

# ON THE DENSITY FUNCTION ON MODULI SPACES OF TORIC 4-MANIFOLDS

ALESSIO FIGALLI

ÁLVARO PELAYO

ABSTRACT. The optimal density function assigns to each symplectic toric manifold  $M$  a number  $0 < d \leq 1$  obtained by considering the ratio between the maximum volume of  $M$  which can be filled by symplectically embedded disjoint balls and the total symplectic volume of  $M$ . In the toric version of this problem,  $M$  is toric and the balls need to be embedded respecting the toric action on  $M$ . The goal of this note is first to give a brief survey of the notion of toric symplectic manifold and the recent constructions of moduli space structure on them, and then recall how to define a natural density function on this moduli space. Then we review previous works which explain how the study of the density function can be reduced to a problem in convex geometry, and use this correspondence to give a simple description of the regions of continuity of the maximal density function when the dimension is 4.

## 1. INTRODUCTION

In symplectic topology the ball packing problem asks how much of the volume of a symplectic manifold  $(M, \omega)$  can be approximated by symplectically embedded disjoint balls, see Figure 1. This is in general a very difficult problem; a lot of progress on it and directly related problems has been made by a number of authors, among them Biran [4, 5, 6], Borman-Li-Wu [7], McDuff–Polterovich [12], Schlenk [18], Traynor [19], and Xu [20]. The *optimal density function*  $\Omega$  assigns to each closed symplectic manifold  $M$  the number  $0 < d \leq 1$  obtained by considering the ratio between the largest volume  $v$  of  $M$  which can be filled by equivariantly and symplectically embedded disjoint balls, and the total symplectic volume  $\text{vol}(M)$  of  $M$ . An *optimal* packing is one for which the sum of these disjoint volumes divided by  $\text{vol}(M)$  is as large as it can be, taking into consideration all possible such packings.

The article [13] discusses a particular instance of the symplectic ball packing problem: *the toric case*. In the toric case both the symplectic manifold  $(M, \omega)$  and the standard open symplectic ball  $\mathbb{B}_r$  in  $\mathbb{C}^n$  are equipped with a Hamiltonian action of an  $n$ -dimensional torus  $\mathbb{T}$ , and the symplectic embeddings of the ball into the manifold  $M$  are equivariant with respect to these actions. In the case of symplectic toric manifolds, there always exists at least an optimal packing [16]. The best way to think of this problem is an

approximation theorem for our integrable system by disjoint integrable systems on balls. This problem is more rigid, because for instance fixed points of the system have to coincide with the origin of the ball, so the symplectic balls mapped this way have less flexibility.

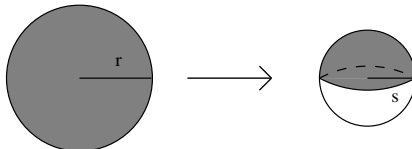


FIGURE 1. A toric symplectic embedding of the 2-ball of radius  $r$  into a 2-sphere of radius  $s$  with  $r/s = \sqrt{2}$ .

In this note we concentrate on symplectic toric manifolds (also called toric integrable systems) because for this we have a good understanding [14, 13, 16], but the question is interesting for any integrable system. These particular systems are usually called *toric*, or *symplectic toric*, and a rich structure theory due to Kostant, Guillemin-Sternberg, Delzant among others led to a complete classification in the 1980s [2, 8, 10, 11]. We refer to Section 3 for the precise definition of symplectic toric manifolds, and to Section 4 for the definition of the moduli space  $\mathcal{M}_{\mathbb{T}}$  of such manifolds, where  $\mathbb{T}$  denotes the standard torus of dimension precisely half the dimension of the manifolds in  $\mathcal{M}_{\mathbb{T}}$ . By using elementary geometric arguments, we shall describe the regions of this moduli space where the optimal density function  $\Omega: \mathcal{M}_{\mathbb{T}} \rightarrow [0, 1]$  is continuous. In particular, as a consequence of this description one can observe that the density function is highly discontinuous.

The structure of the paper is as follows. In Section 2 we state the main theorem of the paper: Theorem A; in Section 3 we review the notion of symplectic toric manifold; in Section 4 we review the construction of the moduli space  $\mathcal{M}_{\mathbb{T}}$  of symplectic toric manifolds; in Section 5 we review the definition of the density function on  $\mathcal{M}_{\mathbb{T}}$ ; in Section 6 we explain how to reduce the proof of Theorem A to the proof of a theorem in convex geometry (Theorem 7.1); in Section 7 we state and prove Theorem 7.1.

*Acknowledgements.* A.P. was partially supported by grants NSF DMS-1055897, NSF DMS-1518420, and a J Tinsley Oden Fellowship at ICES, UT Austin. A.P. thanks Luis Caffarelli and Alessio Figalli for sponsoring his visit. A.F. was partially supported by grants NSF DMS-1262411 and NSF DMS-1361122.

## 2. MAIN THEOREM

Let  $\mathbb{T}$  be a 2-dimensional torus, let  $\mathcal{M}_{\mathbb{T}}$  be the moduli space of symplectic toric 4-manifolds. It is known [15] that  $\mathcal{M}_{\mathbb{T}}$  is neither a locally compact

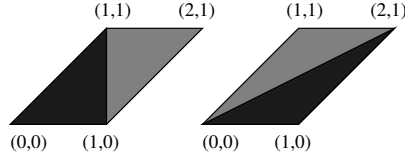


FIGURE 2. Two distinct optimal packings of a symplectic toric manifold  $M$  isomorphic to  $\mathbb{C}\mathbb{P}^1 \times \mathbb{C}\mathbb{P}^1$ , so  $\Omega(M) = 1$ .

nor a complete metric space but its completion is well understood and describable in explicit terms (see Theorem 4.3). Let  $\Omega: \mathcal{M}_{\mathbb{T}} \rightarrow [0, 1]$  be the *optimal density function*, which assigns to a manifold the density of its optimal toric ball packing, cf. Figure 2. By construction  $\Omega$  is an invariant of the symplectic toric type of  $M$ . It is natural to wonder what the precise regions of continuity of  $\Omega$  are; this problem was posed in [15, Problem 30]. The following result solves this problem by giving a characterization of the regions where  $\Omega$  is continuous.

