

On the regularity of the pressure field of Brenier's weak solutions to incompressible Euler equations

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Abstract

In this paper we improve the regularity in time of the gradient of the pressure field arising in Brenier's variational weak solutions [8] to incompressible Euler equations. This improvement is necessary to obtain that the pressure field is not only a measure, but a function in $L^2_{\text{loc}}((0, T); BV_{\text{loc}}(D))$. In turn, this is a fundamental ingredient in the analysis made in [2] of the necessary and sufficient optimality conditions for the variational problem in [5], [8].

1 Introduction

The Euler equations for incompressible fluids flowing inside a d -dimensional domain D relate the evolution of the velocity field u to the spatial gradient of the pressure field p :

$$\begin{cases} \partial_t u + (u \cdot \nabla)u = -\nabla p & \text{in } [0, T] \times D, \\ \operatorname{div} u = 0 & \text{in } [0, T] \times D, \\ u \cdot n = 0 & \text{on } [0, T] \times \partial D. \end{cases} \quad (1)$$

Let us assume that u is smooth, so that it produces a unique flow g , given by

$$\begin{cases} \dot{g}(t, a) = u(t, g(t, a)), \\ g(0, a) = a. \end{cases}$$

(we will also use the notation $e^{tu}(a)$ in the sequel). By the incompressibility condition, we get that at each time t the map $g(t, \cdot) : D \rightarrow D$ is a measure-preserving diffeomorphism of D , that is $g(t, \cdot)_{\#} \mu_D = \mu_D$ (here and in the sequel $f_{\#} \mu$ is the push-forward of a measure μ through a map f , and μ_D is the volume measure of the manifold D). Viewing the space $\text{SDiff}(D)$ of measure-preserving diffeomorphisms of D as an infinite-dimensional manifold with the metric inherited from the embedding in L^2 , and with tangent space made by the divergence-free vector fields,

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Arnold interpreted the equation above, and therefore (1), as a *geodesic* equation on $\text{SDiff}(D)$ [3]. According to this interpretation, one can look for solutions of (1) by minimizing the action

$$\int_0^T \int_D \frac{1}{2} |\dot{g}(t, x)|^2 d\mu_D(x) dt \quad (2)$$

among all paths $g(t, \cdot) : [0, T] \rightarrow \text{SDiff}(D)$ with $g(0, \cdot) = f$ and $g(T, \cdot) = h$ prescribed (typically, by right invariance, f is taken as the identity map \mathbf{i}), and the pressure field arises as a Lagrange multiplier from the incompressibility constraint.

Although in the traditional approach to (1) the initial velocity is prescribed, while in the minimization of (2) it is not, this variational problem has an independent interest and leads to deep mathematical questions, namely existence of relaxed solutions, gap phenomena and optimality conditions, that have been thoroughly investigated by Brenier and Shnirelman in a series of papers (see [5], [6], [7], [8], [9], [10]). Recently, in [2] we compared the different models proposed by Brenier in [5] and [8], and we found necessary and sufficient optimality conditions for minimizers in the more general framework of action-minimizing curves in the space of *measure preserving plans*

$$\Gamma(D) := \{\gamma \in \mathcal{P}(D \times D) : \gamma(B \times D) = \mu_D(B) = \gamma(D \times B) \quad \forall B \in \mathcal{B}(D)\}.$$

This space can be viewed as a kind of closure of the space of *measure-preserving maps*: indeed, if $f : D \rightarrow D$ is measure-preserving, then $(\mathbf{i} \times f)_{\#} \mu_D \in \Gamma(D)$ and, conversely, if $\gamma \in \Gamma(D)$ is concentrated on the graph of f , then f is measure-preserving.

Let us describe briefly the minimization problem considered in [2], that is the natural extension of Brenier's original model in the space of measure-preserving maps. Let us denote, as in [8], by (x, a) the typical variable in $D \times D$ and, for given $\eta, \gamma \in \Gamma(D)$, let us denote by $\eta(dx, da) = \eta_a(dx) d\mu_D(a) \in \Gamma(D)$, $\gamma(dx, da) = \gamma_a(dx) d\mu_D(a) \in \Gamma(D)$ their disintegration with respect to the a variable (in a probabilistic language, η_a and γ_a can be thought as conditional probability distributions given a). For classical solutions a can be thought as the initial position (the Lagrangian variable) of the particle, and x as the actual position of it (the Eulerian variable), and this motivates the terminology *Eulerian-Lagrangian* model adopted in [8].

Now we consider the family of distributional solutions $c_{t,a}$, indexed by $a \in D$, of the continuity equation

$$\partial_t c_{t,a} + \text{div}(v_{t,a} c_{t,a}) = 0 \quad \text{in } \mathcal{D}'((0, T) \times D), \quad \text{for } \mu_D\text{-a.e. } a, \quad (3)$$

with the initial and final conditions

$$c_{0,a} = \eta_a, \quad c_{T,a} = \gamma_a, \quad \text{for } \mu_D\text{-a.e. } a. \quad (4)$$

