High action orbits for Tonelli Lagrangians and superlinear Hamiltonians on compact configuration spaces

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Abstract

Multiplicity results for solutions of various boundary value problems are known for dynamical systems on compact configuration manifolds, given by Lagrangians or Hamiltonians which have quadratic growth in the velocities or in the momenta. Such results are based on the richness of the topology of the space of curves satisfying the given boundary conditions. In this note we show how these results can be extended to the classical setting of Tonelli Lagrangians (Lagrangians which are C^2-convex and superlinear in the velocities), or to Hamiltonians which are superlinear in the momenta and have a coercive action integrand.

Introduction

Let M be a compact manifold, the configuration space of a - possibly non-autonomous - Lagrangian system, given by a regular function L - the Lagrangian - on $\mathbb{R} \times TM$. Here TM is the tangent bundle of M, and its elements are denoted by (q, v), where q is a point in M and v is a tangent vector at q. A classical assumption is that L should be a *Tonelli Lagrangian*, meaning that L is fiberwise C^2-strictly convex (that is $\partial_{vv} L > 0$), and has superlinear growth on each fiber.

On one hand Tonelli assumptions imply that the Legendre transform is a diffeomorphism between the tangent bundle TM and the cotangent bundle T^*M, so that the Hamiltonian vector field on T^*M associated to the Fenchel transform of L induces a vector field on TM. Together with the assumption that such a vector field is complete[1], this produces a dynamical system on TM.

On the other hand Tonelli assumptions imply existence results for minimal orbits, such as the existence of an orbit having minimal action connecting two given points on M, or the existence of a periodic orbit of minimal action in every conjugacy class of the fundamental group of M (if L is assumed to be periodic in time). See for instance [11].

However, more sophisticated existence results for orbits with large action and large Morse index are known only for a smaller class of Lagrangians, namely Lagrangians which behave quadratically in v for $|v|$ large. Their proofs are based on topological methods in critical point theory. The fact that L grows *at most* quadratically allows to find a nice functional setting where the action functional is regular; the fact that L grows *at least* quadratically implies the Palais-Smale condition, the standard compactness property which makes minimax arguments work. For instance, in the case of periodic Lagrangians in this class, Benci [2] has proved the existence of infinitely many contractible periodic orbits, provided that M has finite fundamental group.

The aim of this note is to show how the existence and multiplicity results holding for Lagrangians which behave quadratically in v for $|v|$ large can actually be proved for any Tonelli

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[1]This is always true if L is autonomous, due to the conservation of energy, to the coercivity of the Hamiltonian, and to the compactness of M. In general, completeness follows from a growth condition on $\partial_t L$. See section 1.

Lagrangian which induces a complete vector field. We deal with many boundary conditions at once by fixing a submanifold Q of $M \times M$ and by considering orbits $\gamma : [0,1] \to M$ satisfying the non-local boundary condition

$$(\gamma(0), \gamma(1)) \in Q,$$
$$D_v L(0, \gamma(0), \gamma'(0))[\xi_0] = D_v L(1, \gamma(1), \gamma'(1))[\xi_1], \tag{1}$$

for every vector (ξ_0, ξ_1) in $T(M \times M)$ which is tangent to Q at $(\gamma(0), \gamma(1))$. When Q is the diagonal submanifold, (1) yields to periodic orbits; when Q is the singleton $\{(q_0, q_1)\}$, (1) yields to orbits joining q_0 to q_1. Other choices of Q allow to deal with orbits which are suitably normal to two submanifolds at their end-points, or satisfy Neumann boundary conditions.

Our main result here is that the number of orbits satisfying (1) of a complete Tonelli Lagrangian system is greater than the cuplength of the space $C_Q([0,1], M)$ consisting of all continuous paths $\gamma : [0,1] \to M$ such that $(\gamma(0), \gamma(1))$ belongs to Q (the cuplength of a topological space X is the maximum length of a non-vanishing cup product of elements of degree at least one in the cohomology ring of X). Moreover, if $C_Q([0,1], M)$ has infinitely many non-vanishing Betti numbers[2], there is a sequence of such orbits with diverging action. See section 2 for precise statements and for their consequences. By taking a manifold M with finite fundamental group and by choosing Q to be the diagonal in $M \times M$, we obtain the generalization of Benci's theorem to Tonelli Lagrangians.

The idea of the proof is to modify the Tonelli Lagrangian for $|v|$ large, making it quadratic in v there. A suitable a priori estimate on Lagrangian orbits with bounded action then allows to prove that orbits of the modified Lagrangian found by a minimax argument must lie in the region where the Lagrangian is not modified. The result about the existence of infinitely many orbits uses also Morse index estimates.

The same idea works for first order Hamiltonian systems on the cotangent bundle T^*M of the compact manifold M. The elements of T^*M are denoted by (q, p), where q is a point in M and p is a cotangent vector at q. Here a dynamical system is defined by a Hamiltonian $H : \mathbb{R} \times T^*M \to \mathbb{R}$. If we do not assume that H is convex on the fibers (or that $\partial_{pp} H$ is everywhere invertible), such a Hamiltonian system does not admit a Lagrangian formulation. In this setting, Cieliebak [5] has proved the analogue of Benci's result for Hamiltonians which behave quadratically in p for $|p|$ large and whose action integrand $DH[Y] - H$ grows at least quadratically in p (here Y is the Liouville vector field, expressed in local coordinates by $\sum_i p_i \partial_{p_i}$). His proof is based on the study of the Floer equation, a zero-order perturbation of the equation for J-holomorphic curves on T^*M. The quadratic behavior of H and the growth condition on the action integrand are crucial in the proof of the L^∞ estimates for the solutions of the Floer equation having bounded action.

Here we consider orbits $x : [0,1] \to T^*M$ satisfying the general non-local boundary condition, which in the Hamiltonian setting is just

$$(x(0), -x(1)) \in N^*Q, \tag{2}$$

where N^*Q is the conormal bundle of Q in the cotangent bundle of $M \times M$. We prove that if the Hamiltonian H has superlinear growth in p, its action integrand $DH[Y] - H$ is coercive, and the Hamiltonian vector field is complete, then the number of orbits satisfying (2) is greater than the $\mathbb{Z}_2$-cuplength of $C_Q([0,1], M)$. Moreover, if infinitely many $\mathbb{Z}_2$-homology groups of $C_Q([0,1], M)$ are non-trivial, then there is a sequence of orbits satisfying (2) with diverging action. When Q is the diagonal of $M \times M$, we obtain the generalization of Cieliebak's theorem to a larger class of Hamiltonians. Notice also that if L is a Tonelli Lagrangian then its Fenchel transform H is in the above class, so the Hamiltonian setting considered here includes the Lagrangian one as a particular case. Since the whole argument in the Lagrangian setting is considerably simpler, we prove the Lagrangian statements independently.

We wish to stress the fact that the quadratic modification and the a priori estimates presented here constitute a natural and general argument (see for instance [6], where a similar idea allows

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[2]It may happen that $C_Q([0,1], M)$ has infinitely many non-vanishing Betti numbers but its cuplength is finite, as in the case of the free loop space of the sphere, see [14].

to drop an unnecessary technical assumption on the Hamiltonian in an existence result for homo-clinics). This argument should allow to generalize to Tonelli Lagrangians or to the superlinear Hamiltonians in the above class many other existence results, such as for instance Long's theorem on the existence of infinitely many contractible orbits of integer period for a time-periodic Lagrangian system on the torus, see [10].

1 Basic facts about Lagrangian and Hamiltonian systems

In this section we recall some basic facts about Lagrangian and Hamiltonian dynamics. See for instance [11], [3], [7] for detailed proofs. Let M be a smooth compact connected n-dimensional manifold without boundary. The elements of the tangent bundle TM are denoted by (q, v), where $q \in M$ and $v \in T_q M$. Let $L \in C^2([0, 1] \times TM, \mathbb{R})$ be a Tonelli Lagrangian, meaning that:

(L1) L is C^2-strictly convex on the fibers of TM, that is $\partial_{vv} L > 0$;

(L2) L is superlinear on the fibers of TM, that is for any $K > 0$ there is a finite constant $C(K)$ such that
$$\forall (t, q, v) \in [0, 1] \times TM, \quad L(t, q, v) \geq K|v|_q - C(K)$$

Here $|v|_q$ is a Riemannian norm on M, but by the compactness of M condition (L2) does not depend on the choice of the Riemannian metric on M (up to changing the constant $C(K)$).

Under assumptions (L1) and (L2), the Legendre transform
$$\mathscr{L}_L : [0, 1] \times TM \to [0, 1] \times T^*M, \quad (t, q, v) \mapsto (t, q, D_v L(t, q, v)),$$

is a C^1 diffeomorphism. The Fenchel transform of L is the non-autonomous Hamiltonian on T^*M
$$H(t, q, p) := \max_{v \in T_q M} (p[v] - L(t, q, v)) = p[v(t, q, p)] - L(t, q, v(t, q, p)),$$

where $(t, q, v(t, q, p)) = \mathscr{L}_L^{-1}(t, q, p)$. Under the above assumptions on L, H is a C^2 function on $[0, 1] \times T^*M$, it is C^2-strictly convex and superlinear on the fibers of T^*M. The associated non-autonomous Hamiltonian vector field X_H on T^*M, defined by
$$\omega(X_H(t, q, p), \xi) = -DH(t, q, p)[\xi], \quad \forall (t, q, p) \in [0, 1] \times T^*M, \ \xi \in T_{(q,p)} T^*M, \tag{3}$$

where $\omega = dp \wedge dq$ is the standard symplectic structure on T^*M, is then C^1, so it defines a non-autonomous C^1 local flow on T^*M. We assume that such a flow is complete:

(L3) The solution of
$$\partial_t \phi^H(t, x) = X_H(t, \phi^H(t, x)), \quad \phi^H(0, x) = x$$
exists for every $(t, x) \in [0, 1] \times T^*M$.