**Theorem A.** *Let  $N \geq 1$  be an integer and let  $\mathcal{M}_{\mathbb{T}}^N$  be the set of symplectic toric manifolds with precisely  $N$  points fixed by the  $\mathbb{T}$ -action. Then:*

- (1)  $\Omega$  is discontinuous at every  $(M, \omega, \mathbb{T}) \in \mathcal{M}_{\mathbb{T}}$ , and the restriction  $\Omega|_{\mathcal{M}_{\mathbb{T}}^N}$  is continuous for each  $N \geq 1$ .
- (2) Given  $(M, \omega, \mathbb{T}) \in \mathcal{M}_{\mathbb{T}}^N$ , define  $\Omega_i(M, \omega, \mathbb{T})$ , where  $1 \leq i \leq N$ , to be the optimal density computed along all packings avoiding balls with center at the  $i$ th fixed point of the  $\mathbb{T}$ -action. Then  $\mathcal{M}_{\mathbb{T}}^N$  is the largest neighborhood of  $M$  in  $\mathcal{M}_{\mathbb{T}}$  where  $\Omega$  is continuous if and only if  $\Omega_i(M, \omega, \mathbb{T}) < \Omega(M, \omega, \mathbb{T})$  for all  $i$  with  $1 \leq i \leq N$ .

Note that

$$\mathcal{M}_{\mathbb{T}} = \bigcup_{N \geq 1} \mathcal{M}_{\mathbb{T}}^N.$$

The aforementioned classification of symplectic toric manifolds is in terms of a class of convex polytopes in  $\mathbb{R}^n$ , called now *Delzant polytopes*. In practice this means that the a priori very complicated packing problem for these systems can be formulated and studied in terms of convex geometry, which is what was done in [14, 13, 16]. This is because symplectic toric manifolds have an associated momentum map  $\mu: M \rightarrow \mathbb{R}^n$ , which is a Morse-Bott function with a number of remarkable properties and which captures all the main features of the symplectic geometry of  $(M, \omega)$  and of the Hamiltonian action of  $\mathbb{T}$  on  $M$  (the proof of this is due to Atiyah, Guillemin, Sternberg, and Delzant). In fact  $\mu(M)$  is a convex polytope in  $\mathbb{R}^n$ .

The strategy of this paper is to prove a slightly more general result about polytopes, Theorem 7.1, which implies Theorem A by virtue of a theorem proven in [13] (reviewed in the present paper, see Theorem 6.2).

**Remark 2.1.** The proof of Theorem 7.1 is based on simple geometric arguments that can be easily understood in the case  $n = 2$  (i.e., when dealing with Delzant polygons) but the general approach in the proof should work in any dimension (indeed, in the proof we rely only on results from [16] that hold in any dimension). However the case of symplectic toric 4-manifolds is currently the most interesting as the topology of the moduli space is understood only in that case (see Theorem 4.3).

### 3. SYMPLECTIC TORIC MANIFOLDS

In this section we review the notion of symplectic toric manifold in arbitrary dimension.

Let  $(M, \omega)$  be a closed symplectic  $2n$ -dimensional manifold. Let  $\mathbb{T}^k \cong (S^1)^k$  be the  $k$ -dimensional torus, and write  $\mathbb{T} := \mathbb{T}^n$  to denote the standard  $n$ -dimensional torus.

Let  $\mathfrak{t}$  be the Lie algebra  $\text{Lie}(\mathbb{T})$  of  $\mathbb{T}$  and let  $\mathfrak{t}^*$  be the dual of  $\mathfrak{t}$ . A symplectic action  $\psi : \mathbb{T}^k \times M \rightarrow M$  of a  $k$ -dimensional torus (that is, an action preserving the form  $\omega$ ) is called *Hamiltonian* if there is a map

$$\mu : M \rightarrow \mathfrak{t}^*,$$

known as a *momentum map*, satisfying Hamilton's equation

$$i_{\xi_M} \omega = d\langle \mu, \xi \rangle,$$

for all  $\xi \in \mathfrak{t}$ . The momentum map is defined up to translation by an element of  $\mathfrak{t}^*$ . Nevertheless, we ignore this ambiguity and call it *the momentum map*.

**Definition 3.1.** A symplectic-toric manifold is a quadruple  $(M, \omega, \mathbb{T}, \mu)$  where  $M$  is a  $2n$ -dimensional closed symplectic manifold  $(M, \omega)$  equipped with an effective Hamiltonian action of an  $n$ -dimensional torus  $\mathbb{T}$  with momentum map  $\mu$ .

