ON THE SHAPE OF LIQUID DROPS AND CRYSTALS IN THE SMALL MASS REGIME

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Abstract. We consider liquid drops or crystals lying in equilibrium under the action of a potential energy. For small masses, the proximity of the resulting minimizers from the Wulff shape associated to the surface tension is quantitatively controlled in terms of the smallness of the mass and with respect to the natural notions of distance induced by the regularity of the Wulff shape. Stronger results are proved in the two-dimensional case. For instance, it is shown that a planar crystal undergoing the action of a small exterior force field remains convex, and admits only small translations parallel to its faces.

Contents

1. Introduction

1.1. **The variational problem.** Let us consider a liquid drop or a crystal of mass m subject to the action of a potential. At equilibrium, its shape minimizes (under a volume constraint) the free energy, that consists of a (possibly anisotropic) interfacial surface energy plus a bulk potential energy induced by an external force field [11, 27]. Therefore one is naturally led to consider the variational problem

$$\inf\Big\{\mathcal{E}(E) = \mathcal{F}(E) + \mathcal{G}(E) : |E| = m\Big\}\,. \tag{1.1}$$

Here, for $E \subset \mathbb{R}^n$ ($n \geq 2$), $|E|$ denotes the Lebesgue measure of E, while $\mathcal{F}(E)$ and $\mathcal{G}(E)$ are, respectively, the surface energy and the potential energy of E, that are introduced as follows.

Surface energy: We are given a *surface tension*, that is a convex, positively 1-homogeneous function $f : \mathbb{R}^n \to [0, +\infty)$. Correspondingly we define the surface energy of a set of finite perimeter $E \subset \mathbb{R}^n$ as

$$\mathcal{F}(E) = \int_{\partial^* E} f(\nu_E)\, d\mathcal{H}^{n-1}\,, \tag{1.2}$$

where ν_E is the measure theoretic outer unit normal to E and $\partial^* E$ is its reduced boundary (see section 2.1). The minimization of $\mathcal{F}$ under a volume constraint is described in section 1.2.

Potential energy: The *potential* is a locally bounded Borel function $g : \mathbb{R}^n \to [0, +\infty)$ that is coercive on $\mathbb{R}^n$, i.e., we have

$$g(x) \to +\infty \quad \text{as } |x| \to +\infty\,. \tag{1.3}$$

We also assume that

$$\inf_{\mathbb{R}^n} g = g(0) = 0\,. \tag{1.4}$$

This is done without loss of generality in the study of (1.1), as it amounts to subtract to the free energy a suitable constant and to translate the origin in the system of coordinates. The potential energy of $E \subset \mathbb{R}^n$ is then defined as

$$\mathcal{G}(E) = \int_E g(x)\, dx\,. \tag{1.5}$$

Actually one could also allow g to take the value $+\infty$ in order to include a confinement constraint, since (whenever possible) a minimizer will always avoid the region $\{g = +\infty\}$. Observe that, when g is differentiable on the (open) set $\{g < +\infty\}$, then the energy term $\mathcal{G}(E)$ corresponds to the presence of the force field $-\nabla g$ acting on E.

In this paper we are concerned with the geometric properties of the minimizers for the variational problem (1.1), especially in the small mass regime. Our main results are stated in Theorem 1.1 and Theorem 1.3. We remark that the coercivity assumption (1.3) excludes from our analysis the gravitational case $g(x, y, z) = z$ in $\mathbb{R}^3 \cap \{z \geq 0\}$. We use this assumption just to trivialize some side issues, such as the existence of minimizers. It is very likely that our methods could be adapted to also handle the gravitational case, and other special cases of interest, by exploiting their particular structure.

1.2. **Geometric properties of Wulff shapes.** In the absence of the potential term (g constant), volume-constrained minimizers of the surface energy are obtained by translation and scaling of the open, bounded convex set K known as the *Wulff shape* of f. The set K is explicitly given by the formula

$$K = \bigcap_{\nu \in S^{n-1}} \{x \in \mathbb{R}^n : (x \cdot \nu) < f(\nu)\} = \{x \in \mathbb{R}^n : f_*(x) < 1\}\,, \tag{1.6}$$

where we have introduced $f_* : \mathbb{R}^n \to [0, +\infty)$, defined as

$$f_*(x) = \sup\{x \cdot y : f(y) = 1\}\,, \qquad x \in \mathbb{R}^n\,. \tag{1.7}$$

The minimality property of K is equivalently expressed by the *Wulff inequality*

$$\mathcal{F}(E) \geq n|K|^{1/n}|E|^{(n-1)/n}\,, \tag{1.8}$$

where equality holds if and only if $E = x + K_r$ for some $x \in \mathbb{R}^n$. (Here and in the sequel, we use the notation $K_r = rK$.) Indeed, the right hand side of (1.8) is equal to $\mathcal{F}(K_r) = r^{n-1}\mathcal{F}(K)$, where $r = (|E|/|K|)^{1/n}$ and $\mathcal{F}(K) = n|K|$. It is useful to note that *every* open, bounded convex set K containing the origin is, in fact, the Wulff shape for some surface energy $\mathcal{F} = \mathcal{F}_K$ corresponding to the surface tension $f = f_K$ defined as

$$f(\nu) = \sup\{\nu \cdot x : x \in K\}\,.$$

The geometric properties of a Wulff shape are closely related to the analytic properties of the corresponding surface tension. Two relevant (and somehow complementary) situations are the following ones:

Uniformly elliptic case: The surface tension f is λ-*elliptic*, $\lambda > 0$, if $f \in C^2(\mathbb{R}^n \setminus \{0\})$ and

$$(\nabla^2 f(v)\tau) \cdot \tau \geq \frac{\lambda}{|v|}\left|\tau - \left(\tau \cdot \frac{v}{|v|}\right)\frac{v}{|v|}\right|^2, \tag{1.9}$$

whenever $v, \tau \in \mathbb{R}^n$, $v \neq 0$. Under these hypotheses the boundary of the Wulff shape K is of class C^2 and uniformly convex (see, for instance, [39, page 111]). Moreover, the second fundamental form $\nabla \nu_K$ of K satisfies the identity

$$\nabla^2 f(\nu_K(x))\nabla\nu_K(x) = \mathrm{Id}_{T_x\partial K}\,, \qquad \forall\, x \in \partial K\,. \tag{1.10}$$

(Notice that this makes sense as $\nabla\nu_K(x)\nu_K(x) = 0$ and $\nu_K(x) \cdot (\nabla^2 f(\nu_K(x))v) = 0$ for every $x \in \partial K$ and $v \in \mathbb{R}^n$.) This situation includes of course the *isotropic* case $f(\nu) = \lambda|\nu|$ ($\lambda > 0$). Evidently, in the isotropic case the Wulff shape is the Euclidean ball B_λ and the Wulff inequality reduces to the Euclidean isoperimetric inequality. Isotropic (or smooth, nearly isotropic) surface energies are used to model liquid drops. Moreover, minima of the functional (1.1) with $f(\nu) = \lambda|\nu|$ appear also in phase transition problems, where the mean curvature of the interface is related to the pressure or the temperature on it, represented by g (this is the so-called Gibbs-Thompson relation).

Crystalline case: A surface tension f is *crystalline* if it is the maximum of finitely many linear functions, i.e., if there exists a finite set $\{x_j\}_{j=1}^N \subset \mathbb{R}^n \setminus \{0\}$, $N \in \mathbb{N}$, such that

$$f(\nu) = \max_{1 \leq j \leq N}(x_j \cdot \nu)\,, \qquad \forall\, \nu \in S^{n-1}\,. \tag{1.11}$$

The corresponding Wulff shape is a convex polyhedron. These are the surface tensions used in studying crystals [48].

1.3. **Geometric properties of minimizers.** In the presence of the potential term, the geometric properties of minimizers are much less understood. A noticeable exception to this claim is the case of sessile/pendant (or otherwise constrained) liquid drops under the action of gravity. This situation, that has been extensively considered in the literature (see, for instance, [51, 20, 26]), falls into the variational problem (1.1) for the choice of an isotropic surface tension energy interacting with a potential g of the form

$$g(x) = x_n \quad \text{if } g(x) < +\infty\,.$$

However, if we look to (1.1) in its full generality, then the validity of various natural properties of minimizers is at present unknown. In particular, the following two questions were raised by Almgren. The first question is mentioned in [32], while the second one was communicated to us by Morgan [34].

(Q1) If the potential g is convex (or, more generally, if the sub-level sets $\{g < t\}$ are convex), are minimizers convex or, at least, connected?

(Q2) If the surface energy dominates over the potential energy (e.g., if the potential g is almost constant or if the mass m is sufficiently small), to which extent are minimizers "close" to Wulff shapes?

Let us point out that a question similar to (Q1) was raised by Almgren and Taylor, asking whether a crystal lying on a table under gravity is necessarily convex (see [43, Question 8] and also [10, Problem 8.4]).

About the first question, in [6] the authors prove convexity of minimizers in the two-dimensional case for drops/crystals lying above a table under the action of the gravitational potential, while in [49] convexity is used as an assumption for proving (under additional suitable assumptions) facetting of a minimizing crystal. In [8, 12], by a "level-sets method" approach combined with convexity results for solutions to elliptic PDEs, the convexity of minimizers is proved for general convex potentials in the large mass regime. Finally, in the general planar case, it is shown in [32] that every minimizer is the union of finitely many connected components lying at mutually positive distance, all having different masses, and each component being convex and minimizing the free energy among convex sets with its same volume.

With this paper, we mainly aim to stimulate the investigation of the second question, providing some optimal results, both in the planar case and in general dimension (see, however, Theorem 4.5, Theorem 4.11 and Appendix B for some results that are not related to the small mass regime). Our estimates are quantitative, in the sense that we shall present explicit bounds on the proximity to a Wulff shape in terms of the small mass m. Moreover, the value of the "critical" mass below, which our estimates hold can be made completely explicit from our arguments (though there will be no attempt to find such a explicit expression). Our first main result establishes the connectedness and the uniform proximity of minimizers to Wulff shapes below a critical mass. This is done for very general surface and potential energies.

Theorem 1.1. *There exist positive constants $m_c = m_c(n, f, g)$ and $C = C(n, f, g)$ with the following property: If E is a minimizer in the variational problem* (1.1) *with mass $|E| = m \le m_c$, then E is connected and uniformly close to a Wulff shape, i.e., there exist $x_0 \in \mathbb{R}^n$ and $r_0 > 0$, with*

$$r_0 \le C\, m^{1/n^2}\,,$$

such that

$$x_0 + K_{s(m)(1-r_0)} \subset E \subset x_0 + K_{s(m)(1+r_0)}\,,$$

where we have set

$$s(m) = \left(\frac{m}{|K|}\right)^{1/n}\,.$$

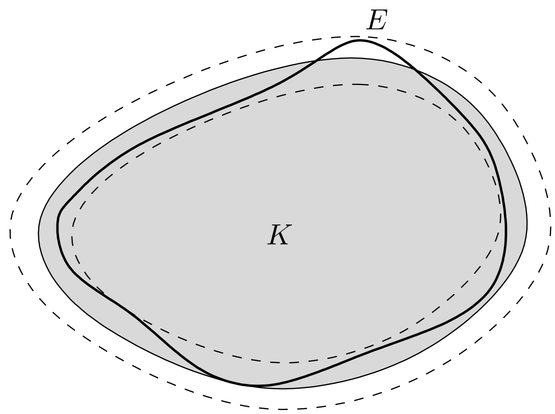

FIGURE 1. In the small mass regime minimizers are connected and uniformly close to a (properly rescaled and translated) Wulff shape, in terms of the smallness of the mass. The convexity of these minimizers remains conjectural, with the exception of the planar case $n = 2$ and of the λ-elliptic case (in general dimension, see Theorem 1.3).

If $n = 2$ then E is a convex set. Moreover, if f is crystalline (or, equivalently, if the Wulff shape K is a convex polygon), then E is a convex polygon with sides parallel to that of K.

Remark 1.2. The above theorem shows that in the planar crystalline case minimizers possess a particularly rigid structure. Although our proof cannot be generalized to higher dimension, this result raises the interesting question whether or not an analogous property should hold in higher dimension (or at least in the physical case $n = 3$). The analogous result in higher dimension should say that, if f is crystalline, then minimizers with sufficiently small mass are polyhedra with sides parallel to that of K. This would show that a minimizer E can be obtained by K by slightly translating the faces of ∂K, and in particular the minimization problem (1.1) would reduce to a finite dimensional problem (the dimension being equal to the number of faces of K).

The main question left open by Theorem 1.1 concerns the convexity of minimizers at small mass in dimension $n \geq 3$. We address this problem in the case of smooth λ-elliptic surface tensions and of potentials of class C^1. In this situation the Wulff shape turns out to be a uniformly convex set with smooth boundary. Correspondingly we prove that minimizers at small mass are not merely convex, but that they are in fact uniformly convex sets with smooth boundary and with principal curvatures uniformly close to that of a (properly rescaled) Wulff shape. To express this last property we shall make use of the second order characterization (1.10) of Wulff shapes.

