On flows of $H^{3/2}$-vector fields on the circle

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Abstract

In this paper we study the regularity of flow maps of $H^{3/2}$-vector fields on the circle in terms of fractional Sobolev spaces. This problem is motivated by the understanding of the geometry of Bers's universal Teichmüller space.

Bers's universal Teichmüller space, classically denoted by $T(1)$, plays a significant role in Teichmüller theory, as it contains as complex submanifolds all the finite-dimensional Teichmüller spaces of Riemann surfaces. The space $T(1)$ is an infinite-dimensional complex manifold modeled on a Banach space, and it is a fundamental object in mathematics and in mathematical physics (for instance, it is a promising geometric environment for a non-perturbative version of bosonic string theory). We refer to [2, 6, 8] for an introduction to the subject and more details.

It turns out that $T(1)$ is a group formed by quasi-symmetric maps on the circle which can be endowed with a well-defined complex Hilbert manifold structure, compatible with the Weil-Petersson metric and making the connected component to the identity into a topological group (see [8]). Moreover the tangent space at the identity consists of $H^{3/2}$-vector fields on the circle $\mathbb{S}^1$. It is not known if the flows of such vector fields are contained in the connected component of the identity (it is believed that this is the case and that they generate the whole connected component). Thus, it is important to characterize the space of such flow maps. The aim of this paper is precisely to give a characterizations of these flows in terms of fractional Sobolev norms.

It is well-known that, if a vector field belongs to $H^{3/2+\varepsilon}(\mathbb{S}^1)$ for some $\varepsilon > 0$, then for any $t \geq 0$ the flow map $f(t,\cdot) : \mathbb{S}^1 \to \mathbb{S}^1$ belongs to $H^{3/2+\varepsilon}(\mathbb{S}^1)$ too (and this is the optimal regularity one can hope for). Hence, one would be tempted to conjecture that the same holds for $\varepsilon = 0$. Unfortunately $s = 3/2$ is the critical exponent for the embedding $H^s_{loc}(\mathbb{R}) \hookrightarrow C^1(\mathbb{R})$, and indeed, as shown in this paper, the above result for $\varepsilon = 0$ is false. We can prove the following:

Theorem 0.1 *Let $f(t,\cdot) : \mathbb{S}^1 \to \mathbb{S}^1$ denote the flow map of a vector field $\boldsymbol{u} \in C([0,T], H^{3/2}(\mathbb{S}^1))$. Then:*

1. *The flow map belongs to $W^{1,p}(\mathbb{S}^1)$ for all $p \in [1,\infty)$.*
2. *The flow map belongs to $W^{1+r,q}(\mathbb{S}^1)$ for all $r \in (0,1/2)$, $q \in [1,1/r)$.*

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On the other hand there exists an autonomous vector field $\boldsymbol{u} \in H^{3/2}(\mathbb{S}^1)$ such that its flow map is neither Lipschitz nor $W^{1+r,1/r}$ for all $r \in (0,1)$. In particular the flow map is not $H^{3/2}$.

As shown in [3], a direct consequence of our theorem is that every element of $T(1)$ in the connected component of the identity belongs to $H^{3/2-\varepsilon}(\mathbb{S}^1)$ for all $\varepsilon > 0$, a fact which is important in the study of the geometry of $T(1)$.

The structure of the paper is the following: in Section 1 we recall some simple well-known facts on $H^{3/2}$-functions and on flows of $H^{3/2}$-vector fields on the circle. Then in Section 2 we prove the regularity part of the theorem above. Finally in Section 3 we construct the counterexample to the $H^{3/2}$-regularity.

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1 Preliminaries

We recall that $H^{3/2}(\mathbb{S}^1) = W^{3/2,2}(\mathbb{S}^1)$ denotes the space of function $\boldsymbol{v} \in L^2(\mathbb{S}^2)$ whose distributional derivative $\boldsymbol{v}_x$ belongs to $H^{1/2}(\mathbb{S}^1) = W^{1/2,2}(\mathbb{S}^1)$ (see (2.4) below for the definition of the $W^{1/2,2}$-norm).

Let $\boldsymbol{u} \in C([0,T], H^{3/2}(\mathbb{S}^1))$, and consider the ODE

$$\begin{cases} \dot{f}(t,x) = \boldsymbol{u}\big(t, f(t,x)\big), & x \in \mathbb{S}^1, \\ f(0,x) = x. \end{cases} \tag{1.1}$$

Since $H^{3/2}$ embeds into logLipschitz (see Lemma 1.1 below), existence and uniqueness for the above ODE are well-known (see for instance [4]). Moreover the flow map $f(t,\cdot) : \mathbb{S}^1 \to \mathbb{S}^1$ is a homeomorphism for all $t \in [0,T]$. We want to study the Sobolev regularity of the maps $f(t,\cdot)$.

First of all we recall some known results on $H^{3/2}$-functions.