Notice that minimization of the kinetic energy $\int_0^T \int_D \frac{1}{2} |v_{t,a}|^2 dc_{t,a} dt$ among all possible solutions of the continuity equation would give, according to [4], the optimal transport problem between η_a and γ_a (for instance, a path of Dirac masses on a geodesic connecting $g(a)$ to $h(a)$ if $\eta_a = \delta_{g(a)}$, $\gamma_a = \delta_{h(a)}$). Here, instead, by averaging with respect to a we minimize the *mean* kinetic energy

$$\int_D \int_0^T \int_D \frac{1}{2} |v_{t,a}|^2 dc_{t,a} dt d\mu_D(a)$$

with the only *global* constraint between the family $\{c_{t,a}\}$ given by the incompressibility of the flow:

$$\int_D c_{t,a} d\mu_D(a) = \mu_D \quad \forall t \in [0, T]. \quad (5)$$

It is useful to rewrite this minimization problem in terms of the the global measure $c(dt, dx, da)$ in $[0, T] \times D \times D$ and the measures $c_t(dx, da)$ in $D \times D$

$$c(dt, dx, da) := c_{t,a}(dx) dt d\mu_D(a), \quad c_t(dx, da) := c_{t,a}(dx) d\mu_D(a)$$

(from whom $c_{t,a}$ can obviously be recovered by disintegration), and the velocity field $v(t, x, a) := v_{t,a}(x)$: the action becomes

$$\int_0^T \int_{D \times D} \frac{1}{2} |v|^2(t, x, a) dc(t, x, a),$$

while (3) is easily seen to be equivalent to

$$\frac{d}{dt} \int_{D \times D} \phi(x, a) dc_t(x, a) = \int_{D \times D} \langle \nabla_x \phi(x, a), v(t, x, a) \rangle dc_t(x, a) \quad (6)$$

for all $\phi \in C_b(D \times D)$ with a bounded continuous gradient with respect to the x variable.

Thus, we can minimize the action on the class of couples measures-velocity fields (c, v) that satisfy (6) and (5), with the endpoint condition (4). We refer to [8] and [2] for general results on the existence of minimizing pairs (c, v) (it holds when, for instance, $D = [0, 1]^d$ or $D = \mathbb{T}^d$ it the flat d -dimensional torus), and describe here the two properties of minimizing pairs (c, v) that play a role in this paper:

- (a) (**Constancy of kinetic energy**) The map $t \mapsto \int |v|^2(t, x, a) dc_t$ coincides a.e. in $(0, T)$ with a constant ($2T^{-1}$ times the minimal action);
- (b) (**Weak solution to Euler's equations**) There exists a distribution p in $(0, T) \times D$ satisfying

$$\nabla p = -\partial_t \left(\int_D v(t, x, a) dc_{t,x}(a) \right) - \operatorname{div} \left(\int_D v(t, x, a) \otimes v(t, x, a) dc_{t,x}(a) \right), \quad (7)$$

in the sense of distributions.

The proof of the first one follows, as for classical action-minimizing curves, by using the minimality among all possible reparameterizations.

The second one follows by a perturbation argument based on [6], see also [2].

In this paper we refine a little bit the deep analysis made in [8] of the regularity of the gradient of the pressure field: Brenier proved that the distributions $\partial_{x_i} p$ are locally finite measures in $(0, T) \times D$, but this information is not sufficient (due to a lack of time regularity) to imply that p is a function. As shown in Corollary 3.3, a sufficient condition, that gives also $p \in L_{\text{loc}}^2((0, T); L_{\text{loc}}^{d/(d-1)}(D))$, is that

$$\partial_{x_i} p \in L_{\text{loc}}^2((0, T); \mathcal{M}_{\text{loc}}(D)), \quad i = 1, \dots, d.$$

The proof of this regularity property is the main scope of this paper. The fact that p is a function at least in some $L^1_{\text{loc}}(L^r_{\text{loc}})$ space, for some $r > 1$, plays an important role in the analysis, developed in [2], of the necessary and sufficient optimality conditions for action-minimizing curves in $\Gamma(D)$. Indeed, these conditions involve the Lagrangian

$$\mathcal{L}_p(\gamma) := \int \frac{1}{2} |\dot{\gamma}(t)|^2 - p(t, \gamma(t)) dt,$$

the (locally) minimizing curves for \mathcal{L}_p and the value function induced by \mathcal{L}_p , and none of these objects makes sense if p is only a measure in the time variable.

Throughout this paper a minimizing pair (c, v) is fixed, and we shall denote by

$$A^* := \int_0^T \int_{D \times D} \frac{1}{2} |v|^2(t, x, a) dc(t, x, a) = \frac{T}{2} \int_{D \times D} |v|^2(t, x, a) dc_t(x, a)$$

its action (the last equality follows from the property (a) stated above). To simplify our notation we just denote by \int the integration on the whole space $(0, T) \times D \times D$, whenever no ambiguity arises. We shall also assume that either D is the closure of a bounded Lipschitz domain in \mathbb{R}^d , or that $D = \mathbb{T}^d$ is the d -dimensional flat torus, and denote by $\int dx$ the integration with respect to μ_D .

2 A difference quotients estimate

In order to proceed to the proof, we recall an approximation of the pressure field obtained in [8] through a dual formulation. The arguments in [8] extend with no change to the more general model described in the introduction, where an initial and final measure-preserving plan (instead of \mathbf{i} and a measure-preserving map f) are considered.