**Example 3.2.** The projective space  $(\mathbb{C}\mathbb{P}^n, \lambda \cdot \omega_{\text{FS}})$ , where  $\omega_{\text{FS}}$  is the Fubini-Study form given by

$$\omega_{\text{FS}} = \frac{1}{2(\sum_{i=0}^n \bar{z}_i z_i)} \sum_{k=0}^n \sum_{j \neq k} (\bar{z}_j z_j dz_k \wedge d\bar{z}_k - \bar{z}_j z_k dz_j \wedge d\bar{z}_k)$$

equipped with the rotational action of  $\mathbb{T}^n$ ,

$$(e^{i\theta_1}, \dots, e^{i\theta_n}) \cdot [z_0 : \dots : z_n] = [z_0 : e^{-2\pi i \theta_1} z_1 : \dots : e^{-2\pi i \theta_n} z_n],$$

is a  $2n$ -dimensional symplectic-toric manifold. The components of the momentum map are

$$\mu_k^{\mathbb{C}\mathbb{P}^n}(z) = \frac{\lambda |z_k|^2}{\sum_{i=0}^n |z_i|^2}.$$

The corresponding momentum polytope is equal to the convex hull in  $\mathbb{R}^n$  of 0 and the scaled canonical vectors  $\lambda e_1, \dots, \lambda e_n$ .

Strictly speaking,  $\mu$  is a map from  $M$  to  $\mathfrak{t}^*$ . However, the presentation is simpler if from the beginning we identify both  $\mathfrak{t}$  and  $\mathfrak{t}^*$  with  $\mathbb{R}^n$  and consider  $\mu$  as a map from  $M$  to  $\mathbb{R}^n$ . The procedure to do this, that we now describe, is standard but not canonical. Choose an epimorphism  $E: \mathbb{R} \rightarrow \mathbb{T}^1$ , for instance,  $x \mapsto e^{2\pi\sqrt{-1}x}$ . This Lie group epimorphism has discrete center  $\mathbb{Z}$  and the inverse of the corresponding Lie algebra isomorphism is given by  $\text{Lie}(\mathbb{T}^1) \ni \frac{\partial}{\partial x} \mapsto \frac{1}{2\pi} \in \mathbb{R}$ . Thus, for  $\mathbb{T}$  we get the non-canonical isomorphism between the corresponding commutative Lie algebras

$$\text{Lie}(\mathbb{T}) = \mathfrak{t} \ni \frac{\partial}{\partial x_k} \mapsto \frac{1}{2\pi} e_k \in \mathbb{R}^n,$$

where  $e_k$  is the  $k^{\text{th}}$  element in the canonical basis of  $\mathbb{R}^n$ . Choosing an inner product  $\langle \cdot, \cdot \rangle$  on  $\mathfrak{t}$ , we obtain an isomorphism  $\mathfrak{t} \rightarrow \mathfrak{t}^*$ , and hence taking its inverse and composing it with the isomorphism  $\mathfrak{t} \rightarrow \mathbb{R}^n$  described above, we get an isomorphism  $\mathcal{I}: \mathfrak{t}^* \rightarrow \mathbb{R}^n$ . In this way, we obtain a momentum map  $\mu = \mu_{\mathcal{I}}: M \rightarrow \mathbb{R}^n$ .

**Example 3.3.** Consider the open ball  $\mathbb{B}_r$  of radius  $r$  in  $\mathbb{C}^n$ , equipped with the standard symplectic form  $\omega_0 = \frac{i}{2} \sum_j dz_j \wedge d\bar{z}_j$  and the Hamiltonian action  $\text{Rot}$  by rotations given by  $(\theta_1, \dots, \theta_n) \cdot (z_1, \dots, z_n) = (\theta_1 z_1, \dots, \theta_n z_n)$ . In this case the components of the momentum map  $\mu^{\mathbb{B}_r}$  are  $\mu_k^{\mathbb{B}_r} = |z_k|^2$ . The image  $\Delta^n(r)$  of the momentum map is

$$(3.1) \quad \Delta^n(r) = \text{ConvHull}(0, r^2 e_1, \dots, r^2 e_n) \setminus \text{ConvHull}(r^2 e_1, \dots, r^2 e_n),$$

where  $\{e_i\}_{i=1}^n$  stands for the canonical basis of  $\mathbb{R}^n$ .

The image of the momentum map of a symplectic-toric manifold is a particular class of convex polytope, a so called *Delzant polytope*, see Figure 3.

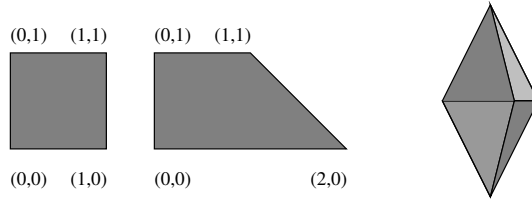


FIGURE 3. Two Delzant polytopes (left) and a non-Delzant polytope (right).

**Definition 3.4.** ([1]) *A convex polytope  $\Delta$  in  $\mathbb{R}^n$  is a Delzant polytope if it is simple, rational and smooth:*

- (i)  $\Delta$  is simple if there are exactly  $n$  edges meeting at each vertex  $v \in V$ ;
- (ii)  $\Delta$  is rational if for every vertex  $v \in V$ , the edges meeting at  $v$  are of the form  $v + tu_i$ ,  $t \geq 0$ , and  $u_i \in \mathbb{Z}^n$ ;

- (iii) A vertex  $v \in V$  is smooth if the edges meeting at  $v$  are of the form  $v + tu_i$ ,  $t \geq 0$ , where the vectors  $u_1, \dots, u_n$  can be chosen to be a  $\mathbb{Z}$  basis of  $\mathbb{Z}^n$ .  $\Delta$  is smooth if every vertex  $v \in V$  is smooth.

#### 4. MODULI SPACES OF TORIC MANIFOLDS

With the conventions above, where  $\mathbb{T}$  and the identification  $\mathcal{I}: \mathfrak{t}^* \rightarrow \mathbb{R}^n$  are fixed, we next define the moduli space of toric manifolds.