Lemma 1.1 *Let $\boldsymbol{v} \in H^{3/2}(\mathbb{S}^1)$. Then, for any $\lambda > 0$, there exists a constant $C_\lambda > 0$ such that*

$$\int_{\mathbb{S}^1} e^{\lambda |\boldsymbol{v}_x| / \|\boldsymbol{v}_x\|_{H^{1/2}(\mathbb{S}^1)}} \, dx \le C_\lambda. \tag{1.2}$$

Furthermore

$$|\boldsymbol{v}(x) - \boldsymbol{v}(y)| \le \|\boldsymbol{v}_x\|_{H^{1/2}(\mathbb{S}^1)} |x-y| \log\left(\frac{C_1}{|x-y|}\right) \qquad \forall\, x, y \in \mathbb{S}^1.$$

Proof. We recall that, if $w \in H^{1/2}(\mathbb{S}^1)$ and $\|w\|_{H^{1/2}(\mathbb{S}^1)} = 1$, then there exist two constants $c, C > 0$ such that

$$\int_{\mathbb{S}^1} e^{cw^2} \, dx \le C$$

(see for instance [7, pag. 26, Exercise 5]). Using the inequality

$$\lambda\frac{|\boldsymbol{v}_x|}{\|\boldsymbol{v}_x\|_{H^{1/2}(\mathbb{S}^1)}}\leq c\frac{|\boldsymbol{v}_x|^2}{\|\boldsymbol{v}_x\|^2_{H^{1/2}(\mathbb{S}^1)}}+\frac{\lambda^2}{4c},$$

we immediately get (1.2) with $C_\lambda = Ce^{\lambda^2/(4c)}$.

To prove the second part of the lemma, by Jensen's inequality applied to $w := \boldsymbol{v}/\|\boldsymbol{v}_x\|_{H^{1/2}(\mathbb{S}^1)}$ we have

$$\begin{aligned}|w(x)-w(y)| &\leq \int_x^y |w_x(z)|\,dz = \int_x^y \log\Big(e^{|w_x(z)|}\Big)\,dz \\ &\leq |x-y|\log\bigg(\frac{1}{|y-x|}\int_x^y e^{|w_x(z)|}\,dz\bigg) \leq |x-y|\log\bigg(\frac{\|e^{|w_x|}\|_{L^1(\mathbb{S}^1)}}{|x-y|}\bigg).\end{aligned}$$

As $\|e^{|w_x|}\|_{L^1(\mathbb{S}^1)} \leq C_1$, the logLipschitz regularity of v follows. □

2 Regularity results

We can now begin the study of the regularity properties of the flow map $f(t,\cdot)$. Since we want to prove a priori estimates on some Sobolev norms of $f(t,\cdot)$, in what follows we can assume without loss of generality that $\boldsymbol{u}$ is smooth (so that f is smooth too).

2.1 The flow map is $W^{1,p}$ for all $p \in [1,\infty)$.

Proposition 2.1 *Let ρ_t denotes the density of the push-forward of the Lebesgue measure $\mathscr{L}^1$ under the flow map $f(t,\cdot)$, i.e.*

$$f(t,\cdot)_\#\mathscr{L}^1 = \rho_t\mathscr{L}^1.$$

Then

$$\rho_t(f(t,x)) = e^{-\int_0^t \boldsymbol{u}_x(s,f(s,x))\,ds}, \qquad f_x(t,x) = e^{\int_0^t \boldsymbol{u}_x(s,f(s,x))\,ds}. \tag{2.1}$$

Moreover, for any $p \in [1,\infty)$ and $T > 0$, there exist two constants $\bar{C}_1$ and $\bar{C}_2$, depending only on the product $pT\|\boldsymbol{u}_x\|_{L^\infty([0,T],H^{1/2}(\mathbb{S}^1))}$, such that

$$\|\rho_t\|_{L^p(\mathbb{S}^1)} \leq \bar{C}_1 \qquad \forall t \in [0,T], \tag{2.2}$$

$$\|f_x(t)\|_{L^p(\mathbb{S}^1)} \leq \bar{C}_2 \qquad \forall t \in [0,T].$$

Proof. The formula for f_x follows easily differentiating (1.1) with respect to x and observing that $f_x(0,x) = 1$, while the formula for ρ_t is a direct consequence of the identity $f_x(t,x)\rho_t(f(t,x)) = 1$ (which follows from the definition of ρ_t and the change of variable formula).