Let us consider the Banach space $E := C^0(\tilde{Q}) \times [C^0(\tilde{Q})]^d$, where $\tilde{Q} := [0, T] \times D \times D$, and we define the convex functions $\alpha : E \rightarrow (-\infty, \infty]$ and $\beta : E \rightarrow (-\infty, \infty]$ given by

$$\alpha(F, \Phi) := \begin{cases} 0 & \text{if } F + \frac{1}{2} |\Phi|^2 \leq 0, \\ +\infty & \text{otherwise,} \end{cases}$$

$$\beta(F, \Phi) := \begin{cases} \langle c, F \rangle + \langle vc, \Phi \rangle & \text{if } F = -\partial_t \phi - p, \Phi = -\nabla_x \phi, \\ & \text{for some } \phi \in C^0(\tilde{Q}) \text{ and } p \in C^0([0, T] \times D), \\ +\infty & \text{otherwise,} \end{cases}$$

where (c, v) is the fixed minimizing pair. By the Fenchel-Rockafeller duality Theorem, Brenier proved in [8, Section 3.2] that

$$\sup_{(F, \Phi) \in E} \{-\alpha(-F, -\Phi) - \beta(F, \Phi)\} = \inf_{(\tilde{c}, \tilde{v}\tilde{c}) \in E^*} \{\alpha^*(\tilde{c}, \tilde{v}\tilde{c}) + \beta^*(\tilde{c}, \tilde{v}\tilde{c})\},$$

where α^* and β^* denote the Legendre-Fenchel transforms of α and β respectively. Writing explicitly the minimization problem appearing in the right hand side, one exactly recovers the

minimization of the action $\frac{1}{2} \int |v|^2 dc$, coupled with the endpoint and incompressibility constraints (4) and (5). Indeed

$$\alpha^*(\tilde{c}, \tilde{v}\tilde{c}) = \frac{1}{2} \langle |\tilde{v}|^2, \tilde{c} \rangle = \frac{1}{2} \int |\tilde{v}|^2 d\tilde{c},$$

and

$$\beta^*(\tilde{c}, \tilde{v}\tilde{c}) := \begin{cases} 0 & \text{if } \langle c - \tilde{c}, \partial_t \phi + p \rangle + \langle vc - \tilde{v}\tilde{c}, \nabla_x \phi \rangle = 0 \quad \forall p, \phi, \\ +\infty & \text{otherwise.} \end{cases}$$

Thus it is simple to check that $\beta^*(\tilde{c}, \tilde{v}\tilde{c}) = 0$ if and only if the two constraints (4) and (5) are satisfied.

One therefore deduces that the minimum of the action coincides with the dual problem $\sup_{(F, \Phi) \in E} \{-\alpha(-F, -\Phi) - \beta(F, \Phi)\}$, which more concretely can be written as

$$\sup_{p, \phi} \langle c, \partial_t \phi + p \rangle + \langle vc, \nabla_x \phi \rangle,$$

with

$$\partial_t \phi + \frac{1}{2} |\nabla_x \phi|^2 + p \leq 0.$$

Thus, the duality tells us that, for any $\varepsilon > 0$, there exist $p_\varepsilon(t, x)$ and $\phi_\varepsilon(t, x, a)$ satisfying

$$\partial_t \phi_\varepsilon + \frac{1}{2} |\nabla_x \phi_\varepsilon|^2 + p_\varepsilon \leq 0$$

and

$$\frac{1}{2} \langle |v|^2, c \rangle \leq \langle c, \partial_t \phi_\varepsilon + p_\varepsilon \rangle + \langle vc, \nabla_x \phi_\varepsilon \rangle + \varepsilon^2.$$

As shown in [8, Section 3.2], from this one deduces the estimate

$$\frac{1}{2} \int |v - \nabla_x \phi_\varepsilon|^2 dc \leq \varepsilon^2. \quad (8)$$

We remark that, up to adding to ϕ_ε a function of time, one can always assume $\int_D p_\varepsilon(t, x) dx = 0$ for all $t \in [0, T]$. As shown in [8, Section 3.4], the family p_ε is compact in the sense of distributions, so that there exists a cluster point p . Moreover, since any limit point p of p_ε is seen to satisfy (7) in the sense of distribution for any minimizing pair (c, v) , ∇p is uniquely determined, and this enforces the convergence of the whole family $(\nabla p_\varepsilon)_{\varepsilon > 0}$ to ∇p in the sense of distributions.

Let us now prove the following regularity result on $\nabla_x \phi_\varepsilon$: we present a proof slightly different from the one in [8].

Proposition 2.1. *Let $\tau \in (0, T)$, let $w : \overline{D} \rightarrow \mathbb{R}^d$ be a smooth divergence-free vector field parallel to ∂D and let $e^{sw}(x)$ be the measure-preserving flow in D generated by w . Then, for $\eta < \tau$ we have*

$$\int_\tau^{T-\tau} \int_{D \times D} |\nabla_x \phi_\varepsilon(t + \eta, e^{\delta w}(x), a) - \nabla_x \phi_\varepsilon(t, x, a)|^2 dc \leq L(\varepsilon^2 + \eta^2 + \delta^2), \quad (9)$$

with L depending only on τ, w, T and A^* .