**4.1. The moduli relation.** Let

$$(M, \omega, \mathbb{T}, \mu: M \rightarrow \mathbb{R}^n)$$

and

$$(M', \omega', \mathbb{T}, \mu': M' \rightarrow \mathbb{R}^n)$$

be symplectic toric manifolds. These two symplectic toric manifolds are *isomorphic* if there exists an equivariant symplectomorphism  $\varphi: M \rightarrow M'$  (i.e.,  $\varphi$  is a diffeomorphism satisfying  $\varphi^*\omega' = \omega$  which intertwines the  $\mathbb{T}$  actions) such that  $\mu' \circ \varphi = \mu$  (see also [3, Definition I.1.16]). We denote by

$$\mathcal{M}_{\mathbb{T}} := \mathcal{M}_{\mathbb{T}}^{\mathcal{I}}$$

the *moduli space* (that is, the set of equivalence classes) of symplectic toric manifolds of a fixed dimension  $2n$  under this equivalence relation.

**Theorem 4.1.** (Delzant [8, Theorem 2.1]) *Let  $(M, \omega, \mathbb{T}, \mu)$  and  $(M', \omega', \mathbb{T}, \mu')$  be symplectic toric manifolds. If the images  $\mu(M)$  and  $\mu'(M')$  are equal, then  $(M, \omega, \mathbb{T}, \mu)$  and  $(M', \omega', \mathbb{T}, \mu')$  are isomorphic.*

The convexity theorem of Atiyah [2] and Guillemin-Sternberg [10] says that the momentum map image is a convex polytope. In addition, if the dimension of the torus is precisely half the dimension of the manifold, this polytope is Delzant (Definition 3.4).

Let  $\mathcal{D}_{\mathbb{T}}$  denote the set of Delzant polytopes. It follows from Theorem 4.1 that

$$(4.1) \quad [(M, \omega, \mathbb{T}, \mu)] \ni \mathcal{M}_{\mathbb{T}} \longmapsto \mu(M) \in \mathcal{D}_{\mathbb{T}}$$

is injective. Delzant also showed how starting from a Delzant polytope it is possible to reconstruct a symplectic toric manifold with momentum image precisely equal to the Delzant polytope, thus implying that (4.1) is a *bijection*. To simplify our notation we often write  $(M, \omega, \mathbb{T}, \mu)$  identifying the representative with the corresponding equivalence class  $[(M, \omega, \mathbb{T}, \mu)]$  in the moduli space  $\mathcal{M}_{\mathbb{T}}$ .

**4.2. Metric on  $\mathcal{M}_{\mathbb{T}}$ .** Endow  $\mathcal{D}_{\mathbb{T}}$  with the distance function given by the volume of the symmetric difference

$$(\Delta_1 \setminus \Delta_2) \cup (\Delta_2 \setminus \Delta_1)$$

of any two polytopes  $\Delta_1$  and  $\Delta_2$ . Using (4.1) we define a metric  $d_{\mathbb{T}}$  on  $\mathcal{M}_{\mathbb{T}}$  as the pullback of the metric defined on  $\mathcal{D}_{\mathbb{T}}$ . In this way we obtain the metric space  $(\mathcal{M}_{\mathbb{T}}, d_{\mathbb{T}})$ . This metric induces a topology  $\nu$  on  $\mathcal{M}_{\mathbb{T}}$ .

Consider the  $\sigma$ -algebra  $\mathfrak{B}(\mathbb{R}^n)$  of Borel sets of  $\mathbb{R}^n$ . Let  $\lambda: \mathfrak{B}(\mathbb{R}^n) \rightarrow \mathbb{R}_{\geq 0} \cup \{\infty\}$  be the Lebesgue measure on  $\mathbb{R}^n$ , and let  $\mathfrak{B}'(\mathbb{R}^n) \subset \mathfrak{B}(\mathbb{R}^n)$  consist of the Borel sets with finite Lebesgue measure. Let  $\chi_C: \mathbb{R}^n \rightarrow \mathbb{R}$  be the characteristic function of  $C \in \mathfrak{B}'(\mathbb{R}^n)$ . Define

$$(4.2) \quad d(A, B) := \|\chi_A - \chi_B\|_{L^1}.$$

This extends the distance function defined above on  $\mathcal{D}_{\mathbb{T}}$ . However, it is not a metric on  $\mathfrak{B}'(\mathbb{R}^n)$ . By identifying the sets  $A, B \in \mathfrak{B}'(\mathbb{R}^n)$  with  $d(A, B) = 0$ , we obtain a metric on the resulting quotient space of  $\mathfrak{B}'(\mathbb{R}^n)$ . Let  $\mathcal{C}$  be the set of convex compact subsets of  $\mathbb{R}^n$  with positive Lebesgue measure,  $\emptyset$  the empty set, and  $\hat{\mathcal{C}} := \mathcal{C} \cup \{\emptyset\}$ . Then  $\hat{\mathcal{C}}$  equipped with  $d$  in (4.2) is a metric space.

**Remark 4.2.** (Other moduli spaces) There is no  $\text{AGL}(n, \mathbb{Z})$  equivalence relation involved in the definition of  $\mathcal{D}_{\mathbb{T}}$ . Such equivalence relation is often put on this space so that it is in one-to-one correspondence with the moduli space of symplectic toric manifolds *up to equivariant isomorphisms*. This is *not* the relation which is relevant to this paper. Nonetheless, to avoid confusion let us recall that two symplectic toric manifolds  $(M, \omega, \mathbb{T}, \mu)$  and  $(M', \omega', \mathbb{T}, \mu')$  are *equivariantly isomorphic* if there exists an automorphism of the torus  $h: \mathbb{T} \rightarrow \mathbb{T}$  and an  $h$ -equivariant symplectomorphism  $\varphi: M \rightarrow M'$ , i.e., such that the following diagram commutes:

$$\begin{array}{ccc} \mathbb{T} \times M & \xrightarrow{\rho^*} & M \\ (h, \varphi) \downarrow & & \varphi \downarrow \\ \mathbb{T} \times M' & \xrightarrow{\rho'^*} & M'. \end{array}$$