Let us first estimate $\|\rho_t\|_{L^p(\mathbb{S}^1)}$. Thanks to (2.1) we have

$$\begin{aligned}
\int_{\mathbb{S}^1} \rho_t^p(x)\,dx &= \int_{\mathbb{S}^1} \rho_t^{p-1}(x)\rho_t(x)\,dx = \int_{\mathbb{S}^1} \rho_t^{p-1}(f(t,x))\,dx \\
&\le \int_{\mathbb{S}^1} e^{(p-1)\int_0^t |\boldsymbol{u}_x(s,f(s,x))|\,ds}\,dx \le \int_{\mathbb{S}^1} \frac{1}{t}\int_0^t e^{t(p-1)|\boldsymbol{u}_x(s,f(s,x))|}\,ds\,dx \\
&= \frac{1}{t}\int_0^t \int_{\mathbb{S}^1} e^{t(p-1)|\boldsymbol{u}_x(s,f(s,x))|}\,dx\,ds \le \frac{1}{t}\int_0^t \int_{\mathbb{S}^1} e^{T(p-1)|\boldsymbol{u}_x(s,x)|}\rho_s(x)\,dx\,ds,
\end{aligned}$$

where at the second line we used Jensen's inequality. Now, set $\Lambda(t) := \int_0^t \|\rho_s\|_{L^p(\mathbb{S}^1)}^p\,ds$ and apply Hölder's inequality to get

$$\begin{aligned}
\Lambda'(t) &\le \frac{1}{t}\left(\int_0^t \int_{\mathbb{S}^1} e^{Tp|\boldsymbol{u}_x(s,x)|}\,dx\,ds\right)^{1/p'} \Lambda^{1/p}(t) \\
&\le K t^{1/p'-1}\Lambda^{1/p}(t) = K t^{-1/p}\Lambda^{1/p}(t),
\end{aligned} \tag{2.3}$$

with $K := \|\int e^{Tp|\boldsymbol{u}_x(t,x)|}\,dx\|_{L^\infty(0,T)}^{1/p'}$. An integration of this differential inequality yields $\Lambda(t) \le K^{p'} t$, which inserted into (2.3) gives

$$\int_{\mathbb{S}^1} \rho_t^p(x)\,dx \le \left\| \int_{\mathbb{S}^1} e^{Tp|\boldsymbol{u}_x(t,x)|}\,dx \right\|_{L^\infty(0,T)} \qquad \forall t \in [0,T].$$

Thanks to (1.2) applied with $\lambda = Tp\|\boldsymbol{u}_x\|_{L^\infty([0,T],H^{1/2}(\mathbb{S}^1))}$, the estimate on $\|\rho_t\|_{L^p(\mathbb{S}^1)}$ follows.

Now that we have the bound on $\|\rho_t\|_{L^p(\mathbb{S}^1)}$, it is not difficult to control $\|f_x(t)\|_{L^p(\mathbb{S}^1)}$:

$$\begin{aligned}
\int_{\mathbb{S}^1} |f_x(t,x)|^p\,dx &\le \int_{\mathbb{S}^1} e^{p\int_0^t |\boldsymbol{u}_x(s,f(s,x))|\,ds}\,dx \le \int_{\mathbb{S}^1} e^{p\int_0^T |\boldsymbol{u}_x(s,f(s,x))|\,ds}\,dx \\
&\le \int_{\mathbb{S}^1} \frac{1}{T}\int_0^T e^{Tp|\boldsymbol{u}_x(s,f(s,x))|}\,ds\,dx = \frac{1}{T}\int_0^T \int_{\mathbb{S}^1} e^{Tp|\boldsymbol{u}_x(s,x)|}\rho_s(x)\,dx\,ds,
\end{aligned}$$

and we conclude applying Holder's inequality, together with (1.2) and (2.2). $\square$

Thanks to the above proposition, we obtain that $f \in L^\infty([0,T], W^{1,p}(\mathbb{S}^1))$ for all $p \in [1,\infty)$.

2.2 The flow map is $W^{1+r,q}$ for all $r \in (0,1/2)$, $q \in [1,1/r)$.

In this paragraph we prove higher spatial regularity on f.

Let us recall the definition of the $W^{s,p}(\mathbb{S}^1)$ norm for $s \in (0,1)$:

$$\|v\|_{W^{s,p}(\mathbb{S}^1)} := \|v\|_{L^p(\mathbb{S}^1)} + \left(\int_{\mathbb{S}^1}\int_{\mathbb{S}^1} \frac{|v(x)-v(y)|^p}{|x-y|^{1+sp}}\,dx\,dy\right)^{1/p}. \tag{2.4}$$

We need a preliminary result:

Lemma 2.2 *For all $r \in (0,1/2)$ and $q \in [1,1/r)$, we have $\boldsymbol{u}_x \circ f \in L^\infty([0,T], W^{r,q}(\mathbb{S}^1))$.*

Proof. We observe that for any $\alpha \in (1,\infty)$, denoting by α' is its dual exponent, we have

$$\int_{\mathbb{S}^1} e^{p|\boldsymbol{u}_x(t,f(t,x))|}\,dx = \int_{\mathbb{S}^1} e^{p|\boldsymbol{u}_x(t,x)|}\rho_t(x)\,dx \leq \|e^{p|\boldsymbol{u}_x(t)|}\|_{L^\alpha(\mathbb{S}^1)}\|\rho_t\|_{L^{\alpha'}(\mathbb{S}^1)}.$$

This, combined with (1.2) and (2.2), implies

$$e^{|\boldsymbol{u}_x(\cdot,f)|} \in L^\infty([0,T],L^p(\mathbb{S}^1)) \qquad \forall\, p \in [1,\infty). \tag{2.5}$$

In particular $\boldsymbol{u}_x(\cdot,f) \in L^\infty([0,T],L^q(\mathbb{S}^1))$.