We also use the notation $\phi_t^H(\cdot) = \phi^H(t, \cdot)$. Assumption (L3) holds, for example, if H satisfies the condition
$$\partial_t H(t, q, p) \leq c(1 + H(t, q, p)) \quad \forall (t, q, p) \in [0, 1] \times T^*M. \tag{4}$$
Indeed, since $DH[X_H] = 0$, (4) implies
$$\frac{d}{dt} H(t, \phi^H(t, x)) = \partial_t H(t, \phi^H(t, x)) \leq c(1 + H(t, \phi^H(t, x))),$$

so, by Gronwall Lemma, H is bounded along the flow, which then exists for every $t \in [0, 1]$, by the coercivity of H. Condition (4), written in terms of L, becomes
$$-\partial_t L(t, q, v) \leq c(1 + D_v L(t, q, v)[v] - L(t, q, v)) \quad \forall (t, q, v) \in [0, 1] \times TM.$$

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Notice that if L - and thus H - is 1-periodic in time, condition (L3) is equivalent to the existence of the flow for every $(t,x) \in \mathbb{R} \times T^*M$. The corresponding flow on TM, obtained by conjugating ϕ^H by the Legendre transform $\mathscr{L}_L$, is denoted by

$$\phi^L : [0,1] \times TM \to TM.$$

Its orbits have the form $t \mapsto (\gamma(t), \gamma'(t))$, where $\gamma \in C^2([0,1], M)$ solves the Euler-Lagrange equation, which in local coordinates is written as

$$\frac{d}{dt} \partial_v L(t, \gamma(t), \gamma'(t)) = \partial_q L(t, \gamma(t), \gamma'(t)). \tag{5}$$

These orbits are precisely the extremal curves of the Lagrangian action functional

$$\mathbb{A}_L(\gamma) := \int_0^1 L(t, \gamma(t), \gamma'(t)) \, dt.$$

Assumptions (L1), (L2), and (L3) imply that solutions with bounded action are C^2-bounded:

LEMMA 1.1 *Assume that the C^2 Lagrangian $L : [0,1] \times TM \to \mathbb{R}$ satisfies (L1), (L2), (L3). For every $A \in \mathbb{R}$, the set*

$$\left\{ \gamma \in C^2([0,1], M) \mid \gamma \text{ is a solution of (5) with } \mathbb{A}_L(\gamma) \leq A \right\}$$

is bounded in $C^2([0,1], M)$.

Proof. Since

$$\int_0^1 L(t, \gamma(t), \gamma'(t)) \, dt \leq A,$$

there exists $t_0 \in [0,1]$ such that

$$L(t_0, \gamma(t_0), \gamma'(t_0)) \leq A.$$

By assumption (L2), we have in particular $L(t, q, v) \geq |v|_q - C(1)$. So we get

$$|\gamma'(t_0))|_{\gamma(t_0)} \leq A + C(1).$$

Since the Lagrangian flow is a globally defined continuous family of homeomorphisms, the set

$$K := \left\{ \phi_t^L \circ (\phi_s^L)^{-1}(q, v) \mid (s, t) \in [0,1]^2, \ (q, v) \in TM, \ |v|_q \leq A + C(1) \right\}$$

is compact. The point $(\gamma(t), \gamma'(t)) = \phi_t^L \circ (\phi_{t_0}^L)^{-1}(\gamma(t_0), \gamma'(t_0))$ belongs to K for every $t \in [0,1]$, so we have an uniform bound in C^1. By computing the time derivative in left-hand side of (5) and by using assumption (L1), we get in local coordinates

$$\gamma''(t) = (\partial_{vv} L)^{-1} \left[\partial_q L - \partial_t \partial_v L - \partial_{qv} L \left[\gamma'(t) \right] \right],$$

everything being evaluated at $(t, \gamma(t), \gamma'(t))$. This gives a uniform bound in C^2. $\qquad \square$

Let Q be a smooth closed submanifold of $M \times M$. We are interested in the solutions $\gamma : [0,1] \to M$ of (5) which satisfy the nonlocal boundary condition

$$\begin{aligned}
(\gamma(0), \gamma(1)) &\in Q, \\
D_v L(0, \gamma(0), \gamma'(0))[\xi_0] &= D_v L(1, \gamma(1), \gamma'(1))[\xi_1], \quad \forall (\xi_0, \xi_1) \in T_{(\gamma(0), \gamma(1))} Q.
\end{aligned} \tag{6}$$

Equivalently, the corresponding orbit $x : [0,1] \to T^*M$ of X_H, that is $(t, x(t)) = \mathscr{L}_L(t, \gamma(t), \gamma'(t))$, is required to satisfy the nonlocal Lagrangian condition

$$(x(0), -x(1)) \in N^*Q,$$

where $N^*Q \subset T^*(M \times M)$ denotes the conormal bundle of Q (we recall that the conormal bundle of a submanifold Q of a manifold N - here $N = M \times M$ - is the Lagrangian submanifold of T^*N consisting of all (q,p) with $q \in Q$ and p vanishing on T_qQ).

For instance, if $Q = Q_0 \times Q_1$ is the product of two submanifolds of M, we are looking at solutions γ which join Q_0 to Q_1, and are normal to Q_0 and Q_1 in a sense specified by the Lagrangian L. In particular, if Q_0 and Q_1 are two points, we are looking at solutions joining them without any further condition. If L is assumed to be 1-periodic in the time variable t, and $Q = \Delta$ is the diagonal submanifold of $M \times M$, the solutions of (5), (6) are exactly the 1-periodic solutions of the Euler-Lagrange equation. Finally, if $Q = M \times M$, condition (6) reduces to the Neumann condition

$$D_v L(0, \gamma(0), \gamma'(0)) = 0, \quad D_v L(1, \gamma(1), \gamma'(1)) = 0, \tag{7}$$

or equivalently, by convexity,

$$L(0, \gamma(0), \gamma'(0)) = \min_{v \in T_{\gamma(0)}M} L(0, \gamma(0), v), \quad L(1, \gamma(1), \gamma'(1)) = \min_{v \in T_{\gamma(1)}M} L(1, \gamma(1), v).$$

In order to write the first variation of the action at a curve $\gamma \in C^1([0,1], M)$, it is convenient to consider an open subset U of $\mathbb{R}^n$ and a smooth local coordinate system

$$[0,1] \times U \to [0,1] \times M, \quad (t,q) \mapsto (t, \varphi(t,q)), \tag{8}$$

such that $\gamma(t) = \varphi(t, \tilde{\gamma}(t))$ for every $t \in [0,1]$, for some $\tilde{\gamma} \in C^1([0,1], U)$ (see the appendix for a possible construction of such a map). Its differential, that is the tangent bundle coordinate system

$$\Phi : [0,1] \times U \times \mathbb{R}^n \to [0,1] \times TM, \quad \Phi(t,q,v) = (t, \varphi(t,q), D\varphi(t,q)[1,v]),$$

allows to pull back the Lagrangian L onto $[0,1] \times U \times \mathbb{R}^n$, by setting $\tilde{L} = L \circ \Phi$. If ξ is a variation of γ, that is a C^1 section of the vector bundle $\gamma^*(TM)$ over $[0,1]$, and $\tilde{\xi}(t) = D_q\varphi(t, \tilde{\gamma}(t))^{-1}[\xi(t)]$ is the corresponding variation of $\tilde{\gamma}$, the first variation of $\mathbb{A}_L$ at γ along ξ is

$$\begin{aligned} d\mathbb{A}_L(\gamma)[\xi] = d\mathbb{A}_{\tilde{L}}(\tilde{\gamma})[\tilde{\xi}] &= \int_0^1 \left(D_q\tilde{L}(t, \tilde{\gamma}, \tilde{\gamma}')[\tilde{\xi}] + D_v\tilde{L}(t, \tilde{\gamma}, \tilde{\gamma}')[\tilde{\xi}'] \right) dt \\ &= \int_0^1 \left(D_q\tilde{L}(t, \tilde{\gamma}, \tilde{\gamma}') - \frac{d}{dt} D_v\tilde{L}(t, \tilde{\gamma}, \tilde{\gamma}') \right) [\tilde{\xi}] \, dt \\ &\quad + D_v L(1, \gamma(1), \gamma'(1))[\xi(1)] - D_v L(0, \gamma(0), \gamma'(0))[\xi(0)], \end{aligned} \tag{9}$$

the last identity holding if γ, and thus also $\tilde{\gamma}$, is C^2. Together with a standard regularity argument, this formula shows that the solutions of (5), (6) are precisely the extremal curves of $\mathbb{A}_L$ on the space of curves $\gamma \in C^1([0,1], M)$ such that $(\gamma(0), \gamma(1)) \in Q$.

Let γ be a solution of (5), (6). The second variation of $\mathbb{A}_L$ at γ is given by the formula:

$$\begin{aligned} d^2\mathbb{A}_L(\gamma)[\xi, \eta] = d^2\mathbb{A}_{\tilde{L}}(\tilde{\gamma})[\tilde{\xi}, \tilde{\eta}] &= \int_0^1 \Big(D_{vv}\tilde{L}(t, \tilde{\gamma}, \tilde{\gamma}')[\tilde{\xi}', \tilde{\eta}'] + D_{qv}\tilde{L}(t, \tilde{\gamma}, \tilde{\gamma}')[\tilde{\xi}, \tilde{\eta}'] \\ &\quad + D_{vq}\tilde{L}(t, \tilde{\gamma}, \tilde{\gamma}')[\tilde{\xi}', \tilde{\eta}] + D_{qq}\tilde{L}(t, \tilde{\gamma}, \tilde{\gamma}')[\tilde{\xi}, \tilde{\eta}] \Big) dt. \end{aligned} \tag{10}$$

It is a continuous symmetric bilinear form on the Hilbert space W consisting of the $W^{1,2}$ sections ξ of $\gamma^*(TM)$ such that $(\xi(0), \xi(1)) \in T_{(\gamma(0), \gamma(1))}Q$. This bilinear form is a compact perturbation of the form

$$(\xi, \eta) \mapsto \int_0^1 \left(D_{vv}\tilde{L}(t, \tilde{\gamma}, \tilde{\gamma}')[\tilde{\xi}', \tilde{\eta}'] + \tilde{\xi} \cdot \tilde{\eta} \right) dt,$$

which is coercive on W by (L1). Therefore, the Morse index $m_Q(\gamma, L)$ (resp. the large Morse index $m_Q^*(\gamma, L)$), i.e. the sum of the multiplicities of the negative (resp. non-positive) eigenvalues of the selfadjoint operator on W representing $d^2\mathbb{A}_L(\gamma)$, is finite. Equivalently, $m_Q(\gamma, L)$ (resp. $m_Q^*(\gamma, L)$) is the dimension of a maximal linear subspace of W on which $d^2\mathbb{A}_L(\gamma)$ is negative

(resp. non-positive). Moreover, since the elements in the kernel of $d^2 \mathbb{A}_L(\gamma)$ solve a system of n second order ODEs, we have the bound

$$0 \le m_Q^*(\gamma, L) - m_Q(\gamma, L) \le 2n.$$

LEMMA 1.2 *Assume that the C^2 Lagrangian $L : [0,1] \times TM \to \mathbb{R}$ satisfies (L1), (L2), (L3). For every $A \in \mathbb{R}$, there exists $N \in \mathbb{N}$ such that, for every γ solution of (5), (6) with $\mathbb{A}_L(\gamma) \le A$, there holds $m_Q^*(\gamma, L) \le N$.*

Proof. By Lemma 1.1 and Ascoli-Arzelà theorem, the set of solutions of (5), (6) with $\mathbb{A}_L(\gamma) \le A$ is compact in the C^1 topology. Formula (10) shows that the map $\gamma \mapsto d^2 \mathbb{A}_L(\gamma)$ is continuous from the C^1 topology of curves to the operator norm topology of $W^{1,2}$. Since the large Morse index is upper semi-continuous in the operator norm topology, the thesis follows. □

2   Statement of the Lagrangian results

Let $C_Q([0,1], M)$ be the space of continuous paths $\gamma : [0,1] \to M$ such that $(\gamma(0), \gamma(1)) \in Q$, endowed with the C^0 topology. We recall that the cuplength of a topological space X is the number

$$\text{cuplength}\,(X) = \sup\{m \in \mathbb{N} \,|\, \exists\, \omega_1, \ldots, \omega_m \in H^*(X), \ \deg \omega_j \ge 1\ \forall j = 1, \ldots, m,$$
$$\text{such that } \omega_1 \cup \cdots \cup \omega_m \ne 0\}.$$

Here $H^*(X)$ is the singular cohomology ring of X with integer coefficients, and $\cup$ denotes the cup product. The main result of this note concerning Lagrangian systems is the following:

THEOREM 2.1 *Let M be a compact manifold, Q a smooth closed submanifold of $M \times M$, and $L \in C^2([0,1] \times TM, \mathbb{R})$ a Lagrangian satisfying (L1), (L2), and (L3).*

(a) *The Lagrangian non-local boundary value problem (5), (6) has at least*

$$\text{cuplength}\,(C_Q([0,1], M)) + 1$$

many solutions.

