Isoperimetric-type inequalities on constant curvature manifolds

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Abstract

By exploiting optimal transport theory on Riemannian manifolds and adapting Gromov's proof of the isoperimetric inequality in the Euclidean space, we prove an isoperimetric-type inequality on simply connected constant curvature manifolds.

1 Introduction

Let $M^n(K)$ be a n-dimensional simply connected Riemannian manifold with constant sectional curvature $K \in \mathbb{R}$, that is $M^n(K)$ is either the sphere ($K > 0$), the Euclidean space ($K = 0$), or the hyperbolic space ($K < 0$). The isoperimetric problem is a very important topic in differential geometry. The first unified solution in the above model spaces of constant sectional curvature was given by Schmidt [17, 18]. Later on, a variety of methods have been found to solve it (see [6] for the more references).

When the dimension n is equal to 2, there is a nice analytic form for the classic isoperimetric inequality: let $E \subset M^2(K)$ be a smooth open set, denote by A the area of E, and by L the length of its boundary. Then

$$L^2 \geq 4\pi A - KA^2, \tag{1.1}$$

with equality if and only if E is a geodesic ball. As the area and the length are invariant under isometries, the domains in question can be identified under the isometry action. For general dimension, the isoperimetric inequality states that geodesic balls minimize the perimeter among all subsets $E \subset M^n(K)$ with fixed volume. Here we present a weighted isoperimetric inequality in any dimension on $M^n(K)$, which generalizes the classic one on the Euclidean space.

Set $c(x,y) := \frac{1}{2}d_g(x,y)^2$, where $d_g(x,y)$ is the geodesic distance between x and y on $M^n(K)$, and for $K \in \mathbb{R}$ define

$$G_K(r) := \begin{cases} \frac{(\sqrt{K}r)\cos(\sqrt{K}r)}{\sin(\sqrt{K}r)} & \text{if } K > 0, \\ 1 & \text{if } K = 0, \\ \frac{(\sqrt{|K|}r)\cosh(\sqrt{|K|}r)}{\sinh(\sqrt{|K|}r)} & \text{if } K < 0, \end{cases} \tag{1.2}$$

$$\ell_K(r) := \begin{cases} \frac{\sqrt{K}r}{\sin(\sqrt{K}r)} & \text{if } K > 0, \\ 1 & \text{if } K = 0, \\ \frac{\sqrt{|K|}r}{\sinh(\sqrt{|K|}r)} & \text{if } K < 0. \end{cases} \tag{1.3}$$

We denote by ω_n the volume of the unit ball in the Euclidean space $\mathbb{R}^n$. Fix a reference point $N \in M^n(K)$ (for instance, the north pole of the sphere when $K > 0$), and define $r_x := d_g(x, N)$. Given $E \subset M^n(K)$ a set of finite perimeter, we denote by $\mathcal{F}E$ its reduced boundary, and by ν_x its measure theoretic outward unit normal at x (see Section 2.2). On the tangent space $T_N M$, we use the polar coordinates. We define a map on $T_N M$ by

$$\begin{array}{rrcl} F_K: & T_N M & \to & T_N M \\ & (r,\theta) & \mapsto & (\alpha_K(r),\theta) \end{array}$$

where

$$\alpha_K(r) := \frac{re^{-G_K(r)}}{\ell_K(r)} \tag{1.4}$$

Observe that $\alpha'_K(r) = \ell_K(r)e^{-G_K(r)}$ is positive on $[0,+\infty)$ when $K \le 0$ (resp. on $[0, \pi/\sqrt{K})$ when $K > 0$), hence F_K is a smooth diffeomorphism from $T_N M$ (resp. $B(0, \pi/\sqrt{K})$) onto $T_N M$. Given $v \in \mathbb{R}^n$ and $R \in (0,+\infty)$, we set $E_{v,R} = \exp_N(F_K^{-1}(B(v, R)))$. Remark when $v = 0$, these sets are just the geodesic balls centered at N. However, this fact is not yet true in the general case. Our isoperimetric-type inequality can be read as follows:

Theorem 1.1 *Let $E \subset M^n(K)$ be set with finite perimeter such that $d_g^2(\cdot, N) : M \to \mathbb{R}$ is smooth in a neighborhood of E. Then*

$$\int_{\mathcal{F}E} e^{(n-1)[G_K(0)-G_K(r_x)]} \left|\nabla_x \nabla_y c(x, N)\cdot \nu_x\right| d\mathscr{H}^{n-1}(x) \geq n\, \omega_n^{1/n} \left(\int_E e^{n[G_K(0)-G_K(r_x)]} \ell_K(r_x)\, d\,\mathrm{vol}(x)\right)^{\frac{n-1}{n}}. \tag{1.5}$$

Furthermore equality holds if and only if $E = E_{v,R}$ for some $v \in \mathbb{R}^n$ and $R \in (0,+\infty)$.

The assumption that $d^2(\cdot, N) : M \to \mathbb{R}$ is smooth in a neighborhood of E is always satisfied if $K \le 0$, while for $K > 0$ it amounts to say that E is at positive distance from the antipodal point of N.

We also observe that, when $K < 0$, one has $\left|\nabla_x \nabla_y c(x, N) \cdot \nu_x\right| \le 1$, and equality holds only when $E = E_{v,R}$ is a geodesic ball centered at N. Hence, we get the following:

Corollary 1.2 *If $K \le 0$ then*

$$\int_{\mathcal{F}E} e^{(n-1)[G_K(0)-G_K(r_x)]}\, d\mathscr{H}^{n-1}(x) \geq n\, \omega_n^{1/n} \left(\int_E e^{n[G_K(0)-G_K(r_x)]} \ell_K(r_x)\, d\,\mathrm{vol}(x)\right)^{\frac{n-1}{n}}.$$

for all $E \subset M^n(K)$ with finite perimeter. Furthermore equality holds if and only if E is a geodesic ball (centered at N if $K < 0$).

We now make some comments on Theorem 1.1. First of all, when $K = 0$ we have $G_K = \ell_K \equiv 1$, and we recover the classical isoperimetric inequality on the Euclidean space. When $K \neq 0$, compared to the classic one (1.1), our inequality is not invariant under isometry action, and it depends on the choice of the pole $N \in M^n(K)$. On the other hand, the extremal domains are transformations of balls on the tangent space $T_N M$ without a constraint on the volume (including all the geodesic balls centered at N), so that our inequality presents a sort of "invariance under dilations/translation" (although it is not really dilation/translation-invariant), which is not present in the classical isoperimetric inequality on $M^n(K)$ for $K \neq 0$. Moreover, when $K < 0$ we can "kill" the "translation invariance" and obtain an inequality whose minima are only the balls centered at the origin.

After proving our result, by "writing it on the tangent space" we realized that the family of inequalities in Theorem 1.1 can be deduced (in a non-trivial way) by the classical isoperimetric inequality on $\mathbb{R}^n$ (see Remark 3.3). However, this is a sort of "a-posteriori" proof: we do not see any easy way to guess a priori what is the right transformation in order to transfer the classical isoperimetric inequality onto $M^n(K)$, and obtain a "geometric" inequality as for instance the one in Corollary 1.2. Moreover, we also believe that our approach has an interest in his own. Indeed, our proof is based on the optimal transport theory. This strategy is due to Gromov in the Euclidean space, but becomes more complicated in the non-flat case. This paper is a tentative to understand rearrangement of sets on manifolds via optimal transport maps. The weights in the above inequalities come from two factors: on one side the tensor field $\nabla_x \nabla_y c(\cdot, N) \in \Gamma(T^*M \otimes T^*_N M)$ appears in the Monge-Ampère type equations satisfied by our optimal transport maps, and its determinant is equal to $\ell_K(r_x)^{n-1}$ (in Fermi charts); on the other hand the tensor field $e^{(n-1)[G_K(0)-G_K(r_x)]}\nabla_x \nabla_y c(x, N)$ is divergence free with respect to x, and this makes it "suitable" for an integration by parts (see Section 3). Because of the weights, the best constant in the above inequalities does not depend on the geometry of the manifolds. This comes from the fact the weights are completely determined by the Riemannian distance and the curvature tensor, that is, all geometric informations are behind the weights. Let us mention that optimal transport theory has already revealed being useful to prove geometric inequalities on manifolds, see for instance [7, 8].

The paper is structured as follows: in Section 2 we collect some preliminaries on Riemannian geometry and BV functions. Then we prove Theorem 1.1. The rigorous proof of the theorem is given in Section 3. However, since the argument may seem a bit mysterious, we will give in Appendix A a more intuitive (but formal) proof which explains the ideas behind our result. In Section 3.1 we write our inequality on the tangent space, from which one can see the strict link with the classical isoperimetric inequality.

2 Preliminary results and notation

2.1 Preliminaries on Riemannian geometry

In this section we recall some basic notions of Riemannian geometry and we introduce some notation, referring to [16] for a general introduction.

Let (M, g) be a smooth complete n-dimensional Riemannian manifold. Given $v, w \in T_x M$ tangent vectors at x, we use the notation $\langle v, w\rangle_x = g_x(v, w)$ and $|v|_x = \sqrt{g_x(v, v)}$. We denote the volume element by $d\,\mathrm{vol}$. In a local chart, it can be expressed as $d\,\mathrm{vol} = \sqrt{\det(g_{ij})}dx$. Let ∇ be the Levi-Civita connection associated to the metric g. We denote by ∇_x the *covariant derivative* with respect to the metric g, that is, $\nabla_x : \Gamma(E) \to \Gamma(E \otimes T^*M)$, where E is a vector bundle on M, and $\Gamma(E)$ denotes the space of sections over E [16, Section 2.1.3]. We respectively denote ∇^x the *contravariant version of covariant derivative*, that is, $\nabla^x : \Gamma(E) \to \Gamma(E \otimes TM)$. More precisely, for any section $S \in \Gamma(E)$ and for any $\xi \in T_x^*M$ we have $\nabla^x S(\xi, \cdot) := \nabla_x S(g_x^{-1}(\xi), \cdot)$, where $g_x^{-1}(\xi) \in T_xM$ satisfies $\langle g_x^{-1}(\xi), \alpha\rangle_x = \xi \cdot \alpha$ for all $\alpha \in T_xM$. In particular, when $f : M \to \mathbb{R}$, $\nabla_x f \in T_x^*M$ is the *differential* of f at x, $\nabla^x f \in T_xM$ is the *gradient* of f at x, and $\nabla_x \nabla^x f \in T_xM \otimes T_x^*M$ is the *hessian* of f at x. Moreover, for $F : M \to M$ we denote by $\nabla_x F \in T_x^*M \otimes T_{F(x)}M$ the differential of F at x. Finally, given a function $h : M \times M \to \mathbb{R}$, we denote by $\nabla_x h(x, y)$ (resp. $\nabla^x h(x, y)$) and $\nabla_y h(x, y)$ (resp. $\nabla^y h(x, y)$) the differential (resp. gradient) of h with respect to the x and the y variable respectively.

Given a geodesic γ and $X(t)$ a vector field along $\gamma(t)$, we use the notation $\dot{J} = \frac{dJ}{dt} := \nabla_{\dot{\gamma}(t)} J(t)$. If $\dot{J} = 0$, we say that the vector field $X(t)$ is parallel transported along $\gamma(t)$. A vector field $J(t)$ along γ is called a *Jacobi field* if it satisfies the Jacobi equations

$$\ddot{J} + \mathrm{Riem}(J, \dot{\gamma})\dot{\gamma} = 0,$$

where Riem denotes the Riemann tensor. Suppose now that $\gamma(t)$ is a unit-speed geodesic, and set $e_1(t) = \dot{\gamma}(t)$. Then e_1 is parallel transported along $\gamma(t)$. By completing $e_1(0)$ to an orthonormal basis $(e_1(0), \cdots, e_n(0))$ at the point $\gamma(0)$, by parallel transport we get a orthonormal basis $(e_1(t), \cdots, e_n(t))$ at the point $\gamma(t)$. In particular, if we assume M to be a complete manifold with constant sectional curvature K, then the family $\left(te_1(t), \frac{t}{\ell_K(t)} e_2(t), \cdots, \frac{t}{\ell_K(t)} e_n(t)\right)$ is a basis for the (n-dimensional) vector space of the Jacobi fields satisfying $J(0) = 0$, with ℓ_K defined in (1.3).

Definition 2.1 (Fermi chart) *Given $x \in M$ and $v \in T_xM \backslash \{0\}$ such that $[0, 1] \ni s \mapsto \exp_x(sv)$ is the unique minimizing geodesic between x and $\exp_x(v)$, let $(e_1, \cdots, e_n)$ be an orthonormal basis of T_xM with $e_1 = \dfrac{v}{|v|}$. The associated Fermi chart $x = (x^1, \cdots, x^n)$ along the unit-speed geodesic*

$$[0, |v|_x] \ni s \to c(s) := \exp_x(se_1)$$

(which is called the axis of the chart) is defined, after parallel transport of the orthonormal basis $(e_1, \cdots, e_n)$ along the axis, by

$$q(x) = (q^1, \cdots, q^n) \iff q = \mathcal{F}(x) := \exp_{c(x^1)}\left(\sum_{\alpha=2}^{n} x^\alpha e_\alpha\right).$$

We remark that the differential of $\mathcal{F}$ on $\left\{q \in \mathbb{R}^n \,|\, 0 \le q^1 \le |v|_x,\, q^2 = \ldots = q^n = 0\right\}$ is readily found equal to the identity; so there exists a neighborhood of the axis on which the map $\mathcal{F}$ defines a chart. Note that in this definition we keep the flexibility of rotating all basis vectors at x but the first one e_1. Along the axis, the geodesic motion $[0, 1] \ni s \to \exp_x(sv)$ simply reads $s \mapsto (s|v|_x, 0 \ldots, 0)$, and the chart is *normal* (in particular, Christoffel symbols vanish), i.e.

$$g_{ij}(q^1, 0) = \delta_{ij}, \qquad \partial_k g_{ij}(q^1, 0) = 0 \quad \forall\, q^n \in [0, |v|_x],\, \forall\, i, j, k \in \{1, \ldots, n\},$$

(see for instance [2] or [16]).