In [15], the space  $\widetilde{\mathcal{M}}_{\mathbb{T}}$  denotes the *moduli space of equivariantly isomorphic  $2n$ -dimensional symplectic toric manifolds*.<sup>1</sup> Let  $\text{AGL}(n, \mathbb{Z}) = \text{GL}(n, \mathbb{Z}) \ltimes \mathbb{R}^n$  be the group of affine transformations of  $\mathbb{R}^n$  given by  $x \mapsto Ax + c$ , where  $A \in \text{GL}(n, \mathbb{Z})$  and  $c \in \mathbb{R}^n$ . We say that two Delzant polytopes  $\Delta_1$  and  $\Delta_2$  are  $\text{AGL}(n, \mathbb{Z})$ -*equivalent* if there exists  $\alpha \in \text{AGL}(n, \mathbb{Z})$  such that  $\alpha(\Delta_1) = \Delta_2$ . Let  $\widetilde{\mathcal{D}}_{\mathbb{T}}$  be the moduli space of Delzant polytopes modulo the equivalence relation given by  $\text{AGL}(n, \mathbb{Z})$ ; we endow this space with the quotient topology induced by the projection map  $\pi: \mathcal{D}_{\mathbb{T}} \rightarrow \widetilde{\mathcal{D}}_{\mathbb{T}} \simeq \mathcal{D}_{\mathbb{T}} / \text{AGL}(n, \mathbb{Z})$ . The map (4.1) induces a bijection between  $\widetilde{\mathcal{M}}_{\mathbb{T}}$  and  $\widetilde{\mathcal{D}}_{\mathbb{T}}$ . Thus  $\widetilde{\mathcal{M}}_{\mathbb{T}}$  is a topological space with the topology  $\tilde{\nu}$  induced by this bijection. The topological space  $(\widetilde{\mathcal{M}}_{\mathbb{T}}, \tilde{\nu})$  is studied in [15]. It would be interesting to give a version of Theorem A where  $\mathcal{M}_{\mathbb{T}}$  is replaced by  $\widetilde{\mathcal{M}}_{\mathbb{T}}$ . It was proven in [15, Theorem 1] that  $(\widetilde{\mathcal{M}}_{\mathbb{T}}, \tilde{\nu})$  is connected.

<sup>1</sup> Two equivariantly isomorphic toric manifolds  $(M, \omega, \mathbb{T}, \mu)$  and  $(M', \omega', \mathbb{T}, \mu')$  are isomorphic if and only if  $h$  in (4.3) is the identity and  $\mu' = \mu \circ \varphi$ .

**4.3. Topological structure in dimension 4.** The following theorem, *which holds in dimension 4*, describes properties of the topology of the aforementioned moduli space.

**Theorem 4.3** ([15]). *Let  $\mathbb{T}$  be a 2-dimensional torus. The space  $(\mathcal{M}_{\mathbb{T}}, d_{\mathbb{T}})$  is neither locally compact nor a complete metric space. Its completion can be identified with the metric space  $(\hat{\mathcal{C}}, d)$  in the following sense: identifying  $(\mathcal{M}_{\mathbb{T}}, d_{\mathbb{T}})$  with  $(\mathcal{D}_{\mathbb{T}}, d)$  via (4.1), the completion of  $(\mathcal{D}_{\mathbb{T}}, d)$  is  $(\hat{\mathcal{C}}, d)$ .*

## 5. DENSITY FUNCTION

Let  $(M, \omega, \mathbb{T}, \mu)$  be a symplectic toric manifold of dimension  $2n$ . Let  $\Lambda$  be an automorphism of  $\mathbb{T}$ . Let  $r > 0$ . We say that a subset  $B$  of  $M$  is a  $\Lambda$ -equivariantly embedded symplectic ball of radius  $r$  if there is a symplectic embedding  $f : \mathbb{B}_r \rightarrow M$  with  $f(\mathbb{B}_r) = B$  and such that the diagram:

$$\begin{array}{ccc} \mathbb{T}^n \times \mathbb{B}_r & \xrightarrow{\Lambda \times f} & \mathbb{T}^n \times M \\ \downarrow \text{Rot} & \circlearrowleft & \downarrow \psi \\ \mathbb{B}_r & \xrightarrow{f} & M \end{array}$$

is commutative. Let  $B$  be a  $\Lambda$ -equivariantly embedded symplectic ball. We say that  $B$  has center  $f(0)$ . We say that another subset  $B'$  of  $M$  is an equivariantly embedded symplectic ball of radius  $r'$  if there exists an automorphism  $\Lambda'$  of  $\mathbb{T}$  such that  $B'$  is a  $\Lambda'$ -equivariantly embedded symplectic ball of radius  $r'$ .

In what follows, the symplectic volume of a subset  $X \subset M$  is given by

$$\text{vol}_{\omega}(X) := \int_X \omega^n.$$

Following [13, Definition 1.6], a *toric ball packing* of  $M$  is given by a disjoint union of the form

$$\mathcal{P} := \bigsqcup_{\alpha \in A} B_{\alpha},$$

where each  $B_{\alpha}$  is an equivariantly embedded symplectic ball (note that we are not saying that all these balls must have the same radii). The *density*  $\Omega(\mathcal{P})$  of a toric ball packing  $\mathcal{P}$  is given by the quotient

$$\Omega(\mathcal{P}) := \frac{\text{vol}_{\omega}(\mathcal{P})}{\text{vol}_{\omega}(M)} \in [0, 1].$$

The *density of a symplectic-toric manifold*  $(M, \omega, \mathbb{T}, \mu)$  is given by

$$\Omega(M, \omega, \mathbb{T}, \mu) := \sup \left\{ \Omega(\mathcal{P}) \mid \mathcal{P} \text{ is a toric ball packing of } M \right\}.$$

An *optimal packing* (also called a *maximal packing*) is a toric ball packing at which this density is achieved. The *optimal density function* is defined as follows: it assigns to a manifold the density of one of its optimal packings



(such an optimal packing always exists, see [16]). The optimal density function is interesting to us because *it is a symplectic invariant*. In this paper we analyze the continuity of the density function on the moduli space  $\mathcal{M}_{\mathbb{T}}$  (see Theorem A), which was posed as an open problem in [15, Problems 4 and 30].