Theorem 1.3. *If $g \in C^1_{\mathrm{loc}}(\mathbb{R}^n)$, $f \in C^{2,\alpha}(\mathbb{R}^n \setminus \{0\})$ for some $\alpha \in (0,1)$, and f is λ-elliptic, then there exist a critical mass $m_0 = m_0(n, g, f)$ and a constant $C = C(n, g, f, \alpha)$ with the following property: If E is a minimizer in (1.1) with $|E| = m \leq m_0$ and if we set*

$$F = \left(\frac{|K|}{m}\right)^{1/n} E\,,$$

then ∂F is of class $C^{2,\alpha}$ and

$$\max_{\partial F} |\nabla^2 f(\nu_F)\nabla\nu_F - \mathrm{Id}_{T_x\partial F}| \le C\, m^{2\alpha/(n+2\alpha)}\,. \tag{1.12}$$

In particular, if m is small enough (the smallness depending on n, f, and g only) then F (and so E) is a convex set.

Remark 1.4. If $g \in C^{1,\beta}_{loc}(\mathbb{R}^n)$ and $f \in C^{3,\beta}(\mathbb{R}^n \setminus \{0\})$ for some $\beta > 0$, then the conclusion of Theorem 1.3 can be strengthened to

$$\max_{\partial F} |\nabla^2 f(\nu_F)\nabla\nu_F - \mathrm{Id}_{T_x\partial F}| \le C\, m^{2/(n+2)}\,,$$

which corresponds to (1.12) with $\alpha = 1$.

Remark 1.5. At a first sight, Theorem 1.3 could be seen as a slight generalization of the fact that small liquid drops lying on a table are asymptotically spherical as volume tends to zero [45], or that in a Riemannian manifold, isoperimetric regions of small volume are smoothly close to being round balls (this fact was first proved by Kleiner as explained in [50], see also [29, 38, 35]). However, our result differs from other results of this kind in the fact of being “quantitative”. Indeed, once uniform $C^{2,\alpha}$-bounds for minimizers at small masses are established (see Theorem 4.6), one usually deduces their convexity by a compactness argument (indeed, as $m \to 0$, minimizers converge to K in the C^2-topology, and the uniform convexity of K entails their convexity). In order to prove Theorem 1.3 one needs a different approach, as the use of compactness arguments rules out the possibility of finding explicit rates of convergence in terms of $m \to 0$. Observe that also the constant C appearing in (1.12) is obtained by a constructive method, and so it is a priori computable.

Remark 1.6. Theorems 1.1 and 1.3 deal with the connectedness and convexity properties of liquid drops and crystals in the small mass regime. Outside this special regime, one

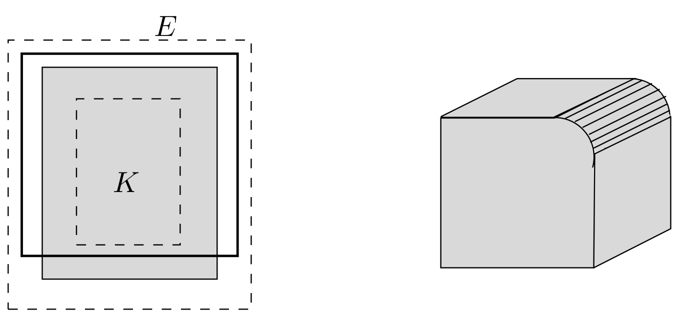

FIGURE 2. In the planar crystalline case, minimizers are convex polygons, with sides parallel to the polygonal Wulff shape associated with the crystalline surface tension (picture on the left). The argument used in the proof of this result, when repeated in three dimensions, seems not sufficient to draw the analogous conclusions. For example, in the case of a cubic crystal, the two dimensional argument used in the proof of Theorem 1.1 allows to exclude that a cube with a rounded vertex is a minimizer, but it is not sufficient to exclude a cube with a rounded edge (see the picture on the right).

expects convexity of minimizers provided g is convex (see question (Q1)). As already mentioned, this was proved in [8, 12] when the mass is large enough. The natural problem of how to fill the gap in between these two results is open. It seems very likely that new ideas are needed to deal with this case.

1.4. **Organization of the paper.** In section 2 we recall some basic definitions about sets of finite perimeter, and collect some useful facts concerning surface energies and volume constrained variations.

In section 3 we introduce and study the class of (ε, R)-minimizers of the surface energy $\mathcal{F}$. Given $\varepsilon, R > 0$, a set of finite perimeter $E \subset \mathbb{R}^n$ is a (volume constrained) (ε, R)-minimizer of $\mathcal{F}$ provided

$$\mathcal{F}(E) \le \mathcal{F}(F) + \varepsilon\,|K|^{1/n}|E|^{(n-1)/n}\frac{|E\Delta F|}{|E|}\,, \tag{1.13}$$

for every set of finite perimeter $F \subset \mathbb{R}^n$ with

$$|F| = |E| \quad \text{and} \quad F \subset I_R(E)\,,$$

where $I_R(E)$ is the R-neighborhood of E with respect to K, i.e.,

$$I_R(E) = \{x \in \mathbb{R}^n : \mathrm{dist}_K(x, E) < R\}\,, \qquad \mathrm{dist}_K(x, E) = \inf_{y\in E} f_*(x - y)\,. \tag{1.14}$$

(In the above definition, neighborhoods are defined in terms of f_* only to deduce cleaner estimates.) After discussing the basic regularity properties of (ε, R)-minimizers of $\mathcal{F}$ (Theorem 3.1), we focus on the geometric properties characteristic to the small ε regime. The L^1-proximity (in terms of the smallness of ε) of every $(\varepsilon, n+1)$-minimizer to a properly rescaled and translated Wulff shape is an almost direct consequence of the main result in [19] (Theorem 3.2 and Lemma 3.3). In Theorem 3.4 and Corollary 3.5 we pave the way to the proof of Theorem 1.1 by proving that, in fact, $(\varepsilon, n + 1)$-minimizers are connected and uniformly close to Wulff shapes. This result may appear to the specialists as a classical application of standard density estimates combined with the above mentioned L^1-estimate. However, at least to our knowledge, there are no universal density estimates available for (ε, R)-minimizers, i.e., density estimates independent of the minimizer. This follows from the fact that, if E is a (ε, R)-minimizer, the class of competitors has to satisfy the constraint $|F| = |E|$, and so we are forced to make a mass adjustments which introduces a dependence on E (see Lemma 2.3 and Theorem 3.1). For this reason the proof of Theorem 3.4, although it follows the lines of many other proofs of the same kind, presents some subtle points. This careful approach allows us to show that the uniform proximity result of Corollary 3.5 holds for every $(\varepsilon, n+1)$-minimizer with $\varepsilon \le \varepsilon(n)$, where $\varepsilon(n)$ depends on the dimension n only, and not on f. In section 3.3 we focus on the planar case $n = 2$, and show that $(\varepsilon, 3)$-minimizers are convex (Theorem 3.6), and that, in the crystalline case, they are convex polygons (Theorem 3.7) provided $\varepsilon \le \varepsilon_0$, where ε_0 is a universal constant independent of f. As a preparatory step towards the proof of Theorem 1.3, in section 3.4 (see also appendix C) we consider λ-elliptic surface tensions and apply the regularity theory for almost minimizing rectifiable currents to show that the boundaries of $(\varepsilon, n+1)$-minimizers of $\mathcal{F}$ satisfy uniform $C^{1,\alpha}$-estimates for every $\alpha \in (0,1)$ (Theorem 3.8).

In section 4 we prove our main results on optimal shapes in the variational problem (1.1). The first step consists in showing that optimal shapes for (1.1) are uniformly bounded in terms of their mass, the dimension n and the way g grows at infinity. For small masses, this boundedness result is true for any potential g (Theorem 4.2). Although it is not needed for the proof of our main results, we thought it conceptually important to provide these bounds for arbitrary masses. We do this in Theorem 4.5, assuming that g is locally Lipschitz. As a by-product of the uniform boundedness result in Theorem 4.2, we immediately see that a minimizer E for (1.1) with $|E| = m$ is also an $(\varepsilon, n+1)$-minimizer for $\varepsilon \le C\, m^{1/n}$, where C is an (explicitly computable) constant depending on n, f, and g only. This fact allows us to apply the results of section 3, to deduce Theorem 1.1 as a corollary. Moreover, when f is λ-elliptic, $f \in C^{2,\alpha}(\mathbb{R}^n \setminus \{0\})$ and $g \in C^{0,\alpha}_{\rm loc}(\mathbb{R}^n)$ for some $\alpha \in (0,1)$, then the regularity Theorem 3.8 for $(\varepsilon, n+1)$-minimizers can be combined with the first variation formula for the free energy and with elliptic regularity theory to prove that the corresponding minimizers satisfy uniform $C^{2,\alpha}$-estimates (Theorem 4.6). Hence, in section 4.3 we apply the second variation formula for the free energy to a suitable normal vector field to show that the second fundamental form of the boundary of a minimizer E is, up to a dilation taking E into $F = (|K|/m)^{1/n} E$, L^2-close to the second fundamental form of ∂K (Theorem 4.9). A simple interpolation between $C^{0,\alpha}$ and L^2 allows combining Theorem 4.9 with the estimates from Theorem 4.6 to show the uniform proximity of the second fundamental form of ∂F to that of ∂K, thus proving Theorem 1.3. Finally, in section 4.4 we use a variant of the argument in section 3.3 to show that, even outside the small mass regime, planar crystals have a remarkably rigid structure. More precisely, if f is crystalline and g is continuous then the boundary of a planar, crystalline minimizer consists of two pieces, one which is included in some level set $\{g = \ell\}$ and the other one which is polygonal, with normal directions chosen among the normal directions to ∂K.

In appendix A we finally gather the first and second variation formulas of the free energy, together with a brief description of an useful bootstrap argument. In appendix B we make a first (small) step towards a positive answer to the convexity question (Q1), by showing that minimizers in (1.1) corresponding to potentials with convex level sets have non-negative anisotropic mean curvature (in fact, a stronger global condition is proven to hold true). Finally, appendix C reviews the regularity theory for almost minimizing currents, and shows how these kinds of results apply to our setting, to prove uniform $C^{1,\alpha}$-regularity for $(\varepsilon, n+1)$-minimizers.

2. Sets of finite perimeter and volume-constrained variations

2.1. **Sets of finite perimeter.** In this section we recall some basic definitions and properties on sets of finite perimeter. We refer to [4] for an extensive introduction to the subject and for a proof to all the properties stated below. A Borel set $E \subset \mathbb{R}^n$ is a *set of* *finite perimeter* in $\mathbb{R}^n$ provided

$$\sup\left\{\int_E \operatorname{div} T(x)\,dx : T \in C^1_c(\mathbb{R}^n;B)\right\} < +\infty\,,$$

where $B = B_1$ denotes the Euclidean unit ball. If this is the case, the distributional gradient $D1_E$ of the characteristic function 1_E of E defines a Radon measure on $\mathbb{R}^n$, with values in $\mathbb{R}^n$, such that the distributional divergence theorem

$$\int_E \operatorname{div} T(x)\,dx = -\int_{\mathbb{R}^n} T\cdot d\,D1_E\,, \qquad \forall\, T \in C^1_c(\mathbb{R}^n;\mathbb{R}^n)\,, \tag{2.1}$$

holds true. The total variation of $D1_E$ is then used to define the perimeter of E relative to a set $A \subset \mathbb{R}^n$ on setting

$$P(E;A) = |D1_E|(A)\,, \qquad P(E) = |D1_E|(\mathbb{R}^n)\,.$$

If A is open, an equivalent definition for $P(E;A)$, which turns out to be very useful when proving lower semicontinuity result, is also given by

$$P(E;A) = \sup\left\{\int_E \operatorname{div} T(x)\,dx : T \in C^1_c(A;B)\right\}\,. \tag{2.2}$$

If E is a bounded open set with C^1 boundary, then E is a set of finite perimeter in $\mathbb{R}^n$ and $D1_E = -\nu_E\,\mathcal{H}^{n-1}\llcorner\partial E$, where ν_E is the outer unit normal to E and $\mathcal{H}^{n-1}$ denotes the $(n-1)$-dimensional Hausdorff measure. In particular,

$$P(E;A) = \mathcal{H}^{n-1}(A\cap\partial E)\,, \qquad P(E) = \mathcal{H}^{n-1}(\partial E)\,,$$

and (2.1) amounts to the classical divergence theorem. Turning back to generic sets of finite perimeter, one see that up to modifying E on a set of measure zero (an operation that leaves $D1_E$ unchanged) it can always be assumed that

$$\operatorname{spt}(D1_E) = \partial E\,,$$

[25, Proposition 3.1]. The reduced boundary $\partial^* E$ of E is then defined as the set of those $x \in \partial E$ such that the limit

$$\nu_E(x) = \lim_{r\to 0^+}\frac{D1_E(B(x,r))}{|D1_E|(B(x,r))}$$

exists and belongs to S^{n-1}. It turns out that $\partial^* E$ is a countably $\mathcal{H}^{n-1}$-rectifiable set in $\mathbb{R}^n$ and that

$$D1_E = -\nu_E\,\mathcal{H}^{n-1}\llcorner\partial^* E\,.$$

In particular, if $A \subset \mathbb{R}^n$

$$P(E;A) = \mathcal{H}^{n-1}(A\cap\partial^* E)\,, \qquad P(E) = \mathcal{H}^{n-1}(\partial^* E)\,,$$

and the distributional divergence theorem (2.1) takes the more appealing form

$$\int_E \operatorname{div} T(x)\,dx = \int_{\partial^* E} T\cdot\nu_E\,d\,\mathcal{H}^{n-1}\,, \qquad \forall\, T \in C^1_c(\mathbb{R}^n;\mathbb{R}^n)\,. \tag{2.3}$$

We also recall the basic lower semicontinuity and approximation results for sets of finite perimeter. Let us say that $E_h \to E$ (resp., $E_h \stackrel{\rm loc}{\to} E$) if $1_{E_h} \to 1_E$ in $L^1(\mathbb{R}^n)$ (resp., $L^1_{\rm loc}(\mathbb{R}^n)$). If $A \subset \mathbb{R}^n$ is open and $E_h \stackrel{\rm loc}{\to} E$ then we have

$$P(E;A) \le \liminf_{h\to\infty} P(E_h;A)\,.$$

Moreover, given a set of finite perimeter E there always exists a sequence $\{E_h\}_{h\in\mathbb{N}}$ of open sets with smooth boundaries such that $E_h \stackrel{\rm loc}{\to} E$ and $|D1_{E_h}| \stackrel{*}{\rightharpoonup} |D1_E|$.