(b) *If the k-th singular homology group $H_k(C_Q([0,1], M))$ is non-trivial, then problem (5), (6) has a solution γ with Morse index estimates*

$$m_Q(\gamma, L) \le k \le m_Q^*(\gamma, L).$$

(c) *If the k-th singular group $H_k(C_Q([0,1], M))$ is non-trivial for infinitely many natural numbers k, then problem (5), (6) has an infinite sequence of solutions with diverging action and diverging Morse index.*

In the case $Q = \Delta$, the diagonal in $M \times M$, $C_\Delta([0,1], M)$ coincides with $\Lambda(M)$, the space of free loops in M. If M is compact and simply connected, $H_k(\Lambda(M)) \ne 0$ for infinitely many natural numbers k (as proved by Sullivan in [14]), so conclusion (c) of the above theorem holds. If M is a compact manifold with finite fundamental group, the above considerations can be applied to its universal covering (which is still compact), producing contractible periodic solutions on M. These considerations yield to the following:

COROLLARY 2.2 *Assume that the compact manifold M has finite fundamental group, and let $L : \mathbb{R} \times TM \to \mathbb{R}$ be a C^2 Lagrangian 1-periodic in time and satisfying (L1), (L2), and (L3). Then the Euler-Lagrange equation (5) has an infinite sequence of 1-periodic contractible solutions, with diverging action and diverging Morse index.*

We remark that, in the autonomous case, the corollary above still gives infinitely many solutions, which are distinct as curves in the phase space TM since their action diverges. However, in the case of the geodesic flow, all these solutions could be the same geodesic parametrized with increasing speed.

When M is simply connected, also $\Omega(M)$ - the space of based loops in M - has infinitely many non-vanishing homology groups (a classical result by Serre, [13]). Moreover, if Q is the singleton $\{(q_0, q_1)\}$, with $q_0, q_1 \in M$, the space $C_Q([0,1], M)$ is homotopically equivalent to $\Omega(M)$. Therefore, conclusion (c) of Theorem 2.1 implies the following:

COROLLARY 2.3 *Assume that the compact manifold M has finite fundamental group, and let $L : [0,1] \times TM \to \mathbb{R}$ be a C^2 Lagrangian satisfying (L1), (L2), and (L3). Then for every pair of points q_0, q_1 in M and for every continuous path $\overline{\gamma}$ joining them, there is an infinite sequence of solutions γ of (5) joining q_0 and q_1, homotopic to $\overline{\gamma}$, with diverging action and diverging Morse index.*

In the case $Q = Q_0 \times M$, with Q_0 a closed submanifold of M, the map

$$C_{Q_0 \times M}([0,1], M) \to Q_0, \quad \gamma \mapsto \gamma(0),$$

is a homotopy equivalence, a homotopy inverse of it being the function mapping each $q \in Q_0$ into the constant path $\gamma(t) \equiv q$. Therefore statement (a) in Theorem 2.1 has the following consequence:

COROLLARY 2.4 *Assume that M is a compact manifold, and let $L : [0,1] \times TM \to \mathbb{R}$ be a C^2 Lagrangian satisfying (L1), (L2), and (L3). Then the Euler-Lagrange equation (5) has at least* cuplength $(Q_0) + 1$ *many solutions γ satisfying the boundary conditions*

$$\gamma(0) \in Q_0, \quad D_v L(0, \gamma(0), \gamma'(0))|_{T_{\gamma(0)} Q_0} = 0, \quad D_v L(1, \gamma(1), \gamma'(1)) = 0.$$

In particular, the Neumann problem (7) for the Euler-Lagrange equation (5) has at least cuplength $(M) + 1$ *many solutions.*

3 Quadratic Lagrangians

Let us assume that the C^2 Lagrangian $L : [0,1] \times M \to \mathbb{R}$ satisfies the conditions:

(L1') There is a constant $\ell_0 > 0$ such that $\partial_{vv} L(t, q, v) \geq \ell_0 I$.

(L2') There is a constant $\ell_1 > 0$ such that

$$\|\partial_{vv} L(t, q, v)\| \leq \ell_1, \quad \|\partial_{qv} L(t, q, v)\| \leq \ell_1 (1 + |v|_q), \quad \|\partial_{qq} L(t, q, v)\| \leq \ell_1 (1 + |v|_q^2).$$

Condition (L1') is expressed in term of a Riemannian metric on M, but the condition does not depend on such a metric (up to changing the constant ℓ_0). Condition (L2') is expressed in terms of a system of local coordinates on M and of a Riemannian metric on M, but the condition does not depend on these choices (up to changing the constant ℓ_1). Assumption (L1') implies that L grows at least quadratically in v, while (L2') implies that L grows at most quadratically.

We observe that assumption (L1') implies both (L1) and (L2). Under these assumptions, Benci [2] has shown that the $W^{1,2}$ functional setting used in the study of geodesics allows to prove existence results for the critical points of the action functional $\mathbb{A}_L$. The aim of this section is to recall Benci's results, extending them from the periodic case to the case of a general boundary condition of the form (6).

Let $W^{1,2}([0,1], M)$ be the space of all curves $\gamma : [0,1] \to M$ of Sobolev class $W^{1,2}$. This space has a natural structure of a Hilbert manifold modeled on $W^{1,2}([0,1], \mathbb{R}^n)$. Indeed, a smooth atlas on $W^{1,2}([0,1], M)$ is defined by composition by diffeomorphisms of the form (8). Its tangent space at $\gamma \in W^{1,2}([0,1], M)$ is identified with the space of $W^{1,2}$ sections of the vector bundle $\gamma^*(TM)$. For sake of completeness, the following proposition, as well as Proposition 3.2 below, are proved in the appendix.

PROPOSITION 3.1 *If L satisfies (L2') then $\mathbb{A}_L$ is of class C^2 on $W^{1,2}([0,1], M)$, and $D\mathbb{A}_L(\gamma)$ coincides with $d\mathbb{A}_L(\gamma)$, given by (9), while $D^2\mathbb{A}_L(\gamma)$ at a critical point γ coincides with $d^2\mathbb{A}_L(\gamma)$, given by (10).*

Let $W_Q^{1,2}([0,1], M)$ be the set of $\gamma \in W^{1,2}([0,1], M)$ such that $(\gamma(0), \gamma(1)) \in Q$. Being the inverse image of Q by the smooth submersion

$$W^{1,2}([0,1], M) \to M \times M, \quad \gamma \mapsto (\gamma(0), \gamma(1)),$$

$W_Q^{1,2}([0,1], M)$ is a closed smooth submanifold of $W^{1,2}([0,1], M)$. By Proposition 3.1 and identity (9), the restriction of $\mathbb{A}_L$ is C^2, and the critical points of such a restriction are exactly the solutions of (5), (6). The second differential of $\mathbb{A}_L|_{W_Q^{1,2}([0,1],M)}$ at a critical point γ is the restriction of $D^2\mathbb{A}_L(\gamma)$ to $T_\gamma W_Q^{1,2}([0,1], M)$, and it coincides with the corresponding restriction of $d^2\mathbb{A}_L(\gamma)$, given by (10). In particular the Hessian of $\mathbb{A}_L|_{W_Q^{1,2}([0,1],M)}$ at a critical point is a Fredholm operator. Therefore, the Morse index and the large Morse index of such a critical point are the numbers $m_Q(\gamma, L)$ and $m_Q^*(\gamma, L)$ defined in section 1.

A Riemannian metric $\langle \cdot, \cdot \rangle$ on M induces a complete Riemannian metric on the Hilbert manifold $W^{1,2}([0,1], M)$, namely

$$\langle \xi, \eta \rangle_{W^{1,2}} := \int_0^1 \left(\langle \nabla_t \xi, \nabla_t \eta \rangle_{\gamma(t)} + \langle \xi, \eta \rangle_{\gamma(t)} \right) dt, \quad \forall \gamma \in W^{1,2}([0,1], M), \ \xi, \eta \in T_\gamma W^{1,2}([0,1], M), \tag{11}$$

where ∇_t denotes the Levi-Civita covariant derivative along γ. We recall that a C^1 functional f on a Riemannian Hilbert manifold $(\mathscr{M}, \|\cdot\|)$ satisfies the Palais-Smale condition if every sequence $(u_h) \in \mathscr{M}$ such that $f(u_h)$ is bounded and $\|Df(u_h)\|_*$ is infinitesimal is compact (here $\|\cdot\|_*$ denotes the dual norm on $T_{u_h}^* \mathscr{M}$).

PROPOSITION 3.2 *If L satisfies (L1') and (L2'), then the restriction of $\mathbb{A}_L$ to $W_Q^{1,2}([0,1], M)$ satisfies the Palais-Smale condition.*

Let $\alpha, \beta \in H_*(X)$ be non-zero singular homology classes in the topological space X, with $\deg \alpha < \deg \beta$. We recall that α is said to be subordinate to β, $\alpha < \beta$, if there exists a singular cohomology class $\omega \in H^*(X)$ such that $\alpha = \beta \cap \omega$, where $\cap : H_{p+q}(X) \otimes H^q(X) \to H_p(X)$ denotes the cap product. If $\alpha < \beta$ and b is a singular cycle representing β, there exists a singular cycle a representing α with support contained in the support of b.

The fact that the Lagrangian action functional $\mathbb{A}_L$ is C^2, bounded from below (because so is L), satisfies the Palais-Smale condition on a complete Riemannian manifold, and has a Fredholm Hessian, implies the following result:

THEOREM 3.3 *Let M be a compact manifold and let Q be a smooth closed submanifold of $M \times M$. Assume that the C^2 Lagrangian $L : [0,1] \times TM \to \mathbb{R}$ satisfies (L1') and (L2').*

(a) *Let $\alpha \in H_k(W_Q^{1,2}([0,1], M))$, $k \in \mathbb{N}$, be a non-zero homology class, and let $\mathscr{K}_\alpha$ be the family consisting of the supports of the singular cycles in $W_Q^{1,2}([0,1], M)$ representing α. Then the number*

$$c_\alpha(L) := \inf_{K \in \mathscr{K}_\alpha} \max_{\gamma \in K} \mathbb{A}_L(\gamma)$$

is a critical level of the action function $\mathbb{A}_L$, and there is a critical point γ with

$$\mathbb{A}_L(\gamma) = c_\alpha(L), \quad m_Q(\gamma, L) \leq k \leq m_Q^*(\gamma, L).$$

(b) *If $\alpha_1 < \cdots < \alpha_m$ are non-zero homology classes in $H_*(W_Q^{1,2}([0,1], M))$, then*

$$c_{\alpha_1}(L) \leq \cdots \leq c_{\alpha_m}(L),$$

and either all the inequalities are strict, or one of the critical levels $c_{\alpha_j}(L)$, $j = 2, \ldots, m$, contains a continuum of critical points.