## 6. CONVEX GEOMETRY

Following [13, 16] we say that a subset  $\Sigma$  of a Delzant polytope  $\Delta$  is an *admissible simplex of radius  $r$  with center at a vertex  $v$  of  $\Delta$*  if there is an element of  $\text{AGL}(n, \mathbb{Z})$  which takes:

- $\Delta^n(r^{1/2})$  to  $\Sigma$ ,
- the origin to  $v$ ,
- the edges of  $\Delta^n(r^{1/2})$  meeting at the origin to the edges of  $\Delta$  meeting at  $v$ .

We write

$$r_v := \max \left\{ r > 0 \mid \exists \text{ an admissible simplex of radius } r \text{ with center } v \right\}$$

for every vertex  $v$ .

In what follows we write  $\text{vol}_{\text{euc}}(A)$  for the Euclidean volume of a subset  $A \subset \Delta$ .

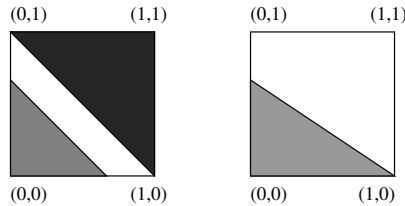


FIGURE 4. The packing on the left is admissible, the one on the right is not.

An *admissible packing of a Delzant polytope  $\Delta$*  is given by a disjoint union

$$\mathcal{P} := \bigsqcup_{\alpha \in A} \Sigma_{\alpha},$$

where each  $\Sigma_{\alpha}$  is an admissible simplex (note that we are not saying that they must have the same radii). The *density* of an admissible packing  $\mathcal{P}$  is defined by the quotient

$$\Omega(\mathcal{P}) := \frac{\text{vol}_{\text{euc}}(\mathcal{P})}{\text{vol}_{\text{euc}}(\Delta)} \in [0, 1],$$

and the *density of  $\Delta$*  is defined by

$$\Omega(\Delta) := \sup \left\{ \Omega(\mathcal{P}) \mid \mathcal{P} \text{ is an admissible packing of } \Delta \right\}.$$

An *optimal packing* (also called a *maximal packing*) is an admissible ball packing at which this density is achieved. The *optimal density function* assigns to a Delzant polytope the density of one of its optimal packings (such packing always exists, see [16]).

Figure 4 shows an example of an admissible packing and a non admissible one; at the manifold level, the admissible one corresponds to a symplectic toric ball packing of  $\mathbb{C}\mathbb{P}^1 \times \mathbb{C}\mathbb{P}^1$  by two disjoint balls, while the non admissible one cannot be interpreted in this way because the shaded triangle does not correspond to an equivariantly and symplectically embedded ball. Both packings in Figure 5 are admissible.

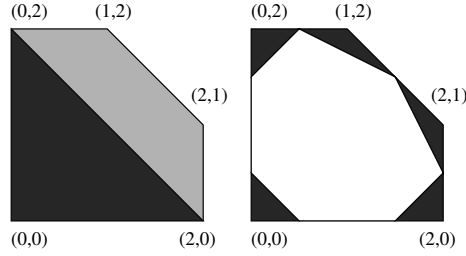


FIGURE 5. Admissible packings of the polygon of a blow up of the product of two symplectic spheres of radius  $1/\sqrt{2}$ .

Let  $I \subset \mathbb{R}^n$  be an interval with rational slope. The *rational-length*  $\text{length}_{\mathbb{Q}}(I)$  of  $I$  is the (unique) number  $\ell$  such that  $I$  is  $\text{AGL}(n, \mathbb{Z})$ -congruent to a length  $\ell$  interval on a coordinate axis. As it is shown by the following result, an admissible simplex is parametrized by its center and its radius.

**Lemma 6.1** ([13]). *Let  $\Delta$  be a Delzant polytope and let  $v \in \Delta$  be a vertex of  $\Delta$ . We denote the  $n$  edges leaving  $v$  by  $e_v^1, \dots, e_v^n$ . Then:*

(i)  $r_v$  is given by

$$r_v = \min \left\{ \text{length}_{\mathbb{Q}}(e_v^1), \dots, \text{length}_{\mathbb{Q}}(e_v^n) \right\}.$$

(ii) *There exists an admissible simplex  $\Sigma(v, r)$  centered at  $v$  of radius  $r$  if and only if we have that  $0 \leq r \leq r_v$ . Moreover  $\Sigma(v, r)$  is the unique such admissible simplex.*

(iii) *The volume of  $\Sigma(v, r)$  in (ii) is given by*

$$\text{vol}_{\text{euc}}(\Sigma(v, r)) = \frac{r^n}{n!}.$$

Now we can state the tool that we will use in the proof of the main theorem of the paper. Consider a symplectic-toric manifold  $(M, \omega, \mathbb{T}, \mu)$  and let  $\Delta := \mu(M)$ .

**Theorem 6.2** ([13]). *The following hold:*

- (i) If  $B$  is an equivariantly embedded symplectic ball with center at  $p \in M$  and radius  $r$ , the momentum map image  $\mu(B)$  is an admissible simplex in  $\Delta$  centered at  $\mu(p)$  of radius  $r^2$ .

