ is a non-trivial minimal hyperbolic Aubry set.

4.3 The case of surfaces of higher genus

A similar construction can be made in a surface of higher genus as follows. Let M_1 be a 2-torus, let a hyperbolic metric on $M_1 \setminus \{P_1\}$ with $P_1 \in M_1$, χ_1, χ_1' two curves surrounding the point P_1 as above, and let $M_2 \setminus \{P_2\}$ be a punctured surface of genus g, $g \geq 1$ equipped with a hyperbolic metric. Construct two curves χ_2 and χ_2' bounding the puncture as in the example above, cut M_2 through χ_2' and join it smoothly to M_1 along χ_1'. Join also smoothly the Riemannian metrics on M_1 and M_2 in the tube between χ_2' and χ_1'. Then define $M := M_1 \# M_2$ and consider the smooth geodesic Lagrangian given by (4.2). By construction, $H_1(M, \mathbb{R}) = H_1(M_1, \mathbb{R}) \oplus H_1(M_2, \mathbb{R})$ and the minimizing measures with homologies in $H_1(M_1, \mathbb{R})$ or in $H_1(M_2, \mathbb{R})$ do not cross $\chi_1' = \chi_2'$. Also the β function satisfies $\beta_L = \beta_{L_1} \oplus \beta_{L_2}$, where L_1 and L_2 denote the geodesic Lagrangians obtained as above on M_1 and M_2.

Take an irrational homology class $h \in H_1(M, \mathbb{R})$ and let $c = [\omega] \in H^1(M, \mathbb{R})$ be a cohomology class such that

$$\beta_L(h \oplus 0) + \alpha_L(c) = \langle c, (h \oplus 0) \rangle,$$

where $(h \oplus 0) \in H_1(M, \mathbb{R}) = H_1(M_1, \mathbb{R}) \oplus H_1(M_2, \mathbb{R})$. Let $\Lambda \subset TM$ be the minimal set obtained in Section 4.2, and let $U \in C^\infty(M, \mathbb{R})$ be a C^2-small smooth non-negative function on M such that $U^{-1}(\{0\}) = \Lambda$. Then Λ is a non-trivial minimal hyperbolic Aubry set for the Lagrangian $L - \omega + U$ on TM.

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