To prove the fractional Sobolev regularity of $\boldsymbol{u}_x(\cdot,f)$, fix $\alpha \in (1,\infty)$, with $\alpha - 1$ small (the smallness, depending on r and q, will be chosen later). Then

$$\begin{aligned}
&\int_{\mathbb{S}^1}\int_{\mathbb{S}^1} \frac{|\boldsymbol{u}_x(t,f(t,x)) - \boldsymbol{u}_x(t,f(t,y))|^q}{|x-y|^{1+rq}}\,dx\,dy \\
&\quad = \int_{\mathbb{S}^1}\int_{\mathbb{S}^1} \frac{|\boldsymbol{u}_x(t,f(t,x)) - \boldsymbol{u}_x(t,f(t,y))|^q}{|f(t,x)-f(t,y)|^{1+rq}}\,\frac{|f(t,x)-f(t,y)|^{1+rq}}{|x-y|^{1+rq}}\,dx\,dy \\
&\quad \leq \left(\int_{\mathbb{S}^1}\int_{\mathbb{S}^1} \frac{|\boldsymbol{u}_x(t,f(t,x)) - \boldsymbol{u}_x(t,f(t,y))|^{\alpha q}}{|f(t,x)-f(t,y)|^{\alpha(1+rq)}}\,dx\,dy\right)^{1/\alpha} \\
&\qquad \left(\int_{\mathbb{S}^1}\int_{\mathbb{S}^1} \frac{|f(t,x)-f(t,y)|^{\alpha'(1+rq)}}{|x-y|^{\alpha'(1+rq)}}\,dx\,dy\right)^{1/\alpha'}.
\end{aligned}$$

Regarding the second term, we have

$$\int_{\mathbb{S}^1}\int_{\mathbb{S}^1} \frac{|f(t,x)-f(t,y)|^{\alpha'(1+rq)}}{|x-y|^{\alpha'(1+rq)}}\,dx\,dy = \|f\|^{\alpha'(1+rq)}_{W^{\tau,\alpha'(1+rq)}(\mathbb{S}^1)}, \qquad \tau = 1 - \frac{1}{\alpha'(1+rq)}.$$

Since $\tau < 1$ and $f \in L^\infty([0,T],W^{1,p}(\mathbb{S}^1))$ for all $p \in [1,\infty)$, the second term is bounded uniformly in time.

We now consider the first term. We have

$$\begin{aligned}
\int_{\mathbb{S}^1}\int_{\mathbb{S}^1} \frac{|\boldsymbol{u}_x(t,f(t,x)) - \boldsymbol{u}_x(t,f(t,y))|^{\alpha q}}{|f(t,x)-f(t,y)|^{\alpha(1+rq)}}\,dx\,dy &= \int_{\mathbb{S}^1}\int_{\mathbb{S}^1} \frac{|\boldsymbol{u}_x(t,x) - \boldsymbol{u}_x(t,y)|^{\alpha q}}{|x-y|^{\alpha(1+rq)}}\rho_t(x)\rho_t(y)\,dx\,dy \\
&\leq \left(\int_{\mathbb{S}^1}\int_{\mathbb{S}^1} \frac{|\boldsymbol{u}_x(t,x) - \boldsymbol{u}_x(t,y)|^{\alpha^2 q}}{|x-y|^{\alpha^2(1+rq)}}\,dx\,dy\right)^{1/\alpha}\left(\int_{\mathbb{S}^1}\int_{\mathbb{S}^1}\rho_t(x)^{\alpha'}\rho_t(y)^{\alpha'}\,dx\,dy\right)^{1/\alpha'} \\
&\leq \|\boldsymbol{u}_x(t)\|^{\alpha q}_{W^{u,\alpha^2 q}(\mathbb{S}^1)}\|\rho_t\|^2_{L^{\alpha'}(\mathbb{S}^1)},
\end{aligned}$$

with

$$u = r + \frac{\alpha^2 - 1}{\alpha^2 q}.$$

We recall that

$$W^{1/2,2}(\mathbb{S}^1) \hookrightarrow W^{u,q}(\mathbb{S}^1) \qquad \text{for } u \in (0,1/2],\ q \in [1,1/u] \tag{2.6}$$