Indeed, everything follows from the abstract results of section I.3.2 in [4], apart from the Morse index estimate which is proved in [15].

4 Convex quadratic modifications

Let $L \in C^2([0,1] \times TM, \mathbb{R})$ be a Lagrangian satisfying (L1), (L2), and (L3).

DEFINITION 4.1 *We say that a Lagrangian $L_0 \in C^2([0,1] \times TM, \mathbb{R})$ is a convex quadratic R-modification of L if:*

(a) *$L_0(t,q,v) = L(t,q,v)$ for $|v|_q \leq R$;*

(b) *L_0 satisfies (L1') and (L2');*

(c) *$L_0(t,q,v) \geq |v|_q - C(1)$, where $C(1)$ is defined in condition (L2).*

Constructing a convex quadratic R-modifications of a Tonelli Lagrangian L is not difficult, although some care is needed in order to preserve the convexity and the linear lower bound. Here is a possible construction.

Let $\varphi : \mathbb{R} \to \mathbb{R}$ be a smooth function such that $\varphi(s) = s$ for $s \leq 1$ and $\varphi(s)$ is constant for $s \geq 2$. We fix a positive number λ such that

$$\lambda \geq \max \left\{ L(t,q,v) \mid (t,q,v) \in [0,1] \times TM, \ |v|_q \leq 2R \right\}, \tag{12}$$

and we define a new Lagrangian L_1 on $[0,1] \times TM$ by

$$L_1 := \lambda \varphi \left(\frac{L}{\lambda} \right).$$

Since L is coercive, L_1 is constant outside a compact set, so we can find a positive number μ such that

$$\mu I \geq -\partial_{vv} L_1, \quad \text{on } [0,1] \times TM. \tag{13}$$

Up to replacing μ with a larger number, we may also assume that

$$4R\mu \geq 1, \quad 2R^2\mu \geq 2R - C(1) - \min L_1. \tag{14}$$

The affine function $s \mapsto \mu s - 2\mu R^2$ is negative on $(-\infty, R^2]$ and positive on $[4R^2, +\infty)$, so we can find a smooth convex function $\psi : \mathbb{R} \to \mathbb{R}$ such that $\psi(s) = 0$ for $s \leq R^2$ and $\psi(s) = \mu s - 2\mu R^2$ for $s \geq 4R^2$. We define the Lagrangian L_0 by

$$L_0(t,q,v) := L_1(t,q,v) + \psi \left(|v|_q^2 \right).$$

By (12),
$$L_0(t,q,v) = L(t,q,v) + \psi \left(|v|_q^2 \right), \quad \text{for } |v|_q \leq 2R, \tag{15}$$

and in particular L_0 coincides with L for $|v|_q \leq R$, proving property (a) of Definition 4.1. By (15) and the convexity and the monotonicity of ψ,

$$\partial_{vv} L_0(t,q,v) \geq \partial_{vv} L(t,q,v), \quad \text{for } |v|_q \leq 2R. \tag{16}$$

On the other hand, by (13),

$$\partial_{vv} L_0(t,q,v) = \partial_{vv} L_1(t,q,v) + 2\mu I \geq \mu I, \quad \text{for } |v|_q \geq 2R. \tag{17}$$

Since L is C^2-strictly convex, (16) and (17) imply that L_0 satisfies (L1'). Since L_1 is constant outside a compact set,
$$L_0(t,q,v) = \mu |v|_q^2 + \text{constant},$$

for $|v|_q$ large, so L_0 satisfies also (L2'), concluding the proof of property (b) in Definition 4.1. By (15) and (L2),
$$L_0(t,q,v) \geq L(t,q,v) \geq |v|_q - C(1), \quad \text{for } |v|_q \leq 2R. \tag{18}$$

On the other hand, by (14),

$$L_0(t, q, v) \geq \min L_1 + \mu|v|_q^2 - 2\mu R^2 \geq |v|_q - C(1), \quad \text{for } |v|_q \geq 2R. \tag{19}$$

Indeed, (14) implies that

$$\mu s^2 - s + \min L_1 - 2\mu R^2 + C(1) \geq 0, \quad \text{for } s \geq 2R,$$

as it easily seen by evaluating the above polynomial and its derivative at $s = 2R$. Inequalities (18) and (19) imply that L_0 satisfies property (c) of Definition 4.1. Therefore, L_0 is a convex quadratic R-modification of L.

If L is 1-periodic in time, the above construction produces a Lagrangian which is also 1-periodic in time.

LEMMA 4.2 *Let M be a compact manifold and $L \in C^2([0,1] \times TM, \mathbb{R})$ be a Lagrangian satisfying (L1), (L2), (L3). For every $A > 0$ there exists a number $R(A)$ such that for any $R > R(A)$ and for any Lagrangian L_0 which is a convex quadratic R-modification of L, the following holds: if γ is a critical point of $\mathbb{A}_{L_0}$ such that $\mathbb{A}_{L_0}(\gamma) \leq A$, then $\|\gamma'\|_\infty \leq R(A)$. In particular, such a γ is an extremal curve of $\mathbb{A}_L$, and $\mathbb{A}_{L_0}(\gamma) = \mathbb{A}_L(\gamma)$.*

Proof. Let $C(1)$ be the constant appearing in assumption (L2). Since the Lagrangian flow of L is globally defined, as in the proof of Lemma 1.1 we find that the set

$$K := \left\{ \phi_t^L \circ (\phi_s^L)^{-1}(q, v) \mid (s, t) \in [0,1]^2, \ (q, v) \in TM, \ |v|_q \leq A + C(1) \right\}$$

is compact. Notice that K contains every $(q, v) \in TM$ with $|v|_q \leq A + C(1)$. Let $R(A)$ be the maximum of $|v|_q$ for $(q, v) \in K$. Let $R > R(A)$, L_0 a convex quadratic R-modification of L, and γ a critical point of $\mathbb{A}_{L_0}$ such that $\mathbb{A}_{L_0}(\gamma) \leq A$. Then there exists $t_0 \in [0,1]$ such that

$$|\gamma'(t_0)|_{\gamma(t_0)} - C(1) \leq L_0(t_0, \gamma(t_0), \gamma'(t_0)) \leq \mathbb{A}_{L_0}(\gamma) \leq A,$$

where the first inequality follows by (c) in Definition 4.1. So we get

$$|\gamma'(t_0))|_{\gamma(t_0)} \leq A + C(1),$$

which implies that $(\gamma(t_0), \gamma'(t_0)) \in K$. Let $I \subset [0,1]$ be the maximal interval containing t_0 such that $(\gamma(t), \gamma'(t)) \in K$ for every $t \in I$. If $t \in I$, then $|\gamma'(t))|_{\gamma(t)} \leq R(A) < R$. Therefore L and L_0 coincide in a neighborhood of $(t, \gamma(t), \gamma'(t))$, and so do the Lagrangian vector fields defined by L and L_0. Hence, by the definition of K, $(\gamma(s), \gamma'(s))$ belongs to K for s in a neighborhood of t in $[0,1]$, proving that I is open in $[0,1]$. Being obviously closed, I coincides with $[0,1]$. Therefore $\|\gamma'\|_\infty \leq R(A)$. $\qquad \square$

5 Proof of Theorem 2.1

We denote by $C_Q^k([0,1], M)$ the space of C^k curves $\gamma : [0,1] \to M$ such that $(\gamma(0), \gamma(1)) \in Q$, endowed with the C^k topology, for $1 \leq k \leq \infty$. Let $\alpha \in H_*(C_Q([0,1], M))$ be a non-zero homology class. Since the inclusions

$$C_Q^k([0,1], M) \hookrightarrow W_Q^{1,2}([0,1], M) \hookrightarrow C_Q([0,1], M), \quad 1 \leq k \leq \infty, \tag{20}$$

are homotopy equivalences, we may regard α as a non-zero homology class on each of these different spaces of curves. Let $\mathscr{K}_\alpha'$ be the (non-empty) family of supports of singular cycles in $C_Q^1([0,1], M)$ representing the homology class α. Since singular chains have compact support, and since $\mathbb{A}_L$ is bounded from below, and continuous on $C_Q^1([0,1], M)$,

$$c_\alpha'(L) := \inf_{K \in \mathscr{K}_\alpha'} \max_{\gamma \in K} \mathbb{A}_L(\gamma)$$

is a finite number.

Let $m \geq 1$ be an integer, and let $\alpha_1 < \cdots < \alpha_m$ be non-zero elements of $H_*(C_Q([0,1], M))$. Let $K_j \in \mathscr{K}'_{\alpha_j}$ be such that

$$\max_{\gamma \in K_j} \mathbb{A}_L(\gamma) \leq c'_{\alpha_j}(L) + 1, \quad \forall j = 1, \ldots, m.$$

Set

$$A := \max \left\{ c'_{\alpha_j}(L) + 1 \mid j = 1, \ldots, m \right\} = c'_{\alpha_m}(L) + 1.$$

We choose $R > 0$ so large that $R > R(A)$, the number provided by Lemma 4.2, and

$$R \geq \sup \left\{ \|\gamma'\|_\infty \; \Big| \; \gamma \in \bigcup_{j=1}^{m} K_j \right\}. \tag{21}$$

Let L_0 be a convex quadratic R-modification of L. Since the first inclusion in (20) is a homotopy equivalence, the family $\mathscr{K}'_{\alpha_j}$ is contained in the family $\mathscr{K}_{\alpha_j}$ appearing in Theorem 3.3. Therefore, taking also (21) into account and the fact that $L_0(t, q, v)$ coincides with $L(t, q, v)$ if $|v|_q \leq R$,

$$c_{\alpha_j}(L_0) := \inf_{K \in \mathscr{K}_{\alpha_j}} \max_{\gamma \in K} \mathbb{A}_{L_0}(\gamma) \leq \max_{\gamma \in K_j} \mathbb{A}_{L_0}(\gamma) = \max_{\gamma \in K_j} \mathbb{A}_L(\gamma) \leq c'_{\alpha_j}(L) + 1 \leq A.$$

By Theorem 3.3 (a) the functional $\mathbb{A}_{L_0}|_{W_Q^{1,2}([0,1], M)}$ has a critical point γ_j with $\mathbb{A}_{L_0}(\gamma_j) = c_{\alpha_j}(L_0)$ and

$$m_Q(\gamma_j; L_0) \leq \deg \alpha_j \leq m_Q^*(\gamma_j; L_0).$$

Since $\mathbb{A}_{L_0}(\gamma_j) = c_{\alpha_j}(L_0) \leq A$ and $R > R(A)$, by Lemma 4.2 $\|\gamma'_j\|_\infty \leq R(A) < R$. Since $L_0(t, q, v)$ coincides with $L(t, q, v)$ if $|v|_q \leq R$, the curve γ_j is a solution of (5), (6) for the Lagrangian L, and

$$m_Q(\gamma_j; L) = m_Q(\gamma_j; L_0) \leq \deg \alpha_j \leq m_Q^*(\gamma_j; L_0) = m_Q^*(\gamma_j; L).$$

By Theorem 3.3 (b), either the critical levels $c_{\alpha_j}(L_0)$ are all distinct - hence the curves γ_j are also distinct - or one of these levels contains a continuum of critical points of $\mathbb{A}_{L_0}$, and thus of $\mathbb{A}_L$. In any case, we deduce that the original problem (5), (6) has at least m solutions. Statement (a) of Theorem 2.1 follows from the identity

$$\text{cuplength}\,(X) + 1 = \sup \left\{ m \in \mathbb{N} \mid \exists\, \alpha_1 < \cdots < \alpha_m \text{ non-zero elements of } H_*(X) \right\},$$

(see [4], Theorem I.1.1).