All the relevant properties of sets of finite perimeter are left invariant by modifications on sets of (Lebesgue) measure zero. The proper notion of connectedness in this framework is then introduced as follows: a set of finite perimeter E (with finite measure) is said *indecomposable* if $E = E_1 \cup E_2$, $P(E) = P(E_1) + P(E_2)$, and $|E| = |E_1| + |E_2|$ imply $|E_1|\,|E_2| = 0$. As a reference for indecomposable set we refer to [5].

2.2. **Basic properties of the surface energy.** We now gather some basic properties of the surface energy that will be useful in the sequel. Given $A \subset \mathbb{R}^n$ we shall define the surface energy of the set of finite perimeter $E \subset \mathbb{R}^n$ relative to A as

$$\mathcal{F}(E;A) = \int_{A\cap\partial^* E} f(\nu_E)\, d\mathcal{H}^{n-1}\,,$$

where, of course, $\mathcal{F}(E;\mathbb{R}^n) = \mathcal{F}(E)$. From the classical Reshetnyak theorems (see [4, Theorems 2.38-2.39], or [44] for a simpler proof) and from our basic assumptions on the surface tension f, we deduce the following lemma about the behavior of the surface energy under local convergence of sets. (The first part of the following result can be easily proven using a suitable duality formula for $\mathcal{F}$ as in (2.2).)

Lemma 2.1. *If $A \subset \mathbb{R}^n$ is open and $E_h \stackrel{\rm loc}{\to} E$ then*

$$\mathcal{F}(E;A) \le \liminf_{h\to\infty} \mathcal{F}(E_h;A)\,.$$

If, moreover, $P(E_h) \to P(E)$, then

$$\mathcal{F}(E;A) = \lim_{h\to\infty} \mathcal{F}(E_h;A)\,.$$

In proving estimates involving the surface tension f we are going to make frequent use of the quantities $0 < \alpha_1 \le \alpha_2 < +\infty$ defined as

$$\alpha_1 = \min_{S^{n-1}} f\,, \qquad \alpha_2 = \max_{S^{n-1}} f\,. \tag{2.4}$$

In particular,

$$\alpha_1 \le \|\nabla f\|_{L^\infty(\mathbb{R}^n;\mathbb{R}^n)} \le \alpha_2\,. \tag{2.5}$$

Indeed, by the positive 1-homogeneity and by the convexity of f we immediately see that f is sub-additive, so that

$$f(x+t\,v) - f(x) \le f(t\,v) \le t\alpha_2$$

for every $x \in \mathbb{R}^n$, $t > 0$, and $v \in S^{n-1}$. On the other hand, if f is differentiable at $x \in \mathbb{R}^n$ then $\nabla f(x) \cdot (x/|x|) = f(x/|x|) \geq \alpha_1$. Thus,

$$\alpha_1 \leq \sup\{\nabla f(x) \cdot v : v \in S^{n-1}\} = |\nabla f(x)| \leq \alpha_2 \,,$$

and (2.5) immediately follows. It is also useful to note that the dual function f_* to f introduced in (1.7), that is still convex and positively 1-homogeneous, satisfies

$$\inf_{S^{n-1}} f_* = \frac{1}{\alpha_2} \,, \qquad \sup_{S^{n-1}} f_* = \frac{1}{\alpha_1} \,, \tag{2.6}$$

$$\frac{1}{\alpha_2} \leq \|\nabla f_*\|_{L^\infty(\mathbb{R}^n;\mathbb{R}^n)} \leq \frac{1}{\alpha_1} \,. \tag{2.7}$$

Eventually we notice from (1.6) that B_{α_1} is the largest Euclidean ball centered at the origin that is contained in K, while B_{α_2} is the smallest Euclidean ball centered at the origin that contains K. Thus,

$$\alpha_1 = \sup\{r > 0 : B_r \subset K\} \,, \qquad \alpha_2 = \inf\{r > 0 : K \subset B_r\} \,. \tag{2.8}$$

Because of (2.4), (2.5), (2.6), (2.7), and (2.8) we shall often produce estimates depending on the ratio α_2/α_1. One is usually able to rule out such a dependence by means of the following lemma.

Lemma 2.2 (A normalization lemma)**.** *If K is an open, bounded convex set with $0 \in K$, then there exist an affine map $L : \mathbb{R}^n \to \mathbb{R}^n$ with $\det L = 1$ and $r = r(n, |K|) > 0$, such that*

$$B_r \subset L(K) \subset B_{n\,r} \,. \tag{2.9}$$

If $f_{L(K)}$ is the surface tension associated to $L(K)$, then

$$\frac{\sup_{S^{n-1}} f_{L(K)}}{\inf_{S^{n-1}} f_{L(K)}} \leq n \,, \qquad \mathcal{F}_K(E) = \mathcal{F}_{L(K)}(L(E)) \,, \tag{2.10}$$

for every set of finite perimeter $E \subset \mathbb{R}^n$. Moreover, if g is a bounded Borel function and we set $g_L = g \circ L^{-1}$, then

$$\int_E g = \int_{L(E)} g_L \,. \tag{2.11}$$

Proof. By John's Lemma [28, Theorem III], we may associate to K an affine map $L_0 : \mathbb{R}^n \to \mathbb{R}^n$ such that $\det L_0 > 0$ and $B_1 \subset L_0(K) \subset B_n$. Therefore, up to the multiplication of L by a constant, we can achieve (2.9). Clearly, (2.11) is a trivial consequence of the fact that $\det L = 1$. Finally, to show (2.10) let us now recall that, if E is a bounded open set with smooth boundary, then

$$\mathcal{F}_K(E) = \lim_{\varepsilon \to 0^+} \frac{|E + \varepsilon K| - |E|}{\varepsilon} \,.$$

Since L is affine with $\det L = 1$,

$$|L(E) + \varepsilon L(K)| - |L(E)| = |L(E + \varepsilon K)| - |E| = |E + \varepsilon K| - |E| \,,$$

hence $\mathcal{F}_K(E) = \mathcal{F}_{L(K)}(L(E))$. By a density argument and Lemma 2.1 we immediately get (2.10). $\square$

2.3. **Volume-constrained variations.** In studying minimizers to the variational problem 1.1, we will often construct suitable comparison sets, that are typically obtained by a "cut and paste" operation followed by a mass adjustment (usually done by a dilation). Having in mind to work with the notion of (ε, R)-minimizers introduced in (1.13), we have to be careful to control the surface energy variation and the L^1-distance variation created in the mass adjustment. We will adjust mass in two ways: either by a first variation argument (where the surface energy variation depends on the set itself in a quite involved way), see Lemma 2.3 (this lemma is sometimes referred to as "Almgren's Lemma", see [36, Lemma 13.5]); or by a scaling operation (in this case the surface energy variation is trivial but the L^1-distance variation requires an estimate), see Lemma 2.4. We now prove these technical lemmas.

Lemma 2.3. *If E is a set of finite perimeter in $\mathbb{R}^n$ and A is an open set such that $A \cap \partial^* E$ is nonempty, then there exist $s_0 = s_0(E, A)$ and $C = C(E, A, \alpha_2)$ such that, for every $s \in (-s_0, s_0)$, there exists a set of finite perimeter F with the following properties:*

$$E \Delta F \subset\subset A \,,$$
$$|E| - |F| = s \,,$$
$$|\mathcal{F}(E) - \mathcal{F}(F)| \le C\, s \,.$$

Proof. Let $T \in C_c^\infty(A; \mathbb{R}^n)$ and $\Phi_t(x) = x + t\,T(x)$, $x \in \mathbb{R}^n$. There exists $t_0 > 0$ such that Φ_t is a diffeomorphism of $\mathbb{R}^n$ whenever $|t| < t_0$. Hence $\Phi_t(E)$ is a set of finite perimeter for every $|t| < t_0$, with $\Phi_t(E) \Delta E \subset\subset A$. By the first variation formulae in Appendix A.1 we have

$$\begin{aligned} |\Phi_t(E)| &= |E| + t \int_{\partial^* E} T \cdot \nu_E \, d\mathcal{H}^{n-1} + O(t^2)\,, \\ \mathcal{F}(\Phi_t(E)) &= \int_{\partial^* E} (1 + t \operatorname{div} T + O(t^2))\, f(\nu_E - t(\nabla T)^* \nu_E + O(t^2))\, d\mathcal{H}^{n-1}\,, \end{aligned} \tag{2.12}$$

where, here and in the rest of the proof, we denote by $O(s)$ a function of s such that $|O(s)| \le C|s|$ for a constant C depending on T only. In order to estimate $\mathcal{F}(\Phi_t(E)) - \mathcal{F}(E)$ we now notice that by (2.5)

$$|f(\nu_E - t(\nabla T)^* \nu_E + O(t^2)) - f(\nu_E)| \le \alpha_2(|t||\nabla T| + O(t))\,, \tag{2.13}$$

while, thanks to (2.4) and the simple inequality

$$|\nu_E - t(\nabla T)^* \nu_E + O(t^2)| \le 1 + |t||\nabla T| + O(t^2)\,,$$

we also have

$$|t \operatorname{div} T + O(t^2)|\, f(\nu_E - t(\nabla T)^* \nu_E + O(t^2)) \le \alpha_2(|t||\nabla T| + O(t^2))\,. \tag{2.14}$$

By combining (2.12), (2.13), and (2.14), we find that

$$|\mathcal{F}(E) - \mathcal{F}(\Phi_t(E))| \le 2\alpha_2 \left(|t| \int_{\partial^* E} |\nabla T| d\mathcal{H}^{n-1} + O(t^2) \right)\,.$$

Since $A\cap\partial^*E$ is non-empty, it is easily seen that there exists $T\in C^\infty_c(A;\mathbb{R}^n)$ such that

$$\gamma=\int_{\partial^*E}T\cdot\nu_E\,d\mathcal{H}^{n-1}>0\,,$$

for instance on setting $T=\varphi\,\nu_E(x_0)$ for $x_0\in A\cap\partial^*E$ and $\varphi\in C^\infty_c(A)$ such that $1_{B(x,r/2)}\le\varphi\le 1_{B(x,r)}$ (with r sufficiently small). Therefore, the function $t\mapsto|\Phi_t(E)|=|E|+t\gamma+O(t^2)$ is injective on some open interval $(-t_0,t_0)$ where

$$\begin{aligned}\frac{\gamma}{2}|t| &\le ||E|-|\Phi_t(E)||\,,\\ |\mathcal{F}(E)-\mathcal{F}(\Phi_t(F))| &\le 4\alpha_2|t|\int_{\partial^*E}|\nabla T|\,d\mathcal{H}^{n-1}\,.\end{aligned}$$

We conclude by choosing $s_0>0$ such that the interval $(|E|-s_0,|E|+s_0)$ is contained in the image of $(-t_0,t_0)$ through $t\mapsto|\Phi_t(E)|$, proving the result with the constant C defined as

$$C=\frac{8\,\alpha_2}{\gamma}\int_{\partial^*E}|\nabla T|\,d\mathcal{H}^{n-1}\,.$$

□

Lemma 2.4. *There exists a constant $C(n)$ with the following property: If E is a set of finite perimeter with $E\subset B_R$, then*

$$|E\Delta(\lambda\,E)|\le C(n)|\lambda-1|\,R\;P(E)\,,\tag{2.15}$$

whenever $\lambda\in(1/2,2)$.