(see [5, pag. 350, Theorem 4]). Thus

$$\|\boldsymbol{u}_x(t)\|_{W^{u,\alpha^2 q}(\mathbb{S}^1)} \leq C\|\boldsymbol{u}_x(t)\|_{W^{1/2,2}(\mathbb{S}^1)}$$

provided $u \leq 1/2$ and $u\alpha^2 q \leq 1$. Since both α and q are greater or equal than 1, by the definition of u these two conditions are true provided that

$$r + (\alpha^2 - 1) \leq \frac{1}{2}, \qquad \alpha^2(1 + rq) \leq 2.$$

As $r < 1/2$ and $rq < 1$, for $\alpha - 1$ small enough both inequalities hold, and so also the first term is bounded uniformly in time. $\square$

We can now prove the fractional Sobolev estimate on f_x. Considering the equation satisfied by $|f_x(t,x) - f_x(t,y)|$, we easily get

$$\begin{aligned}\frac{d}{dt}|f_x(t,x) - f_x(t,y)| \leq &|\boldsymbol{u}_x(t, f(t,x)) - \boldsymbol{u}_x(t, f(t,y))||f_x(t,x)| \\ &+ |\boldsymbol{u}_x(t, f(t,y))||f_x(t,x) - f_x(t,y)|\end{aligned}$$

Since $f_x(0,x) = 1$ we have $|f_x(0,x) - f_x(0,y)| = 0$, and so by Duhamel's formula

$$|f_x(t,x) - f_x(t,y)| \leq \int_0^t e^{\int_s^t |\boldsymbol{u}_x(\tau, f(\tau,y))|\, d\tau} |\boldsymbol{u}_x(s, f(s,x)) - \boldsymbol{u}_x(s, f(s,y))||f_x(s,x)|\, ds.$$

Hence, using Holder's and Jensen's inequalities, we obtain

$$\begin{aligned}\frac{|f_x(t,x) - f_x(t,y)|^q}{|x-y|^{1+rq}} &\leq t^{1-1/q} \int_0^t e^{q\int_s^t |\boldsymbol{u}_x(\tau, f(\tau,y))|\, d\tau} \frac{|\boldsymbol{u}_x(s, f(s,x)) - \boldsymbol{u}_x(s, f(s,y))|^q}{|x-y|^{1+rq}} |f_x(s,x)|^q\, ds \\ &\leq t^{1-1/q} \int_0^T \left(\frac{1}{T}\int_0^T e^{Tq|\boldsymbol{u}_x(\tau, f(\tau,y))|}\, d\tau\right) \frac{|\boldsymbol{u}_x(s, f(s,x)) - \boldsymbol{u}_x(s, f(s,y))|^q}{|x-y|^{1+rq}} |f_x(s,x)|^q\, ds.\end{aligned}$$

We already proved in (2.5) that $e^{|\boldsymbol{u}_x(\cdot,f)|} \in L^\infty([0,T], L^p(\mathbb{S}^1))$ for all $p < \infty$. Moreover $f_x \in L^\infty([0,T], L^p(\mathbb{S}^1))$ for all $p < \infty$. Finally, since $\boldsymbol{u}_x(\cdot, f) \in L^\infty([0,T], W^{\tau,\alpha}(\mathbb{S}^1))$ for all $\tau < 1/2$ and $\alpha < 1/\tau$, as $rq < 1$ there exists $\beta > 1$, with $\beta - 1$ small, such that

$$\frac{|\boldsymbol{u}_x(s, f(s,x)) - \boldsymbol{u}_x(s, f(s,y))|^q}{|x-y|^{1+rq}} \in L^\beta(\mathbb{S}^1 \times \mathbb{S}^1),$$

uniformly in time. Indeed this amounts to say that

$$\int_{\mathbb{S}^1}\int_{\mathbb{S}^1} \frac{|\boldsymbol{u}_x(s, f(s,x)) - \boldsymbol{u}_x(s, f(s,y))|^{\beta q}}{|x-y|^{(1+rq)\beta}}\, dx\, dy = \|\boldsymbol{u}_x(s,f)\|^{\beta q}_{W^{\tau,\beta q}(\mathbb{S}^1)} \leq C \qquad \forall\, s \in [0,T],$$

where $\tau = r + \frac{\beta-1}{q\beta}$. By (2.6) this is true provided $q\beta \leq 1/\tau$, or equivalently $(1+rq)\beta \leq 2$, which is clearly true for β sufficiently close to 1.

Combining all these facts together and using Holder's inequality on $\mathbb{S}^1 \times \mathbb{S}^1$, we easily deduce that

$$\frac{|f_x(t,x) - f_x(t,y)|^q}{|x-y|^{1+rq}} \in L^1(\mathbb{S}^1 \times \mathbb{S}^1)$$

uniformly in time. Thus $f_x \in L^\infty([0,T], W^{r,q}(\mathbb{S}^1))$ for all $r \in (0, 1/2)$ and $q \in [1, 1/r)$, as wanted.

3 A counterexample to higher regularity

In this section we show the optimality of the results proved before. As we will see, the flow of the vector field constructed below is neither Lipschitz nor $W^{1+r,1/r}$ for all $r \in (0,1)$. In particular, taking $r = 1/2$, the flow map is not $H^{3/2}$.