Moreover, if  $\Sigma$  is an admissible simplex in  $\Delta$  of radius  $r$ , then there is an equivariantly embedded symplectic ball  $B$  in  $M$  of radius  $r^{1/2}$  and such that

$$\mu(B) = \Sigma.$$

- (ii) Let  $\mathcal{P}$  be a toric ball packing of  $M$ . Then  $\mu(\mathcal{P})$  is an admissible packing of  $\Delta$ . Furthermore, we have the equality

$$\Omega(\mathcal{P}) = \Omega(\mu(\mathcal{P})).$$

Moreover, for any admissible packing  $\mathcal{Q}$  of  $\Delta$  there is a toric ball packing  $\mathcal{P}$  of  $M$  such that

$$\mu(\mathcal{P}) = \mathcal{Q}.$$

## 7. THE COMBINATORIAL CONVEXITY STATEMENT

In this section we state and prove a theorem in convex geometry, that in view of Theorem 6.2 implies Theorem A.

**Theorem 7.1.** *Let  $N \geq 1$  be an integer and let  $\mathcal{P}^N$  be the set of Delzant polygons of  $N$  vertices, and let  $\mathcal{P}$  be the set of all Delzant polygons, so that  $\mathcal{P} = \bigcup_{N \geq 1} \mathcal{P}^N$ . Then:*

- (1)  $\Omega$  is discontinuous at every  $\Delta \in \mathcal{P}$ , and the restriction  $\Omega|_{\mathcal{P}^N}$  is continuous for each  $N \geq 1$ .
- (2) Given  $\Delta \in \mathcal{P}^N \subset \mathcal{P}$ , define  $\{\Omega_i(\Delta)\}_{1 \leq i \leq N}$  to be the maximal density avoiding vertex  $i$ . Then  $\mathcal{P}^N$  is the largest set containing  $\Delta$  where  $\Omega$  is continuous if and only if  $\Omega_i(\Delta) < \Omega(\Delta)$  for all  $1 \leq i \leq N$ .

*Proof.* In the proof of this result we will constantly make use of the following simple observation (see also the discussion below): given  $\Delta \in \mathcal{P}$ , a neighborhood of  $\Delta$  is made up of polygons where either we translate the sizes in a parallel way, or we chop some small corners of  $\Delta$ . In particular the number of vertices can only increase.

We prove (1) first. To show that  $\Omega$  is discontinuous at any  $\Delta \in \mathcal{P}^N$ , we fix  $\epsilon > 0$  small and we chop all the corners, adding around each of them a small side of length  $\leq \epsilon$ , making sure that the new polygon is still a Delzant polygon.

More precisely, if two consecutive sides are given by  $(0, a)$  and  $(b, 0)$  for some  $a, b > 0$ , because the polygon is a Delzant polygon it follows that  $a/b \in \mathbb{Q}$ . Hence, if we chop the corner at the origin with the segment connecting  $(a/M, 0)$  to  $(0, b/M)$  where  $M \in \mathbb{N}$  is very large, we obtain a new Delzant polygon. For the general case, given a couple of consecutive sides, there is a map in  $\text{AGL}(2, \mathbb{Z})$  sending  $(0, 0)$  to the common vertex, and

$(a, 0)$  and  $(0, b)$  to the other two vertices. Hence, it suffices to chop from the polygon the image of the triangle determined by  $(a/M, 0)$ ,  $(0, b/M)$ ,  $(0, 0)$ .

This construction gives us a polygon with  $2N$  vertices with the following property: when considering the definition of density, any simplex will have at least one side of length at most  $\epsilon$  while the others are universally bounded, so the total volume will be at most  $CN\epsilon$ . Since  $\epsilon$  can be arbitrarily small, this proves that  $\Omega$  cannot be continuous.

On the other hand, we show that  $\Omega$  is continuous when restricted to  $\mathcal{P}^N$ . To see this, we call an angle  $\alpha$  smooth if it can be obtained as the angle of a smooth triangle having the origin as one of its vertices,  $\alpha$  being the angle at the origin. Since the other two vertices belong to  $\mathbb{Z}^2$  (call them  $v_1, v_2$ ), it follows that the sides  $(0, v_1)$  and  $(0, v_2)$  have lengths  $\ell_1, \ell_2 \geq 1$ . Also, since such triangle is the image of  $\Delta^2(1)$  under a map in  $\text{AGL}(2, \mathbb{Z})$  its area must be  $1/2$ , that is

$$\frac{1}{2} = \frac{\ell_1 \ell_2 \sin \alpha}{2}.$$

These facts imply that there exists a constant  $C_\alpha$ , depending only on  $\sin \alpha$ , such that

$$\ell_1, \ell_2 \leq C_\alpha.$$

Now, fix  $\Delta \in \mathcal{P}_N$ , and let  $\alpha_1, \dots, \alpha_N$  be the angles of  $\Delta$ . Then, if  $\Delta' = (\alpha'_1, \dots, \alpha'_N) \in \mathcal{P}^N$ , since the set of smooth angles is discrete it follows that  $\alpha'_i = \alpha_i$  whenever  $d(\Delta', \Delta) \ll 1$ . Hence, for every  $\Delta'$  close to  $\Delta$  we are just translating the sides of the polygon in a parallel way, and since  $\Omega$  is continuous along this family of transformations [16] this proves that  $\Omega$  is continuous on the whole  $\mathcal{P}^N$ , proving (1).