Taking $m = 1$ and α_1 a non-zero element of $H_k(C_Q([0,1], M))$ in the above argument proves statement (b) of Theorem 2.1. Under the assumption of Theorem 2.1 (c), problem (5), (6) has then a sequence of solutions with diverging large Morse index. By Lemma 1.2, the fact that the large Morse index diverges implies that also the action diverges. This concludes the proof of Theorem 2.1.

REMARK 5.1 *A natural question is whether $c'_{\alpha_j}(L)$ is a critical value of $\mathbb{A}_L$ or not. We observe that, choosing compact sets $K_j \in \mathscr{K}'_{\alpha_j}$ such that*

$$\max_{\gamma \in K_j} \mathbb{A}_L(\gamma) \leq c'_{\alpha_j}(L) + \epsilon \quad \forall j = 1, \ldots, m$$

for a certain $\epsilon > 0$ small, by the above argument we get

$$\mathbb{A}_{L_\epsilon}(\gamma_\epsilon) = c_{\alpha_j}(L_\epsilon) \leq c'_{\alpha_j}(L) + \epsilon,$$

where L_ϵ is a R_ϵ-modification of L and γ_ϵ is a critical point for both L and L_ϵ. Now, since $\mathbb{A}_L(\gamma_\epsilon) = \mathbb{A}_{L_\epsilon}(\gamma_\epsilon) \leq c'_{\alpha_j}(L) + \epsilon$, the family of critical points (γ_ϵ) is bounded in C^2 (by Lemma 1.1), so, up to subsequences, it converges in the C^1 topology to a critical point γ_∞ which satisfies $\mathbb{A}_L(\gamma_\infty) \leq c'_{\alpha_j}(L)$.

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Let us now assume that L has at most quadratic growth in v, that is $L(t, q, v) \leq C(1 + |v|_q^2)$ for a certain constant C. In this case, up to enlarging R, we can assume that every R-modification L_0 of L satisfies $L_0 \geq L$. This implies that $\mathbb{A}_{L_0} \geq \mathbb{A}_L$, so we get

$$c_{\alpha_j}(L_0) = \inf_{K \in \mathscr{K}_{\alpha_j}} \max_{\gamma \in K} \mathbb{A}_{L_0}(\gamma) = \inf_{K \in \mathscr{K}'_{\alpha_j}} \max_{\gamma \in K} \mathbb{A}_{L_0}(\gamma) \geq \inf_{K \in \mathscr{K}'_{\alpha_j}} \max_{\gamma \in K} \mathbb{A}_L(\gamma) = c'_{\alpha_j}(L),$$

where in the second equality we used the density of $C_Q^1([0, 1], M)$ in $W_Q^{1,2}([0, 1], M)$ in the $W^{1,2}$ topology, and the continuity of $\mathbb{A}_{L_0}$ with respect to this topology. So, if γ_{ϵ_k} converges to γ_∞ as $k \to \infty$, we get in this case

$$c'_{\alpha_j}(L) \geq \mathbb{A}_L(\gamma_\infty) = \lim_k \mathbb{A}_L(\gamma_{\epsilon_k}) = \lim_k c_{\alpha_j}(L_{\epsilon_k}) \geq c'_{\alpha_j}(L),$$

which implies that the number $c'_{\alpha_j}(L)$ is indeed a critical value for the action of L. It is not clear to the authors whether this remains true for any Tonelli Lagrangian.

6 Statement of the Hamiltonian Results

Let $H \in C^2([0, 1] \times T^*M, \mathbb{R})$ be a time-dependent Hamiltonian on the cotangent bundle of the smooth compact connected n-dimensional manifold M. We denote by $\lambda = p\,dq$ the Liouville 1-form on T^*M, whose differential $d\lambda = dp \wedge dq$ is the standard symplectic form ω on T^*M. The Liouville vector field Y on T^*M is defined by

$$\omega(Y, \cdot) = \lambda, \tag{22}$$

which in local coordinates becomes $Y = \sum_i p_i \partial_{p_i}$. We denote by X_H the Hamiltonian vector field defined by (3), and by ϕ^H its non-autonomous local flow.

We assume that the Hamiltonian H satisfies the following conditions:

(H1) The action integrand $DH[Y] - H$ is coercive on $[0, 1] \times T^*M$, that is

$$DH(t, q, p)[Y(q, p)] - H(t, q, p) \geq a(|p|_q), \quad \text{where} \quad \lim_{s \to +\infty} a(s) = +\infty;$$

(H2) The function H is superlinear on the fibers of T^*M, that is

$$H(t, q, p) \geq h(|p|_q), \quad \text{where} \quad \lim_{s \to +\infty} \frac{h(s)}{s} = +\infty;$$

(H3) The non-autonomous flow ϕ^H is globally defined on $[0, 1] \times T^*M$.

REMARK 6.1 *For fixed $t \in [0, 1]$ and $(q, p) \in T^*M$ with $|p|_q = 1$, let us consider the function $f : [0, +\infty) \to \mathbb{R}$ defined as*

$$f(s) := DH(t, q, sp)[Y(q, sp)] - H(t, q, sp).$$

Then, if we set $g(s) := H(t, q, sp)$, we get $g'(s)s - g(s) = f(s)$, from which

$$\frac{g(s)}{s} = g(1) + \int_1^s \frac{f(\sigma)}{\sigma^2} \, d\sigma.$$

Thus we see that $g(s)$ is superlinear if and only if $f(s)/s^2$ is not integrable at infinity. Therefore, assumption (H2) implies that $f(s)/s^2$ is not integrable at infinity. Hence (H2) is in a certain sense "stronger" than (H1), and the function $a(s)$ appearing in (H1) is expected to grow at least linearly.

Let Q be a smooth closed submanifold of $M \times M$. We are interested in the solutions $x : [0,1] \to T^*M$ of

$$x'(t) = X_H(t, x(t)), \tag{23}$$

satisfying the boundary condition

$$(x(0), -x(1)) \in N^*Q, \tag{24}$$

where X_H is the Hamiltonian vector field defined by (3), and $N^*Q \subset T^*(M \times M)$ denotes the conormal bundle of Q (see section 1).

The Hamiltonian action functional acts on paths on T^*M, and it is defined as

$$\mathbb{A}_H(x) = \int_0^1 \left(\lambda[x'(t)] - H(t, x(t)) \right) dt,$$

where $x : [0,1] \to T^*M$. Notice that if $x : [0,1] \to T^*M$ is an orbit of X_H then by (22), (23) and (3) we have

$$\lambda[x'] - H(t, x) = \omega(Y(x), X_H(t, x)) - H(t, x) = DH(t, x)[Y(x)] - H(t, x). \tag{25}$$

The above identity explains why we refer to the quantity $DH[Y] - H$ as to the *action integrand*. Notice also that if H is the Fenchel transform of a Tonelli Lagrangian L, then the value of the action integrand along an orbit coincides pointwise with the value of the Lagrangian along the corresponding orbit on the tangent bundle.

The first variation of $\mathbb{A}_H$ on the space of free paths on T^*M is

$$d\mathbb{A}_H(x)[\xi] = \int_0^1 \left(\omega(\xi, x') - D_x H(t, x)[\xi] \right) dt + \lambda(x(1))[\xi(1)] - \lambda(x(0))[\xi(0)], \tag{26}$$

where ξ is a section of $x^*(TT^*M)$. The functional $\mathbb{A}_H$ is unbounded from above and from below. However, we have the following:

LEMMA 6.2 *Assume that the C^2 Hamiltonian $H : [0,1] \times T^*M \to \mathbb{R}$ satisfies (H1), (H3). For every $A \in \mathbb{R}$ the set*

$$\left\{ x \in C^2([0,1], T^*M) \mid x \text{ is a solution of (23) with } \mathbb{A}_H(x) \leq A \right\}$$

is bounded in $C^2([0,1], T^*M)$.

Proof. Since

$$\int_0^1 \left(\lambda[x'(t)] - H(t, x(t)) \right) dt \leq A,$$

there exists $t_0 \in [0,1]$ such that

$$DH(t_0, x(t_0))[Y(x(t_0))] - H(t_0, x(t_0)) = \lambda[x'(t_0)] - H(t_0, x(t_0)) \leq A,$$

where we have used (25) in the first equality. By assumption (H1), $x(t_0) = (q(t_0), p(t_0))$ belongs to the compact set $\{(p, q) \in T^*M \mid a(|p|_q) \leq A\}$. Since the Hamiltonian flow is a globally defined continuous family of homeomorphisms, also the set

$$K := \left\{ \phi_t^H \circ (\phi_s^H)^{-1}(p, q) \mid (s, t) \in [0,1]^2, \; (q, p) \in T^*M, \; a(|p|_q) \leq A \right\}$$

is compact. The point $x(t) = \phi_t^H \circ (\phi_{t_0}^H)^{-1}(x(t_0))$ belongs to K for every $t \in [0,1]$, so we have a uniform bound for x in C^0. Since x solves an ordinary differential equation with C^1 coefficients, we deduce a uniform bound in C^2. □

The aim of the remaining part of this note is to prove the following:

THEOREM 6.3 *Let M be a compact manifold, Q a smooth closed submanifold of $M \times M$, and $H \in C^2([0,1] \times T^*M, \mathbb{R})$ a Hamiltonian satisfying (H1), (H2), and (H3).*

(a) *The non-local boundary value problem (23), (24) has at least*

$$\text{cuplength}_{\mathbb{Z}_2}\left(C_Q([0,1], M)\right) + 1$$

many solutions.

(b) *If the k-th singular group $H_k(C_Q([0,1], M), \mathbb{Z}_2)$ is non-trivial for infinitely many natural numbers k, then problem (23), (24) has an infinite sequence of solutions with diverging action.*

The Hamiltonian analogues of Corollaries 2.2, 2.3, 2.4 are easily deduced from the above theorem.