Let us consider the function $\boldsymbol{u} : [0,1] \to \mathbb{R}$ given by

$$\boldsymbol{u}(x) = \left(\int_0^x \int_y^{1/2} \frac{1}{z\sqrt{|\log(z)|}\log|\log(z)|}\, dz\, dy \right) \varphi(x),$$

where φ is a smooth function such that

$$0 \leq \varphi \leq 1, \qquad \varphi(x) = \begin{cases} 1 & \text{for } x \in [0,1/4], \\ 0 & \text{for } x \in [1/2,1]. \end{cases}$$

We observe that $\boldsymbol{u}$ is a periodic autonomous vector field on $[0,1]$ (and so it can be considered as a vector field on $\mathbb{S}^1$) such that

$$\boldsymbol{u}_x(x) = \int_x^{1/2} \frac{1}{z\sqrt{|\log(z)|}\log|\log(z)|}\, dz$$

for $x \in (0,1/4)$. Moreover $\boldsymbol{u}$ is smooth away from 0.

Since the function

$$\mathbb{R}^2 \ni w \mapsto \int_{|w|}^{1/2} \frac{1}{z\sqrt{|\log(z)|}\log|\log(z)|}\, dz$$

belongs to $H^1_{loc}(\mathbb{R}^2)$, its trace on the line $\{z_2 = 0\}$ belongs to $H^{1/2}_{loc}(\mathbb{R})$ (see [1, Paragraph 7.56]). From this fact we easily deduce that $\boldsymbol{u}_x \in H^{1/2}(\mathbb{S}^1)$, so that $\boldsymbol{u} \in H^{3/2}(\mathbb{S}^1)$

We now want to prove that flow map generated by $\boldsymbol{u}$ is neither Lipschitz nor in $W^{1+r,1/r}$ for all $r \in (0,1)$. To this aim we will use that the flow map is Hölder continuous uniformly in time (for instance, this is a consequence of the $W^{1,p}$ regularity proved in Proposition 2.1). More precisely, since $f(t,0) = 0$ for all t, there exist $C > 0$ and $0 < \alpha < 1$ such that

$$0 \leq f(t,x) \leq Cx^\alpha \qquad \text{for } t \in [0,1],\ x \in [0,1]. \tag{3.1}$$

For simplicity of notation, we define

$$U(y) := \int_y^{1/2} \frac{1}{z\sqrt{|\log(z)|}\log|\log(z)|}\, dz$$

(so that $U(y) = \boldsymbol{u}_x(y)$ for $y > 0$ close to 0). Observe that $U'(y) < 0$ and $U(y) \to +\infty$ as $y \to 0^+$.

3.1 The flow map is not Lipschitz.

From the equation

$$\dot{f}(t,x) = \int_0^{f(t,x)} U(y)\,dy \qquad \text{for } x>0 \text{ small}$$

we have

$$\frac{d}{dt}\log\bigl(f(t,x)\bigr) = \fint_0^{f(t,x)} U(y)\,dy,$$

where $\fint$ denotes the averaged integral. We want to prove that, for $t\in(0,1]$, the flow map $f(t,\cdot):[0,1]\to[0,1]$ is not Lipschitz at 0.

Since $y\mapsto U(y)$ is decreasing, by (3.1) we get

$$\frac{d}{dt}\log\bigl(f(t,x)\bigr) = \fint_0^{f(t,x)} U(y) \geq \fint_0^{Cx^\alpha} U(y)\,dy := V(x) \qquad \text{for } x>0 \text{ small.}$$

This implies that for $x>0$ small

$$f(t,x) \geq xe^{tV(x)} \qquad \forall\, t\in[0,1].$$

Since $V(x)\to+\infty$ as $x\to 0^+$, we obtain

$$\lim_{x\to 0^+}\frac{|f(t,x)-f(t,0)|}{x} = \lim_{x\to 0^+}\frac{f(t,x)}{x} \geq \lim_{x\to 0^+} e^{tV(x)} = +\infty \qquad \forall\, t\in(0,1].$$

This proves the desired result.

3.2 The flow map is not in $W^{1+r,1/r}$ for all $r\in(0,1)$.

Fix $t>0$ small, and let $q:=1/r>1$. We want to prove that

$$\int_{\mathbb{S}^1}\int_{\mathbb{S}^1}\frac{|f_x(t,x)-f_x(t,x+h)|^q}{h^2}\,dx\,dh = +\infty.$$

Obviously it suffices to show that

$$\int_0^\varepsilon\int_0^\varepsilon\frac{|f_x(t,x)-f_x(t,x+h)|^q}{h^2}\,dx\,dh = +\infty.$$

for some $\varepsilon>0$ small (the smallness of ε to be chosen later).