Proof. In the sequel we fix a cut-off function $\zeta : [0, T] \rightarrow [0, 1]$ identically equal to 1 on $[\tau, T - \tau]$. We recall the following estimate (Proposition 3.1 in [8]), which follows by the “quasi optimality” of $(p_\varepsilon, \phi_\varepsilon)$ in the dual problem:

$$\begin{aligned} & \frac{1}{2} \int \left| (\partial_t + v^\eta \cdot \nabla_x) e^{\delta\zeta w} - \nabla_x \phi_\varepsilon \circ e^{\delta\zeta w} \right|^2 dc^\eta \\ & \leq \varepsilon^2 + \frac{1}{2} \int \left| (\partial_t + v^\eta \cdot \nabla_x) e^{\delta\zeta w} \right|^2 dc^\eta - \frac{1}{2} \int |v|^2 dc, \end{aligned} \quad (10)$$

(here $e^{\delta\zeta w}(x)$ is the flow generated by w starting from x , at time $\delta\zeta$) where (v^η, c^η) is the “reparameterization” of (v, c) given by

$$c^\eta = c^\eta(t) dt = c_{t+\eta\zeta(t)} dt, \quad v^\eta(t, x, a) = (1 + \eta\zeta'(t))v(t + \eta\zeta(t), x, a).$$

The minimality of (v, c) gives $\int |v^\eta|^2 dc^\eta \geq \int |v|^2 dc$, and the constancy of $t \mapsto \int |v|^2(t, x, a) dc_t$ gives

$$\int |v^\eta|^2 dc^\eta - \int |v|^2 dc = \int (\eta^2(\zeta')^2 + 2\eta\zeta') dc \leq C\eta^2, \quad (11)$$

with C depending on T, A^* , and ζ .

Since c is a weak solution to the incompressible Euler equations and w is divergence-free, we have

$$\int v \cdot (\partial_t + v \cdot \nabla_x)(\zeta w) dc = 0.$$

As a consequence, performing a change of variable in time, it is simple to check that

$$\int v^\eta \cdot (\partial_t + v^\eta \cdot \nabla_x)(\zeta w) dc^\eta = O(\eta). \quad (12)$$

If we now add and subtract v^η , we can rewrite (10) as

$$\begin{aligned} & \int \left| (\partial_t + v^\eta \cdot \nabla_x)(e^{\delta\zeta w}(x) - x) + (v^\eta - \nabla_x \phi_\varepsilon \circ e^{\delta\zeta w}) \right|^2 dc^\eta \\ & \leq 2\varepsilon^2 + \int \left| (\partial_t + v^\eta \cdot \nabla_x)(e^{\delta\zeta w}(x) - x) + v^\eta \right|^2 dc^\eta - \int |v|^2 dc. \end{aligned}$$

Rearranging the squares we get

$$\begin{aligned} & \int \left| v^\eta - \nabla_x \phi_\varepsilon \circ e^{\delta\zeta w} \right|^2 dc^\eta \leq -2 \int \left[(\partial_t + v^\eta \cdot \nabla_x)(e^{\delta\zeta w}(x) - x) \right] \cdot \left[v^\eta - \nabla_x \phi_\varepsilon \circ e^{\delta\zeta w} \right] dc^\eta \\ & \quad + 2\varepsilon^2 + 2 \int v^\eta \cdot (\partial_t + v^\eta \cdot \nabla_x)(e^{\delta\zeta w}(x) - x) dc^\eta \\ & \quad + \int |v^\eta|^2 dc^\eta - \int |v|^2 dc. \end{aligned}$$

Defining

$$\begin{aligned} f(\delta, \varepsilon, \eta) &:= \int \left| v^\eta - \nabla_x \phi_\varepsilon \circ e^{\delta \zeta w} \right|^2 dc^\eta = \int \left| (1+\eta \zeta') v(1+\eta \zeta, x, a) - \nabla_x \phi_\varepsilon(1+\eta \zeta, e^{\delta \zeta w}(x), a) \right|^2 dc \\ &\geq \int_\tau^{T-\tau} \int_{D \times D} \left| v - \nabla_x \phi_\varepsilon(t + \eta, e^{\delta w}(x), a) \right|^2 dc \end{aligned}$$

we see that it suffices to bound f from above. Since $e^{\delta \zeta w} x - x = \delta \zeta(t) w(x) + O(\delta^2)$ (in the C^1 norm in spacetime), by Schwartz inequality, (11) and (12) we get

$$f \leq C \sqrt{f} \delta + 2\varepsilon^2 + C(\delta \eta + \delta^2) + C\eta^2,$$

which implies $f(\delta, \varepsilon, \eta) \leq C(\delta^2 + \varepsilon^2 + \eta^2)$, with C depending on T, A^*, ζ , and w . This, together with $\int |v - \nabla_x \phi_\varepsilon|^2 dc \leq 2\varepsilon^2$, gives (9). \square

3 Proof of the main result

Theorem 3.1. *Let $\tau \in (0, T)$ and let $w : \overline{D} \rightarrow \mathbb{R}^d$ be a smooth divergence-free vector field parallel to ∂D . Then there exists a constant $C = C(w, \tau, T, A^*)$ such that*

$$|\langle \nabla p \cdot w, \zeta f \rangle| \leq C \|f\|_\infty \|\zeta\|_{L^2(0, T)} \quad \forall \zeta \in C_c^\infty((\tau, T - \tau); [0, +\infty)), f \in C_c^\infty((0, 1) \times D). \quad (13)$$