Now fix $y \in M$, and let $r_x := d_g(x, y)$ be the distance function from y. The basic observation is that $\exp_y(-\nabla^y c(x, y)) = x$ for all $x \in M$ provided r_x is regular at the point x, that is, x is not in the focal locus of y. Differentiating this relation we get

$$-d_v \exp_y \cdot \nabla_x \nabla^y c(x, y) = I_x, \qquad \text{with } v \in T_y M,\ x = \exp_y(v),$$

where $I_x : T_x M \to T_x M$ is the identity map. Equivalently

$$d_v \exp_y^{-1} = -\nabla_x \nabla^y c(x, y). \tag{2.1}$$

Using the Fermi chart associated to the minimizing geodesic between x and y, we have $g_x = g_y = I_n$ the identity matrix. We claim that

$$-\nabla_x \nabla^y c(x, y) = -\nabla^x \nabla_y c(x, y) = -\nabla_x \nabla_y c(x, y) = -\nabla^x \nabla^y c(x, y) = \begin{pmatrix} 1 & 0 \\ 0 & \ell_K(r_x) I_{n-1} \end{pmatrix}. \tag{2.2}$$

Indeed, for all $v, w \in T_x M$ we set $J(t) := d_{tv} \exp_x \cdot (tw)$ for $t \in [0, 1]$. As J is a variation of a family of geodesics $\{\exp_x(t(v + sw))\}$ for all $t \in [0, 1]$ and for some parameter s, we can see J is a Jacobi field along the geodesic $\exp_x(tv)$ verifying $J(0) = 0$ and $\dot{J}(0) = w$. Therefore, thanks to the discussion right before Definition 2.1 the desired claim follows from (2.2). In particular we get

$$\begin{aligned} \det(-\nabla_x \nabla^y c(x, y)) &= \det(-\nabla^x \nabla_y c(x, y)) = \det(-\nabla^x \nabla^y c(x, y)) \\ &= \det(-\nabla_x \nabla_y c(x, y)) = \ell_K(r_x)^{n-1} \end{aligned} \tag{2.3}$$

in the Fermi chart. As a consequence we obtain

$$\det(-\nabla^x \nabla^y c(x, y)) \det(-\nabla_x \nabla_y c(x, y)) = \ell_K(r_x)^{2n-2},$$
$$\det(-\nabla^x \nabla_y c(x, y)) \det(-\nabla_x \nabla^y c(x, y)) = \ell_K(r_x)^{2n-2}.$$

Observe that both terms above are intrinsic and independent of the choice of coordinates.

Finally we consider the *Laplace-Betrami operator* Δ on the space of real functions. In local coordinates it has the expression

$$\Delta = \frac{1}{\sqrt{\det g}} \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i} \left(g^{ij} \sqrt{\det g} \frac{\partial}{\partial x_j} \right).$$

Fix $y \in M$, and as before let $r_x := d_g(x, y)$ be the distance function from y. Assuming (M, g) be a complete manifold with constant sectional curvature K and using polar coordinates around the point y, the metric matrix g_{ij} at the point x can be expressed as a diagonal one $\big(1, \frac{r_x^2}{\ell_K^2(r_x)}, \ldots, \frac{r_x^2}{\ell_K^2(r_x)}\big)$, and a direct calculation leads to

$$\Delta\Big(\frac{1}{2} r_x^2\Big) = (n-1) G_K(r_x) + 1, \tag{2.4}$$

with $G_K(r)$ defined in (1.2).

2.2 Preliminaries on BV functions

We recall here some basic properties of functions of bounded variation, referring to [1] for an exhaustive introduction to the subject.

In the following, vol denotes the volume measure with respect to the Riemannian metric g. We distinguish between the measure vol and the volume element $d\,\mathrm{vol}$. The following presentation in Euclidean spaces can be found in [12] (see also [9, Section 2]).

Given a vector valued Borel measure μ on M (or more generally a vector bundle valued Borel measure μ on M), we define its *total variation* as the non-negative Borel measure $|\mu|$ defined on the Borel set E by the formula

$$|\mu|(E) = \sup\left\{\sum_{n\in\mathbb{N}} \|\mu(E_n)\| \,:\, E_m \cap E_n = \emptyset, \bigcup_{n\in\mathbb{N}} E_n = E\right\},$$

where $\|\cdot\|$ denotes a norm on the vector bundle on which the measure μ takes values. Given a Borel set E, we say E has *finite perimeter* if the distributional gradient $D\chi_E$ of its characteristic function χ_E is a vector valued Borel measure on M with finite total variation on M, that is, $|D\chi_E|(M) < \infty$. For a smooth bounded domain E, we denote by ν_E the outer unit normal vector field on the boundary ∂E. It follows from the Divergence Theorem that E is a set of finite perimeter with $D\chi_E = \nu_E d\mathscr{H}^{n-1}\lfloor\partial E$ and $|D\chi_E|(M) = \mathscr{H}^{n-1}(\partial E)$. Let us denote $B_r(x)$ the geodesic ball around x with the radius equal to r. Given $E \subset M$ be set with finite perimeter, we denote its *reduced boundary* by

$$\begin{aligned}\mathcal{F}E := \{x \in M \mid \ & |D\chi_E|(B_r(x)) > 0\ \forall\, r > 0 \text{ and} \\ & \nu_E(x) := \lim_{r\to 0^+} -\tfrac{D\chi_E(B_r(x))}{|D\chi_E|(B_r(x))} \text{ exists and is a unit tangent vector at } x\}.\end{aligned}$$

Here $\nu_E(x) \in T_xM$ is called the *measure theoretic outwards unit normal* to E. We say a sequence of sets E_n converges to E in L^1_{loc}, and we write $E_n \xrightarrow{L^1_{loc}} E$, if $\mathrm{vol}\big(C \cap ((E_n \setminus E) \cup (E \setminus E_n))\big) \to 0$ for any compact set $C \subset M$. A geometric description of the reduced boundary is the following structure result(for a proof, see for instance [1, Theorems 3.59]):

Theorem 2.2 (De Giorgi Rectifiability Theorem) *Let $E \subset M$ be a set with finite perimeter. Then, for any $x \in \mathcal{F}E$ we have*

$$\frac{1}{r}\exp_x^{-1}(B_{r_0}(x) \cap E) \xrightarrow{L^1_{loc}} \{z \in T_xM \,:\, \nu_E(x)\cdot z < 0\} \qquad \text{as } r \to 0^+$$

on T_xM. Moreover, there holds

$$D\chi_E = -\nu_E(x) d\mathscr{H}^{n-1}\lfloor\mathcal{F}E, \qquad |D\chi_E|(M) = \mathscr{H}^{n-1}(\mathcal{F}E).$$

Let $E \subset M$ be a Borel set and $\lambda \in [0,1]$, we denote by $E^{(\lambda)}$ the set of points x of M having density λ with respect to E, that is, $x \in E^{(\lambda)}$ if

$$\lim_{r\to 0} \frac{\mathrm{vol}(E \cap B_r(x))}{\mathrm{vol}(B_r(x))} = \lambda.$$

We define the *essential boundary* $\partial^* E$ of E by setting $\partial^* E = M \setminus (E^{(0)} \cup E^{(1)})$. By virtue of a result due to Federer (see [1, Theorems 3.61]), if E is a Borel set of finite perimeter then

$$\mathcal{F}E \subseteq E^{(\frac{1}{2})} \subseteq \partial^* E,$$

and these three sets are $\mathscr{H}^{n-1}$-equivalent.

Now, suppose E and F two Borel sets of finite perimeter. By [1, Proposition 3.38, Example 3.68, Example 3.97] $E \cap F$ is a set of finite perimeter. Moreover, if we set

$$J_{E,F} = \{x \in \mathcal{F}E \cap \mathcal{F}F \ : \ \nu_E(x) = \nu_F(x)\},$$

then up to $\mathscr{H}^{n-1}$-null sets we have

$$\mathcal{F}(E \cap F) = J_{E,F} \cup (\mathcal{F}E \cap F^{(1)}) \cup (\mathcal{F}F \cap E^{(1)}).$$

Furthermore

$$\nu_{E\cap F}(x) = \begin{cases} \nu_E(x), & \text{if } x \in \mathcal{F}E \cap F^{(1)}, \\ \nu_F(x), & \text{if } x \in \mathcal{F}F \cap E^{(1)}, \\ \nu_E(x) = \nu_F(x), & \text{if } x \in J_{E,F}, \end{cases} \qquad \text{at } \mathscr{H}^{n-1}\text{-a.e. } x \in \mathcal{F}(E \cap F).$$

We also recall the following technical result (see [12, Lemma2.2]):

Lemma 2.3 *Let $E \subset M$ and $F \subset M$ be sets of finite perimeter with* $\mathrm{vol}(E \cap F) = 0$. *Then*

$$\nu_{E\cup F} d\mathscr{H}^{n-1} \lfloor \mathcal{F}(E \cup F) = \nu_E d\mathscr{H}^{n-1} \lfloor (\mathcal{F}E \setminus \mathcal{F}F) + \nu_F d\mathscr{H}^{n-1} \lfloor (\mathcal{F}F \setminus \mathcal{F}E)$$

and $\nu_E(x) = -\nu_F(x)$ at $\mathscr{H}^{n-1}$-a.e. $x \in \mathcal{F}E \cap \mathcal{F}F$

Fix $N \in M$, and let $S \in BV(M, T_N M)$, that is, $S \in L^1(M, T_N M)$ and *distributional derivative* $D_x S$ of S has finite total variation on M. By [1, Theorems 3.59 and 3.77], for $\mathcal{H}^{n-1}$-a.e. $x \in \mathcal{F}E$ there exists a vector $\mathrm{tr}_E(S)(x) \in T_N M$ such that

$$\lim_{r\to 0} \frac{1}{r^n} \int_{B_r(x)\cap E} |S(y) - \mathrm{tr}_E(S)(x)| \, d\,\mathrm{vol}(y) = 0$$

Such $\mathrm{tr}_E(S)$ is called the *inner trace* of S on E. Moreover, by the Lebesgue Decomposition Theorem the *distributional derivative* $D_x S$ of S can be decomposed as

$$D_x S = \nabla^c_x S \ \mathrm{vol} + D^s_x S,$$

where $\nabla_x S$ denotes the density of the absolutely continuous part of $D_x S$ with respect to the volume measure, and $D^s_x S$ is singular with respect to the volume measure. Similarly, for any smooth section A of $TM \otimes T^*_N M$, we decompose the *covariant distributional derivative* $D_x(A \cdot S)$ of $A \cdot S$

$$D_x(A \cdot S) = \nabla_x(A \cdot S)) \ \mathrm{vol} + D^s_x(A \cdot S),$$

where $(\nabla_x(A\cdot S))^c$ is the density of $\nabla_x(A\cdot S)$ with respect to the volume measure, and $\nabla_x^s(A\cdot S)$ is the singular part of $\nabla_x(A\cdot S)$ with respect to the volume measure. In particular the *distributional divergence* $\mathrm{Div}_x(A\cdot S)$ of $A\cdot S$ decomposes (with obvious notation) as

$$\mathrm{Div}_x(A\cdot S) = \mathrm{div}_x(A\cdot S)\,\mathrm{vol} + \mathrm{Div}_x^s(A\cdot S).$$

We recall the *generalized Divergence Theorem* in the following form (which is a consequence of [1, Example 3.97] applied to the pair of functions $A\cdot S$ and χ_E):

$$\int_{E^{(1)}} d\big(\mathrm{Div}_x(A\cdot S)\big) = \int_{\mathcal{F}E} \big(A(x)\cdot \mathrm{tr}_E(S)(x)\big)\cdot \nu_E(x)\, d\mathcal{H}^{n-1}(x). \tag{2.5}$$

Suppose now that $D_x(A\cdot S)$ is a positive definite and symmetric endomorphism on TM. Then $\mathrm{Div}_x(A\cdot S)$ is a non negative Radon measure on M, and it is equivalent to the total variation of $D_x(A\cdot S)$: for all Borel set $E\subset M$,

$$\frac{1}{\sqrt{n}}\mathrm{Div}_x(A\cdot S)(E) \le |D_x(A\cdot S)|(E) \le \mathrm{Div}_x(A\cdot S)(E). \tag{2.6}$$

Here we recall that the vector bundle $TM\otimes T^*M$ (or $TM\otimes T^*_N M$) is endowed the induced metric by g, that is, for $b\in T_xM\otimes T^*_yM$ with $x,y\in M$, if we write $b=\sum_{ij} b^i{}_j \frac{\partial}{\partial x_i}\otimes dy^j$ in a local chart then $|b|^2 := \sum_{ijkl} g_{ik}g^{jl}b^i{}_j b^k{}_l$. (Let us observe that if everything is smooth then (2.6) follows from $\frac{1}{\sqrt{n}}\sum_{i=1}^n \lambda_i \le \sqrt{\sum_{i=1}^n \lambda_i^2} \le \sum_{i=1}^n \lambda_i$, with $\lambda_i\ge 0$. In the general case one argues by approximation.) Thanks to (2.6) we get that both $\mathrm{div}_x(A\cdot S)$ vol and Div_x^s are non-negative measures, and in particular

$$\mathrm{Div}_x(A\cdot S) - \mathrm{div}_x(A\cdot S)\,\mathrm{vol} \ge \mathrm{Div}_x^s(A\cdot S) \ge 0. \tag{2.7}$$

2.3 Preliminaries on optimal transport

Let (M,g), (M',g') be two complete connected n-dimensional Riemannian manifold equipped with their volume measures vol and vol$'$. Given two probability measures $\mu=\rho_0\,\mathrm{vol}$ and $\nu=\rho_1\,\mathrm{vol}'$ on M and M' respectively, a Borel map $\Phi: M\to M'$ is called a *transport map* if $\Phi_\#\mu=\nu$, that is

$$\nu(B') = \mu[\Phi^{-1}(B')] \qquad \forall\, B'\subset M' \text{ Borel}.$$

Given now a function $c: M\times M'\to\mathbb{R}$ (called *cost*), a transport map $\mathcal{T}$ is called an *optimal transport map* if it minimizes the total cost functional

$$\mathcal{C}(\Phi) := \int_M c\big(x,\Phi(x)\big)\ d\mu(x)$$

In this paper, we will consider three cases:

(1) $M'=M$ and $c(x,y)=\frac{1}{2}d_g(x,y)^2$ (d_g standing for the geodesic distance on (M,g)).