Proof. If $u\in C^1_c(\mathbb{R}^n)$ and $\lambda\in[1/2,2]$, then for every $x\in\mathbb{R}^n$

$$|u(x)-u(x/\lambda)|\le 2\,|\lambda-1||x|\int_0^1|\nabla u(x+t(1-1/\lambda)x)|dt\,.$$

If $\operatorname{spt}(u)\subset B_R$, then by Fubini theorem we have

$$\begin{aligned}\int_{\mathbb{R}^n}|u(x)-u(x/\lambda)|\,dx &\le |\lambda-1|R\int_0^1dt\int_{\mathbb{R}^n}|\nabla u(x+t(1-1/\lambda)x)|\,dx\\ &\le |\lambda-1|\,R\left(\int_0^1\frac{dt}{(1+t(1-1/\lambda))^n}\right)\int_{\mathbb{R}^n}|\nabla u|\\ &\le C(n)\,|\lambda-1|\,R\int_{\mathbb{R}^n}|\nabla u|\,.\end{aligned}$$

We prove (2.15) by testing this inequality on $u_\varepsilon=1_E*\rho_\varepsilon$ and letting $\varepsilon\to0^+$. □

3. Stability properties of (ε,R)-minimizers

3.1. **Basic properties of (ε,R)-minimizers.** This section is devoted to the study of geometric properties of (ε,R)-minimizers. Given $\varepsilon,R>0$, let us recall that a set of finite perimeter $E\subset\mathbb{R}^n$ is a (volume constrained) (ε,R)-minimizer of the surface energy $\mathcal{F}=\mathcal{F}_K$ provided

$$\mathcal{F}(E)\le\mathcal{F}(F)+\varepsilon\,|K|^{1/n}|E|^{(n-1)/n}\frac{|E\Delta F|}{|E|}\,,$$

for every set of finite perimeter $F \subset \mathbb{R}^n$ with $|F| = |E|$ and $F \subset I_R(E)$, where $I_R(E)$ is defined as the R-neighborhood of E with respect to f_*, see (1.14). Note that (ε, R)-minimality is formulated to be a scale invariant property with respect to ε. Indeed, if E is an (ε, R)-minimizer of $\mathcal{F}_K$, then for every $\lambda > 0$ the rescaled set $\lambda\, E$ is an $(\varepsilon, \lambda\, R)$-minimizer of $\mathcal{F}_K$, or equivalently an (ε, R)-minimizer of $\mathcal{F}_{\lambda K}$. More in general, if L is an affine transformation with $\det L > 0$, Lemma 2.2 and the discussion above gives that $L(E)$ is a (ε, R)-minimizer of $\mathcal{F}_{L(K)}$. Of course, the Wulff shape K is an (ε, R)-minimizer of $\mathcal{F}_K$ for every ε and R.

Theorem 3.1 (Basic regularity estimates for (ε, R)-minimizers)**.** *If E is an (ε, R)-minimizer of $\mathcal{F}$ then $\mathcal{H}^{n-1}(\partial E \setminus \partial^* E) = 0$ and E is equivalent to its interior. Moreover, ∂E is differentiable at every point of $\partial^* E$, i.e.,*

$$\lim_{r \to 0^+} \sup \left\{ \frac{|(x - x_0) \cdot \nu_E(x_0)|}{|x - x_0|} : x \in B(x_0, r) \cap \partial E\,, x \neq x_0 \right\} = 0\,, \tag{3.1}$$

for every $x_0 \in \partial^* E$.

Proof. Step one. As stated in section 2.1, up to modifying E on a set of measure zero, we can assume that

$$\partial E = \mathrm{spt}(D1_E) = \{x \in \mathbb{R}^n : 0 < |E \cap B(x, r)| < \omega_n r^n \text{ for every } r > 0\}\,.$$

We now claim that there exist positive constants $\kappa = \kappa(n, f, E, R) < 1$ and $\overline{r} = \overline{r}(n, f, E, R)$, such that if $x \in \partial E$ and $r < \overline{r}$, then

$$\mathcal{H}^{n-1}(B(x, r) \cap \partial^* E) \geq \kappa\, r^{n-1}\,, \tag{3.2}$$

$$\omega_n r^n (1 - \kappa) \geq |B(x, r) \cap E| \geq \kappa \omega_n r^n\,. \tag{3.3}$$

Of course, it will suffice to show that these estimates hold for a.e. $r < \overline{r}$. Hence, thanks to the coarea formula [4, Theorem 2.93] applied with the Lipschitz function $x \mapsto |x|$ and the countably $\mathcal{H}^{n-1}$-rectifiable set $\partial^* E$, we may restrict to consider values of $r < \overline{r}$ such that

$$\mathcal{H}^{n-1}(\partial B(x, r) \cap \partial^* E) = 0\,. \tag{3.4}$$

Fix $x_1 \neq x_2 \in \partial E$, and let $r_0 = r_0(E) > 0$ be such that

$$2r_0 < \alpha_1 R\,, \qquad B(x_1, 2r_0) \cap B(x_2, 2r_0) = \emptyset\,,$$

(recall that, by definition (2.4) of α_1, we have $B_{\alpha_1 R} \subset K_R$). Let $s_1, s_2 > 0$ and C_1, C_2 be the constants given by Lemma 2.3 applied to E on the open sets $B(x_1, r_0)$ and $B(x_2, r_0)$ respectively, and notice that $s_k < |B(x_k, r_0)| = \omega_n r_0^n$. We set,

$$\overline{s} = \min\{s_1, s_2\}\,, \qquad \overline{C} = \max\{C_1, C_2\}\,,$$

and require $\overline{r}$ to satisfy

$$\omega_n \overline{r}^n < \overline{s}\,,$$

so that, in particular, $\overline{r} < r_0$. Since the balls $B(x_1, 2r_0)$ and $B(x_2, 2r_0)$ are disjoint, we can decompose ∂E as $M_1 \cup M_2$, where

$$M_k = \{x \in \partial E : B(x, \overline{r}) \cap B(x_k, r_0) = \emptyset\}\,, \quad k = 1, 2\,,$$

so that $\partial E \setminus B(x_k, 2r_0) \subset M_k$. We are now in the position to prove (3.2) and (3.3). Let $x \in M_1$ and consider the function

$$u(r) = |E \cap B(x,r)| = \int_0^r \mathcal{H}^{n-1}(E \cap \partial B(x,s))\, ds\,.$$

If we set $G = E \setminus B(x,r)$ for some $r < \overline{r}$ such that (3.4) holds true, then we have

$$0 < |E| - |G| = u(r) < \omega_n r^n < \overline{s}\,.$$

Since $E \cap B(x_2, r_0) = G \cap B(x_2, r_0)$, we can apply Lemma 2.3 to find a set of finite perimeter F such that $F\Delta G \subset\subset B(x_2, r_0)$, $|F| = |G| + (|E| - |G|) = |E|$, and

$$\mathcal{F}(F) \le \mathcal{F}(G) + \overline{C}\big||G| - |F|\big| = \mathcal{F}(G) + \overline{C}\big(|E| - |G|\big) = \mathcal{F}(G) + \overline{C}u(r)\,.$$

We can now test the (ε, R)-minimality of E against F to find

$$\mathcal{F}(E) \le \mathcal{F}(G) + \overline{C}u(r) + \varepsilon\,|K|^{1/n}\frac{|E\Delta F|}{|E|^{1/n}} = \mathcal{F}(G) + \overline{C}u(r) + 2\varepsilon\,|K|^{1/n}\frac{u(r)}{|E|^{1/n}}\,,$$

where we used that by construction $|E\Delta F| = 2u(r)$. Moreover, by (3.4), we have that

$$\begin{aligned} \mathcal{F}(E) - \mathcal{F}(G) &= \int_{B(x,r)\cap\partial^* E} f(\nu_E)\, d\mathcal{H}^{n-1} - \int_{\partial B(x,r)\cap E^1} f(\nu_{B(x,r)})\, d\mathcal{H}^{n-1} \\ &\ge \alpha_1 \mathcal{H}^{n-1}(B(x,r)\cap\partial^* E) - \alpha_2\, u'(r)\,. \end{aligned}$$

Hence we get

$$P(E; B(x,r)) \le C(u'(r) + u(r))$$

for some constant $C = C(E, f)$. Since $P(E \cap B(x,r)) = P(E; B(x,r)) + u'(r)$, due to the Euclidean isoperimetric inequality, we also have

$$n\omega_n^{1/n} u(r)^{(n-1)/n} \le P(E \cap B(x,r)) \le C(u'(r) + u(r))\,, \tag{3.5}$$

for every $r < \overline{r}$ such that (3.4) holds. Since $u(r) \le \omega_n \overline{r}^n$, up to further decrease $\overline{r}$ (depending on n and on the constant C appearing in (3.5)), we may assume that

$$Cu(r) = Cu(r)^{(n-1)/n}u(r)^{1/n} \le \frac{n\omega_n^{1/n}}{2}u(r)^{(n-1)/n}\,,$$

whenever $r < \overline{r}$, so that (3.5) implies

$$c\, u(r)^{(n-1)/n} \le u'(r)\,,$$

for a.e. $r < \overline{r}$ and for some $c = c(n, f, E, R)$. Since $u(r) > 0$ for every $r > 0$ we deduce that $(u(r)^{1/n})' \ge c$ for a.e. $r < \overline{r}$. Hence $u(r) \ge c\, r^n$ for $r < \overline{r}$, that is the lower bound in (3.3). The upper bound in (3.3) follows by an entirely similar argument, where we consider $G = E \cup B(x,r)$ instead of $G = E \setminus B(x,r)$. This remark completes the proof of (3.3). Now that (3.3) has been proved we can apply the relative isoperimetric inequality in $B(x,r)$ (see, for instance, [4, Remark 3.45]) to see that

$$\mathcal{H}^{n-1}(B(x,r)\cap\partial^* E) \ge \tau(n)\min\{|E \cap B(x,r)|, |B(x,r)\setminus E|\}^{(n-1)/n} \ge \tau(n) c r^{n-1}\,,$$

and (3.2) is proved.

Step two. We now conclude the proof of the theorem. First of all, (3.2) combined with a standard covering argument (see, e.g., [3, Corollario 4.2.4]) implies that $\mathcal{H}^{n-1}(\partial E \setminus \partial^* E) = 0$, hence that E is equivalent to its interior. To show the differentiability of ∂E at the points of $\partial^* E$, let us recall that if $x_0 \in \partial^* E$ then (see, e.g., [4, Theorem 3.59])

$$E_{x_0,r} = \frac{E - x_0}{r} \stackrel{\text{loc}}{\to} \{x \in \mathbb{R}^n : \nu_E(x_0) \cdot x \le 0\}$$

as $r \to 0^+$ (recall that the above convergence of sets means that their characteristic functions converge in L^1_{loc}). We now show that this L^1_{loc}-convergence combined with the density estimate (3.3) implies that convergence holds in the Hausdorff sence, which is actually equivalent to (3.1). Indeed, for every $\sigma > 0$ we have

$$\lim_{r \to 0^+} \Big| E_{x_0,r} \cap \{x \in B : x \cdot \nu_E(x_0) \ge -\sigma\} \Big| = \Big| \{x \in B : 0 \ge x \cdot \nu_E(x_0) \ge -\sigma\} \Big| \le \omega_{n-1}\sigma \,,$$

so there exists $r_\sigma = r_\sigma(E, x_0) > 0$ such that

$$|\{x \in E \cap B(x_0, r) : (x - x_0) \cdot \nu_E(x_0) \ge -\sigma r\}| \le 2\omega_{n-1}\sigma r^n \,, \tag{3.6}$$

for every $r < r_\sigma$. Let us now chose $L = L(n, f, E, R) > 0$ such that

$$\kappa L^n > 2\omega_{n-1} \,, \tag{3.7}$$

where κ is the constant found in Step one. We claim that if $\sigma > 0$ is small enough with respect to L, then for every $r < r_\sigma$ we have

$$\{x \in \partial E \cap B(x_0, r/2) : (x - x_0) \cdot \nu_E(x_0) \ge L\sigma^{1/n} r\} = \emptyset \,. \tag{3.8}$$

Indeed if (3.8) is not true, then there exists $x_1 \in \partial E$ such that, provided σ is small enough,

$$\begin{aligned} E \cap B(x_1, L\sigma^{1/n} r) \quad &\subset \quad \{x \in E \cap B(x_0, r) : \nu_E \cdot (x - x_0) \ge -\sigma r\} \,, \\ |E \cap B(x_1, L\sigma^{1/n} r)| \quad &\ge \quad \kappa L^n \sigma r^n \,, \end{aligned}$$

where we have also taken (3.3) into account. Since the combination of these two facts would lead to a contradiction with (3.6) and (3.7), we conclude that (3.8) holds true. By an analogous argument one proves that, for the same values of L and σ,

$$\{x \in \partial E \cap B(x_0, r/2) : (x - x_0) \cdot \nu_E(x_0) \le -L\sigma^{1/n} r\} = \emptyset \,, \tag{3.9}$$

whenever $r < r_\sigma$. On combining (3.8) and (3.9) we conclude that for every $\sigma > 0$ there exists $r_\sigma > 0$ such that

$$|(x - x_0) \cdot \nu_E(x_0)| \le L\sigma^{1/n} |x - x_0| \,,$$

for every $x \in \partial E \cap B(x_0, r_\sigma/2)$, so that (3.1) is proved. $\square$

3.2. **Stability properties of** (ε, R)**-minimizers at small** ε**.** Given a set of finite perimeter E, we define its *Wulff deficit* (with respect to $\mathcal{F}$) as

$$\delta(E) = \frac{\mathcal{F}(E)}{n|K|^{1/n}|E|^{(n-1)/n}} - 1\,. \tag{3.10}$$

By the Wulff inequality (1.8) we have $\delta(E) \geq 0$, and the characterization of the equality case gives that $\delta(E) = 0$ if and only if $|E\Delta(x + K_r)| = 0$ for some $x \in \mathbb{R}^n$ and $r > 0$. In [19] we have proved the following theorem, that gives a (sharp) strengthened form of the Wulff inequality (1.8).