Proof. For $\zeta \in C_c^\infty(\tau, T - \tau)$ nonnegative, $\eta \in (0, \tau/2)$ and $\delta, \varepsilon > 0$ we consider the following expression:

$$\begin{aligned} I = I(\zeta, \delta, \eta, \varepsilon) &:= \int_0^T \int_D \zeta(t) \left| \int_0^1 [p_\varepsilon(t + \eta\theta, e^{\delta w}(x)) - p_\varepsilon(t + \eta\theta, x)] d\theta \right| dx dt \\ &= \int \zeta(t) \left| \int_0^1 [p_\varepsilon(t + \eta\theta, e^{\delta w}(x)) - p_\varepsilon(t + \eta\theta, x)] d\theta \right| dc(t, x, a). \end{aligned}$$

Our goal is to bound I from above. This will be achieved in the following (many) steps: $I \leq I_1 + I_2 + I_3$ and estimate of I_2, I_3 ; $I_1 \leq 2\|\zeta\|_\infty \varepsilon^2 - (I_4 + I_5 + I_6)$ and estimate of I_5 and I_6 ; $I_4 = I_7 + I_8$ and estimate of I_8 ; $I_7 = 2I_9 + I_{10}$ and estimate of I_9 ; $I_{10} = I_{11} + I_{12}$ and estimate of I_{12} ; finally $I_{11} = I_{13} + I_{14}$ and estimate of I_{13} and I_{14} . In order to avoid a cumbersome notation, during this proof we denote by C a generic constant depending only on (w, τ, T, A^*) , whose specific value can change from line to line.

We now consider $\lambda_\varepsilon(t, x, a) := -(\partial_t \phi_\varepsilon + \frac{1}{2} |\nabla_x \phi_\varepsilon|^2 + p_\varepsilon) \geq 0$, and we recall that $\int \lambda_\varepsilon dc \leq \varepsilon^2$. We have

$$I \leq I_1 + I_2 + I_3,$$

where

$$\begin{aligned}
I_1 &:= \int \zeta(t) \left| \int_0^1 [\lambda_\varepsilon(t + \eta\theta, e^{\delta w}(x), a) - \lambda_\varepsilon(t + \eta\theta, x, a)] d\theta \right| dc, \\
I_2 &:= \int \zeta(t) \left| \int_0^1 [\partial_t \phi_\varepsilon(t + \eta\theta, e^{\delta w}(x), a) - \partial_t \phi_\varepsilon(t + \eta\theta, x, a)] d\theta \right| dc, \\
I_3 &:= \int \zeta(t) \left| \int_0^1 \left[\frac{1}{2} |\nabla_x \phi_\varepsilon|^2(t + \eta\theta, e^{\delta w}(x), a) - \frac{1}{2} |\nabla_x \phi_\varepsilon|^2(t + \eta\theta, x, a) \right] d\theta \right| dc.
\end{aligned}$$

By (9) we have

$$\|\nabla_x \phi_\varepsilon(t + \eta\theta, e^{\delta w}(x), a)\|_{L^2(\zeta^2 c)} \leq \|\nabla_x \phi_\varepsilon(t, x, a)\|_{L^2(\zeta^2 c)} + \sqrt{L} \|\zeta\|_\infty (\varepsilon + \eta + \delta) \quad \forall \theta \in (0, 1).$$

Therefore writing $|A|^2 - |B|^2$ as $(A - B) \cdot (A + B)$ and using (9) once more, we can estimate

$$I_3 \leq C(\varepsilon + \eta + \delta) \left(\int \zeta^2(t) |\nabla_x \phi_\varepsilon|^2(t, x, a) dc + C \|\zeta\|_\infty^2 (\varepsilon^2 + \eta^2 + \delta^2) \right)^{1/2}. \quad (14)$$

For I_2 we first integrate with respect to θ and then use the mean value theorem to obtain

$$\begin{aligned}
I_2 &\leq \frac{\delta}{\eta} \int \zeta(t) \int_0^1 \left| [\nabla_x \phi_\varepsilon(t + \eta, e^{\sigma \delta w}(x), a) - \nabla_x \phi_\varepsilon(t, e^{\sigma \delta w}(x), a)] \cdot w(e^{\sigma \delta w}(x)) \right| d\sigma dc \\
&\leq C \frac{\delta}{\eta} \int_0^1 \int \zeta(t) \left| [\nabla_x \phi_\varepsilon(t + \eta, e^{\sigma \delta w}(x), a) - \nabla_x \phi_\varepsilon(t, e^{\sigma \delta w}(x), a)] \right| dcd\sigma \\
&\leq C \frac{\delta}{\eta} (\varepsilon + \eta + \delta) \|\zeta\|_{L^2(0, T)}.
\end{aligned} \quad (15)$$