(2) $M=M'=\mathbb{R}^n$ and $c(v,w)=\frac{1}{2}|v-w|^2$.

(3) $M' = T_N M$ for some (fixed) point $N \in M$, and $c(x, v) = \frac{1}{2}\nabla_y d_g(x, N)^2 \cdot v$.

The first two cases have been studied by Brenier [3] and McCann [13, 14, 15], while the third case (and indeed also the first two) is covered by a result of Fathi and the first author [10] (see also [11] or [20, Chapter 10]).

In all these cases, it is known that there is a unique optimal transport map $\mathcal{T}$. Moreover, whenever the target measure ν is compactly supported, there exists a locally semiconvex function $\varphi : M \to \mathbb{R}$ (i.e. φ can be locally written in charts as the sum of a convex and a smooth function) such that, for μ-a.e. $x \in M$,

$$z \mapsto \varphi(z) + c(z, \mathcal{T}(x)) \quad \text{attains a global minimum at } x, \tag{2.8}$$

and

$$\nabla^x \varphi + \nabla^x c(x, \mathcal{T}(x)) = 0 \quad \text{holds } \mu\text{-a.e.} \tag{2.9}$$

Finally, $\nabla_x \nabla^x \varphi + \nabla_x \nabla^x c(x, \mathcal{T}(x))$ exists and is a symmetric matrix μ-a.e., and satisfies the Monge-Ampère type equation

$$\det\big(\nabla_x \nabla^x \varphi + \nabla_x \nabla^x c(x, \mathcal{T}(x))\big) = \frac{\rho_0(x)\sqrt{\det g_x}\,\big|\det\big(\nabla_y \nabla^x c(x, \mathcal{T}(x))\big)\big|}{\rho_1(\mathcal{T}(x))\sqrt{\det g'_{\mathcal{T}(x)}}} \tag{2.10}$$

at μ-a.e. x.

3 Proof of Theorem 1.1

Let us first explain our strategy. Fix a point $N \in M$ as the pole of the manifold, and denote by $B_r(N) \subset M$ the geodesic ball of radius r centered at N. Recall that $r_x := d_g(x, N)$. Given $E \subset M$, we denote by $\mathrm{vol}(E)$ its volume. Moreover, let $R \in (0, +\infty)$ be the unique positive number such that $\mathrm{vol}(E) = \mathrm{vol}(B_R(N))$. The natural adaptation of Gromov argument (see Section A.1) would be to consider the optimal transport map sending the uniform density on E onto the uniform density on $B_R(N)$. However, as explained in Appendix A this choice presents some difficulties which is not clear how to bypass. For this reason we want to keep the freedom of choosing the radius λ of the target ball, and consider transport E onto the ball $B_\lambda(N)$, with λ arbitrary. Moreover, we want also to be free to choose both the source and the target density. As shown in Appendix A, the best solution in the choice of the radius seems to be to consider the limit as $\lambda \to 0$. This leads to consider the "infinitesimal" transport maps, where we transport a suitable density on E onto the uniform measure on the unit ball in the tangent plane $T_N M$. Since the proof below may somehow look obscure and it is difficult to understand the ideas behind it, we suggest the reader to first read the formal argument given in Appendix A.

Proof of Theorem 1.1
Let us define for $K \in \mathbb{R}$

$$\tilde\rho_K(r) := \begin{cases} \left(\frac{\sin(\sqrt{K}r)}{\sqrt{K}r}\right)^{n-2} e^{n(G_K(0)-G_K(r))} & \text{if } K > 0,\\ 1 & \text{if } K = 0,\\ \left(\frac{\sinh(\sqrt{|K|}r)}{\sqrt{|K|}r}\right)^{n-2} e^{n(G_K(0)-G_K(r))} & \text{if } K < 0,\end{cases}$$

$$H_K(r) := 1 - e^{(n-1)[G_K(0)-G_K(r)]},$$

and

$$c(x,y) := \frac{1}{2} d_g(x,y)^2.$$

Thanks to (2.4), we have

$$\nabla_y \Delta_x c(x,N) = (n-1)\nabla_y G_K(r_x) = (n-1)\nabla_y \nabla_x c(x,N) \cdot \nabla^x G_K(r_x). \tag{3.1}$$

This gives the following key identity:

$$\mathrm{div}_x\big(H_K(r_x)\nabla_y\nabla_x c(x,N)\big) = \mathrm{div}_x\big(\nabla_y\nabla_x c(x,N)\big). \tag{3.2}$$

Indeed, expanding both sides we get

$$\nabla_y\nabla_x c(x,N)\cdot\nabla^x H_K(r_x) + H_K(r_x)\nabla_y\Delta_x c(x,N) = \nabla_y\Delta_x c(x,N),$$

and so (3.2) follows from (3.1) and the definition of H_K. In the following, we will use $\cdot$ both for denoting the action of covector on a vector, and for the composition of endomorphisms (or equivalently the product of matrices). Moreover, given A a section of $T^*M \otimes TM$, we denote by $\mathrm{tr}_x[A]$ the trace of $A(x) \in T_x^*M \otimes T_xM$. We will need the following result:

Lemma 3.1 *Let F be a bounded set with finite perimeter such that $c(\cdot,N) : M \to R$ is smooth in a neighborhood of F, and let $S \in BV_{loc}(M, T_NM)$. Then*

$$\int_{\mathcal{F}F} \big(1 - H_K(r_x)\big)\big(\nabla_y\nabla_x c(x,N)\cdot \mathrm{tr}_F(S)(x)\big)\cdot \nu_x \, d\mathscr{H}^{n-1}(x)$$
$$= \int_{F^{(1)}} \big(1 - H_K(r_x)\big)\, d\big(\mathrm{tr}_x[\nabla_y\nabla^x c(x,N)\cdot D_xS]\big).$$

Proof. Applying (3.2), we infer

$$\mathrm{div}_x\big((1 - H_K(r_x))\nabla_y\nabla^x c(x,N)\big) = 0 \tag{3.3}$$

so that

$$\mathrm{Div}_x\big((1 - H_K(r_x))\nabla_y\nabla^x c(x,N)\cdot S\big) = \big(1 - H_K(r_x)\big)\,\mathrm{tr}_x\big[\nabla_y\nabla^x c(x,N)\cdot D_xS\big]. \tag{3.4}$$

Now the result follows from (2.5).

□

- **1: Proof of** (1.5)**.** For simplicity of notation, we denote by M the manifold $M^n(K)$. Moreover, without loss of generality we assume that E coincides with the set of its density points, i.e. $E = E^{(1)}$.

Since by assumption $d_g^2(\cdot,N) : M \to \mathbb{R}$ is smooth in a neighborhood of E, we can consider the set

$$\tilde{E} := -\nabla^y c(E,N) = (\exp_N)^{-1}(E) \subset T_NM.$$

We now want to transport $\tilde{E}$ onto the unit ball $B(N,1) \subset T_N M$ using the Euclidean quadratic cost. More precisely, we consider $\tilde{\mathcal{T}} : T_N M \to T_N M$ the optimal transport map from

$$\tilde{\mu}(dv) := \tilde{c}_E \tilde{\rho}_K(|v|_N)\chi_{\tilde{E}}(v)dv, \qquad \tilde{c}_E := \frac{1}{\int_{\tilde{E}} \tilde{\rho}_K(|v|_N))\, dv},$$

to

$$\tilde{\nu}(dw) := \frac{1}{|B(N,1)|}\chi_{B(N,1)}(w)dw = \frac{1}{\omega_n}\chi_{B(N,1)}(w)dw$$

for the cost function $-\langle v, w\rangle_N$ (or equivalently for the cost $\frac{1}{2}|v-w|_N^2$).

Let $\mathcal{T} : M \to T_N M$ be defined as

$$\mathcal{T}(x) := \tilde{\mathcal{T}}(-\nabla^y c(x,N)), \qquad x \in M,$$

set $\mu := (\exp_N)_\# \tilde{\mu}$, and write $\mu = \rho_E(r_x)\chi_E(x)\operatorname{vol}(dx)$. Then, denoting by $\bar{g}$ the Euclidean metric on the vector tangent bundle $T(T_N M)$,

$$\det\bigl(d_v \exp_N\bigr) = \frac{\tilde{c}_E\, \tilde{\rho}_K(r_x)\, \sqrt{\det \bar{g}_v}}{\rho_E(r_x)\, \sqrt{\det g_x}} \qquad \text{for } x = \exp_N(v),$$

which as $d_v \exp_N = (-\nabla_x \nabla^y c(x,N))^{-1}$ implies

$$\rho_E(r_x) = \frac{\tilde{c}_E\, \tilde{\rho}_K(r_x)\, \bigl|\det\bigl(\nabla_x \nabla^y c(x,N)\bigr)\bigr|\, \sqrt{\det \bar{g}_v}}{\sqrt{\det g_x}}.$$

It is clear that $\mathcal{T}$ is the optimal transport map from μ to $\tilde{\nu}$ for the cost $c_0(x,w) := \nabla_y c(x,N) \cdot w$. Moreover, since $c_0(\cdot, w)$ is smooth in a neighborhood of E for all w, thanks to the local semiconvexity of φ the relation (2.9) can be smoothly inverted in terms of $\mathcal{T}$, and we get $\mathcal{T} \in BV_{loc}(M, T_N M)$. We can therefore apply the above lemma with $S = \mathcal{T}$ and $F = E \cap B_\rho(N)$ for some $\rho > 0$, obtaining

$$\begin{aligned}\int_{\mathcal{F}(E\cap B_\rho(N))} & e^{(n-1)[G_K(0)-G_K(r_x)]}\bigl(-\nabla_y \nabla_x c(x,N) \cdot \operatorname{tr}_E(\mathcal{T})(x)\bigr) \cdot \nu_x\, d\mathscr{H}^{n-1}(x) \\ &= \int_{E\cap B_\rho(N)} e^{(n-1)[G_K(0)-G_K(r_x)]}\, d\bigl(\operatorname{tr}_x[-\nabla_y \nabla^x c(x,N) \cdot D_x \mathcal{T}]\bigr),\end{aligned}$$

where we recall that D_x denotes the distributional derivative. Recalling that (2.9) holds for some semiconvex function $\varphi : M \to \mathbb{R}$, we get that

$$-\nabla_y \nabla^x c(x,N) \cdot D_x \mathcal{T} = D_x \nabla^x \varphi + \nabla_x \nabla^x c(x, \mathcal{T}(x))$$

is symmetric. Moreover by (2.8) we deduce that

$$-\nabla_y \nabla^x c(x,N) \cdot D_x \mathcal{T} \geq -\nabla_y \nabla^x c(x,N) \cdot \nabla_x \mathcal{T}\, d\operatorname{vol}(x) \geq 0 \qquad \mu\text{-a.e.},$$

i.e. the singular part of $-\nabla_y \nabla^x c(x,N) \cdot D_x \mathcal{T}$ is non-negative, and the density of its absolutely continuous part $-\nabla_y \nabla^x c(x,N) \cdot \nabla_x \mathcal{T}$ is non-negative definite μ-a.e. Therefore

$$\int_{\mathcal{F}(E\cap B_\rho(N))} e^{(n-1)[G_K(0)-G_K(r_x)]} \big(-\nabla_y \nabla_x c(x,N) \cdot \operatorname{tr}_E(\mathcal{T})\big) \cdot \nu_x \, d\mathscr{H}^{n-1}(x) \\ \geq \int_{E\cap B_\rho(N)} e^{(n-1)[G_K(0)-G_K(r_x)]} \operatorname{tr}_x[-\nabla_y \nabla^x c(x,N) \cdot \nabla_x \mathcal{T}] \, d\operatorname{vol}(x).$$