REMARK 6.4 *The appearance of cohomology with $\mathbb{Z}_2$ coefficients in the above theorem is related to the fact that the proof makes use of degree theory for Fredholm maps. The extension to integer valued cohomology could be probably obtained by considering the concept of determinant bundle over the space of Fredholm operators (see [8]). The statements concerning the Morse index of solutions in Theorem 2.1 can also be extended to the Hamiltonian setting, by replacing the Morse index by the Maslov index of suitable paths of Lagrangian subspaces (see for instance [12], [16], and [1]). Since the Fenchel transform of a Tonelli Lagrangian satisfies (H1) and (H2), and since the Hamiltonian Maslov index coincides with the Lagrangian Morse index in this case, such an extension of Theorem 6.3 implies Theorem 2.1.*

7 Quadratic Hamiltonians

In [5] Cieliebak extended Floer's idea of using J-holomorphic curves in the study of periodic orbits of Hamiltonian systems on compact manifolds to the case of cotangent bundles. In this section we describe Cieliebak's results, in the setting of our more general boundary conditions.

Let us assume that the Hamiltonian $H : [0,1] \times T^*M \to \mathbb{R}$ is smooth and satisfies the conditions:

(H1') There are constants $h_0 > 0$, $h_1 \geq 0$, such that

$$DH(t,q,p)[Y(q,p)] - H(t,q,p) \geq h_0|p|_q^2 - h_1,$$

for every $(t,q,p) \in T^*M$.

(H2') There is a constant $h_2 \geq 0$ such that

$$\|\partial_q H(t,q,p)\| \leq h_2(1 + |p|_q^2), \quad \|\partial_p H(t,q,p)\| \leq h_2(1 + |p|_q),$$

for every $(t,q,p) \in T^*M$.

Condition (H1') is expressed in term of a norm on the vector bundle T^*M, but the condition does not depend on such a norm (up to changing the constants). Condition (H2') uses also a system of local coordinates on M and the induced system on T^*M, but the condition does not depend on these choices. Condition (H1') implies that H grows at least quadratically on the fibers (see Remark 6.1), while (H2') implies that H grows at most quadratically. In particular, (H1') implies both (H1) and (H2). Let us fix a superlinear function $h : [0, +\infty) \to \mathbb{R}$ such that

$$H(t,q,p) \geq h(|p|_q), \quad \forall (t,q,p) \in [0,1] \times T^*M. \tag{27}$$

Let us fix a Riemannian metric on M. Such a metric induces metrics on TM and T^*M, together with a horizontal-vertical splitting

$$TT^*M = T^h T^*M \oplus T^v T^*M,$$

and isomorphism $T^h_{(q,p)}T^*M \cong T_qM$, $T^v_{(q,p)}T^*M \cong T^*_qM \cong T_qM$. Let us consider the almost-complex structure J on T^*M which is represented by the matrix

$$J = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix},$$

with respect to such a splitting. It is ω-compatible, in the sense that $\omega(J\cdot,\cdot)$ is a Riemannian metric on T^*M (actually, it is the Riemannian metric induced by the one on M).

Let us consider the Floer equation

$$\partial_s u + J(u)[\partial_t u - X_H(t,u)] = 0, \tag{28}$$

where $u : \mathbb{R} \times [0,1] \to T^*M$, and (s,t) are the coordinates on $\mathbb{R} \times [0,1]$. It is a Cauchy-Riemann type first order elliptic PDE. The solutions of (28) which do not depend on s are precisely the orbits of the Hamiltonian vector field X_H. If u solves (28), formula (26) and an integration by parts yield to the energy identity

$$\int_a^b \int_0^1 |\partial_s u|^2 \, dt \, ds = \mathbb{A}_H(u(a,\cdot)) - \mathbb{A}_H(u(b,\cdot)) + \int_a^b \big(\lambda(u(s,1))[\partial_s u(s,1)] - \lambda(u(s,0))[\partial_s u(s,0)]\big)\, ds.$$

In particular, if u satisfies the non-local boundary condition

$$(u(s,0), -u(s,1)) \in N^*Q, \quad \forall s \in \mathbb{R}, \tag{29}$$

the fact that the Liouville form $\lambda \times \lambda$ of $T^*(M \times M)$ vanishes on conormal bundles implies that

$$\int_a^b \int_0^1 |\partial_s u|^2 \, dt \, ds = \mathbb{A}_H(u(a,\cdot)) - \mathbb{A}_H(u(b,\cdot)), \tag{30}$$

so the function $s \mapsto \mathbb{A}_H(u(s,\cdot))$ is decreasing.

Let $A \in \mathbb{R}$. We denote by $Z^A(H)$ the space of smooth maps $u : \mathbb{R} \times [0,1] \to T^*M$ solving the Floer equation (28), satisfying the boundary condition (29), having finite energy,

$$\int_{-\infty}^{+\infty} \int_0^1 |\partial_s u|^2 \, dt \, ds < +\infty$$

and action upper bound

$$\mathbb{A}_H(u(s,\cdot)) \leq A, \quad \forall s \in \mathbb{R}.$$

The space $Z^A(H)$, endowed with the C^∞_{loc} topology, is a metrizable space.

LEMMA 7.1 *For every $A \in \mathbb{R}$, the space $Z^A(H)$ is compact.*

The main step in the proof of the above lemma is to show that the solutions $u \in Z^A(H)$ are uniformly bounded in C^0 (see Theorem 5.4 in [5]). Here is where the quadraticity assumptions (H1') and (H2') are used. Actually, Cieliebak replaces (H2') by a stronger condition on the second derivatives of H. A proof of the C^0 estimate under the above assumptions is contained in [1], Theorem 1.14. Then the facts that ω is exact and that the Liouville form $\lambda \times \lambda$ vanishes on N^*Q allow to exclude bubbling of J-holomorphic spheres or disks, and this yields to uniform C^1 bounds for the elements of $Z^A(H)$. The conclusion follows from a standard elliptic bootstrap. See [5], Theorem 6.3.

We denote by $\overline{H}^*(\cdot, \mathbb{Z}_2)$ the Alexander-Spanier cohomology with coefficients in $\mathbb{Z}_2$. We denote by ev_A the evaluation map

$$\mathrm{ev}_A : Z^A(H) \to C_Q([0,1], M), \quad u \mapsto \pi \circ u(0,\cdot),$$

where $\pi : T^*M \to M$ is the projection. The following result says that if A is large enough then $Z^A(H)$ is non-empty, and that actually its topology is quite rich.

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LEMMA 7.2 *Let α be a non-zero class in $\overline{H}^*(C_Q([0,1],M),\mathbb{Z}_2)$. Then there exists $A_0 \in \mathbb{R}$,*

$$A_0 = A_0\Big(\alpha, h, \max_{(t,q)\in[0,1]\times M} H(t,q,0)\Big),$$

such that for every $A \geq A_0$ the cohomology class $\mathrm{ev}^*_A(\alpha)$ is not zero in $\overline{H}^*(Z^A(H))$.

This fact is proved in [5], Theorem 7.6. Let us sketch the argument, in order to show how A_0 depends on H and α (this dependence is not explicitly stated in [5]).

Let us fix numbers $S > 0$ and $p > 1$. Let $\mathscr{B}$ be the Banach manifold consisting of the maps u in the Sobolev space $W^{1,p}([-S,S]\times[0,1],T^*M)$ such that $u(S,t)\in M$ for every $t\in[0,1]$, where M denotes the image of the zero section in T^*M. Let $\mathscr{E}$ be the Banach bundle over $\mathscr{B}$ whose fiber at u is the space of L^p sections of $u^*(TT^*M)$. Let $\mathscr{W}$ be the Banach manifold of paths γ in $W^{1-1/p,p}([0,1],M)$ such that $(\gamma(0),\gamma(1))\in Q$, and consider the product manifold $\mathscr{E}\times\mathscr{W}$ as the total space of a bundle over $\mathscr{B}$. The section

$$F : \mathscr{B} \to \mathscr{E}\times\mathscr{W}, \quad u \mapsto (\partial_s u + J(u)[\partial_t u - X_H(t,u)], \pi\circ u(-S,\cdot)),$$

is smooth. If $F(u) \in \{0\}\times\mathscr{W}$, then F is a Fredholm section of index zero at u, meaning that the fiberwise derivative $D^f F(u) : T_u\mathscr{B} \to \mathscr{E}_u \times T_{\pi\circ u(-S,\cdot)}\mathscr{W}$ is a Fredholm operator of index zero. This follows from elliptic estimates and from the fact that $W^{1-1/p,p}$ is the space of traces of $W^{1,p}$.

Let $K \subset C^\infty_Q([0,1],M)$ be a compact set such that $i^*_K(\alpha)\neq 0$, where $i_K : K \hookrightarrow C_Q([0,1],M)$ denotes the inclusion. If $u = (q,p)\in\mathscr{B}$ is such that $F(u)\in\{0\}\times K$, then the boundary conditions on u and (27) imply the estimates

$$\mathbb{A}_H(u(-S,\cdot)) = \int_0^1 p(-S,t)[\partial_t q(-S,t)]\,dt - \int_0^1 H(t,q(-S,t),p(-S,t))\,dt$$

$$\leq \int_0^1 \Big(\|\partial_t q(-S,\cdot)\|_\infty |p(-S,t)|_{q(-S,t)} - h\big(|p(-S,t)|_{q(-S,t)}\big)\Big)\,dt,$$

$$\mathbb{A}_H(u(S,\cdot)) = -\int_0^1 H(t,q(S,t),0)\,dt \geq -\max_{(t,q)\in[0,1]\times M} H(t,q,0),$$

Since $q(-S,\cdot)$ belongs to the compact set K and h is superlinear, we deduce that

$$\mathbb{A}_H(u(-S,\cdot)) - \mathbb{A}_H(u(S,\cdot)) \leq A_0, \tag{31}$$

where

$$A_0 := \max_{s\geq 0}\Big(\max_{\gamma\in K}\|\gamma'\|_\infty s - h(s)\Big) + \max_{(t,q)\in[0,1]\times M} H(t,q,0). \tag{32}$$

The above action bound and the argument used in the proof of Lemma 7.1 show that $F^{-1}(\{0\}\times K)$ is a compact subset of $\mathscr{B}$. It can be shown that $Z_{S,K} := F^{-1}(\{0\}\times K)$ has an open neighborhood $\mathscr{U}$ in $\mathscr{B}$ such that

$$\deg(F|_{\mathscr{U}}, \mathscr{U}, (0,\gamma)) = 1, \quad \forall\gamma\in K, \tag{33}$$

where $\deg$ denotes the Smale $\mathbb{Z}_2$-degree of a Fredholm section of index zero. Statement (33), together with the fact that maps between finite dimensional manifolds of non-zero degree induce injective homomorphisms in cohomology, can be used to show that $f^*(i^*_K(\alpha))\neq 0$, where $f : Z_{S,K} \to K$ is the map $f(u) = \pi\circ u(-S,\cdot)$. If we now let S tend to infinity, the action bound (31) implies that the elements of $Z_{S,K}$ converge to elements of $Z^{A_0}(H)$ in C^∞_{loc}, and the conclusion of Lemma 7.2 follows by the tautness of the Alexander-Spanier cohomology.