Let us now consider I_1 : using $\lambda_\varepsilon \geq 0$ and $\int \lambda_\varepsilon dc \leq \varepsilon^2$, we obtain

$$\begin{aligned}
I_1 &\leq \int \zeta(t) \int_0^1 [\lambda_\varepsilon(t + \eta\theta, e^{\delta w}(x), a) + \lambda_\varepsilon(t + \eta\theta, x, a)] d\theta dc \\
&\leq 2\|\zeta\|_\infty \varepsilon^2 + \int \zeta(t) \int_0^1 [\lambda_\varepsilon(t + \eta\theta, e^{\delta w}(x), a) + \lambda_\varepsilon(t + \eta\theta, x, a) - 2\lambda_\varepsilon(t, x, a)] d\theta dc \\
&\leq 2\|\zeta\|_\infty \varepsilon^2 - I_4 - I_5 - I_6,
\end{aligned}$$

where

$$\begin{aligned}
I_4 &:= \int \zeta(t) \int_0^1 [\partial_t \phi_\varepsilon(t + \eta\theta, e^{\delta w}(x), a) + \partial_t \phi_\varepsilon(t + \eta\theta, x, a) - 2\partial_t \phi_\varepsilon(t, x, a)] d\theta dc, \\
I_5 &:= \frac{1}{2} \int \zeta(t) \int_0^1 [|\nabla_x \phi_\varepsilon|^2(t + \eta\theta, e^{\delta w}(x), a) + |\nabla_x \phi_\varepsilon|^2(t + \eta\theta, x, a) - 2|\nabla_x \phi_\varepsilon|^2(t, x, a)] d\theta dc, \\
I_6 &:= \int \zeta(t) \int_0^1 [p_\varepsilon(t + \eta\theta, e^{\delta w}(x)) + p_\varepsilon(t + \eta\theta, x) - 2p_\varepsilon(t, x)] d\theta dc.
\end{aligned}$$

Now we notice that

$$I_6 = 0, \quad (16)$$

since $\int c(t, x, da) = 1$ (by the incompressibility constraint (5)), $e^{\delta w}$ is measure-preserving, and $\int p_\varepsilon(t, x) dx = 0$. For I_5 , we have the same bound as for I_3 , that is

$$|I_5| \leq C(\varepsilon + \eta + \delta) \left(\int \zeta^2(t) |\nabla_x \phi_\varepsilon|^2(t, x, a) dc + C \|\zeta\|_\infty^2 (\varepsilon^2 + \eta^2 + \delta^2) \right)^{1/2}. \quad (17)$$

We continue splitting I_4 as $I_7 + I_8$, with

$$\begin{aligned} I_7 &:= \int \int_0^1 [\zeta(t) (\partial_t \phi_\varepsilon(t + \eta\theta, e^{\delta w}(x), a) + \partial_t \phi_\varepsilon(t + \eta\theta, x, a)) - 2\zeta(t - \theta\eta) \partial_t \phi_\varepsilon(t, x, a)] d\theta dc, \\ I_8 &:= 2 \int \int_0^1 [\zeta(t - \theta\eta) - \zeta(t)] \partial_t \phi_\varepsilon(t, x, a) d\theta dc. \end{aligned}$$

For I_8 , using once more that $\lambda_\varepsilon \geq 0$ and $\int \lambda_\varepsilon dc \leq \varepsilon^2$, we have the bound

$$\begin{aligned} |I_8| &\leq 2 \left| \int \int_0^1 [\zeta(t - \theta\eta) - \zeta(t)] \lambda_\varepsilon(t, x, a) d\theta dc \right| + \left| \int \int_0^1 [\zeta(t - \theta\eta) - \zeta(t)] |\nabla_x \phi_\varepsilon|^2(t, x, a) d\theta dc \right| \\ &\quad + 2 \left| \int \int_0^1 [\zeta(t - \theta\eta) - \zeta(t)] p_\varepsilon(t, x, a) d\theta dc \right| \\ &\leq 4 \|\zeta\|_\infty \varepsilon^2 + \left| \int \int_0^1 [\zeta(t - \theta\eta) - \zeta(t)] |\nabla_x \phi_\varepsilon|^2(t, x, a) d\theta dc \right| \end{aligned}$$

where in the last inequality we used that $\int p_\varepsilon dc_t = \int p_\varepsilon dx = 0$. Using also the fact that $t \mapsto \int |v|^2(t, x, a) dc_t$ does not depend on t we get

$$|I_8| \leq 4 \|\zeta\|_\infty \varepsilon^2 + 2 \|\zeta\|_\infty \int \left| |\nabla_x \phi_\varepsilon|^2(t, x, a) - |v|^2(t, x, a) \right| dc. \quad (18)$$

We now consider $I_7 = I_9 + 2I_{10}$, where

$$\begin{aligned} I_9 &:= \int \zeta(t) \int_0^1 [\partial_t \phi_\varepsilon(t + \eta\theta, e^{\delta w}(x), a) - \partial_t \phi_\varepsilon(t + \eta\theta, x, a)] d\theta dc, \\ I_{10} &:= \int \int_0^1 [\zeta(t) \partial_t \phi_\varepsilon(t + \eta\theta, x, a) - \zeta(t - \theta\eta) \partial_t \phi_\varepsilon(t, x, a)] d\theta dc. \end{aligned}$$

We have, as for I_2 ,

$$\begin{aligned} |I_9| &= \frac{1}{\eta} \left| \int \zeta(t) [(\phi_\varepsilon(t + \eta, e^{\delta w}(x), a) - \phi_\varepsilon(t + \eta, x, a)) - (\phi_\varepsilon(t, e^{\delta w}(x), a) - \phi_\varepsilon(t, x, a))] dc \right| \\ &= \frac{\delta}{\eta} \left| \int \zeta(t) \int_0^1 [\nabla_x \phi_\varepsilon(t + \eta, e^{\sigma \delta w}(x), a) - \nabla_x \phi_\varepsilon(t, e^{\sigma \delta w}(x), a)] \cdot w(e^{\sigma \delta w}(x)) d\sigma dc \right| \\ &\leq C \frac{\delta}{\eta} (\varepsilon + \eta + \delta) \|\zeta\|_{L^2(0, T)}. \end{aligned} \quad (19)$$