Moreover by (2.10) $\mathcal{T}$ satisfies the Monge-Ampère equation

$$\det\big(-\nabla_w \nabla^x c_0(x,w) \cdot \nabla_x \mathcal{T}\big) = \frac{\omega_n \, \rho_E(r_x) \, \sqrt{\det g_x} \big|\det\big(\nabla_w \nabla^x c_0(x,w)\big)\big|}{\sqrt{\det \bar{g}_v}} \tag{3.5}$$

vol-a.e. in E, where $c_0(x,w) := \nabla_y c(x,N) \cdot w$. Therefore, since $\nabla_w \nabla^x c_0(x,w) = \nabla_y \nabla^x c(x,N)$, by the arithmetic-geometric inequality we get

$$\int_{E\cap B_\rho(N)} e^{(n-1)[G_K(0)-G_K(r_x)]} \operatorname{tr}_x[-\nabla_y \nabla^x c(x,N) \cdot \nabla_x \mathcal{T}] \, d\operatorname{vol}(x) \\ \geq n \int_{E\cap B_\rho(N)} e^{(n-1)[G_K(0)-G_K(r_x)]} \det\big(-\nabla_y \nabla^x c(x,N) \cdot \nabla_x \mathcal{T}\big)^{1/n} \, d\operatorname{vol}(x) \\ = n \, \omega_n^{1/n} \tilde{c}_E^{-1/n} \int_{E\cap B_\rho(N)} e^{(n-1)[G_K(0)-G_K(r_x)]} \tilde{\rho}_K(r_x)^{1/n} \\ \big|\det\big(\nabla_y \nabla^x c(x,N)\big) \det\big(\nabla_x \nabla^y c(x,N)\big)\big|^{1/n} \, d\operatorname{vol}(x).$$

Finally, as $\mathcal{T}(x) \in B(N,1)$ for vol-a.e. $x \in E$, we have $|\operatorname{tr}_E(\mathcal{T})| \leq 1$ and we get

$$\int_{\mathcal{F}(E\cap B_\rho(N))} e^{(n-1)[G_K(0)-G_K(r_x)]} \big(-\nabla_y \nabla_x c(x,N) \cdot \operatorname{tr}_E(\mathcal{T})\big) \cdot \nu_x \, d\mathscr{H}^{n-1}(x) \\ = \int_{\mathcal{F}(E\cap B_\rho(N))} e^{(n-1)[G_K(0)-G_K(r_x)]} \big(-\nabla_x \nabla_y c(x,N) \cdot \nu_x\big) \cdot \operatorname{tr}_E(\mathcal{T}) \, d\mathscr{H}^{n-1}(x) \\ \leq \int_{\mathcal{F}(E\cap B_\rho(N))} e^{(n-1)[G_K(0)-G_K(r_x)]} \big|\nabla_x \nabla_y c(x,N) \cdot \nu_x\big| \, d\mathscr{H}^{n-1}(x),$$

Combining all together, we obtain

$$\int_{\mathcal{F}(E\cap B_\rho(N))} e^{(n-1)[G_K(0)-G_K(r_x)]} \big|\nabla_x \nabla_y c(x,N) \cdot \nu_x\big| \, d\mathscr{H}^{n-1}(x) \\ \geq n \, \omega_n^{1/n} \tilde{c}_E^{-1/n} \int_{E\cap B_\rho(N)} e^{n[G_K(0)-G_K(r_x)]} \ell_K(r_x) \, d\operatorname{vol}(x).$$

Letting $\rho \to +\infty$ and observing that

$$\frac{1}{\tilde{c}_E} = \int_{\tilde{E}} \tilde{\rho}_K(|v|) \, dv = \int_E e^{n[G_K(0)-G_K(r_x)]} \ell_K(r_x) \, d\operatorname{vol}(x)$$

the result follows.

- **2: The equality case.** From the above proof, we see that equality hold if and only if

$$D_s\mathcal{T} = 0 \quad (\text{that is, } \mathcal{T} \in W^{1,1}(M, T_N M)), \tag{3.6}$$

$$-\nabla_y \nabla^x c(x,N) \cdot \nabla_x \mathcal{T} = \lambda(x) I \qquad \text{vol-a.e. in } E \tag{3.7}$$

and

$$\big(-\nabla_y \nabla_x c(x,N) \cdot \mathrm{tr}_E(\mathcal{T})\big) \cdot \nu_x = \big|\nabla_x \nabla_y c(x,N) \cdot \nu_x\big| \qquad \mathscr{H}^{n-1}\text{-a.e. on } \mathcal{F}E. \tag{3.8}$$

• *Equality holds if $E = E_{w,R}$.* We observe that, for μ-a.e. x, we have

$$\nabla_v \tilde{\mathcal{T}} = \nabla_x \mathcal{T} \cdot d_v(\exp_N) \qquad \text{for } x = \exp_N(v),$$

so that

$$\begin{aligned}-\nabla_y \nabla^x c(x,N) \cdot \nabla_x \mathcal{T} &= \nabla_y \nabla^x c(x,N) \cdot \nabla_x \mathcal{T} \cdot d_v(\exp_N) \cdot \nabla_x \nabla^y c(x,N) \\ &= \nabla_y \nabla^x c(x,N) \cdot \nabla_v \tilde{\mathcal{T}} \cdot \nabla_x \nabla^y c(x,N) : T_x M \to T_x M.\end{aligned}$$

In particular (3.5) gives

$$\det\big(\nabla_v \tilde{\mathcal{T}}\big) = \omega_n \tilde{c}_E \tilde{\rho}_K(|v|_N). \tag{3.9}$$

We consider a radially symmetric function $\tilde{\mathcal{T}}_1 : \exp_N^{-1}(E_{w,R}) \to B(w,R) \subset T_N M$ defined in polar coordinates by $\tilde{\mathcal{T}}_1(r,\theta) = (\alpha_K(r), \theta)$, where $\alpha_K(r)$ is defined in (1.4). Now using the Euclidean coordinates on $T_N M$, we define $\tilde{\mathcal{T}}_2 : \exp_N^{-1}(E_{w,R}) \to B(0,1) \subset T_N M$ by $\tilde{\mathcal{T}}_2(z) = (\tilde{\mathcal{T}}_1(z) - w)/R$, and $\mathcal{T}_2(x) := \tilde{\mathcal{T}}_2(-\nabla^y c(x,N))$. We want to show that $\mathcal{T} = \mathcal{T}_2$.

By using Fermi coordinates along the geodesic jointing N to x, g_x and g_y can be read as the identity matrix. Moreover, $\nabla_v \tilde{\mathcal{T}}_2$ (resp. $-\nabla_y \nabla^x c(x,N)$) can be expressed as the diagonal matrix $\big(\frac{\alpha_K'(r_x)}{R}, \frac{\alpha_K(r_x)}{R r_x}, \cdots, \frac{\alpha_K(r_x)}{R r_x}\big)$ (resp. $\big(1, \ell_K(r_x), \cdots, \ell_K(r_x)\big)$, see (2.2)). Using the identification $T_v(T_N M) = T_{\tilde{\mathcal{T}}(v)}(T_N M) = T_N M$, we have

$$\ell_K(r) = \Big(\big|\det\big(\nabla_x \nabla_y c(x,N)\big)\big|\, \big|\det\big(\nabla^x \nabla^y c(x,N)\big)\big|\Big)^{1/2(n-1)},$$

and we easily get that the eigenvalues of $-\nabla_y \nabla^x c(x,N) \cdot \nabla_x \mathcal{T}_2$ are all equal since α_K solves the ODE

$$\alpha_K'(r) = \frac{\alpha_K(r) l_K^2(r)}{r}. \tag{3.10}$$

Thanks to the choice of $\tilde{\rho}_K$, the map $\tilde{\mathcal{T}}_2$ solves (3.9). Moreover, $\tilde{\mathcal{T}}_2$ is the gradient of some convex function on $T_N M$. To see this, we have $\tilde{\mathcal{T}}_2(z) = \frac{1}{R}\nabla(\varphi(|z|) - z \cdot w)$, where $\varphi(t) = \displaystyle\int_0^t \alpha_K(s)ds$. Hence, the convexity of $\frac{1}{R}(\varphi(|z|) - z \cdot w)$ comes from its Hessian matrix $\nabla \tilde{\mathcal{T}}_2$ being positive definite. By uniqueness of the optimal transport map, we get $\tilde{\mathcal{T}} = \tilde{\mathcal{T}}_2$. Hence, (3.6) and (3.7) are verified.

To check the last condition (3.8), we write in the Fermi coordinates $\nu_x = (\nu_x^1, \nu_x') \in \mathbb{R} \times \mathbb{R}^{n-1}$. Let us denote $\nu_v = (\nu_v^1, \nu_v') \in \mathbb{R} \times \mathbb{R}^{n-1}$ the unit outward normal vector of $\mathcal{F}\tilde{E}$ at v and $\tilde{\nu}_{v_1} = (\tilde{\nu}_{v_1}^1, \tilde{\nu}_{v_1}') \in \mathbb{R} \times \mathbb{R}^{n-1}$ the unit outward normal vector of $\mathcal{F}\big(F_K(\tilde{E})\big)$ at $v_1 = F_K(v)$. Here we use the orthonormal basis with respect to the polar coordinates on $T_N M$. Recall $F_K(\tilde{E}) = B(w, R)$ and $\alpha_K(|v|) = |v_1|$. Using (3.10), we have

$$\tilde{\nu}_{v_1} = \frac{1}{\sqrt{(\nu_v^1)^2 + |\nu_v'|^2 \ell_K^4(|v|)}} (\nu_v^1, \nu_v' \ell_K^2(|v|)).$$

Similarly, we infer (see also Section 4)

$$\nu_x = \frac{1}{\sqrt{(\nu_v^1)^2 + |\nu_v'|^2 \ell_K^2(|v|)}} (\nu_v^1, \nu_v' \ell_K(|v|)).$$

Therefore, we obtain

$$\nu_x = \frac{1}{\sqrt{(\tilde{\nu}_{v_1}^1)^2 + |\tilde{\nu}_{v_1}'|^2 / \ell_K^2(|v|)}} \Big(\tilde{\nu}_{v_1}^1, \frac{\tilde{\nu}_{v_1}'}{\ell_K(|v|)}\Big),$$

and as a consequence

$$-\nabla_x \nabla_y c(x, N) \cdot \nu_x = \frac{1}{\sqrt{(\tilde{\nu}_{v_1}^1)^2 + |\tilde{\nu}_{v_1}'|^2 / \ell_K^2(|v|)}} \big(\tilde{\nu}_{v_1}^1, \tilde{\nu}_{v_1}'\big).$$

Hence, since $\mathrm{tr}_E(\mathcal{T})(x) = \tilde{\nu}_{v_1}$, (3.8) is verified and we have proved the desired result.

- *If equality holds, then $E = E_{w,R}$.* We prove the result in two steps.

Step 1: E is indecomposable. We have to show that, for any $F \subset E$ having finite perimeter and such that

$$\mathscr{H}^{n-1}(\mathcal{F}E) = \mathscr{H}^{n-1}(\mathcal{F}F) + \mathscr{H}^{n-1}(\mathcal{F}(E \setminus F)),$$

we have

$$\mathrm{vol}(F) = 0 \quad \text{or} \quad \mathrm{vol}(E \setminus F) = 0.$$

To this aim, we first remark that $E \setminus F$ has also finite perimeter. Moreover it is not difficult to check that $\mathscr{H}^{n-1}(\mathcal{F}F \cap E) = \mathscr{H}^{n-1}(\mathcal{F}(E \setminus F) \cap E) = 0$. Hence, by applying Theorem 1.1

we deduce

$$
\begin{aligned}
\int_{\mathcal{F}E} & e^{(n-1)[G_K(0)-G_K(r_x)]}\big|\nabla_x\nabla_y c(x,N)\cdot\nu_x\big|\,d\mathscr{H}^{n-1}(x)\\
&=\int_{\mathcal{F}F} e^{(n-1)[G_K(0)-G_K(r_x)]}\big|\nabla_x\nabla_y c(x,N)\cdot\nu_x\big|\,d\mathscr{H}^{n-1}(x)\\
&\quad+\int_{\mathcal{F}(E\setminus F)} e^{(n-1)[G_K(0)-G_K(r_x)]}\big|\nabla_x\nabla_y c(x,N)\cdot\nu_x\big|\,d\mathscr{H}^{n-1}(x)\\
&\geq n\,\omega_n^{1/n}\Big(\int_F e^{n[G_K(0)-G_K(r_x)]}\ell_K(r_x)\,d\,\mathrm{vol}(x)\Big)^{\frac{n-1}{n}}\\
&\quad+n\,\omega_n^{1/n}\Big(\int_{E\setminus F} e^{n[G_K(0)-G_K(r_x)]}\ell_K(r_x)\,d\,\mathrm{vol}(x)\Big)^{\frac{n-1}{n}}\\
&\geq n\,\omega_n^{1/n}\Big(\int_E e^{n[G_K(0)-G_K(r_x)]}\ell_K(r_x)\,d\,\mathrm{vol}(x)\Big)^{\frac{n-1}{n}}\\
&=\int_{\mathcal{F}E} e^{(n-1)[G_K(0)-G_K(r_x)]}\big|\nabla_x\nabla_y c(x,N)\cdot\nu_x\big|\,d\mathscr{H}^{n-1}(x).
\end{aligned}
$$

By the strict concavity of the function $t\mapsto t^{(n-1)/n}$ we obtain $\min\{\mathrm{vol}(F),\mathrm{vol}(E\setminus F)\}=0$, and the claim is proved.