Theorem 3.2. *If E is a set of finite perimeter in $\mathbb{R}^n$ with $|E| = |K|$ then there exists $x_0 \in \mathbb{R}^n$ such that*

$$\delta(E) \geq C(n)\left(\frac{|E\Delta(x_0 + K)|}{|K|}\right)^2, \tag{3.11}$$

or, equivalently,

$$\mathcal{F}(E) \geq n|K|^{1/n}|E|^{1/n'}\left\{1 + C(n)\left(\frac{|E\Delta(x_0 + K)|}{|K|}\right)^2\right\},$$

where $C(n)$ is a constant depending on the dimension n only.

The above result says that, if $\delta(E)$ is small, then E is close in L^1-norm to a translation of K. As the next lemma shows, $(\varepsilon, n+1)$-minimizers have small deficit for ε sufficiently small (the choice $R = n + 1$ comes from the fact that we need all the translations $x + K$ with $E \cap (x + K) \neq \emptyset$ to be admissible competitors).

Lemma 3.3. *If E is an $(\varepsilon, n + 1)$-minimizer of $\mathcal{F}$ with $|E| = |K|$ then*

$$\delta(E) \leq C(n)\,\varepsilon^2. \tag{3.12}$$

Moreover, there exists $x_0 \in \mathbb{R}^n$ such that

$$|E\Delta(x_0 + K)| \leq C(n)|K|\varepsilon\,, \tag{3.13}$$

where $C(n)$ is a constant depending on the dimension n only.

Proof. Step one. If L is as in Lemma 2.2, then $L(E)$ is an $(\varepsilon, n+1)$-minimizer of $\mathcal{F}_{L(K)}$, with $\delta(E) = \delta_{L(K)}(L(E))$, $|E| = |L(E)|$, $|K| = |L(K)|$ and $|E\Delta(x+K)| = |L(E)\Delta(L(x)+L(K))|$ for every $x \in \mathbb{R}^n$. Therefore we may assume without loss of generality that K satisfies

$$B_r \subset K \subset B_{rn}\,,$$

for some $r = r(n, |K|)$. In particular, $\alpha_2/\alpha_1 \leq n$ (see (2.4)).

Step two. If $E \cap (x + K) \neq \emptyset$, then $x + K \subset I_{n+1}(E)$. Indeed, given $z \in E \cap (x + K)$, $y \in x + K$, and taking into account that $K = \{f_* < 1\}$, then we have

$$f_*(y - z) \leq f_*(y - x) + f_*(x - z) \leq 1 + \frac{\alpha_2}{\alpha_1} f_*(z - x) \leq 1 + n\,.$$

Therefore, if $x \in \mathbb{R}^n$ is such that $E \cap (x+K) \neq \emptyset$, then by the $(\varepsilon, n+1)$-minimality of E we find

$$\mathcal{F}(E) \leq \mathcal{F}(x+K) + \varepsilon\,|E\Delta(x+K)| = \mathcal{F}(K) + \varepsilon|E\Delta(x+K)|\,.$$

Since $\mathcal{F}(K) = n|K|$ and $|E| = |K|$, this implies

$$\delta(E) = \frac{\mathcal{F}(E)}{n|K|} - 1 \leq \frac{|E\Delta(x+K)|}{n|K|}\,\varepsilon\,.$$

On choosing $x = x_0$ such that (3.11) holds, we deduce (3.13). Next, by inserting (3.13) in the above estimate, we also find (3.12). □

We now prove an uniform stability estimate together with a connectedness results. In order to apply this result in the case of minimizers to the variational problem (1.1) we work with a slightly different notion of minimality rather than (ε, R)-minimality. The theorem is then applied to (ε, R)-minimizers in Corollary 3.5 below.

Theorem 3.4. *There exist constants $C(n)$ and $\varepsilon(n)$ with the following property: If $0 < \varepsilon < \varepsilon(n)$ and if E is a set of finite perimeter with $|E| = |K|$, such that*

$$\delta(E) \leq C(n)\varepsilon^2\,, \tag{3.14}$$

$$|E\Delta K| \leq C(n)|K|\varepsilon\,, \tag{3.15}$$

$$\mathcal{F}(E) \leq \mathcal{F}(F) + \varepsilon|E\Delta F|\,, \tag{3.16}$$

whenever $|F| = |E|$, $F \setminus E \subset K_3$, then E is indecomposable and for some $r_0 \leq C(n)\varepsilon^{1/n}$,

$$K_{1-r_0} \subset E \subset K_{1+r_0}\,.$$

Corollary 3.5 (Uniform proximity to the Wulff shape)**.** *There exist constants $C(n)$ and $\varepsilon(n)$ with the following property: If E is an $(\varepsilon, n+1)$-minimizer of $\mathcal{F}$ with $|E| = |K|$ and $\varepsilon < \varepsilon(n)$, then E is connected and there exists $x_0 \in \mathbb{R}^n$ and $r_0 \leq C(n)\varepsilon^{1/n}$ such that*

$$x_0 + K_{1-r_0} \subset E \subset x_0 + K_{1+r_0}\,. \tag{3.17}$$

Proof of Corollary 3.5. It follows immediately from Theorem 3.1, Lemma 3.3, Theorem 3.4, and the fact that an open indecomposable set with $\mathcal{H}^{n-1}(\partial E \setminus \partial^* E) = 0$ is connected [5, Theorem 2]. □

Proof of Theorem 3.4. We can apply Lemma 2.2 and assume without loss of generality that, for a constant $\rho = \rho(n, |K|)$, we have $\rho \leq \alpha_1 \leq \alpha_2 \leq n\,\rho$ (see (2.4)). Since $\rho^{-1}E$ satisfies the minimality condition (3.16) with $\mathcal{F}_{\rho^{-1}K}$ and $\rho^{-1}K$ in place of $\mathcal{F}$ and K, up to scale both E and K by the factor $1/\rho$, we may work under the additional assumptions that

$$B \subset K \subset B_n\,, \tag{3.18}$$

$$1 \leq \alpha_1 \leq \alpha_2 \leq n\,, \tag{3.19}$$

$$\frac{1}{n} \leq \|\nabla f_*\|_{L^\infty(\mathbb{R}^n;\mathbb{R}^n)} \leq 1\,, \tag{3.20}$$

where (3.20) follows from (3.19) and (2.7). We notice that by (3.14) and by taking into account that $\mathcal{F}(K) = n|K| \le n|B_n| \le C(n)$ we have

$$\mathcal{F}(E) \le \big(1 + C(n)\varepsilon^2\big)\mathcal{F}(K) \le C(n)|K| \le C(n)\,. \tag{3.21}$$

By [18, 4.2.25], [5, Theorem 1], there are countably many disjoint sets of finite perimeter $\{E_h\}_{h\in\mathbb{N}}$ such that

$$E = \bigcup_{h\in\mathbb{N}} E_h\,, \qquad P(E) = \sum_{h\in\mathbb{N}} P(E_h)\,,$$

and each E_h is indecomposable, in the sense that if $F \subset E_h$ is a set of finite perimeter with $P(E_h) = P(F) + P(E_h \setminus F)$ then $|F||E_h \setminus F| = 0$. In fact, the reduced boundaries of the E_h's are pairwise disjoint mod-$\mathcal{H}^{n-1}$, so that we also have

$$\mathcal{F}(E) = \sum_{h\in\mathbb{N}} \mathcal{F}(E_h)\,. \tag{3.22}$$

Without loss of generality we may assume that $|E_1| \ge |E_h|$ for every $h \in \mathbb{N}$.

Step one: L^1-estimates for E_1. We claim that

$$\begin{aligned} |E \setminus E_1| &\le C(n)\delta(E)^{n'}\,, & (3.23)\\ |E_1 \Delta K| &\le C(n)\varepsilon\,. & (3.24)\end{aligned}$$

Since $|E_1\Delta K| \le |E\Delta K| + |E \setminus E_1|$, (3.24) is an immediate consequence of (3.15), (3.23), and (3.14). Hence we directly focus on the proof of (3.23). Without loss of generality we may assume that $E_h \ne \emptyset$ for some $h > 1$. Then for every $k \ge 1$ we introduce the sets of finite perimeter

$$F_k = \bigcup_{h=1}^{k} E_h\,, \qquad G_k = \bigcup_{h=k+1}^{\infty} E_h\,.$$

By (3.22), by the Wulff inequality and by concavity of $t^{1/n'}$ on $t > 0$, we find that

$$\mathcal{F}(E) = \mathcal{F}(F_k) + \mathcal{F}(G_k) \ge n|K|^{1/n}(|F_k|^{1/n'} + |G_k|^{1/n'}) \ge n|K|^{1/n}|E|^{1/n'}\,,$$

i.e., by the definition of $\delta(E)$ (3.10),

$$\delta(E) \ge \left(\frac{|F_k|}{|E|}\right)^{1/n'} + \left(1 - \frac{|F_k|}{|E|}\right)^{1/n'} - 1\,.$$

Observe now that there exists a constant $c_0(n) > 0$ such that

$$t^{1/n'} + (1-t)^{1/n'} - 1 \ge c_0(n)t^{1/n'}\,, \qquad \forall\, t \in [0, 1/2]\,.$$

Hence, if $|E_1| \ge |E|/2$ and we chose $k = 1$, then we find

$$\delta(E) \ge c_0 \left(1 - \frac{|E_1|}{|E|}\right)^{1/n'}\,,$$

that is (3.23) as required. Let us now assume on the contrary that $|E_1| < |E|/2$, then there exists $k \ge 2$ such that

$$|F_{k-1}| < \frac{|E|}{2}\,, \quad |G_{k-1}| \ge \frac{|E|}{2}\,, \quad |F_k| \ge \frac{|E|}{2}\,, \quad |G_k| < \frac{|E|}{2}\,.$$

Therefore, by the above argument we deduce that

$$\delta(E)\geq c_0\left(\frac{|F_{k-1}|}{|E|}\right)^{1/n'}\,,\qquad \delta(E)\geq c_0\left(\frac{|G_k|}{|E|}\right)^{1/n'}\,,$$

so that

$$C(n)\delta(E)^{n'}\geq\frac{|F_{k-1}|+|G_k|}{|E|}=1-\frac{|E_k|}{|E|}\geq 1-\frac{|E_1|}{|E|}\geq\frac{1}{2}\,.$$

Hence, if $|E_1|<|E|/2$, then $\delta(E)\geq\delta(n)>0$, and this can be excluded by (3.14) provided $\varepsilon\leq\varepsilon(n)$ for $\varepsilon(n)$ small enough.

Step two: Uniform outer estimate for E_1. We now prove that, if $\varepsilon(n)$ is sufficiently small, then there exists $r\in(1,1+C(n)\varepsilon^{1/n})$ such that

$$E_1\subset K_r\,.$$

We consider the decreasing function $u:[0,+\infty)\to[0,+\infty)$ defined as

$$u(r)=\int_{E_1\setminus K_r}|\nabla f_*(x)|dx\,,\quad r>0\,.$$

By the coarea formula u is absolutely continuous, with

$$\begin{aligned}u(r)=\int_r^{+\infty}\mathcal{H}^{n-1}((\partial K_s)\cap E_1)\,ds\,,&\qquad\text{for every } r>0\,,\\ u'(r)=-\mathcal{H}^{n-1}((\partial K_r)\cap E_1)\,,&\qquad\text{for a.e. } r>0\,,\end{aligned}$$

and moreover (using (3.20))

$$\frac{1}{n}|E_1\setminus K_r|\leq u(r)\leq|E_1\setminus K_r|\,.\tag{3.25}$$

We define

$$r_1=\sup\{r\geq 1:u(r)>0\}\,,$$

and prove that

$$r_1-1\leq C(n)\varepsilon^{1/n}\,.\tag{3.26}$$

For every $r\in(1,\min\{r_1,2\})$ we have $|(E_1\cap K_r)\cup(E\setminus E_1)|<|E|$, therefore we can find $s=s(r)>1$ such that $|F|=|E|$, where we have defined

$$F=s(E_1\cap K_r)\cup(E\setminus E_1)\,.$$

Set

$$v(r)=|s(E_1\cap K_r)\cap(E\setminus E_1)|\,,\quad r>1\,.$$

Observe that, by (3.25) and by (3.24),

$$u(r)\leq|E_1\setminus K|\leq C(n)\varepsilon\,,\tag{3.27}$$

while, (3.23) and (3.14) give

$$v(r)\leq|E\setminus E_1|\leq C(n)\varepsilon^{2n'}\leq C(n)\varepsilon\,.\tag{3.28}$$