For I_{10} , we use the continuity equation $\partial_t c + \operatorname{div}_x(vc) = 0$ (see (6)) and add and subtract $\zeta(t)$ to get

$$\begin{aligned}
I_{10} &= \int \int_0^1 \int_0^1 \partial_t [\zeta(t - (1 - \sigma)\eta\theta) \partial_t \phi_\varepsilon(t + \eta\theta\sigma, x, a)] \eta\theta \, d\sigma d\theta dc \\
&= - \int \int_0^1 \int_0^1 \zeta(t - (1 - \sigma)\eta\theta) \partial_t \nabla_x \phi_\varepsilon(t + \eta\theta\sigma, x, a) \cdot v(t, x, a) \eta\theta \, d\sigma d\theta dc \\
&= - \int \int_0^1 \int_0^1 [\zeta(t - (1 - \sigma)\eta\theta) - \zeta(t)] \partial_t \nabla_x \phi_\varepsilon(t + \eta\theta\sigma, x, a) \cdot v(t, x, a) \eta\theta \, d\sigma d\theta dc \\
&\quad - \int \int_0^1 \int_0^1 \zeta(t) \partial_t \nabla_x \phi_\varepsilon(t + \eta\theta\sigma, x, a) \cdot v(t, x, a) \eta\theta \, d\sigma d\theta dc \\
&= - \int \int_0^1 \int_0^1 [\zeta(t - (1 - \sigma)\eta\theta) - \zeta(t)] \partial_t \nabla_x \phi_\varepsilon(t + \eta\theta\sigma, x, a) \cdot v(t, x, a) \eta\theta \, d\sigma d\theta dc \\
&\quad - \int \zeta(t) \int_0^1 [\nabla_x \phi_\varepsilon(t + \eta\theta, x, a) - \nabla_x \phi_\varepsilon(t, x, a)] \cdot v(t, x, a) \, d\theta dc \\
&=: I_{11} + I_{12}.
\end{aligned}$$

Now we see that, using (9) and the Schwarz inequality, we easily get

$$|I_{12}| \leq C(\varepsilon + \eta) \left(\int \zeta^2(t) |\nabla_x \phi_\varepsilon|^2 \, dc + C \|\zeta\|_\infty^2 (\varepsilon^2 + \eta^2) \right)^{\frac{1}{2}}. \quad (20)$$

For I_{11} , it can be written as $I_{13} + I_{14}$, where

$$\begin{aligned}
I_{13} &:= \int \int_0^1 \int_0^1 \partial_t [[\zeta(t - (1 - \sigma)\eta\theta) - \zeta(t)] \nabla_x \phi_\varepsilon(t + \eta\theta\sigma, x, a)] \cdot v(t, x, a) \eta\theta \, d\sigma d\theta dc \\
&= \int \int_0^1 [\zeta(t - \eta\theta) - \zeta(t)] \nabla_x \phi_\varepsilon(t, x, a) \cdot v(t, x, a) \, d\theta dc, \\
&= \int \int_0^1 [\zeta(t - \eta\theta) - \zeta(t)] [\nabla_x \phi_\varepsilon(t, x, a) - v(t, x, a)] \cdot v(t, x, a) \, d\theta dc \\
&\quad - \int \int_0^1 [\zeta(t - \eta\theta) - \zeta(t)] |v|^2(t, x, a) \, d\theta dc
\end{aligned}$$

and

$$I_{14} := \int \int_0^1 \int_0^1 [\zeta'(t - (1 - \sigma)\eta\theta) - \zeta'(t)] \nabla_x \phi_\varepsilon(t + \eta\theta\sigma, x, a) \cdot v(t, x, a) \eta\theta \, d\sigma d\theta dc.$$

Recalling that $t \mapsto \int |v|^2(t, x, a) \, dc_t$ is constant, by (8) we have

$$|I_{13}| \leq \left| \int \int_0^1 [\zeta(t - \eta\theta) - \zeta(t)] (\nabla_x \phi_\varepsilon(t, x, a) - v(t, x, a)) \cdot v(t, x, a) \, d\theta dc \right| \leq C \|\zeta\|_\infty \varepsilon. \quad (21)$$

Finally, by (9) we can bound I_{14} with

$$\begin{aligned} |I_{14}| &\leq \|\zeta''\|_\infty \eta^2 \int_{\tau/2}^{T-\tau/2} \int_{D \times A} \int_0^1 \int_0^1 |\nabla_x \phi_\varepsilon(t + \eta\theta\sigma, x, a) \cdot v(t, x, a)| d\sigma d\theta dc \\ &\leq \|\zeta''\|_\infty \eta^2 C (\|\nabla_x \phi_\varepsilon\|_2 + C(\varepsilon + \eta)). \end{aligned} \quad (22)$$

Collecting (14), (15), (16), (17), (19), (20), (21), (22) we can bound from above I as follows:

$$\begin{aligned} &C(\varepsilon + \eta + \delta) \left(\int \zeta^2(t) |\nabla_x \phi_\varepsilon|^2 dc + C\|\zeta\|_\infty^2 (\varepsilon^2 + \eta^2 + \delta^2) \right)^{\frac{1}{2}} \\ &+ I_8 + C \frac{\delta}{\eta} (\varepsilon + \eta + \delta) \|\zeta\|_{L^2(0,T)} + \|\zeta''\|_\infty \eta^2 C (\|\nabla_x \phi_\varepsilon\|_2 + C(\varepsilon + \eta)) + 2\|\zeta\|_\infty \varepsilon^2 + C\|\zeta\|_\infty \varepsilon. \end{aligned}$$