Next we show (2). Assume first that

$$\Omega(\Delta) = \Omega_i(\Delta) \text{ for some } i \in \{1, \dots, N\}.$$

This implies that we can find an optimal family in the definition of  $\Omega$  with no simplex centered at  $i$  (simply take an optimal family in the definition of  $\Omega_i(\Delta)$ ). Then, by slightly reducing the radius of each simplex we can keep the density arbitrarily close to  $\Omega(\Delta)$  making sure that no simplex touches the vertex  $i$ . This gives us the possibility to chop a small corner around the vertex  $i$ , obtaining Delzant polytope  $\Delta' \in \mathcal{P}^{N+1}$  keeping the density still arbitrarily close  $\Omega(\Delta)$ . This procedure shows that we can find a family  $\{\Delta_k\}_{k \geq 1} \subset \mathcal{P}^{N+1}$  with

$$(7.1) \quad d(\Delta_k, \Delta) \rightarrow 0 \text{ and } \Omega(\Delta_k) \rightarrow \Omega(\Delta),$$

proving that there is a set larger than  $\mathcal{P}^N$  where  $\Omega$  is continuous at  $\Delta$ . Viceversa, assume there exists  $\eta > 0$  such that

$$\Omega_i(\Delta) \leq \Omega(\Delta) - \eta \text{ for all } i = 1, \dots, N.$$

Since it is impossible to approximate  $\Delta$  with polygons  $\Delta'$  that have strictly less than  $N$  vertices and we already proved that  $\Omega$  is continuous on  $\mathcal{P}^N$ , we

have to exclude that we can find some sequence  $\{\Delta_k\}_{k \in \mathbb{N}} \subset \cup_{j > N} \mathcal{P}^j$  such that (7.1) hold.

To see this notice that if we chop any corner around one of the vertices (say around  $i$ ) adding a small side, then necessarily the density will be close to  $\Omega_i(\Delta)$ , hence less than

$$\Omega(\Delta) - \eta/2,$$

which proves that (7.1) cannot hold. This concludes the proof of (2).  $\square$

#### REFERENCES

- [1] A. Cannas da Silva: *Lectures on Symplectic Geometry* (book), Lecture Notes in Mathematics 1764, Corrected 2nd Printing, Springer, 2008.
- [2] M. Atiyah: Convexity and commuting Hamiltonians, *Bull. London Math. Soc.* **14**, 1–15 (1982).
- [3] M. Audin, A. Cannas da Silva, E. Lerman: *Symplectic geometry of integrable systems*, Advanced Courses in Mathematics. CRM Barcelona. Birkhäuser Verlag, Basel (2003).
- [4] P. Biran: A stability property of symplectic packing, *Invent. Math.* **136** (1999), no. 1, 123–155.
- [5] P. Biran: Symplectic packing in dimension 4. *Geom. Funct. Anal.* **7** (1997), no. 3, 420–437.
- [6] P. Biran: Geometry of symplectic intersections, *Proc. ICM Beijing III* (2004) 1-3.
- [7] M.S. Borman, T.-J. Li, W. Wu: Spherical Lagrangians via ball packings and symplectic cutting, *Selecta Mathematica* **1**(2013) 261-283.
- [8] T. Delzant: Hamiltoniens périodiques et images convexes de l’application moment, *Bull. Soc. Math. France* **116**, 315–339 (1988).
- [9] W. Fulton: *Introduction to Toric Varieties*, Annals of Mathematical Studies, 131. The William H. Roever Lectures in Geometry. Princeton University Press, Princeton, NJ (1993).
- [10] V. Guillemin and S. Sternberg: Convexity properties of the moment mapping, *Invent. Math.* **67**, 491–513 (1982).
- [11] V. Guillemin: *Moment Maps and Combinatorial Invariants of Hamiltonian  $T^n$ -spaces*. Progress in Mathematics, 122. Birkhäuser Boston, Inc., Boston, MA, 1994.
- [12] D. McDuff, L. Polterovich: Symplectic packings and algebraic geometry. With an appendix by Y. Karshon. *Invent. Math.* **115** (1994), no. 3, 405–434.
- [13] Á. Pelayo: Toric symplectic ball packing, *Topology Appl.* **153** (2006) 3633–3644.
- [14] A. Pelayo: Topology of spaces of equivariant symplectic embeddings, *Proc. Amer. Math. Soc.* **135** (2007) 277–288.
- [15] Á. Pelayo, A. Pires, T. Ratiu, and S. Sabatini: Moduli spaces of toric manifolds, *Geom. Dedicata* **169** (2014) 323-341.
- [16] Á. Pelayo and B. Schmidt: Maximal ball packings of symplectic-toric manifolds, *Int. Math. Res. Not.*, ID rnm139, 24 pp (2008).
- [17] Á. Pelayo and S. Vũ Ngọc: First steps in symplectic and spectral theory of integrable systems, *Discrete and Cont. Dyn. Syst., Series A* **32**, 3325–3377 (2012).
- [18] F. Schlenk: Embedding Problems in Symplectic Geometry. de Gruyter Expositions in Mathematics, vol. 40. Walter de Gruyter, Berlin (2005).
- [19] L. Traynor: Symplectic packing constructions. *J. Diff. Geom.* **41** (1995), no. 3, 735–751.
- [20] G. Xu: Curves in  $P^2$  and symplectic packings. *Math. Ann.* **299** (1994), no. 4, 609–613.

**Alessio Figalli**

The University of Texas at Austin  
Mathematics Dept. RLM 8.100  
2515 Speedway Stop C1200  
Austin, Texas 78712-1202, USA  
*E-mail:* figalli@math.utexas.edu

**Álvaro Pelayo**

Department of Mathematics  
University of California, San Diego  
9500 Gilman Drive # 0112  
La Jolla, CA 92093-0112, USA  
*E-mail:* alpelayo@math.ucsd.edu