Differentiating the equation satisfied by f with respect to x, for $x>0$ small we have

$$\begin{cases} \dot{f}_x(s,x) = f_x(s,x)U(f(s,x)), \\ f_x(0,x) = 1, \end{cases}$$

which gives

$$f_x(t,x) = e^{\int_0^t U(f(s,x))\,ds}.$$

Therefore, if we define $F(x) := \int_0^t U(f(s,x))\,ds$ (recall that t is fixed), for $x, y > 0$ small we get

$$\begin{aligned} f_x(t,x) - f_x(t,y) = e^{F(x)} - e^{F(y)} &= \int_0^1 \frac{d}{d\tau} e^{\tau F(x)+(1-\tau)F(y)}\,d\tau \\ &= [F(x) - F(y)] \int_0^1 e^{\tau F(x)+(1-\tau)F(y)}\,d\tau, \end{aligned} \tag{3.2}$$

and

$$F(x) - F(y) = \int_0^t \big(U(f(s,x)) - U(f(s,y))\big)\,ds = \int_0^t \int_{f(s,x)}^{f(s,y)} \frac{1}{z\sqrt{|\log(z)|\log|\log(z)|}}\,dz\,ds.$$

Assume $0 < x < y$, with y small. Then, since $f(s,x) \le f(s,y)$ for all s, we have $F(x) \ge F(y)$. Hence

$$\int_0^1 e^{\tau F(x)+(1-\tau)F(y)}\,d\tau \ge e^{F(y)} \tag{3.3}$$

and

$$|F(x) - F(y)| = F(x) - F(y) = \int_0^t \int_{f(s,x)}^{f(s,y)} \frac{1}{z\sqrt{|\log(z)|\log|\log(z)|}}\,dz\,ds.$$

We now have the following:

Lemma 3.1 *There exists $\delta > 0$ such that, if $0 < x < y < \delta$, then the function*

$$s \mapsto \int_{f(s,x)}^{f(s,y)} \frac{1}{z\sqrt{|\log(z)|\log|\log(z)|}}\,dz$$

is increasing on $[0,t]$.

Proof. Recalling that $U'(u) = -\frac{1}{u\sqrt{|\log(u)|\log|\log(u)}}$, we have

$$\begin{aligned} \frac{d}{ds}\int_{f(s,x)}^{f(s,y)} \frac{1}{z\sqrt{|\log(z)|\log|\log(z)|}}\,dz &= \frac{\dot f(s,y)}{f(s,y)\sqrt{|\log(f(s,y))|\log|\log(f(s,y))|}} \\ &\quad - \frac{\dot f(s,x)}{f(s,x)\sqrt{|\log(f(s,x))|\log|\log(f(s,x))|}} = -\frac{\int_0^{f(s,y)} U(z)\,dz}{U'(f(s,y))} + \frac{\int_0^{f(s,x)} U(z)\,dz}{U'(f(s,x))}. \end{aligned}$$

Considering the function

$$u \mapsto G(u) := \frac{\int_0^u U(z)\,dz}{U'(u)},$$

we observe that

$$G'(u) = \frac{U(u)}{U'(u)} - \frac{U''(u)\int_0^u U(z)\,dz}{U'(u)^2} = \frac{U(u)U'(u) - U''(u)\int_0^u U(z)\,dz}{U'(u)^2}.$$

Since $U'(u) < 0$ for u small, while $\int_0^u U(z)\,dz, U(u), U''(u) > 0$ for u small, we immediately get $G'(u) < 0$. This implies that $G(u)$ is decreasing for u small, and since $f(s,x) \le f(s,y)$ the result follows. $\square$

Thanks to the lemma above we get

$$F(x) - F(y) \geq \int_0^t \int_{f(0,x)}^{f(0,y)} \frac{1}{z\sqrt{|\log(z)|}\log|\log(z)|}\, dz\, ds = t[U(x) - U(y)]$$

for $0 < x < y$ small, which combined with (3.2) and (3.3) gives

$$|f_x(t,x) - f_x(t,y)| \geq t[U(x) - U(y)]e^{F(y)}. \tag{3.4}$$

We now want to give a lower bound on $e^{F(y)}$. By (3.1) we have $U(f(s,y)) \geq U(Cy^\alpha)$, and so

$$F(y) \geq \int_0^t U(Cy^\alpha)\, ds = tU(Cy^\alpha).$$

Moreover we observe that, for $u > 0$ sufficiently small,

$$\begin{aligned} U(u) &= \int_u^{1/2} \frac{1}{z\sqrt{|\log(z)|}\log|\log(z)|}\, dz = -2\int_u^{1/2} \frac{d}{dz}\Big(\sqrt{|\log(z)|}\Big)\frac{1}{\log|\log(z)|}\, dz \\ &= -2\sqrt{|\log(z)|}\frac{1}{\log|\log(z)|}\bigg|_u^{1/2} + 2\int_u^{1/2} \frac{1}{z\sqrt{|\log(z)|}\big(\log|\log(z)|\big)^2}\, dz \\ &\geq \sqrt{|\log(u)|}\frac{1}{\log|\log(u)|}, \end{aligned}$$

and so, if $y > 0$ is small enough (the smallness depending on C, α, t, q),

$$e^{F(y)} \geq e^{tU(Cy^\alpha)} \geq e^{2t\sqrt{|\log(Cy^\alpha)|}\frac{1}{\log|\log(Cy^\alpha)|}} \geq e^{\frac{\log|\log(y)|}{2}} = \sqrt{|\log(y)|}.$$