Now, recalling the definition of I , we integrate $p_\varepsilon \zeta$ against a function $f \in C_c^\infty((0, T) \times D)$ and pass to the limit as $\varepsilon \rightarrow 0$, with $\eta = \delta$ frozen, to obtain

$$\frac{1}{\delta} \left| \int_0^1 \langle q, \zeta(t) [f(t - \delta\theta, e^{-\delta w}(x)) - f(t - \delta\theta, x)] d\theta \right| \leq C \|f\|_\infty (\|\zeta\|_{L^2(0,T)} + \delta \|\zeta''\|_\infty + \delta \|\zeta\|_\infty)$$

for any limit point q of p_ε in the sense of distributions, thanks to the fact that, by (18), $I_8 \rightarrow 0$ as $\varepsilon \rightarrow 0$ (here we use again that $t \mapsto \int |v|^2(t, x, a) dc_t$ is constant). So, letting $\delta \rightarrow 0$, we finally obtain (13), with ∇q in place of ∇p . But $\nabla p_\varepsilon \rightarrow \nabla p$ implies that $\nabla p = \nabla q$ and concludes the proof. \square

Remark 3.2. In the case $D = \mathbb{T}^d$ one can also consider constant vector fields w and therefore (13) holds in a stronger (and simpler) form:

$$|\langle \partial_{x_i} p, \zeta f \rangle| \leq C \|f\|_\infty \|\zeta\|_{L^2(0,T)} \quad \forall \zeta \in C_c^\infty((\tau, T - \tau); [0, +\infty)), f \in C^\infty([0, 1] \times \mathbb{T}^d) \quad (23)$$

with C depending only on τ , T and A^* .

A simple localization and smoothing argument based on (13) gives that the pressure field is locally (globally, in the case $D = \mathbb{T}^d$) induced by a function.

Corollary 3.3. Let $d \geq 2$. Then for all smooth subdomains $D' \subset\subset D$ there exists

$$q \in L_{\text{loc}}^2((0, T); BV(D')) \subset L_{\text{loc}}^2((0, T); L^{1^*}(D'))$$

(here $1^* = d/(d-1)$) with $\nabla q = \nabla p$ in the sense of distributions in $(0, T) \times D'$. In the case $D = \mathbb{T}^d$ the same statement holds globally in space, i.e. with $D' = D$. Moreover, in this case the result holds also for $d = 1$ (with $1^* = \infty$).

Proof. We first notice that for $d \geq 2$ any constant vector field \bar{w} in D' can be extended to a divergence-free, smooth and compactly supported vector field in D : indeed, if $D' \subset\subset D_1 \subset\subset$

$D_2 \subset\subset D$, with D_1 and D_2 smooth, we may set $\hat{w} = \bar{w}$ in D_1 , $\hat{w} = 0$ in $D \setminus D_2$, and $\hat{w} = \nabla\psi$ in $D_2 \setminus \bar{D}_1$, where ψ is a solution of

$$\begin{cases} \Delta\psi = 0 & \text{in } D_2 \setminus \bar{D}_1, \\ \frac{\partial\psi}{\partial\nu} = 0 & \text{on } \partial D_2, \\ \frac{\partial\psi}{\partial\nu} = \bar{w} \cdot \nu & \text{on } \partial D_1, \end{cases}$$

(existence of ψ can be obtained by minimizing $\frac{1}{2} \int_{D_2 \setminus \bar{D}_1} |\nabla\phi|^2 - \int_{\partial D_1} \phi \bar{w} \cdot \nu$ in $H^1(D_2 \setminus \bar{D}_1)$). By construction \hat{w} is divergence-free (in the sense of distributions) in D , compactly supported and coincides with \bar{w} in a neighbourhood of \bar{D}' , so that a suitable mollification of \hat{w} provides the required extension.

Thanks to this remark, (13) yields

$$|\langle \partial_{x_i} p, \zeta f \rangle| \leq L \|f\|_\infty \|\zeta\|_{L^2(0,T)} \quad \forall \zeta \in C_c^\infty((\tau, T - \tau); [0, +\infty)), \quad f \in C_c^\infty((0, 1) \times D'), \quad (24)$$

with L depending only on τ , T , D' and A^* . If we denote by q_ε the mollified functions of p , this easily implies that $|\nabla q_\varepsilon|$ is uniformly bounded in $L^2_{\text{loc}}((0, T); L^1(D'))$. In particular, if we denote by \bar{q}_ε the mean value of q_ε on D' , $q_\varepsilon - \bar{q}_\varepsilon$ is uniformly bounded in the space $L^2_{\text{loc}}((0, T); L^{1^*}(D'))$, and if q is any weak limit point (in the duality with $L^2_{\text{loc}}((0, T); L^d(D'))$) we easily get $\nabla q = \nabla p$ and $q \in L^2_{\text{loc}}((0, T); BV(D'))$.

In the case $D = \mathbb{T}^d$ the proof is analogous: it suffices to apply Remark 3.2. \square

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