Step 2: $E=E_{w,R}$. As before, we use the Fermi coordinates along the geodesic jointing N to x. From (3.7), $\nabla_v\tilde{\mathcal{T}}$ can be expressed as the diagonal matrix $\big(\lambda(x),\frac{\lambda(x)}{\ell_K^2(r_x)},\cdots,\frac{\lambda(x)}{\ell_K^2(r_x)}\big)$. Together with (3.9), we deduce

$$
\lambda(x)=\omega_n^{1/n}\,\tilde{c}_E^{1/n}\,\tilde{\rho}_K(r_x)^{1/n}\,\ell_K^{2-2/n}(r_x)=\omega_n^{1/n}\,\tilde{c}_E^{1/n}\,\ell_K(r_x)\,e^{G_K(0)-G_K(r_x)}
$$

which implies that $\lambda(x)$ is radial. Let $s(r):=\frac{\omega_n^{1/n}\,\tilde{c}_E^{1/n}\,r\,e^{G_K(0)-G_K(r)}}{\ell_K(r)}$, so that it satisfies $s'(r)=\frac{s(r)\ell_K^2(r)}{r}$. We define the map $\tilde{\mathcal{T}}_1:T_NM\to T_NM$ in polar coordinates as $\tilde{\mathcal{T}}_1(r,\theta):=(s(r),\theta)$. We can easily check that $\nabla_v\tilde{\mathcal{T}}_1$ can be expressed as the diagonal matrix $\big(s'(r_x),\frac{s(r_x)}{r_x},\cdots,\frac{s(r_x)}{r_x}\big)$ in Fermi coordinates. Hence $\nabla_v\tilde{\mathcal{T}}_1=\nabla_v\tilde{\mathcal{T}}$ on $\tilde{E}$. On the other hand E is indecomposable by Step 1, which implies that also $\tilde{E}$ is indecomposable. Thanks to [9, Proposition 2.12] (see also [12, Lemma A.2]), we deduce that there exists some constant vector $w_1\in T_NM$ such that $\tilde{\mathcal{T}}_1=\tilde{\mathcal{T}}+w_1$. By construction $\tilde{\mathcal{T}}_1(\tilde{E})=B(w_1,1)$ and $s(r)$ is a multiple of $\alpha_K(r)$ (see (1.4)), so $E=E_{w,R}$ for some $R>0$ and $w\in T_NM$. This gives the desired result. and concludes the proof of the theorem.

3.1 Theorem 1.1 seen from the tangent space at N

Let us define $\tilde{E}=\exp_N^{-1}(E)$. The following result holds:

Proposition 3.2 *We have*

$$\int_{\mathcal{F}\tilde{E}} e^{(n-1)[G_K(0)-G_K(|v|)]}\ell_K(|v|)^{1-n}\sqrt{(\nu_v^1)^2+\ell_K(|v|)^4|\nu_v'|^2}\,d\mathscr{H}^{n-1}(v)$$
$$\geq n\omega_n^{1/n}\bigg(\int_{\tilde{E}} e^{n[G_K(0)-G_K(|v|)]}\ell_K(|v|)^{2-n}\,dv\bigg)^{(n-1)/n}, \quad (3.11)$$

where $\nu_v=(\nu_v^1,\nu_v')\in\mathbb{R}\times\mathbb{R}^{n-1}$ *is the unit outward normal vector of* $\mathcal{F}\tilde{E}$ *at* v *expressed in polar coordinates on* $T_N\big(M^n(K)\big)\simeq\mathbb{R}^n$*. Moreover, if* $\tilde{E}$ *is a smooth strictly star-shaped domain in* $\mathbb{R}^n$ *with smooth boundary, that is* $\partial E=\{f(\theta)\theta\,|\,\theta\in\mathbb{S}^{n-1}\}$ *with* $f:\mathbb{S}^{n-1}\to(0,+\infty)$ *smooth, then:*

- *if* $K>0$,

$$\int_{\mathbb{S}^{n-1}} e^{-(n-1)G_K(f)}\bigg(\frac{\sin(\sqrt{K}f)}{\sqrt{K}}\bigg)^{n-1}\sqrt{1+\ell_K(f)^4\frac{|\nabla f|^2}{f^2}}\,d\mathscr{H}^{n-1}$$
$$\geq (n\omega_n)^{1/n}\bigg(\int_{\mathbb{S}^{n-1}} e^{-nG_K(f)}\bigg(\frac{\sin(\sqrt{K}f)}{\sqrt{K}}\bigg)^{n}\,d\mathscr{H}^{n-1}\bigg)^{(n-1)/n}; \quad (3.12)$$

(recall $f:\mathbb{S}^{n-1}\to(0,\pi/\sqrt{K})$*)*

- *if* $K=0$,

$$\int_{\mathbb{S}^{n-1}} f^{n-1}\sqrt{1+\frac{|\nabla f|^2}{f^2}}\,d\mathscr{H}^{n-1}\geq(n\omega_n)^{1/n}\bigg(\int_{\mathbb{S}^{n-1}} f^n\,d\mathscr{H}^{n-1}\bigg)^{(n-1)/n}; \quad (3.13)$$

- *if* $K<0$,

$$\int_{\mathbb{S}^{n-1}} e^{-(n-1)G_K(f)}\bigg(\frac{\sinh(\sqrt{|K|}f)}{\sqrt{|K|}}\bigg)^{n-1}\sqrt{1+\ell_K(f)^4\frac{|\nabla f|^2}{f^2}}\,d\mathscr{H}^{n-1}$$
$$\geq (n\omega_n)^{1/n}\bigg(\int_{\mathbb{S}^{n-1}} e^{-nG_K(f)}\bigg(\frac{\sinh(\sqrt{|K|}f)}{\sqrt{|K|}}\bigg)^{n}\,d\mathscr{H}^{n-1}\bigg)^{(n-1)/n}. \quad (3.14)$$

We observe that, arguing by approximation, the above inequalities hold also when f is non-smooth and vanishes somewhere, provided all the terms in the integrals are suitably interpreted.

Remark 3.3 As can be easily seen by a direct computation, replacing f by $\alpha_K(f)$ in (3.13) (which corresponds to send $\tilde{E}$ onto $F_K(\tilde{E})$), inequalities (3.14) and (3.12) are equivalent to (3.13).

Proof. Since by (2.1)

$$d_v\exp_N^{-1}=-\nabla_x\nabla^y c(x,N)\qquad\text{for } x=\exp_N(v),$$

we have

$$(\exp_N)_\#(dv) = \frac{\left|\det\left(\nabla_x \nabla^y c(x,N)\right)\right|}{\sqrt{\det(g_x)}} d\,\mathrm{vol}(x).$$

Therefore, recalling that in Fermi coordinates $\left|\det\left(\nabla_x \nabla^y c(x,N)\right)\right| = \ell_K(r_x)^{n-1}$ (see (2.3)) and $\det(g_x) = 1$, we get

$$\begin{aligned}
&\int_E e^{n[G_K(0)-G_K(r_x)]} \ell_K(r_x)\, d\,\mathrm{vol}(x) \\
&\quad = \int_E e^{n[G_K(0)-G_K(r_x)]} \ell_K(r_x) \frac{\sqrt{\det(g_x)}}{\left|\det\left(\nabla_x \nabla^y c(x,N)\right)\right|} \frac{\left|\det\left(\nabla_x \nabla^y c(x,N)\right)\right|}{\sqrt{\det(g_x)}}\, d\,\mathrm{vol}(x) \\
&\quad = \int_E e^{n[G_K(0)-G_K(r_x)]} \ell_K(r_x)^{2-n} \frac{\left|\det\left(\nabla_x \nabla^y c(x,N)\right)\right|}{\sqrt{\det(g_x)}}\, d\,\mathrm{vol}(x) \\
&\quad = \int_{\tilde E} e^{n[G_K(0)-G_K(|v|)]} \ell_K(|v|)^{2-n}\, dv.
\end{aligned}$$

Regarding the boundary term, we observe that the surface measure change with the tangential Jacobian of $-\nabla_x \nabla^y c(x,N)$, that is

$$(\exp_N)_\#\left(d\mathscr{H}^{n-1}(v)\right) = \frac{\left|\det\left(\nabla_x \nabla^y c(x,N)_{|_{\nu_x^\perp}}\right)\right|}{\sqrt{\det\left((g_x)_{|_{\nu_x^\perp}}\right)}} d\mathscr{H}^{n-1}(x)$$

where $\nabla_x \nabla^y c(x,N)_{|_{\nu_x^\perp}}$ is the restriction of $\nabla_x \nabla^y c(x,N)$ to $\nu_x^\perp \subset T_x\left(M^n(K)\right)$. Hence

$$\begin{aligned}
&\int_{\mathcal{F}E} e^{(n-1)[G_K(0)-G_K(r_x)]} \left|\nabla_x \nabla_y c(x,N)\cdot \nu_x\right| d\mathscr{H}^{n-1}(x) \\
&\quad = \int_{\mathcal{F}E} e^{(n-1)[G_K(0)-G_K(r_x)]} \frac{\left|\nabla_x \nabla_y c(x,N)\cdot \nu_x\right| \sqrt{\det\left((g_x)_{|_{\nu_x^\perp}}\right)}}{\left|\det\left(\nabla_x \nabla^y c(x,N)_{|_{\nu_x^\perp}}\right)\right|} \frac{\left|\det\left(\nabla_x \nabla^y c(x,N)_{|_{\nu_x^\perp}}\right)\right|}{\sqrt{\det\left((g_x)_{|_{\nu_x^\perp}}\right)}} d\mathscr{H}^{n-1}(x) \\
&\quad = \int_{\mathcal{F}\tilde E} e^{(n-1)[G_K(0)-G_K(|v|)]} \frac{\left|\nabla_x \nabla_y c(x,N)\cdot \nu_x\right|}{\left|\det\left(\nabla_x \nabla^y c(x,N)_{|_{\nu_x^\perp}}\right)\right|} d\mathscr{H}^{n-1}(v).
\end{aligned}$$

Let us write $\nu_x = (\nu_x^1, \nu_x') \in \mathbb{R}\times\mathbb{R}^{n-1}$ using Fermi coordinates. Since

$$-\nabla_x \nabla_y c(x,N) = \begin{pmatrix} 1 & 0 \\ 0 & \ell_K(r_x) I_{n-1} \end{pmatrix},$$

we have $-\nabla_x \nabla_y c(x,N)\cdot \nu_x = (\nu_x^1, \ell_K(r_x)\nu_x')$, and so

$$\left|\nabla_x \nabla_y c(x,N)\cdot \nu_x\right| = \sqrt{(\nu_x^1)^2 + \ell_K(r_x)^2 |\nu_x'|^2}.$$

We now denote by $(\nu_x^1, \dots, \nu_x^n)$ the coordinates of ν_x. To compute $\left|\det\left(\nabla_x \nabla^y c(x,N)_{|_{\nu_x^\perp}}\right)\right|$, we consider the basis of $\nu_x^\perp$ given by

$$\tau_i := (\nu_x^i, 0, \dots, 0, -\nu_x^1, 0, \dots, 0) = \nu_x^i e_1 - \nu_x^1 e_i, \qquad i = 2, \dots, n$$

(up to slightly perturbing ν_x, we can assume $\nu_x^1 \neq 0$). Then, if we denote $L_x := \nabla_x \nabla^y c(x, N)$, we have

$$\left|\det\left(\nabla_x \nabla^y c(x, N)_{|_{\nu_x^\perp}}\right)\right| = \frac{\left|(L_x \tau_2) \wedge \ldots \wedge (L_x \tau_n)\right|}{\left|\tau_2 \wedge \ldots \wedge \tau_n\right|}.$$

Since

$$L_x \tau_i = \nu_x^i e_1 - \ell_K(r_x) \nu_x^1 e_i, \qquad i = 2, \ldots, n,$$

we get

$$\left|(L_x \tau_2) \wedge \ldots \wedge (L_x \tau_n)\right|^2 = \ell_K(r_x)^{2(n-2)} (\nu_x^1)^{2(n-2)} \left[\ell_K(r_x)^2 (\nu_x^1)^2 + \sum_{i=2}^n (\nu_x^i)^2\right],$$

$$\left|\tau_2 \wedge \ldots \wedge \tau_n\right|^2 = (\nu_x^1)^{2(n-2)} \left[(\nu_x^1)^2 + \sum_{i=2}^n (\nu_x^i)^2\right] = (\nu_x^1)^{2(n-2)}.$$

So

$$\left|\det\left(\nabla_x \nabla^y c(x, N)_{|_{\nu_x^\perp}}\right)\right| = \ell_K(r_x)^{(n-2)} \sqrt{\ell_K(r_x)^2 (\nu_x^1)^2 + |\nu_x'|^2}$$

which gives

$$\frac{\left|\nabla_x \nabla_y c(x, N) \cdot \nu_x\right|}{\left|\det\left(\nabla_x \nabla^y c(x, N)_{|_{\nu_x^\perp}}\right)\right|} = \ell_K(r_x)^{2-n} \frac{\sqrt{(\nu_x^1)^2 + \ell_K(r_x)^2 |\nu_x'|^2}}{\sqrt{\ell_K(r_x)^2 (\nu_x^1)^2 + |\nu_x'|^2}}.$$

We now observe that, if $x = \exp_N(v)$, the normal vector ν_v at $\mathcal{F}\tilde{E}$ is given by

$$\nu_v = \frac{(\ell_K(|v|) \nu_x^1, \nu_x')}{\sqrt{\ell_K(|v|)^2 (\nu_x^1)^2 + |\nu_x'|^2}} = (\nu_v^1, \nu_v'),$$

so that

$$\frac{\left|\nabla_x \nabla_y c(x, N) \cdot \nu_x\right|}{\left|\det\left(\nabla_x \nabla^y c(x, N)_{|_{\nu_x^\perp}}\right)\right|} = \ell_K(|v|)^{1-n} \sqrt{(\nu_v^1)^2 + \ell_K(|v|)^4 |\nu_v'|^2}.$$

Hence we finally get

$$\int_{\mathcal{F}E} e^{(n-1)[G_K(0) - G_K(r_x)]} \left|\nabla_x \nabla_y c(x, N) \cdot \nu_x\right| d\mathscr{H}^{n-1}(x)$$
$$= \int_{\mathcal{F}\tilde{E}} e^{(n-1)[G_K(0) - G_K(|v|)]} \ell_K(|v|)^{1-n} \sqrt{(\nu_v^1)^2 + \ell_K(|v|)^4 |\nu_v'|^2}\, d\mathscr{H}^{n-1}(v),$$

and combining all together our isoperimetric inequality read on the tangent space becomes (3.11).