Combining the above inequality with (3.4) we finally obtain

$$|f_x(t,x) - f_x(t,y)| \geq t\sqrt{|\log(y)|}\,[U(x) - U(y)] \qquad \text{for } 0 < x < y \text{ small.} \tag{3.5}$$

Since $u \mapsto \frac{1}{u\sqrt{|\log(u)|}\log|\log(u)|}$ is decreasing near 0, for $0 < x < y$ small we also have

$$\begin{aligned} U(x) - U(y) &= -(y-x)\int_0^1 U'(x + r(y-x))\, dr \\ &= (y-x)\int_0^1 \frac{1}{(x + r(y-x))\sqrt{|\log(x + r(y-x))|}\log|\log(x + r(y-x))|}\, dr \\ &\geq \frac{(y-x)}{y\sqrt{|\log(y)|}\log|\log(y)|}. \end{aligned} \tag{3.6}$$

Thus we are left with proving that

$$\int_0^\varepsilon \int_0^\varepsilon \frac{|f_x(t,x) - f_x(t,x+h)|^q}{h^2}\, dx\, dh = +\infty,$$

with $\varepsilon > 0$ small enough such that both (3.5) and (3.6) hold for $y \in [0, 2\varepsilon]$. Thanks to (3.5) and (3.6) we obtain

$$\begin{aligned}
&\int_0^\varepsilon \int_0^\varepsilon \frac{|f_x(t,x) - f_x(t,x+h)|^q}{h^2}\,dx\,dh \\
&\qquad \geq t^q \int_0^\varepsilon \int_0^\varepsilon |\log(x+h)|^{q/2} \frac{|U(x) - U(x+h)|^q}{h^2}\,dx\,dh \\
&\qquad \geq t^q \int_0^\varepsilon \int_0^\varepsilon |\log(x+h)|^{q/2} \frac{h^{q-2}}{(x+h)^q |\log(x+h)|^{q/2} \log^q |\log(x+h)|}\,dx\,dh \\
&\qquad = t^q \int_0^\varepsilon \int_0^\varepsilon \frac{h^{q-2}}{(x+h)^q \log^q |\log(x+h)|}\,dx\,dh,
\end{aligned}$$

and as $q > 1$

$$\begin{aligned}
\int_0^\varepsilon \frac{h^{q-2}}{(x+h)^q \log^q |\log(x+h)|}\,dx &= \int_h^{h+\varepsilon} \frac{h^{q-2}}{y^q \log^q |\log(y)|}\,dy \\
&\sim \frac{h^{q-2}}{h^{q-1} \log^q |\log(h)|} = \frac{1}{h \log^q |\log(h)|} \qquad \text{as } h \to 0^+,
\end{aligned}$$

where the estimate for the second integral follows easily by an integration by parts. Thus we finally conclude that

$$\int_0^\varepsilon \int_0^\varepsilon \frac{|f_x(t,x) - f_x(t,x+h)|^q}{h^2}\,dx\,dh \gtrsim t^q \int_0^\varepsilon \frac{1}{h \log^q |\log(h)|}\,dh = +\infty,$$

as desired.

References

[1] R. A. Adams: *Sobolev spaces.* Pure and Applied Mathematics, Vol. 65. Academic Press, New York-London, 1975.

[2] L. Bers: *Universal Teichmller space.* Analytic methods in mathematical physics (Sympos., Indiana Univ., Bloomington, Ind., 1968), pp. 65–83. Gordon and Breach, New York, 1970.

[3] F. Gay-Balmaz, J. E. Marsden & T. S. Ratiu: *The geometry of the universal Teichmüller space and the Euler-Weil-Petersson equations.* Preprint.

[4] P. Hartman: *Ordinary differential equations.* Reprint of the second edition. Birkhüser, Boston, Mass., 1982.

[5] V. G. Maz'ja: *Sobolev spaces.* Springer Series in Soviet Mathematics. Springer-Verlag, Berlin, 1985.

[6] O. Pekonen: *Universal Teichmüller space in geometry and physics.* J. Geom. Phys. 15 (1995), no. 3, 227–251.

[7] M. E. TAYLOR: *Partial differential equations. III. Nonlinear equations.* Applied Mathematical Sciences, 117. Springer-Verlag, New York, 1997.

[8] L. A. TAKHTAJAN & L.-P. TEO: *Weil-Petersson metric on the universal Teichmller space.* Mem. Amer. Math. Soc. 183 (2006), no. 861.