We conclude this section by recalling how Lemmas 7.1 and 7.2 imply the existence of solutions of (23), (24), in the case of Hamiltonians satisfying (H1'), (H2'). Let $A \in \mathbb{R}$, and consider the continuous flow

$$(s,u) \mapsto u(s+\cdot,\cdot),$$

on the compact metrizable space $Z^A(H)$. The continuous function

$$u \mapsto \mathbb{A}_H(u(0,\cdot))$$

is a Lyapunov function for such a flow. Then we have the following:

THEOREM 7.3 *Let M be a compact manifold and let Q be a smooth submanifold of $M \times M$. Assume that the smooth Hamiltonian $H : [0,1] \times T^*M \to \mathbb{R}$ satisfies (H1'), (H2'), and (27).*

(a) *Let $\alpha \in \overline{H}^*(C_Q([0,1], M), \mathbb{Z}_2) \cong \overline{H}^*(C_Q^\infty([0,1], M), \mathbb{Z}_2)$ be a non-zero Alexander-Spanier cohomology class, let $i_K : K \hookrightarrow C_Q^\infty([0,1], M)$ be a compact inclusion such that $i_K^*(\alpha) \neq 0$, and let $A \geq A_0$, where A_0 is given by (32). Then there is a solution x of (23), (24), whose action $\mathbb{A}_H(x)$ equals*

$$c_\alpha(H) := \inf_{Z \in \mathscr{Z}_\alpha} \max_{u \in Z} \mathbb{A}_H(u(0, \cdot)),$$

where $\mathscr{Z}_\alpha$ is the set of compact subsets $Z \subset Z^A(H)$ such that the Alexander-Spanier cohomology class $\mathrm{ev}_A|_Z^*(\alpha)$ is not zero.

(b) *The number of solutions x of (23), (24), with $\mathbb{A}_H(x) \leq A$, is greater or equal than the Alexander-Spanier $\mathbb{Z}_2$-cuplength of $Z^A(H)$ plus one.*

(c) *If $\overline{H}^k(C_Q([0,1], M), \mathbb{Z}_2) \neq 0$ for infinitely many indices k, then the set of critical levels*

$$\left\{ c_\alpha(H) \mid \alpha \in \overline{H}^*(C_Q([0,1], M), \mathbb{Z}_2), \ \alpha \neq 0 \right\}$$

is not bounded above.

The first two statements follow from Lusternik-Schnirelman theory on compact metric spaces, see [9], section 6.3. Statement (c) is proved in [5], Proposition 8.4.

8 Proof of Theorem 6.3

Let $H \in C^2([0,1] \times T^*M, \mathbb{R})$ be a Hamiltonian satisfying (H1), (H2), (H3). By an easy regularization argument together with the a priori estimates given by Lemma 6.2, we may assume that H is smooth. Without loss of generality, we may assume that the function a appearing in (H1) grows at most linearly, while the function h appearing in (H2) grows at most as $s \log s$.

DEFINITION 8.1 *We say that $H_0 \in C^\infty([0,1] \times T^*M, \mathbb{R})$ is a quadratic R-modification of H if:*

(a) *$H_0(t,q,p) = H(t,q,p)$ for $|p|_q \leq R$;*

(b) *H_0 satisfies (H1') and (H2');*

(c) *$DH_0(t,q,p)[Y(q,p)] - H_0(t,q,p) \geq a(|p|_q)$;*

(d) *$H_0(t,q,p) \geq h(|p|_q)$.*

It is easy to build a quadratic R-modification of a Hamiltonian H satisfying (H1) and (H2), at least for R big enough. Indeed let us consider a Hamiltonian H_0 of the form

$$H_0(t,q,p) := \varphi(|p|_q)H(t,q,p) + (1 - \varphi(|p|_q))C|p|_q^2,$$

with $C \geq 1$ constant, and $\varphi : [0, +\infty) \to [0,1]$ a smooth decreasing function taking value 1 on $[0, R]$ and 0 on $[R+1, +\infty)$.

Obviously H_0 satisfies (a) and (b). Also (d) is satisfied for R big enough, because H satisfies (H2) and thanks to the assumption that h grows at most as $s \log s$. Finally, as $D(|p|_q^2)[Y] = 2|p|_q^2$, we have

$$DH_0[Y] - H_0 = \varphi (DH[Y] - H) + (1 - \varphi)C|p|_q^2 - |p|_q \varphi' \left(C|p|_q^2 - H \right).$$

Since φ' is non-positive and it vanishes outside the interval $[R, R+1]$, it suffices first to take R large enough so that $|p|_q^2 \geq a(|p|_q)$ for $|p|_q \geq R$ (such R exists since c grows at most linearly at infinity), and then $C \geq 1$ large enough so that $C|p|_q^2 \geq H(t,q,p)$ for $R \leq |p|_q \leq R+1$.

LEMMA 8.2 *Let M be a compact manifold and $H \in C^{\infty}([0, 1] \times T^*M, \mathbb{R})$ be a Hamiltonian satisfying (H1), (H2), (H3). For every $A \in \mathbb{R}$ there exists a number $R(A)$ such that for any $R > R(A)$ and for any Hamiltonian H_0 which is a quadratic R-modification of H, the following holds: if $x = (q, p) : [0, 1] \rightarrow T^*M$ is an orbit of X_{H_0} such that $\mathbb{A}_{H_0}(x) \leq A$, then $\|p\|_{\infty} \leq R(A)$. In particular, such an x is an orbit of X_H, and $\mathbb{A}_{H_0}(x) = \mathbb{A}_H(x)$.*

Proof. Since the Hamiltonian flow of H is globally defined, arguing as in the proof of Lemma 6.2 we find that the set

$$K := \left\{ \phi_t^H \circ (\phi_s^H)^{-1}(q, p) \mid (s, t) \in [0, 1]^2, \ (q, p) \in T^*M, \ a(|p|_q) \leq A \right\}$$

is compact. Let $R(A)$ be the maximum of $|p|_q$ for (q, p) in K. Let $R > R(A)$, H_0 a quadratic R-modification of L, and $x = (q, p)$ an orbit of X_{H_0} such that $\mathbb{A}_{H_0}(x) \leq A$. Then there exists $t_0 \in [0, 1]$ such that

$$a(|p(t_0)|) \leq DH_0(t, x(t_0))[Y(x(t_0))] - H_0(t_0, x(t_0)) \leq A,$$

where the first inequality follows by (c) in Definition 8.1, which implies that $x(t_0) \in K$, and thus $|p(t_0)| \leq R(A)$. Let $I \subset [0, 1]$ be the maximal interval containing t_0 such that $x(t) \in K$ for every $t \in I$. If $t \in I$, then $|p(t)| \leq R(A) < R$. Therefore H and H_0 coincide in a neighborhood of $(t, x(t))$, and so do the Hamiltonian vector fields defined by H and H_0. Hence, by the definition of K, $x(s)$ belongs to K for s in a neighborhood of t in $[0, 1]$, proving that I is open in $[0, 1]$. Being obviously closed, I coincides with $[0, 1]$. Therefore $\|p\|_{\infty} \leq R(A)$. □

Let $\alpha_1, \dots, \alpha_m$, $m \geq 1$, be cohomology classes of degree at least one in $H^*(C_Q([0, 1], M), \mathbb{Z}_2)$ such that $\alpha_1 \cup \dots \cup \alpha_m \neq 0$. Since singular cohomology and Alexander-Spanier cohomology coincide on Banach manifolds, we may regard $\alpha_1, \dots, \alpha_m$ as Alexander-Spanier cohomology classes. Let

$$A = A_0 \Big(\alpha_1 \cup \dots \cup \alpha_m, h, \max_{(t, q) \in [0, 1] \times M} H(t, q, 0) \Big),$$

as given by (32), and let $R > R(A)$, where $R(A)$ is given by Lemma 8.2. Let H_0 be a quadratic R-modification of H (up to taking R larger, we know that a quadratic R-modification always exists). Since $H_0(t, q, p) \geq h(|p|_q)$ and H_0 coincides with H on $[0, 1] \times M$, Lemma 7.2 implies that

$$\mathrm{ev}_A^*(\alpha_1) \cup \dots \cup \mathrm{ev}_A^*(\alpha_m) = \mathrm{ev}_A^*(\alpha_1 \cup \dots \cup \alpha_m) \neq 0 \quad \text{in } \overline{H}^*(Z^A(H_0)).$$

Therefore, the $\mathbb{Z}_2$-Alexander Spanier cuplength of $Z^A(H_0)$ is at least m. By Theorem 7.3 (b), there are at least $m + 1$ orbits x of X_{H_0} satisfying the boundary conditions (24) and the action estimate $\mathbb{A}_{H_0}(x) \leq A$. By Lemma 8.2 these curves are also orbits of X_H. This proves statement (a) of Theorem 6.3.

The proof of statement (b) is analogous to the proof of statement (c) of Theorem 2.1, the Morse index being replaced by the Maslov index. Indeed, the above argument together with the fact that infinitely many $\mathbb{Z}_2$-homology groups of $C_Q([0, 1], M)$ are non-trivial produces a sequence of solutions of (23), (24) with diverging Maslov index. See for instance [12] for the definition of the Maslov index and for the argument yielding to existence of orbits with Maslov index estimates. By Lemma 6.2, the analogue of Lemma 1.2 for the Maslov index holds, so orbits with diverging Maslov index have diverging action, concluding the proof.

A   Regularity of the Lagrangian action and the Palais-Smale condition

Let us say a bit more about the localization argument used in section 1. Let $\gamma_0 : [0, 1] \rightarrow M$ be a continuous curve. Let U be an open subset of $\mathbb{R}^n$, and let

$$[0, 1] \times U \rightarrow [0, 1] \times M, \quad (t, q) \mapsto (t, \varphi(t, q)), \tag{34}$$

be a smooth coordinate system such that $\gamma_0(t) = \varphi(t, \tilde{\gamma}_0(t))$ for every $t \in [0,1]$, for some continuous curve $\tilde{\gamma}_0 : [0,1] \to U$. For instance, such a diffeomorphism can be constructed by choosing a smooth curve $\gamma_1 : [0,1] \to M$ such that $\|\gamma_1 - \gamma_0\|_\infty < \rho$, ρ denoting the injectivity radius of M, and by setting $\varphi(t,q) = \exp_{\gamma_1(t)}[\Psi(t)q]$, where $q \in \mathbb{R}^n$, $|q| < \rho$, and Ψ is a smooth orthogonal trivialization of the vector bundle $\gamma_1^*(TM)$ over $[0,1]$.