By the definition of F we have that

$$|E|=|F|\;=\;|E\setminus E_1|+|s(E_1\cap K_r)|-|s(E_1\cap K_r)\cap(E\setminus E_1)|$$

$$= \quad |E\setminus E_1| + s^n(|E_1| - |E_1\setminus K_r|) - v(r)\,,$$

which by (3.25) gives

$$|E_1| + v(r) \ge s^n(|E_1| - n\,u(r))\,.$$

Taking (3.23) and (3.27) into account, we find that

$$1 < s \le 1 + C(n)\big(u(r)+v(r)\big)\,. \tag{3.29}$$

Hence by a suitable choice of $\varepsilon(n)$ and by taking (3.18) into account, we can bound s through (3.29) so to entail $F\setminus E\subset K_3$. Observing that $|E\Delta F| = 2|F\setminus E|$, the minimality condition (3.16) implies

$$\mathcal{F}(E) \le \mathcal{F}(F) + 2\varepsilon|s\,(K_r\cap E_1)\setminus E_1|\,. \tag{3.30}$$

On the one hand, we remark that

$$\begin{aligned}\mathcal{F}(E) &= \mathcal{F}(E_1) + \mathcal{F}(E\setminus E_1)\\ \mathcal{F}(F) &\le \mathcal{F}(E\setminus E_1) + s^{n-1}\mathcal{F}(K_r\cap E_1) - \mathcal{F}(s(E_1\cap K_r)\cap(E\setminus E_1)),\end{aligned}$$

so that (3.30) gives

$$\mathcal{F}(E_1) + \mathcal{F}(s(E_1\cap K_r)\cap(E\setminus E_1)) \le s^{n-1}\mathcal{F}(K_r\cap E_1) + 2\varepsilon|s\,(K_r\cap E_1)\setminus E_1|\,. \tag{3.31}$$

On the other hand, as $r\in(1,\min\{r_1,2\})$ we have $K_r\subset B_{2n}$, and by Lemma 2.4 we find

$$|s\,(K_r\cap E_1)\setminus E_1| \le |s(K_r\cap E_1)\setminus(K_r\cap E_1)| \le C(n)(s-1)\mathcal{F}(K_r\cap E_1)\,. \tag{3.32}$$

We now combine (3.31) and (3.32): taking also (3.29) into account and applying the Wulff inequality to $s(E_1\cap K_r)\cap(E\setminus E_1)$,

$$\mathcal{F}(E_1) + n|K|^{1/n}\,v(r)^{1/n'} \le \Big(1 + C(n)\big[u(r)+v(r)\big]\Big)\mathcal{F}(K_r\cap E_1)\,. \tag{3.33}$$

We notice that for a.e. $r>0$

$$\mathcal{F}(E_1) = \int_{(\partial E_1)\setminus K_r} f(\nu_E)\,d\mathcal{H}^{n-1} + \int_{(\partial E_1)\cap K_r} f(\nu_E)\,d\mathcal{H}^{n-1}\,, \tag{3.34}$$

$$\begin{aligned}\mathcal{F}(K_r\cap E_1) &= \int_{(\partial E_1)\cap K_r} f(\nu_E)\,d\mathcal{H}^{n-1} + \int_{(\partial K_r)\cap E_1} f(\nu_{K_r})\,d\mathcal{H}^{n-1}\\ &\le \int_{(\partial E_1)\cap K_r} f(\nu_E)\,d\mathcal{H}^{n-1} + n\,|u'(r)|\,,\end{aligned} \tag{3.35}$$

(as $\alpha_2\le n$). Moreover, as $r\in(1,\min\{r_1,2\})$, by (3.21)

$$\mathcal{F}(E_1\cap K_r) \le \mathcal{F}(E) + \mathcal{F}(K_r) \le C(n)\,.$$

We combine this last estimate with (3.33), (3.34), and (3.35) to obtain

$$\int_{(\partial E_1)\setminus K_r} f(\nu_E)\,d\mathcal{H}^{n-1} + c(n)v(r)^{1/n'} \le n\,|u'(r)| + C(n)(u(r)+v(r))\,. \tag{3.36}$$

By applying Wulff's inequality on $E_1\setminus K_r$, thanks to (3.18) and (3.19) we find

$$\begin{aligned}c(n)u(r)^{1/n'} &\le n|K|^{1/n}|E_1\setminus K_r|^{1/n'} \le \mathcal{F}(E_1\setminus K_r)\\ &= \int_{(\partial E_1)\setminus K_r} f(\nu_E)\,d\mathcal{H}^{n-1} + \int_{(\partial K_r)\setminus E_1} f(-\nu_{K_r})\,d\mathcal{H}^{n-1}\end{aligned}$$

$$\leq \int_{(\partial E_1)\setminus K_r} f(\nu_E)\, d\mathcal{H}^{n-1} + n\,|u'(r)|\,,$$

that combined with (3.36) leads to

$$u(r)^{1/n'} + v(r)^{1/n'} \leq C(n)\Big\{|u'(r)| + \big(u(r)+v(r)\big)\Big\}\,, \tag{3.37}$$

for every $r \in (1, \min\{r_1, 2\})$. By (3.27) and (3.28) we can chose $\varepsilon(n)$ small enough to ensure that

$$v(r)^{1/n'} \geq C(n) v(r)\,, \qquad u(r)^{1/n'} \geq 2\,C(n)\,u(r)\,,$$

where $C(n)$ is the constant appearing on the right hand side of (3.37). As a consequence,

$$u(r)^{1/n'} \leq C(n)\,|u'(r)|\,,$$

for every $r \in (1, \min\{r_1, 2\})$. Thus,

$$\begin{aligned} \min\{r_1, 2\} - 1 &\leq C(n) \int_1^{\min\{r_1,2\}} \frac{-u'(r)}{u(r)^{1/n'}}\, dr = C(n)\big(u(1)^{1/n} - u(\min\{r_1,2\})^{1/n}\big) \\ &\leq C(n) u(1)^{1/n} \leq C(n)\varepsilon^{1/n}\,, \end{aligned}$$

where in the last step we have applied (3.15). Hence, if $\varepsilon(n)$ is chosen sufficiently small, this last estimate implies $r_1 \leq 1 + C(n)\varepsilon^{1/n}$, that is (3.26), as required.

Step three: Inner estimate. We now set

$$r_0 = \sup\{r \in [0,1] : |K_r \setminus E| = 0\}\,,$$

and show that

$$1 - r_0 \leq C(n)\varepsilon^{1/n}\,. \tag{3.38}$$

To this end we notice that for every $r \in (r_0, 1)$ we have $|(E_1 \cup K_r) \cup (E \setminus E_1)| > |E|$. Thus $s = s(r) \in (0,1)$ can be defined with the property that, if we set

$$F = s(E_1 \cup K_r) \cup (E \setminus E_1)\,,$$

then $|F| = |E|$. Since we have proved in Step two that $E_1 \subset K_2$, we clearly have $F \setminus E \subset K_3$. Hence we can exploit the minimality condition (3.16) to compare E and F, and deduce by the very same argument used in Step two that

$$u(r)^{1/n'} \leq C(n) u'(r)\,, \qquad \text{for a.e. } r \in (r_0, 1)\,.$$

Of course, now $u : [0, +\infty) \to [0, +\infty)$, is the absolutely continuous, increasing function defined as

$$u(r) = \int_{K_r \setminus E} |\nabla f_*(x)| dx = \int_0^r \mathcal{H}^{n-1}((\partial K_s) \setminus E)\, ds\,, \quad r > 0\,.$$

We leave the details to the interested reader.

Step four: E is indecomposable. We are going to prove that $E = E_1$. Indeed let us now set $F = s\,E_1$ where $s^n|E_1| = |E|$. Recalling that $E_1 \subset K_2$, since $|E \setminus E_1| \leq C(n)\delta(E)^{n'} \leq C(n)\varepsilon^{2n'}$ and

$$1 \leq s \leq 1 + C(n)|E \setminus E_1|\,, \tag{3.39}$$

if $\varepsilon(n)$ is small enough then we clearly have $F \setminus E \subset K_3$. By the minimality condition (3.16),

$$\mathcal{F}(E_1) + \mathcal{F}(E \setminus E_1) = \mathcal{F}(E) \le s^{n-1}\mathcal{F}(E_1) + \varepsilon|(s\,E_1)\Delta E|\,,$$

so that by (3.39)

$$\begin{aligned}\mathcal{F}(E \setminus E_1) &= (s^{n-1}-1)\mathcal{F}(E_1) + \varepsilon|(s\,E_1)\Delta E| \\ &\le C(n)|E \setminus E_1|\mathcal{F}(E_1) + \varepsilon|(s\,E_1)\Delta E|\,.\end{aligned}$$

As $\mathcal{F}(E_1) \le \mathcal{F}(E) \le C(n)$ (see (3.21)), by applying the Wulff inequality to $E \setminus E_1$ and taking into account that

$$|(s\,E_1)\Delta E| \le |(s\,E_1)\Delta E_1| + |E \setminus E_1|\,,$$

we have

$$n|K|^{1/n}|E \setminus E_1|^{1/n'} \le C(n)|E \setminus E_1| + \varepsilon|(s\,E_1)\Delta E_1|\,.$$

By Lemma 2.4 and since $E_1 \subset K_2 \subset B_{2n}$,

$$|(s\,E_1)\Delta E_1| \le C(n)(s-1)P(E_1) \le C(n)(s-1) \le C(n)|E \setminus E_1|\,.$$

Hence,

$$|E \setminus E_1|^{1/n'} \le C(n)|E \setminus E_1|\,.$$

By (3.23) this is impossible for $\varepsilon(n)$ small enough, unless $|E \setminus E_1| = 0$. □

3.3. **Geometric properties of planar (ε, R)-minimizers at small ε.** In this section we restrict our analysis to the planar case $n = 2$. This allows us to take advantage of the fact that the surface energy $\mathcal{F}$ does not increase under convexification to show some strong stability results of (ε, R)-minimizers. Indeed, we will prove that (ε, R)-minimizers are always convex for ε small enough (Theorem 3.6). Moreover, if the surface tension is crystalline, then (ε, R)-minimizers enjoy exactly the same crystalline structure of K (see Theorem 3.7 below).

Theorem 3.6 (Convexity of planar $(\varepsilon, 3)$-minimizers at small ε)**.** *Let $n = 2$. There exists a positive constant $\varepsilon_0 > 0$ such that, if E is an $(\varepsilon, 3)$-minimizer of $\mathcal{F}$ with $|E| = |K|$ and $\varepsilon \le \varepsilon_0$, then E is convex.*

Proof. As in the proof of Theorem 3.4, we can assume without loss of generality that $B_1 \subset K \subset B_2$. By that theorem, provided ε_0 is small enough and up to a translation, we also know that $K_{1-r_0} \subset E \subset K_{1+r_0}$, where $r_0 \le C\,\varepsilon^{1/2}$. Let now $F = \mathrm{co}(E)$ denote the convex hull of E, and assume by contradiction that $\delta = |F \setminus E|/|E| > 0$. Since, by construction, $K_{1-r_0} \subset F \subset K_{1+r_0}$, we find that

$$\delta = \frac{|F \setminus E|}{|K|} \le \frac{|K_{1+r_0} \setminus K_{1-r_0}|}{|K|} = 2r_0 \le C\,\varepsilon^{1/2}\,. \tag{3.40}$$

Therefore, if we rescale F and define

$$F' = (1+\delta)^{-1/2}F\,,$$

then $|F'| = |E|$ and, provided ε_0 is small enough, $F' \subset I_3(E)$. Since F' is obtained by a contraction of the convex set F with respect to $0 \in F$, we have $F' \subset F$. Hence,

$$\begin{aligned} |E \setminus F'| &\leq |F \setminus F'| = |F| - |F'| = \frac{\delta}{1+\delta}\,|F| = \delta|E|\,, \\ |F' \setminus E| &\leq |F \setminus E| = \delta|E|\,. \end{aligned}$$

Moreover, as $E \subset \mathbb{R}^2$, the convexity of f ensures that $\mathcal{F}(F) \leq \mathcal{F}(E)$ [32, Corollary 2.8]. In conclusion, the $(\varepsilon, 3)$-minimality of E implies that

$$\begin{aligned} \mathcal{F}(E) &\leq \mathcal{F}(F') + \varepsilon|E\Delta F'| \leq \frac{\mathcal{F}(E)}{(1+\delta)^{1/2}} + 2\varepsilon\,\delta\,|E| \\ &\leq \left(1 - \frac{\delta}{2} + o(\delta)\right)\mathcal{F}(E) + 2\varepsilon\,\delta\,|E|\,. \end{aligned}$$

By the Wulff inequality (1.8), $\mathcal{F}(E) \geq 2|E|$, hence

$$\frac{\delta}{2} + o(\delta) \leq \varepsilon\,\delta\,,$$

which combined with (3.40) leads to a contradiction for ε_0 small enough. □

Theorem 3.7 (Crystalline structure of $(\varepsilon, 3)$-minimizers at small ε)**.** *Let $n = 2$ and let f be a crystalline surface tension, so that the Wulff shape K is a convex polygon with outer unit normals $\{\nu_i\}_{i=1}^N$. There exists a positive constant ε_0 such that, if E is an $(\varepsilon, 3)$-minimizer with $\varepsilon \leq \varepsilon_0$, then E is a convex polygon with*

$$\nu_E(x) \in \{\nu_i\}_{i=1}^N \qquad \textit{for } \mathcal{H}^1\textit{-a.e. } x \in \partial E.$$