We now prove (3.12). Let $v \in \partial\tilde{E}$. We fix $(e_1, \ldots, e_n)$ a orthonormal basis at the point v, where $e_1 = \frac{v}{|v|}$, and denote by $\nu_v = (\nu_v^1, \ldots, \nu_v^n)$ the outward normal vector at v. Then, a basis for $\nu_v^\perp$ if given by

$$\tau_i := (\nabla_{e_i} f, 0, \ldots, 0, f, 0, \ldots, 0) = \nabla_{e_i} f e_1 + f e_i, \qquad i = 2, \ldots, n.$$

Therefore ν_v is given by

$$\nu_v = \frac{1}{\sqrt{f^2 + |\nabla f|^2}} \big(f e_1 - \sum_{i=2}^{n} \nabla_{e_i} f e_i \big),$$

and we have

$$\big|\tau_2 \wedge \ldots \wedge \tau_n\big|^2 = f^{2(n-2)} \Big[f^2 + \sum_{i=2}^{n} |\nabla_{e_i} f|^2 \Big],$$

As a consequence, we infer

$$\int_{\mathcal{F}\tilde{E}} e^{(n-1)[G_K(0)-G_K(|v|)]} \ell_K(|v|)^{1-n} \sqrt{(\nu_v^1)^2 + \ell_K(|v|)^4 |\nu_v'|^2}\, d\mathscr{H}^{n-1}(v)$$

$$= \int_{\mathbb{S}^{n-1}} e^{-(n-1)(G_K(f)-G_K(0))} \left(\frac{\sin(\sqrt{K} f)}{\sqrt{K}} \right)^{n-1} \sqrt{1 + \ell_K(f)^4 \frac{|\nabla f|^2}{f^2}}\, d\mathscr{H}^{n-1}$$

On the other hand, since

$$\big(e^{-nG_K(r)} \sin^n(\sqrt{K} r)\big)' = nKre^{-nG_K(r)} \sin^{n-2}(\sqrt{K} r),$$

we get

$$\int_{\tilde{E}} e^{n[G_K(0)-G_K(|v|)]} \ell_K(|v|)^{2-n}\, dv = \frac{1}{n} \int_{\mathbb{S}^{n-1}} e^{-n(G_K(f)-G_K(0))} \left(\frac{\sin(\sqrt{K} f)}{\sqrt{K}} \right)^n d\mathscr{H}^{n-1}.$$

This proves (3.12). The proof of (3.13) and (3.14) is analogous. □

A Formal proof of Theorem 1.1

In order to explain the idea behind Theorem 1.1, we first consider the case $K = 0$ (so that $G_K = \ell_K \equiv 1$), and we recall Gromov's proof of the isoperimetric inequality.

A.1 Gromov's proof of the Euclidean isoperimetric inequality

Without loss of generality, we can assume $E \subset \mathbb{R}^n$ bounded and smooth. Let $B_r = B(0, r)$ be the ball centered at the origin with radius $r > 0$. By Brenier's Theorem [3], there exists a unique convex, Lipschitz continuous function $\varphi : \mathbb{R}^n \to \mathbb{R}$ such that its gradient $\mathcal{T} = \nabla\varphi$ pushes forward the probability density $\frac{1}{|E|}\chi_E(x)dx$ onto the probability density $\frac{1}{\omega_n r^n}\chi_{B_r}(y)dy$, where $|E|$ denotes the volume of E. By Caffarelli's regularity result [4, 5] we can assume $\mathcal{T} \in C^\infty(E, B_r)$. Moreover φ solves the following Monge-Ampère equation

$$\det \nabla^2\varphi = \frac{\omega_n r^n}{|E|} \qquad \text{on } E, \tag{A.1}$$

where $\nabla^2\varphi$ is the Hessian matrix of φ. As φ is convex, the Hessian matrix $\nabla^2\varphi$ is a positive definite symmetric matrix, and so by the arithmetic-geometric inequality we get

$$n(\det \nabla^2\varphi)^{1/n} \le \Delta\varphi$$

Thus from the Divergence Theorem we infer

$$n\omega_n^{1/n}|E|^{\frac{n-1}{n}} = \frac{1}{r}\int_E n(\det \nabla^2\varphi)^{1/n}dx \leq \frac{1}{r}\int_E \Delta\varphi dx = \frac{1}{r}\int_{\partial E}\frac{\partial\varphi}{\partial\nu_x}d\mathscr{H}^{n-1} \leq \mathscr{H}^{n-1}(\partial E),$$

where at the last step we used that $|\frac{\partial\varphi}{\partial\nu_x}| = |\langle\mathcal{T},\nu_x\rangle| \leq r$, as $\mathcal{T}(x) \in B_r$ for all $x \in E$.

This concludes the proof of the classic isoperimetric inequality on $\mathbb{R}^n$. Moreover it is easily seen (at least formally) that equality holds at each step if and only if $\mathcal{T}(x) = c_1x + c_2$ for some constants c_i, and this is possible if and only if E is a ball (see [12, Appendix A] for a rigorous proof of the equality case).

We observe that in the above proof the choice of r plays no role, and this fact is due to the invariance under scaling of the isoperimetric inequality on $\mathbb{R}^n$.

A.2 The non-flat case

In the following, we will mimic the same strategy to prove an isoperimetric-type inequality on $M := M^n(K)$ for $K \neq 0$. As before, fix a point $N \in M$ as the pole of the manifold, and denote by $B_r(N) \subset M$ the geodesic ball of radius r centered at N. Recall that $r_x := d_g(x, N)$. For simplicity, we consider the case when M is the unit sphere, that is $K = 1$. Let $E \subset M$ be a bounded and smooth open set which is at positive distance from the antipodal point of N, and denote by $\mathrm{vol}(E)$ the volume of E. Moreover, let $R \in (0, \pi)$ be the unique positive number such that $\mathrm{vol}(E) = \mathrm{vol}(B_R(N))$. Since the isoperimetric inequality on the sphere is not scale-invariant, the most natural choice would be to consider the optimal transport map from E to $B_\lambda(N)$ with $\lambda = R$. However, as we will explain below, this choice presents some difficulties which it is not clear how to bypass, and for this reason it turns out to be simpler to play also with the choice of the radius λ. Therefore, we will transport E onto the ball $B_\lambda(N)$, with $\lambda \in (0, \pi)$ arbitrary. Another degree of freedom that we have is the possibility of considering non-constant densities on both E and $B_\lambda(N)$. For symmetry reasons, we assume that the densities depend only on r_x. Moreover, we also assume that the density on E depends only on its volume (that is, on R only).

More precisely, let $\rho_R^0, \rho_{\lambda,R}^1 : \mathbb{R} \to \mathbb{R}^+$ be non-negative functions to be fixed later, depending respectively on R, and on λ and R. We want to transport the probability measure

$$\mu_E = \frac{c_E}{\mathrm{vol}(E)}\rho_R^0(r_x)\chi_E(x)\,\mathrm{vol}(dx), \qquad c_E = \frac{\mathrm{vol}(E)}{\int_E c_E\rho_R^0(r_x)\,d\,\mathrm{vol}(x)},$$

onto

$$\nu_{\lambda,R}(dy) = \frac{1}{\mathrm{vol}(B_\lambda(N))}\rho_{\lambda,R}^1(r_y)\chi_{B_\lambda(N)}(y)\,\mathrm{vol}(dy).$$

Let $\mathcal{T}_\lambda : M \to M$ denote the optimal transport from μ_E to $\nu_{\lambda,R}$ for the cost $c(x,y) = \frac{1}{2}d_g(x,y)^2$. We recall that by (2.9) we have

$$\nabla_x\varphi_\lambda + \nabla_x c(x, \mathcal{T}_\lambda(x)) = 0, \tag{A.2}$$

and the Monge-Ampère equation (2.10) becomes

$$\left|\det\left(\nabla_y\nabla^x c(x,\mathcal{T}_\lambda(x))\cdot\nabla_x\mathcal{T}_\lambda(x)\right)\right| = \frac{c_E\ \mathrm{vol}(B_\lambda)\,\rho_R^0(r_x)\,\sqrt{\det g_x}\,\left|\det\left(\nabla_y\nabla^x c(x,\mathcal{T}_\lambda(x))\right)\right|}{\mathrm{vol}(B_R)\,\rho_{\lambda,R}^1(r_{\mathcal{T}_\lambda(x)})\,\sqrt{\det g_{\mathcal{T}_\lambda(x)}}},$$

with

$$-\nabla_y\nabla^x c(x,\mathcal{T}_\lambda(x))\cdot\nabla_x\mathcal{T}_\lambda(x) = \nabla_x\nabla^x\varphi_\lambda + \nabla_x\nabla^x c(x,\mathcal{T}_\lambda(x))$$

symmetric and non-negative definite (the above identity follows by differentiating (A.2), while the fact that $\nabla_x\nabla^x\varphi_\lambda + \nabla_x\nabla^x c(x,\mathcal{T}_\lambda(x))$ is non-negative follows from (2.8)). We now remark that, if we want to apply Gromov's strategy in this case, at some moment we will use the arithmetic-geometric inequality with the eigenvalues of $-\nabla_y\nabla^x c(x,\mathcal{T}_\lambda(x))\cdot\nabla_x\mathcal{T}_\lambda(x)$. Therefore we will end up with an expression involving its determinant, which will always depend on $\mathcal{T}_\lambda$ via the term $\left|\det\left(\nabla_y\nabla^x c(x,\mathcal{T}_\lambda(x))\right)\right|$, and there is no hope that we can use the freedom in the choice of ρ_R^0 and $\rho_{\lambda,R}^1$ to cancel this term.

The key observation is now the following: if we choose λ small, since $\mathcal{T}_\lambda(x)\in B_\lambda(N)$, then $\left|\det\left(\nabla_y\nabla^x c(x,\mathcal{T}_\lambda(x))\right)\right| \sim \left|\det\left(\nabla_y\nabla^x c(x,N)\right)\right|$. Thus we may try to take a limit as $\lambda\to 0$ of the transports $\mathcal{T}_\lambda$ in such a way that in the limit we still have something non-trivial. At this step the choice of ρ_R^0 and $\rho_{\lambda,R}^1$ will be crucial: if we want to find an inequality which becomes equality when E is a ball, in the case $E = B_R(N)$ we need to impose that

$$-\nabla_y\nabla^x c(x,\mathcal{T}_\lambda(x))\cdot\nabla_x\mathcal{T}_\lambda(x)$$

is a multiple of the identity at each point (since we want the arithmetic-geometric inequality to become an equality). As we will see, this condition will tell us how to choose ρ_R^0 and $\rho_{\lambda,R}^1$.

A.3 The choice of the densities: the case $E = B_R(N)$

We want to transport

$$\mu(dx) = \frac{1}{\mathrm{vol}(B_R)}\rho_R^0(r_x)\chi_{B_R(N)}(x)\,\mathrm{vol}(dx)$$

onto

$$\nu_\lambda(dy) = \frac{1}{\mathrm{vol}(B_\lambda)}\rho_{\lambda,R}^1(r_y)\chi_{B_\lambda(N)}(y)\,\mathrm{vol}(dy).$$

In order μ and ν_λ to be probability measures, we need

$$\int_{B_R(N)}\rho_R^0(r_x)\,d\,\mathrm{vol}(x) = \mathrm{vol}(B_R(N)), \qquad \int_{B_\lambda(N)}\rho_{\lambda,R}^1(r_y)\,d\,\mathrm{vol}(y) = \mathrm{vol}(B_\lambda(N)).$$

Since ν_λ will be constructed from μ through a push-forward, the above condition on $\rho_{\lambda,R}^1$ will follow from the one on ρ_R^0.