Let Φ denote the differential of (34), that is the tangent bundle coordinate system

$$\Phi : [0,1] \times U \times \mathbb{R}^n \to [0,1] \times TM, \quad \Phi(t,q,v) = (t, \varphi(t,q), D\varphi(t,q)[(1,v)]).$$

Up to replacing U by a smaller open set, we may assume that Φ is bounded together with its inverse. Then the map

$$\varphi_* : W^{1,2}([0,1], U) \to W^{1,2}([0,1], M), \quad \varphi_*(\tilde{\gamma})(t) = \varphi(t, \gamma(t)),$$

is a smooth local coordinate system on the Hilbert manifold $W^{1,2}([0,1], M)$, such that $D\varphi_*$ is bounded together with its inverse from the standard metric of $W^{1,2}([0,1], \mathbb{R}^n)$ to the Riemannian metric on $W^{1,2}([0,1], M)$ defined in (11). If moreover the curve $\gamma_0 : [0,1] \to M$ is in $W^{1,2}$, then so is $\tilde{\gamma}_0$, and $\varphi_*(\tilde{\gamma}_0) = \gamma_0$.

The Lagrangian L on $[0,1] \times TM$ can be pulled back onto $[0,1] \times U \times \mathbb{R}^n$ by setting $\tilde{L} = L \circ \Phi$. Notice that conditions (L1') and (L2') are invariant with respect to coordinates transformations of the form Φ (up to changing the constants), so $\tilde{L}$ satisfies these conditions if L does. Since $L(t, \varphi_*(\tilde{\gamma})(t), \varphi_*(\tilde{\gamma})'(t)) = \tilde{L}(t, \tilde{\gamma}(t), \tilde{\gamma}'(t))$, we have

$$\mathbb{A}_L(\varphi_*(\tilde{\gamma})) = \mathbb{A}_{\tilde{L}}(\tilde{\gamma}).$$

These facts allow to reduce all the arguments on $W^{1,2}([0,1], M)$ which are C^0-local, in the sense that they involve only a C^0-neighborhood of some curve, to the case of a Lagrangian on an open subset of the Euclidean space.

Proof of Proposition 3.1. By the above localization arguments, we may assume that L is a Lagrangian on $[0,1] \times U \times \mathbb{R}^n$, with U an open subset of $\mathbb{R}^n$. If $\gamma \in W^{1,2}([0,1], U)$, $\xi \in W^{1,2}([0,1], \mathbb{R}^n)$, and $h \in \mathbb{R}$ has a small absolute value, we have

$$\frac{1}{h}\Big(\mathbb{A}_L(\gamma + h\xi) - \mathbb{A}_L(\gamma)\Big) = \int_0^1 \int_0^1 \Big(D_q L(t, \gamma + hs\xi, \gamma' + hs\xi')[\xi] + D_v L(t, \gamma + hs\xi, \gamma' + hs\xi')[\xi']\Big)\, dt\, ds. \tag{35}$$

Assumption (L2') implies that

$$|D_v L(t,q,v)| \le \ell_2(1 + |v|_q), \quad |D_q L(t,q,v)| \le \ell_2(1 + |v|_q^2), \tag{36}$$

for some constant ℓ_2. These bounds and the dominated convergence theorem show that the quantity (35) converges to

$$d\mathbb{A}_L(\gamma)[\xi] = \int_0^1 \Big(D_q L(t, \gamma, \gamma')[\xi] - D_v L(t, \gamma, \gamma')[\xi']\Big)\, dt$$

for $h \to 0$. Since $d\mathbb{A}_L(\gamma)$ is a bounded linear functional on $W^{1,2}([0,1], \mathbb{R}^n)$, $\mathbb{A}_L$ is Gateaux differentiable and $d\mathbb{A}_L(\gamma)$ is its Gateaux differential at γ. Since $d\mathbb{A}_L(\gamma)$ continuously depends on $\gamma \in W^{1,2}([0,1], U)$, the total differential theorem implies that $\mathbb{A}_L$ is continuously Fréchét differentiable, with $D\mathbb{A}_L = d\mathbb{A}_L$.

Similarly, by (L2') and by the dominated convergence theorem, the quantity

$$\frac{1}{h}\big(D\mathbb{A}_L(\gamma + h\eta)[\xi] - D\mathbb{A}_L(\gamma)[\xi]\big)$$

$$= \int_0^1 \int_0^1 \Big(D_{vv}L(t, \gamma + hs\eta, \gamma' + hs\eta')[\xi', \eta'] + D_{vq}L(t, \gamma + hs\eta, \gamma' + hs\eta')[\xi', \eta] \tag{37}$$

$$+ D_{qv}L(t, \gamma + hs\eta, \gamma' + hs\eta')[\xi, \eta'] + D_{qq}L(t, \gamma + hs\eta, \gamma' + hs\eta')[\xi, \eta]\Big)\, dt\, ds,$$

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converges to

$$d^2 \mathbb{A}_L(\gamma)[\xi, \eta] = \int_0^1 \Big(D_{vv}L(t, \gamma, \gamma')[\xi', \eta'] + D_{qv}L(t, \gamma, \gamma')[\xi, \eta']$$

$$+ D_{vq}L(t, \gamma, \gamma')[\xi', \eta] + D_{qq}L(t, \gamma, \gamma')[\xi, \eta] \Big) \, dt,$$

for $h \to 0$. Since $d^2 \mathbb{A}_L(\gamma)$ is a bounded symmetric bilinear form on $W^{1,2}([0,1], \mathbb{R}^n)$ which depends continuously on γ, we conclude that $\mathbb{A}_L$ is C^2, and that its second differential is $D^2 \mathbb{A}_L = d^2 \mathbb{A}_L(\gamma)$. Since the second differential at critical points is invariantly defined, in the sense that

$$D^2 \mathbb{A}_L(\psi(\gamma))[D\psi(\gamma)\xi, D\psi(\gamma)\eta] = D^2(\mathbb{A}_L \circ \psi)(\gamma)[\xi, \eta],$$

this concludes the proof of Proposition 3.1.

Proof of Proposition 3.2. Let (γ_h) be a sequence in $W_Q^{1,2}([0,1], M)$ such that $\mathbb{A}_L(\gamma_h)$ is bounded and $\|D\mathbb{A}_L(\gamma_h)|_{T_\gamma W_Q^{1,2}([0,1], M)}\|_*$ is infinitesimal. We have to show that (γ_h) is compact in $W_Q^{1,2}([0,1], M)$. First we observe that by assumption (L1'), $L(t, q, v) \geq \ell_0 |v|_q^2/2 - C$, for some constant C. Together with the bound on the action, this implies that the sequence (γ_h') is bounded in L^2. Therefore,

$$\text{dist}\,(\gamma_h(t), \gamma_h(s)) \leq \int_s^t |\gamma_h'(\sigma)| \, d\sigma \leq |s - t|^{1/2} \left(\int_0^1 |\gamma_h'(\sigma)|^2 \, d\sigma \right)^{1/2},$$

so (γ_h) is equi-1/2-Hölder continuous, and up to a subsequence we may assume that it converges uniformly to some $\gamma \in C([0,1], M)$ with $(\gamma(0), \gamma(1)) \in Q$. If φ_* is a smooth coordinate system on $W^{1,2}([0,1], M)$ constructed as above with $\gamma_0 := \gamma$, γ_h eventually belongs to the image of φ_*, so we may assume that L is a Lagrangian on $[0,1] \times U \times \mathbb{R}^n$ and $\gamma_h \in W^{1,2}([0,1], U)$, where U is an open subset of $\mathbb{R}^n$. Up to choosing the diffeomorphism $(t, q) \mapsto (t, \varphi(t, q))$ properly, we may also assume that there is an affine subspace $V = \zeta + V_0$ of $\mathbb{R}^n \times \mathbb{R}^n$ such that

$$[\varphi(0, \cdot) \times \varphi(1, \cdot)]^{-1}Q = (U \times U) \cap V.$$

Notice also that, since the coordinate system φ_* is C^1-bounded together with its inverse, $D\mathbb{A}_L(\gamma_h)$ converges to zero strongly in the dual of

$$W_{V_0}^{1,2}([0,1], \mathbb{R}^n) := \left\{ \xi \in W^{1,2}([0,1], \mathbb{R}^n) \mid (\xi(0), \xi(1)) \in V_0 \right\}.$$

The fact that (γ_h') is bounded in L^2 now implies that $\gamma \in W^{1,2}([0,1], U)$, and up to a subsequence, (γ_h) converges weakly to γ in $W^{1,2}([0,1], \mathbb{R}^n)$ and strongly in L^2. We must show that this convergence is strong in $W^{1,2}$. Since $D\mathbb{A}_L(\gamma_h)$ is infinitesimal in the dual of $W_{V_0}^{1,2}([0,1], \mathbb{R}^n)$, (γ_h) is bounded in $W^{1,2}$, and $\gamma_h - \gamma \in W_{V_0}^{1,2}([0,1], \mathbb{R}^n)$, the sequence $D\mathbb{A}_L(\gamma_h)[\gamma_h - \gamma]$ is infinitesimal, that is

$$\int_0^1 D_q L(t, \gamma_h, \gamma_h')[\gamma_h - \gamma] \, dt + \int_0^1 D_v L(t, \gamma_h, \gamma_h')[\gamma_h' - \gamma'] \, dt \to 0 \quad \text{for } h \to \infty.$$

Since $D_q L(t, \gamma_h, \gamma_h')$ is bounded in L^2 by (36), and $\gamma_h - \gamma$ converges strongly to 0 in L^2, the first integral in the above expression tends to zero, so

$$\int_0^1 D_v L(t, \gamma_h, \gamma_h')[\gamma_h' - \gamma'] \, dt \to 0 \quad \text{for } h \to \infty. \tag{38}$$

For a.e. $t \in [0,1]$ we have by (L1')

$$D_v L(t, \gamma_h, \gamma_h')[\gamma_h' - \gamma_h] - D_v L(t, \gamma_h, \gamma')[\gamma_h' - \gamma_h]$$

$$= \int_0^1 D_{vv}L(t, \gamma_h, \gamma' + s(\gamma_h' - \gamma'))[\gamma_h' - \gamma', \gamma_h' - \gamma'] \, ds \geq \ell_0 |\gamma_h'(t) - \gamma'(t)|^2.$$

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Integrating this inequality over $[0,1]$ we get

$$\ell_0 \int_0^1 |\gamma_h' - \gamma'|^2 \, dt \leq \int_0^1 D_v L(t, \gamma_h, \gamma_h')[\gamma_h' - \gamma'] \, dt - \int_0^1 D_v L(t, \gamma_h, \gamma')[\gamma_h' - \gamma'] \, dt. \qquad (39)$$

The first integral on the right-hand side tends to 0 for $h \to \infty$ by (38). Since $D_v L(\cdot, \gamma_h, \gamma')$ converges strongly to $D_v L(\cdot, \gamma, \gamma')$ in L^2 by (36), and since $(\gamma_h' - \gamma')$ converges weakly to 0 in L^2, the last integral in (39) is also infinitesimal. Therefore, (39) implies that (γ_h) converges strongly to γ in $W^{1,2}([0,1], \mathbb{R}^n)$, concluding the proof.

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