Proof. Every affine transformation L maps a convex polygon K into a convex polygon $L(K)$, and an $(\varepsilon, 3)$-minimizer E of $\mathcal{F}_K$ into an $(\varepsilon, 3)$-minimizer $L(E)$ of $\mathcal{F}_{L(K)}$. Hence, thanks to Lemma 2.2 and up to a dilation, we can assume without loss of generality that

$$B_1 \subset K \subset B_2\,.$$

In particular, provided ε_0 is sufficiently small and up to a translation, we can apply Corollary 3.5 to entail

$$E \subset B_{2(1+r_0)} \subset B_4\,. \tag{3.41}$$

We now order the normal directions $\{\nu_i\}_{i=1}^N$ in the clockwise direction (see Figure 3), and write, with abuse of notation, $\nu_1 < \nu_2 < \ldots < \nu_N < \nu_1 = \nu_{N+1}$. For every $j \in \{1, ..., N\}$, we see from (1.11) that there exists $x_j \in \mathbb{R}^n \setminus \{0\}$ such that

$$f(\nu) = x_j \cdot \nu\,, \quad \text{if } \nu_j \leq \nu \leq \nu_{j+1}. \tag{3.42}$$

If ε_0 is small enough, then by Theorem 3.6 E is a convex open set. In particular, there exists a continuous injective function $\gamma : [0, 1) \to \mathbb{R}^2$ with $\gamma(1^-) = \gamma(0)$ and $\gamma([0, 1)) = \partial E$, such that γ is of class C^1 over $J = (0, 1)\setminus I$, where I consists of at most countably many points. Correspondingly, the classical outer unit normal ν_E is defined as a continuous vector field over $\gamma(J) \subset \partial E$. We now claim that if $t_0 \in J$, $x_0 = \gamma(t_0)$, then $\nu_E(x_0) \in \{\nu_i\}_{i=1}^N$. We can argue by contradiction, assuming on the contrary (and without loss of

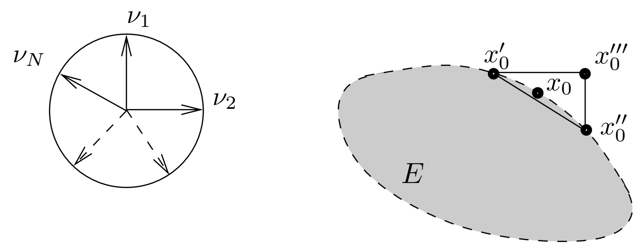

FIGURE 3. The definition of the triangle T, that uses ν_1 and ν_2 as outer unit normals. Notice that, provided σ is small enough, $\gamma((t_0', t_0'')) = T \cap \partial E$.

generality) that $\nu_1 < \nu(x_0) < \nu_2$. By the continuity of ν_E at $\gamma(t_0)$ for every $\sigma > 0$ there exists $t_0'\,, t_0'' \in (0,1)$ such that $|t_0' - t_0''| < \sigma$ and

$$t_0' < t_0 < t_0''\,, \qquad \nu_1 < \nu_E(x_0') \le \nu_E(x_0) \le \nu_E(x_0'') < \nu_2\,, \tag{3.43}$$

where we have set $x_0' = \gamma(t_0')$ and $x_0'' = \gamma(t_0'')$. Correspondingly we define the closed triangle T with vertices at x_0', x_0'' and x_0''', where x_0''' is uniquely identified by the identity

$$(x_0''' - x_0') \cdot \nu_1 = (x_0''' - x_0'') \cdot \nu_2\,,$$

see Figure 3. Notice that, as $\sigma \to 0$, the triangle T shrinks to $\{x_0\}$. We now claim that

$$\mathcal{F}(E \cup T) = \mathcal{F}(E)\,. \tag{3.44}$$

Indeed, we first remark that

$$\mathcal{F}(E \cup T) - \mathcal{F}(E) = \int_{\partial T \setminus \overline{E}} f(\nu_T(x))\, d\mathcal{H}^1(x) - \int_{T \cap \partial E} f(\nu_E(x))\, d\mathcal{H}^1(x)\,.$$

Since $\nu_T(x) \in \{\nu_1, \nu_2\}$ for $x \in \partial T \setminus \overline{E}$ and $\nu_1 \le \nu_E(x) \le \nu_2$ for $x \in T \cap \partial E$, by (3.42) and by the divergence theorem we find that

$$\begin{aligned}
\mathcal{F}(E \cup T) - \mathcal{F}(E) &= \int_{\partial T \setminus \overline{E}} \nu_T(x) \cdot x_1\, d\mathcal{H}^1(x) - \int_{T \cap \partial E} \nu_E(x) \cdot x_1\, d\mathcal{H}^1(x) \\
&= \int_{\partial (T \setminus E)} \nu_{T \setminus E}(x) \cdot x_1\, d\mathcal{H}^1(x) \\
&= \int_{T \setminus E} \mathrm{div}(x_1)\, dx = 0\,,
\end{aligned}$$

as desired. We are now in the position to conclude the proof of the theorem. Indeed, if we let $\delta = |T \setminus E|/|E|$, then $\delta > 0$, $\delta \to 0$ as $\sigma \to 0$, and the set

$$F = (1+\delta)^{-1/2}\,(E \cup T)\,,$$

satisfies $|E| = |F|$. If σ is small enough then $F \subset I_3(E)$ and, by (3.41), $E \cup T \subset B_5$. Thus by Lemma 2.4 we find

$$\begin{aligned}
|E \Delta F| &\le |F \Delta (E \cup T)| + |(E \cup T) \Delta E| \le C\delta\, P(E \cup T) + |T \setminus E| \\
&\le C\delta\, \mathcal{F}(E \cup T) + \delta |E| = C\delta\, \mathcal{F}(E) + \delta |E|\,,
\end{aligned} \tag{3.45}$$

where we applied $P(E\cup T)\le \mathcal{F}(E\cup T)$ (as $B\subset K$) and (3.44). By the $(\varepsilon,3)$-minimality of E and by (3.45)

$$\mathcal{F}(E)\le \frac{\mathcal{F}(E\cup T)}{(1+\delta)^{1/2}}+\varepsilon|E\Delta F|\le \left(1-\frac{\delta}{2}+\varepsilon\, C\delta+o(\delta)\right)\mathcal{F}(E)+\varepsilon\delta|E|\,.$$

By the Wulff inequality $\mathcal{F}(E)\ge 2|E|$ we conclude that

$$(1-2\varepsilon C)\delta+o(\delta)\le\varepsilon\delta\,.$$

If ε_0 is small enough, we obtain a contradiction letting σ (and so δ) converge to 0. □

3.4. **$C^{1,\alpha}$-regularity of (ε,R)-minimizers in the uniform elliptic case.** In the previous section we have shown that $(\varepsilon,n+1)$-minimizers are L^∞-close to K. Assume now that f is λ-elliptic in the sense of (1.9), so that, in particular, ∂K is of class C^2. Then it is not difficult to show that (ε,R)-minimizers of $\mathcal{F}$ are "almost minimizers" of an elliptic integrand (see Lemma C.2). Hence, we can combine the L^∞-closeness of ∂E to ∂K, together with standard regularity theory for almost minimal currents, to show uniform $C^{1,\alpha}$-bounds on ∂E. More precisely, introduce the orthogonal projections $\mathbf{p}:\mathbb{R}^n\to\mathbb{R}^{n-1}$ and $\mathbf{q}:\mathbb{R}^n\to\mathbb{R}$ so that $x=(\mathbf{p}x,\mathbf{q}x)$, and set

$$\mathbf{C}(r,s)=\{x\in\mathbb{R}^n:|\mathbf{p}x|<r\,,|\mathbf{q}x|<s\}\,,\quad \mathbf{C}(r)=\mathbf{C}(r,\infty)\,,\quad \mathbf{D}(r)=\mathbf{C}(r,0)\,.$$

We have the following result:

Theorem 3.8 (Uniform $C^{1,\bar\alpha}$-regularity of $(\varepsilon,n+1)$-minimizers at small ε)**.** *Assume that f is λ-elliptic. Then for every $\bar\alpha\in(0,1)$ there exist positive constants $\eta_0=\eta_0(f)$, $\varepsilon=\varepsilon(n,f,\bar\alpha)$, $r_0=r_0(n,f,\bar\alpha)$, $L=L(n,f,\bar\alpha)$, and $N=N(n,f,\bar\alpha)\in\mathbb{N}$ with the following property:*

If E is an $(\varepsilon,n+1)$-minimizer of $\mathcal{F}$ with $|E|=|K|$, then there exist $u_i:\mathbb{R}^{n-1}\to\mathbb{R}$ and $Q_i:\mathbb{R}^n\to\mathbb{R}^n$ $(i=1,\dots,N)$ such that:

(i) *each Q_i is an isometry of $\mathbb{R}^n$;*
(ii) $\|u_i\|_{C^{1,\bar\alpha}(\mathbf{D}(r_0))}\le L$*;*
(iii) *if we set* $\mathrm{graph}(u_i)=\{(z,u_i(z)):z\in\mathbb{R}^{n-1}\}$*, then*

$$\partial E=\bigcup_{i=1}^N Q_i\big(\mathrm{graph}(u_i)\cap\mathbf{C}(r_0,\eta_0)\big)=\bigcup_{i=1}^N Q_i\big(\mathrm{graph}(u_i)\cap\mathbf{C}(r_0/2,\eta_0)\big)\,.$$

In particular, ∂E is a $C^{1,\bar\alpha}$-manifold.

Remark 3.9. Condition (iii) says that ∂E can be covered by the Q_i-images of the graphs of the functions u_i's not only inside the cylinder $\mathbf{C}(r_0,\eta_0)$, but also inside the smaller cylinder $\mathbf{C}(r_0/2,\eta_0)$. This fact is going to play a role in section 4.3, where we shall apply Theorem 3.8 to the minimizers in (1.1) in the small mass regime. Indeed, when E is a minimizer then each function u_i given by Theorem 3.8 satisfies an elliptic partial differential equation (determined by f and Q_i) inside the disk $\mathbf{D}(r_0)$. Under the natural smoothness assumptions on f and g, the bootstrap argument described in Appendix A.2 will then allow us to bound the $C^{2,\alpha}$-norm of each u_i inside $\mathbf{D}(r_0/2)$, leading thus to establish uniform $C^{2,\alpha}$-estimates for ∂E.

The proof of the above result is well-known to specialists, but we have been unable to find a precise reference. Since we hope to make this paper accessible to a large audience, we have decide to provide a complete proof of Theorem 3.8 in Appendix C.

4. Stability properties of minimizers at small mass

We now turn to the study of minimizers in (1.1), with particular emphasis on the small mass regime. Let us recall that we shall *always* assume as a minimal requirement for the potential $g:\mathbb{R}^n\to[0,+\infty)$ to be a locally bounded Borel function such that

$$g(x)\to+\infty\quad\text{as } |x|\to+\infty\,, \tag{4.1}$$

$$\inf_{\mathbb{R}^n} g = g(0)=0\,. \tag{4.2}$$

We shall frequently refer to the Borel functions $\Psi_g,\Phi_g:(0,+\infty)\to[0,+\infty)$ defined as

$$\Psi_g(R)=\sup_{K_R} g\,,\qquad \Phi_g(R)=\inf_{\mathbb{R}^n\setminus K_R} g\,,\qquad R>0\,.$$

Note that Ψ_g and Φ_g take finite values by the local boundedness of g, and, by (4.1),

$$\lim_{R\to+\infty}\Psi_g(R)=\lim_{R\to+\infty}\Phi_g(R)=+\infty\,.$$

Moreover, if g is continuous in a neighborhood of $x=0$, then by (4.2) we find

$$\lim_{R\to 0^+}\Psi_g(R)=\lim_{R\to 0^+}\Phi_g(R)=0\,.$$

We begin our analysis with a trivial existence result.

Lemma 4.1 (Existence of minimizers)**.** *For every $m>0$ there exists a minimizer for the variational problem* (1.1)*.*

Proof. Let us consider a minimizing sequence $\{E_h\}_{h\in\mathbb{N}}$ for (1.1), so that in particular

$$C=\sup_{h\in\mathbb{N}}\mathcal{F}(E_h)+\mathcal{G}(E_h)<+\infty\,. \tag{4.3}$$

By standard lower semicontinuity and compactness theorems for sets of finite perimeter, the existence of a minimizer is proved by showing that for every $\varepsilon>0$ there exists $R>0$ such that

$$\sup_{h\in\mathbb{N}}|E_h\setminus K_R|<\varepsilon\,.$$

This follows easily from the bound

$$|E_h\setminus K_R|\,\Phi_g(R)\le\int_{|E_h\setminus K_R|} g(x)\,dx\le\mathcal{G}(E_h)\le C\,,$$

together with the fact that $\Phi_g(R)\to+\infty$ as $R\to+\infty$. $\square$

4.1. **Equilibrium shapes as (ε, R)-minimizers.** The application of the results of section 3 to the minimizers of the variational problem (1.1) requires, roughly speaking, to prove that for every $m_0 > 0$ there exists $t(m_0) > 0$ such that every minimizer E with $|E| = m \le m_0$ satisfies the confinement $E \subset \{g \le t(m_0)\}$. The proof of this property will be different depending on the size of m_0. If m_0 is small then we have to exploit the domination of the surface energy $\mathcal{F}$ over the potential energy $\mathcal{G}$ in combination with Theorem 3.4. On the other hand, when the mass m_0 increases the potential energy plays a stronger role and it is the coerciveness of g that prevents minimizers to spread any mass far away from the origin (recall the normalization (4.2)). The small mass regime is addressed in Theorem 4.2 and Corollary 4.3, thus leading to the proof of Theorem 1.1. The complementary case is discussed separately in Section 4.2, Theorem 4.5.