Let $\bar{\mathcal{T}}_\lambda$ denote the optimal transport from μ to ν_λ for the cost $c(x,y)=\frac{1}{2}d_g(x,y)^2$. We are going to choose ρ_R^0 and ρ_λ^1 in such a way that the matrix

$$-\nabla_y\nabla^x c(x,\bar{\mathcal{T}}_\lambda(x))\cdot\nabla_x\bar{\mathcal{T}}_\lambda(x) : T_xM\to T_xM \tag{A.3}$$

is a multiple of the identity map at each point. To this aim, we first remark that by symmetry the optimal transport is "radial", that is it depends only on the distance of x from N. More precisely, if we consider the polar coordinates induced by the exponential map at N, we can write

$$x = (r_x, \theta_x) \longmapsto \bar{\mathcal{T}}_\lambda(x) = (t_\lambda(r_x), \theta_x)$$

with $t_\lambda(r_x) = d_g(\bar{\mathcal{T}}_\lambda(x), N)$. To compute the expression in (A.3), since it is an intrinsic quantity (i.e. independent of the system of coordinates), for each term we can use the Fermi chart around the point x along the geodesic

$$[0,1] \ni s \longmapsto \exp_x\big(s\nabla^x \varphi_\lambda\big)$$

connecting x to $\bar{\mathcal{T}}_\lambda(x)$. In this way we have $g_x = g_{\bar{\mathcal{T}}_\lambda(x)} = I_n$. Moreover, by (2.1) and the Jacobi field equation, one easily has

$$\big[-\nabla_y \nabla^x c(x, \bar{\mathcal{T}}_\lambda(x))\big]^{-1} = d_{\nabla^x \varphi_\lambda}\big(\exp_x\big) = \begin{pmatrix} 1 & 0 \\ 0 & \frac{\sin(d(x,\bar{\mathcal{T}}_\lambda(x)))}{d(x,\bar{\mathcal{T}}_\lambda(x))} I_{n-1} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & \frac{\sin|r_x - t_\lambda(r_x)|}{|r_x - t_\lambda(r_x)|} I_{n-1} \end{pmatrix}.$$

Let $d\theta^2 = \sum_{i=1}^{n-1} d\theta_i^2$ denote the standard metric on the Euclidean sphere $\mathbb{S}^{n-1}$. We observe that, in the polar coordinates induced by the exponential map at N, the metric at $x = (r_x, \theta)$ is $dr^2 + \sin^2(r_x) d\theta^2$, while the metric at $\bar{\mathcal{T}}_\lambda(x) = (t_\lambda(r_x), \theta)$ is $dr^2 + \sin^2(t_\lambda(r_x)) d\theta^2$. Therefore, since in these coordinates

$$\nabla_x \bar{\mathcal{T}}_\lambda(x)\left(\frac{\partial}{\partial r}\right) = t'_\lambda(r_x)\frac{\partial}{\partial r}, \qquad \nabla_x \bar{\mathcal{T}}_\lambda(x)\left(\frac{\partial}{\partial \theta}\right) = \frac{\partial}{\partial \theta},$$

it is not difficult to see that in Fermi coordinates we have

$$\nabla_x \bar{\mathcal{T}}_\lambda(x) = \begin{pmatrix} t'_\lambda(r_x) & 0 \\ 0 & \frac{\sin(t_\lambda(r_x))}{\sin(r_x)} I_{n-1} \end{pmatrix}.$$

Combining all together, we finally obtain

$$-\nabla_y \nabla^x c(x, \bar{\mathcal{T}}_\lambda(x)) \cdot \nabla_x \bar{\mathcal{T}}_\lambda(x) = \begin{pmatrix} t'_\lambda(r_x) & 0 \\ 0 & \frac{\sin(t_\lambda(r_x))\,|r_x - t_\lambda(r_x)|}{\sin(r_x)\,\sin|r_x - t_\lambda(r_x)|} I_{n-1} \end{pmatrix}.$$

Hence the above matrix equals a multiple of the identity if and only if

$$\begin{cases} t'_\lambda(r) = \frac{\sin(t_\lambda(r))\,|r - t_\lambda(r)|}{\sin(r)\,\sin|r - t_\lambda(r)|} & r \in [0, R], \\ t_\lambda(R) = \lambda. \end{cases} \tag{A.4}$$

Since in the sequel we will take the limit as $\lambda \to 0$, we can assume $\lambda \leq R$. Using a standard comparison principle for ODE, it is not difficult to prove that the unique solution of (A.4) is a diffeomorphism from $(0, R]$ onto $(0, \lambda]$, and a homeomorphism from $[0, R]$ onto $[0, \lambda]$ (indeed one can prove that $0 \leq t_\lambda(r) \leq r$, which implies that $t_\lambda(0) = 0$).

We now want to study the Monge-Ampère equation in order to understand how we should choose the densities. We have

$$\left|\det\left(\nabla_y\nabla^x c(x,\bar{\mathcal{T}}_\lambda(x))\nabla_x\bar{\mathcal{T}}_\lambda(x)\right)\right| = \frac{\mathrm{vol}(B_\lambda)\,\rho^0_R(r_x)\,\sqrt{\det g_x}\left|\det\left(\nabla_y\nabla^x c(x,\bar{\mathcal{T}}_\lambda(x))\right)\right|}{\mathrm{vol}(B_R)\,\rho^1_{\lambda,R}(t_\lambda(r_x))\,\sqrt{\det g_{\bar{\mathcal{T}}_\lambda(x)}}}.$$

In the Fermi chart, the left hand side coincides with $\left(\frac{\sin(t_\lambda(r_x))\,|r_x-t_\lambda(r_x)|}{\sin(r_x)\,\sin|r_x-t_\lambda(r_x)|}\right)^n$, while for the right hand side we have $\left|\det\left(\nabla_y\nabla^x c(x,\bar{\mathcal{T}}_\lambda(x))\right)\right| = \left(\frac{|r_x-t_\lambda(r_x)|}{\sin|r_x-t_\lambda(r_x)|}\right)^{n-1}$ and $\det g_x = \det g_{\bar{\mathcal{T}}_\lambda(x)} = 1$. Hence

$$\rho^1_{\lambda,R}(t_\lambda(r_x)) = \frac{\mathrm{vol}(B_\lambda)\,\sin^n(r_x)\,\sin|r_x-t_\lambda(r_x)|\,\rho^0_R(r_x)}{\mathrm{vol}(B_R)\,\sin^n(t_\lambda(r_x))\,|r_x-t_\lambda(r_x)|},$$

or equivalently

$$\rho^1_{\lambda,R}(r) = \frac{\mathrm{vol}(B_\lambda(N))\,\sin^n(t_\lambda^{-1}(r))\,\sin|t_\lambda^{-1}(r)-r|\,\rho^0_R(t_\lambda^{-1}(r))}{\mathrm{vol}(B_R(N))\,\sin^n(r)\,|t_\lambda^{-1}(r)-r|}. \tag{A.5}$$

As t_λ is known (thanks to (A.4)), we see from the above equation that $\rho^1_{\lambda,R}$ is uniquely determined once we fix ρ^0_R. We now remark the following fact: when we will consider a general domain E, the term $\rho^1_{\lambda,R}(d(\mathcal{T}_\lambda(x),N))$ will appear in the Monge-Ampère equation . Therefore, if we want in the limit something independent of the transport map, we need to choose ρ^0_R in such a way that $\rho^1_{\lambda,R}$ converges uniformly to 1 on $[0,\lambda]$ as $\lambda\to 0$.

To this aim, let us define $s(r) := \partial_\lambda t_\lambda(r)|_{\lambda=0}$. Differentiating the ODE for t_λ with respect to λ at $\lambda=0$ we get

$$\begin{cases} s'(r) = s(r)\frac{r}{\sin^2(r)}, & r\in[0,R],\\ s(R)=1, \end{cases}$$

which gives

$$s(r) = e^{-\int_r^R \frac{s}{\sin^2(s)}\,ds}.$$

Since

$$\int_r^R \frac{s}{\sin^2(s)} = \log(\sin u) - \frac{u\cos(u)}{\sin(u)}\Big|_r^R = \log\left(\frac{\sin R}{\sin r}\right) - \frac{R\cos(R)}{\sin(R)} + \frac{r\cos(r)}{\sin(r)},$$

we obtain

$$s(r) = \frac{\sin r}{\sin R}\,e^{\frac{R\cos(R)}{\sin(R)}-\frac{r\cos(r)}{\sin(r)}}.$$

Having in mind that $t_\lambda(r)\sim\lambda s(r)\ll r$ for $\lambda>0$ small, we see that

$$\rho^1_{\lambda,R}(t_\lambda(r_x)) \sim \frac{\mathrm{vol}(B_\lambda)\,\sin^{n+1}(r_x)\,\rho^0_R(r_x)}{\mathrm{vol}(B_R)\,\sin^n(t_\lambda(r_x))\,r_x} \sim \frac{\omega_n\,\sin^{n+1}(r_x)\,\rho^0_R(r_x)}{\mathrm{vol}(B_R)\,s^n(r_x)\,r_x},$$

Thus, in order to have $\rho^1_{\lambda,R}$ almost constant we finally set

$$\rho^0_R(r) := \frac{\mathrm{vol}(B_R)}{\omega_n\sin^n(R)}\,\frac{r}{\sin(r)}\,e^{n\left(\frac{R\cos(R)}{\sin(R)}-\frac{r\cos(r)}{\sin(r)}\right)}. \tag{A.6}$$

It can be checked by an explicit computation that

$$\fint_{B_R(N)} \rho_R^0(r_x)\, d\operatorname{vol}(x) = 1.$$

Moreover, with this choice of ρ_R^0, we have the following uniform convergence result: for any $\varepsilon > 0$ there exists $\lambda_\varepsilon > 0$ such that, if $0 < \lambda \leq \lambda_\varepsilon$, then

$$1 - \varepsilon \leq \rho^1_{\lambda,R}(r) \leq 1 + \varepsilon \qquad \text{on } [0, \lambda]. \tag{A.7}$$

(Since we are just giving a formal argument, we will not prove this fact in details.)

A.4 Back to a general domain

Now that we have chosen the densities ρ_R^0 and $\rho^1_{\lambda,R}$, we come back to a general domain E. We recall that the maps $\mathcal{T}_\lambda : M \to M$ were constructed as the optimal transport maps from

$$\mu_E = \frac{c_E}{\operatorname{vol}(E)} \rho_R^0(r_x) \chi_E(x) \operatorname{vol}(dx)$$

onto

$$\nu_{\lambda,R}(dy) = \frac{1}{\operatorname{vol}(B_\lambda(N))} \rho^1_{\lambda,R}(r_y) \chi_{B_\lambda(N)}(y) \operatorname{vol}(dy).$$

Let us write $\mathcal{T}_\lambda(x) = \exp_x(\nabla^x \varphi_\lambda)$. Then

$$\begin{aligned}
\frac{1}{\lambda} \int_{\mathcal{F}E} (\nabla_x \varphi_\lambda + \nabla_x c(x, N)) \cdot \nu_x \, d\mathscr{H}^{n-1}(x) &= \frac{1}{\lambda} \int_E \Delta_x \varphi_\lambda + \Delta_x c(x, N)\, d\operatorname{vol}(x) \\
&= \frac{1}{\lambda} \int_E \big[\Delta_x \varphi_\lambda + \Delta_x c(x, \mathcal{T}_\lambda(x))\big]\, d\operatorname{vol}(x) \\
&- \frac{1}{\lambda} \int_E \big[\Delta_x c(x, \mathcal{T}_\lambda(x)) - \Delta_x c(x, N)\big]\, d\operatorname{vol}(x) \qquad \text{(A.8)} \\
&= \frac{1}{\lambda} \int_E \big[\Delta_x \varphi_\lambda + \Delta_x c(x, \mathcal{T}_\lambda(x))\big]\, d\operatorname{vol}(x) \\
&- \frac{n-1}{\lambda} \int_E \big[G(d(x, \mathcal{T}_\lambda(x))) - G(r_x)\big]\, d\operatorname{vol}(x)
\end{aligned}$$

where in the last step we used (2.4) with $K = 1$, and $G(r) := G_1(r) = \frac{r\cos(r)}{\sin(r)}$. We now want to take the limit as $\lambda \to 0$.

A.4.1 Properties of the limit transport map

Let us define $\mathcal{T} : M \to T_N M$ as $\mathcal{T} := (\partial_\lambda \mathcal{T}_\lambda)|_{\lambda=0}$. (Again, this is just a formal argument.) Recalling that

$$-\nabla_y \nabla^x c(x, \mathcal{T}_\lambda(x)) \cdot \nabla_x \mathcal{T}_\lambda = [\nabla_x \nabla^x \varphi_\lambda + \nabla_x \nabla^x c(x, \mathcal{T}_\lambda(x))]$$

is symmetric and non-negative definite, letting $\lambda \to 0$ we get that $-\nabla_y \nabla^x c(x, N) \cdot \nabla_x \mathcal{T}$ is symmetric and non-negative definite. On the other hand, as

$$\det\bigl(-\nabla_y \nabla^x c(x, \mathcal{T}_\lambda(x)) \cdot \nabla_x \mathcal{T}_\lambda\bigr) = \frac{c_E \operatorname{vol}(B_\lambda(N))\, \rho_R^0(r_x) \sqrt{\det g_x} \det\bigl(-\nabla_y \nabla^x c(x, \mathcal{T}_\lambda(x))\bigr)}{\operatorname{vol}(E)\, \rho_{\lambda,R}^1(r_{\mathcal{T}_\lambda(x)}) \sqrt{\det g_{\mathcal{T}_\lambda(x)}}},$$

since $\operatorname{vol}(B_\lambda(N)) = \omega_n \lambda^n + o(\lambda^n)$ we infer

$$\lim_{\lambda \to 0} \frac{1}{\lambda^n} \det\bigl(-\nabla_y \nabla^x c(x, \mathcal{T}_\lambda(x)) \cdot \nabla_x \mathcal{T}_\lambda\bigr) = \det\bigl(-\nabla_y \nabla^x c(x, N) \cdot \nabla_x \mathcal{T}\bigr)$$

and by (A.7) we get

$$\lim_{\lambda \to 0} \frac{\operatorname{vol}(B_\lambda(N))\, \rho_R^0(r_x) \sqrt{\det g_x} \det\bigl(-\nabla_y \nabla^x c(x, \mathcal{T}_\lambda(x))\bigr)}{\lambda^n \operatorname{vol}(E)\, \rho_{\lambda,R}^1(r_{\mathcal{T}_\lambda(x)}) \sqrt{\det g_{\mathcal{T}_\lambda(x)}}}$$
$$= \frac{\omega_n\, \rho_R^0(r_x) \sqrt{\det g_x}\, \det\bigl(-\nabla_y \nabla^x c(x, N)\bigr)}{\operatorname{vol}(E)} = \frac{\omega_n\, \rho_R^0(r_x)\, r_x^{n-1}}{\operatorname{vol}(E) \sin^{n-1}(r_x)}.$$

Therefore $\mathcal{T}$ satisfies

$$\det\bigl(-\nabla_y \nabla^x c(x, N) \cdot \nabla_x \mathcal{T}\bigr) = \frac{c_E\, \omega_n\, \rho_R^0(r_x)\, r_x^{n-1}}{\operatorname{vol}(E)\, \sin^{n-1}(r_x)}, \tag{A.9}$$

with ρ_R^0 defined in (A.6). This implies that $\mathcal{T} : M \to T_N M$ is a transport map from μ to

$$\nu_0 = \frac{1}{\omega_n} \chi_{B(N,1)}(v)\, dv,$$

where $B(N, 1) \subset T_N M$ denotes the unit ball in the tangent space, and dv is the Lebesgue measure. Moreover one can show that $\mathcal{T}$ is the optimal transport map from μ to ν_0, where the optimality is with respect to the cost function $c_0 : M \times T_N M \to \mathbb{R}$ defined as

$$c_0(x, v) := \lim_{\lambda \to 0} \frac{d_g(x, \exp_N(\lambda v))^2 - d_g(x, N)^2}{2\lambda} = \nabla_y c(x, N) \cdot v.$$

Observe that, if we write $x = \exp_N(v_x)$ and define $\bar{c}_0 : T_N M \times T_N M \to \mathbb{R}$ as

$$\bar{c}_0(v_x, v) := c_0(x, v),$$

then $\bar{c}_0(v_x, v) = -v_x \cdot v$, so that the cost function c_0 seen on $T_N M \times T_N M$ is equivalent to the Euclidean cost $\frac{1}{2}|v_x - v|^2$.

A.4.2 Back to the proof

Regarding the second term in (A.8), we have

$$\lim_{\lambda \to 0} \frac{G(d(x, \mathcal{T}_\lambda(x))) - G(r_x)}{\lambda} = \nabla_y G(d(x, y))|_{y=N} \cdot \mathcal{T}(x),$$

while for the first term we have

$$\frac{1}{\lambda}\big[\Delta_x\varphi_\lambda + \Delta_x c(x,\mathcal{T}_\lambda(x))\big] = -\frac{1}{\lambda}\,\mathrm{tr}_x[\nabla_y\nabla^x c(x,\mathcal{T}_\lambda(x))\cdot\nabla_x\mathcal{T}_\lambda(x)] \to -\,\mathrm{tr}_x[\nabla_y\nabla^x c(x,N)\cdot\nabla_x\mathcal{T}]$$

as $\lambda\to 0$. Moreover

$$\frac{\nabla_x\varphi_\lambda + \nabla_x c(x,N)}{\lambda} \to -\nabla_y\nabla_x c(x,N)\cdot\mathcal{T}(x).$$

Therefore

$$\begin{aligned}\int_{\mathcal{F}E} -\big(\nabla_y\nabla_x c(x,N)\cdot\mathcal{T}(x)\big)\cdot\nu_x\, d\mathscr{H}^{n-1}(x) = &-\int_E \mathrm{tr}_x[\nabla_y\nabla^x c(x,N)\cdot\nabla_x\mathcal{T}]\,d\,\mathrm{vol}(x)\\ &-(n-1)\int_E \nabla_y G(d(x,y))|_{y=N}\cdot\mathcal{T}(x)\,d\,\mathrm{vol}(x).\end{aligned} \tag{A.10}$$

Now, let $H:[0,\pi]\to\mathbb{R}$ be a smooth function, with $H(0)=0$. We have

$$\begin{aligned}\int_E \nabla_y H(d(x,y))|_{y=N}\cdot\mathcal{T}(x)\,d\,\mathrm{vol}(x) &= \int_E \big(\nabla_x\nabla_y c(x,N)\cdot\nabla^x H(r_x)\big)\cdot\mathcal{T}(x)\,d\,\mathrm{vol}(x)\\ &= \int_E \big(\nabla_y\nabla_x c(x,N)\cdot\mathcal{T}(x)\big)\cdot\nabla^x H(r_x)\,d\,\mathrm{vol}(x)\\ &= \int_{\mathcal{F}E} H(r_x)\big(\nabla_y\nabla_x c(x,N)\cdot\mathcal{T}(x)\big)\cdot\nu_x\,d\mathscr{H}^{n-1}(x)\\ &\quad-\int_E H(r_x)\mathrm{div}_x\big(\nabla_y\nabla^x c(x,N)\cdot\mathcal{T}(x)\big)\,d\,\mathrm{vol}(x).\end{aligned}$$

We now remark that

$$\mathrm{div}_x\big(\nabla_y\nabla^x c(x,N)\cdot\mathcal{T}(x)\big) = \nabla_y\Delta_x c(x,y)|_{y=N}\cdot\mathcal{T}(x) + \mathrm{tr}_x[\nabla_y\nabla^x c(x,N)\cdot\nabla_x\mathcal{T}]$$

Moreover

$$\nabla_y\Delta_x c(x,y)|_{y=N} = \nabla_y\left[(n-1)\frac{d_g(x,y)\,\cos(d_g(x,y))}{\sin(d_g(x,y))}+1\right]\Bigg|_{y=N} = (n-1)\nabla_y G(d(x,y))|_{y=N}.$$

This implies

$$\begin{aligned}\int_E \big[\nabla_y H(d(x,y)) + (n-1)H(d(x,y))\nabla_y G(d(x,y))\big]\big|_{y=N}\cdot\mathcal{T}(x)\,d\,\mathrm{vol}(x)&\\ = \int_{\mathcal{F}E} H(r_x)\big(\nabla_y\nabla_x c(x,N)\cdot\mathcal{T}(x)\big)\cdot\nu_x\,d\mathscr{H}^{n-1}(x)&\\ -\int_E H(r_x)\,\mathrm{tr}_x[\nabla_y\nabla^x c(x,N)\cdot\nabla_x\mathcal{T}]\,d\,\mathrm{vol}(x).&\end{aligned}$$

Let us impose

$$\big[\nabla_y H(d(x,y)) + (n-1)H(d(x,y))\nabla_y G(d(x,y))\big]\big|_{y=N} = (n-1)\nabla_y G(d(x,y)).$$

This is equivalent to say that

$$H'(r) + (n-1)HG'(r) = (n-1)G'(r) \qquad \text{for } r > 0,$$

i.e.

$$\frac{H'(r)}{1-H(r)} = (n-1)G'(r).$$

As $H(0) = 0$, integrating the above differential equation we get

$$H(r) = 1 - e^{(n-1)[G(0)-G(r)]}.$$

Observing that $G' < 0$, we immediately obtain $H(r) \le 0$, so that with this choice of H

$$\begin{aligned}(n-1)\int_E \nabla_y G(d(x,y))|_{y=N} \cdot \mathcal{T}(x)\, d\,\mathrm{vol}(x) \\ &= \int_{\mathcal{F}E} H(r_x)\big(\nabla_y\nabla_x c(x,N)\cdot \mathcal{T}(x)\big)\cdot \nu_x\, d\mathscr{H}^{n-1}(x) \\ &- \int_E H(r_r)\,\mathrm{tr}_x[\nabla_y\nabla^x c(x,N)\cdot \nabla_x \mathcal{T}]\, d\,\mathrm{vol}(x).\end{aligned} \tag{A.11}$$

Collecting all together we finally have

$$\begin{aligned}\int_{\mathcal{F}E} -\big(\nabla_y\nabla_x c(x,N)\cdot\mathcal{T}(x)\big)\cdot\nu_x\, d\mathscr{H}^{n-1}(x) &= -\int_E \mathrm{tr}_x[\nabla_y\nabla^x c(x,N)\nabla_x\mathcal{T}]\, d\,\mathrm{vol}(x) \\ &- \int_{\mathcal{F}E} H(r_x)\big(\nabla_y\nabla_x c(x,N)\cdot\mathcal{T}(x)\big)\cdot\nu_x\, d\mathscr{H}^{n-1}(x) \\ &+ \int_E H(r_x)\,\mathrm{tr}_x[\nabla_y\nabla^x c(x,N)\cdot\nabla_x\mathcal{T}]\, d\,\mathrm{vol}(x),\end{aligned}$$

that is

$$\begin{aligned}\int_{\mathcal{F}E} \big(1-H(r_x)\big)\big(-\nabla_y\nabla_x c(x,N)\cdot\mathcal{T}(x)\big)\cdot\nu_x\, d\mathscr{H}^{n-1}(x) \\ = \int_E \big(1-H(r_x)\big)\mathrm{tr}_x[-\nabla_y\nabla^x c(x,N)\cdot\nabla_x\mathcal{T}]\, d\,\mathrm{vol}(x),\end{aligned}$$

which recalling the formula for H and G means

$$\begin{aligned}\int_{\mathcal{F}E} e^{(n-1)\left[1-\frac{r_x\cos(r_x)}{\sin(r_x)}\right]}\big(-\nabla_y\nabla_x c(x,N)\cdot\mathcal{T}(x)\big)\cdot\nu_x\, d\mathscr{H}^{n-1}(x) \\ = \int_E e^{(n-1)\left[1-\frac{r_x\cos(r_x)}{\sin(r_x)}\right]}\,\mathrm{tr}_x[-\nabla_y\nabla^x c(x,N)\cdot\nabla_x\mathcal{T}]\, d\,\mathrm{vol}(x),\end{aligned}$$

Using now the arithmetic-geometric inequality, we get

$$\begin{aligned}\int_E & e^{(n-1)[G(0)-G(r_x)]}\,\mathrm{tr}\big(-\nabla_y\nabla^x c(x,N)\cdot\nabla_x\mathcal{T}\big)\,d\,\mathrm{vol}(x)\\ &\geq n\int_E e^{(n-1)[G(0)-G(r_x)]}\,\det\big|\nabla_y\nabla^x c(x,N)\cdot\nabla_x\mathcal{T}\big|^{1/n}\,d\,\mathrm{vol}(x)\\ &= n\int_E e^{(n-1)[G(0)-G(r_x)]}\,\frac{c_E^{1/n}\,\omega_n^{1/n}\,\rho_R^0(r_x)^{1/n}\,r_x^{(n-1)/n}}{\mathrm{vol}(E)^{1/n}\,\sin^{(n-1)/n}(r_x)}\,d\,\mathrm{vol}(x).\end{aligned}$$

Moreover, as $|\mathcal{T}|\leq 1$ in E,

$$\begin{aligned}\int_{\mathcal{F}E} & e^{(n-1)[G(0)-G(r_x)]}\big(-\nabla_y\nabla_x c(x,N)\cdot\mathcal{T}(x)\big)\cdot\nu_x\,d\mathscr{H}^{n-1}(x)\\ &= \int_{\mathcal{F}E} e^{(n-1)[G(0)-G(r_x)]}\big(-\nabla_x\nabla_y c(x,N)\cdot\nu_x\big)\cdot\mathcal{T}(x)\,d\mathscr{H}^{n-1}(x)\\ &\leq \int_{\mathcal{F}E} e^{(n-1)[G(0)-G(r_x)]}\big|\nabla_x\nabla_y c(x,N)\cdot\nu_x\big|\,d\mathscr{H}^{n-1}(x),\end{aligned}$$

and so we conclude that

$$\begin{aligned}\int_{\mathcal{F}E} & e^{(n-1)[G(0)-G(r_x)]}|\nabla_x\nabla_y c(x,N)\cdot\nu_x|\,d\mathscr{H}^{n-1}(x)\\ &\geq n\int_E e^{(n-1)[G(0)-G(r_x)]}\,\frac{c_E^{1/n}\,\omega_n^{1/n}\,\rho_R^0(r_x)^{1/n}\,r_x^{(n-1)/n}}{\mathrm{vol}(E)^{1/n}\,\sin^{(n-1)/n}(r_x)}\,d\,\mathrm{vol}(x)\\ &= n\,\omega_n^{1/n}\left(\int_E e^{n[G(0)-G(r_x)]}\left(\frac{r_x}{\sin(r_x)}\right)d\,\mathrm{vol}(x)\right)^{\frac{n-1}{n}},\end{aligned}$$

as wanted.

Remark A.1 The main difficulties to make this proof rigorous are in differentiating $\mathcal{T}_\lambda$ with respect to λ in order to define $\mathcal{T}$, and then to deduce the properties of the limit transport map from the ones of the maps $\mathcal{T}_\lambda$. Although we believe that all this could be done using some refined argument of measure theory and BV functions, we preferred in Section 3 to prove the result working directly with $\mathcal{T}$, without any mention to $\mathcal{T}_\lambda$.

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