Theorem 4.2. *There exist positive constants $m_0 = m_0(n, f, g) > 0$ and $C = C(n, f, g)$ with the following property: If E is a minimizer in (1.1) with mass $|E| = m \le m_0$, then E is connected and there exists $x_0 \in \mathbb{R}^n$ and $r_0 > 0$ with*

$$r_0 \le C(n, f, g) m^{1/n^2},$$

such that

$$x_0 + K_{s(m)(1-r_0)} \subset E \subset x_0 + K_{s(m)(1+r_0)},$$

where we have set

$$s(m) = \left(\frac{m}{|K|}\right)^{1/n}.$$

Moreover, either $f_*(x_0) \le n\, s(m)$ or

$$\Phi_g(f_*(x_0) - n\, s(m)) \le 2\Psi_g(s(m)). \tag{4.4}$$

Proof. We set for brevity $s = s(m)$, so that $|K_s| = m$. In the first two steps below, we assume that $B_r \subset K \subset B_{nr}$ for some $r = r(n, |K|) > 0$. Then in Step three we will show how to remove such assumption.

Step one: Bound on x_0. If E is a minimizer for (1.1) at mass m, then, taking also the Wulff inequality into account, we find

$$\mathcal{F}(E) + \mathcal{G}(E) \le \mathcal{F}(K_s) + \mathcal{G}(K_s) \le \mathcal{F}(E) + \mathcal{G}(K_s).$$

In particular,

$$\mathcal{F}(E) \le \mathcal{F}(K_s) + \mathcal{G}(K_s), \qquad \mathcal{G}(E) \le \mathcal{G}(K_s).$$

From the first inequality, recalling the definition of the Wulff deficit $\delta(E)$ given in (3.10), we get

$$\delta(E) \le \frac{\mathcal{G}(K_s)}{\mathcal{F}(K_s)} = \frac{\int_{K_s} g}{n\,|K|^{1/n} m^{1/n'}} \le \frac{\Psi_g(s)\, s}{n}. \tag{4.5}$$

Hence, thanks to (3.11), there exists a point $x_0 \in \mathbb{R}^n$ such that

$$\frac{|E \Delta (x_0 + K_s)|}{m} \le C(n) \sqrt{\Psi_g(s)\, s}. \tag{4.6}$$

Let us show that either $f_*(x_0)\le n\,s$ or (4.4) holds true. Since Ψ_g is locally bounded and $s\to 0$ as $m\to 0$, by (4.5) there exists $m_0=m_0(n,f,g)>0$ such that, if $m<m_0$, then

$$\frac{|E\setminus(x_0+K_s)|}{m}\le\frac{1}{2}\,,\qquad\text{i.e.}\qquad |E\cap(x_0+K_s)|\ge\frac{m}{2}\,.$$

Moreover, if $f_*(x_0)>n\,s$ then $(x_0+K_s)\cap K_{f_*(x_0)-n\,s}=\emptyset$ (recall that $K=\{f_*<1\}$ and that $-K\subset nK$). Thus, when $f_*(x_0)>n\,s$, we have $g\ge\Phi_g(f_*(x_0)-n\,s)$ on $E\cap(x_0+K_s)$, and so

$$\frac{m}{2}\Phi_g(f_*(x_0)-n\,s)\le\mathcal{G}(E\cap(x_0+K_s))\le\mathcal{G}(E)\le\mathcal{G}(K_s)\le\Psi_g(s)\,m\,,$$

that is (4.4), as required. Since we are assuming that $B_r\subset K\subset B_{n\,r}$, f_* is comparable to the euclidean norm, and so this argument implies the existence of a constant $R_0=R_0(n,f,g)$ such that $|x_0|\le R_0$.

Step two: Connectedness and uniform proximity to x_0+K_s. Let us set

$$E'=s^{-1}\,(E-x_0)\,,$$

so that $|E'|=|K|$. By (4.5) and (4.6), and using that $\Psi_g(s)\le\Psi_g((m_0/|K|)^{1/n})$ we get

$$\delta(E')=\delta(E)\le C(n,f,g)\,s\,,\qquad |E'\Delta K|\le C(n,f,g)\sqrt{s}\,.$$

We observe that if F is such that $|F|=|E'|$ and $F\setminus E'\subset K_3$, then $|x_0+s\,F|=|E|$ with $(x_0+s\,F)\setminus E\subset x_0+K_{3\,s}$. Hence $x_0+K_{3\,s}\subset K_{R_1}$ for some $R_1=R_1(n,f,g)$, so that

$$\mathcal{F}(E)\le\mathcal{F}(x_0+s\,F)+\int_{(x_0+s\,F)\setminus E}g\le\mathcal{F}(x_0+s\,F)+\Psi_g(R_1)|(x_0+s\,F)\Delta E|\,,$$

i.e., since $\mathcal{F}(x_0+s\,F)=s^{n-1}\mathcal{F}(F)$, $\mathcal{F}(E')=s^{1-n}\mathcal{F}(E)$, and $|(x_0+s\,F)\Delta E|=s^n|F\Delta E'|$,

$$\mathcal{F}(E')\le\mathcal{F}(F)+\Psi_g(R_1)s\,|F\Delta E'|=\mathcal{F}(F)+C(n,f,g)s|F\Delta E'|\,.$$

Therefore, provided m (and so s) is small enough, by Theorem 3.4 we conclude that E' is connected, with

$$K_{1-r_0}\subset E'\subset K_{1+r_0}$$

for some $r_0>0$ with $r_0\le C(n,f,g)s^{1/n}$. Thus E is connected and

$$x_0+K_{s(1-r_0)}\subset E\subset x_0+K_{s(1+r_0)}\,,$$

as required.

Step three: Renormalization argument. In the two steps above we were assuming that $B_r\subset K\subset B_{n\,r}$ for some $r=r(n,|K|)>0$. Let us now consider the general case. By Lemma 2.2, $L(E)$ is a minimizer for $\mathcal{F}_{L(K)}+\mathcal{G}_L$ at mass $m=|L(E)|=|E|$. Hence, if $m_0=m_0(n,f,g)>0$ is as in Steps one and two above, then $L(E)$ is connected and there exists $x_1\in\mathbb{R}^n$ such that

$$x_1+L(K)_{s(1-r)}\subset L(E)\subset x_1+L(K)_{s(1+r)}\,,$$

where $r \le C(n,f,g)s^{1/n}$, $s = (|L(E)|/|L(K)|)^{1/n} = (|E|/|K|)^{1/n}$, and $f_{L(K),*}(x_1)$ satisfies the bounds in the statement. Hence, E is also connected and

$$x_0 + K_{s^{1/n}(1-r)} \subset E \subset x_0 + K_{s^{1/n}(1+r)} \qquad x_0 = L^{-1}x_1 \,.$$

Since $f_{K,*}(x_0) = f_{L(K),*}(x_1)$, this concludes the proof. □

We now apply Theorem 4.2 to prove a confinement result in the small mass regime. From the confinement property we shall deduce in Corollary 4.4 that minimizers in (1.1) are (ε, R)-minimizers, with ε small in terms of the mass, eventually proving Theorem 1.1.

Theorem 4.3. *Let $m_0 = m_0(n,f,g)$ as in Theorem 4.2. There exists a locally bounded increasing function $t : (0,+\infty) \to (0,+\infty)$, that is defined in terms of n, f, and g, with the property that if E is a minimizer in (1.1) with $|E| = m \le m_0$, then*

$$E \subset \{g < t(m)\} \,.$$

Moreover, if g is continuous and $\omega_g : [0,+\infty) \to [0,+\infty)$ denotes a modulus of continuity for g over the compact set $\{g \le t(m_0)\}$, then

$$t(m) \le \omega_g\left(Cm^{1/n}\right) \,,$$

where $C = C(n,f,g)$. In particular, $t(m) \to 0$ as $m \to 0^+$.

Proof. We let $\tilde\Phi_g : (0,+\infty) \to (0,+\infty)$ be a locally bounded increasing function such that $\tilde\Phi_g(\Phi_g(r)) \ge r$ for every $r > 0$. By Theorem 4.2, the minimizer E has to satisfy

$$E \subset x_0 + K_{\left(1+C(n,f,g)m^{1/n^2}\right)s(m)} \,, \tag{4.7}$$

where x_0 is such that

$$f_*(x_0) \le n\, s(m) + \tilde\Phi_g(2\Psi_g(s(m))) \,.$$

Since $K_r = \{f_* < r\}$ and $K_r + K_s = K_{r+s}$, we conclude that $E \subset K_{r(m)}$ for

$$r(m) = n\, s(m) + \tilde\Phi_g(2\Psi_g(s(m))) + (1 + C(n,f,g)m^{1/n^2})s(m) \,.$$

This shows that the minimizers in (1.1) of mass m are uniformly bounded in $\mathbb{R}^n$, and thus allows defining $t(m)$ as the infimum of those $t > 0$ such that every minimizer in (1.1) with mass m is contained in $\{g < t\}$.

Finally, let us assume that g is continuous. As in the Step one of the proof of Theorem 4.2 we have $\mathcal{G}(E) \le \mathcal{G}(K_{s(m)})$, so that since $g(0) = 0$ we get

$$\mathcal{G}(E) \le \mathcal{G}(K_{s(m)}) \le \omega_g\big(\mathrm{diam}(K_{s(m)})\big)\, m \le \omega_g\left(C(f)m^{1/n}\right)\, m \,.$$

Moreover, since $E \subset \{g \le t(m_0)\}$ and $\mathrm{diam}(E) \le \left(1 + C(n,f,g)m^{1/n^2}\right) s(m)$ by (4.7), we obtain

$$\begin{aligned}
\sup_E g &\le \omega_g(\mathrm{diam}(E)) + \frac{\mathcal{G}(E)}{|E|} \\
&\le \omega_g\left(\left(1 + C(n,f,g)m^{1/n^2}\right) s(m)\right) + \omega_g\left(C(f)m^{1/n}\right) \\
&\le \omega_g\left(C(n,f,g)m^{1/n}\right) \,,
\end{aligned}$$

as desired. □

Corollary 4.4. *Let $m_0 = m_0(n, f, g)$ be as in Theorem 4.2. For every $R > 0$ there exists a constant $C = C(n, f, g, R)$ such that, if E is a minimizer in (1.1) with $|E| = m \le m_0$, then E is a (ε, R)-minimizer for $\mathcal{F}$, with*

$$\varepsilon = C(n, f, g, R)\, m^{1/n} .$$

In particular

$$\mathcal{F}(E) \le \mathcal{F}(K_{s(m)})(1 + C(n, f, g, R)m^{2/n}) , \tag{4.8}$$

where $s(m) = (m/|K|)^{1/n}$.

Proof. By Theorem 4.3, $E \subset \{g < t(m)\}$, where $t : (0, \infty) \to (0, \infty)$ is an increasing function that depends on n, f, and g only. If now $F \subset I_R(E)$ and $|F| = m$, then $F \subset I_R(\{g \le t(m_0)\})$. Since g is locally bounded and $I_R(\{g \le t(m_0)\})$ is a bounded set, we find that

$$M(n, f, g, R) = \sup_{I_R(\{g\le t(m_0)\})} g < \infty .$$

Since $\mathcal{E}(E) \le \mathcal{E}(F)$ and $2|F \setminus E| = |E\Delta F|$ we thus deduce that

$$\begin{aligned} \mathcal{F}(E) &\le \mathcal{F}(F) + \int_{F\setminus E} g \le \mathcal{F}(F) + M\,|F \setminus E| \\ &= \mathcal{F}(F) + \frac{M}{2|K|^{1/n}}\, m^{1/n}|K|^{1/n} \frac{|F\Delta E|}{|E|^{1/n}} . \end{aligned}$$

Hence E is a (ε, R)-minimizer with $\varepsilon = C(n, f, g, R)\, m^{1/n}$ and

$$C(n, f, g, R) = \frac{M(n, f, g, R)}{2|K|^{1/n}} .$$

Then (4.8) is an immediate consequence of (3.12). □

Proof of Theorem 1.1. Let $m_0 = m_0(n, f, g)$ be the constant of Theorem 4.2. If E is a minimizer in (1.1) with $|E| = m$ and $m \le m_0$, then the first part of the theorem follows immediately by Theorem 4.2. Let us now assume that $n = 2$ and let ε_0 be a constant such that Theorem 3.6 and Theorem 3.7 hold true. If $C(n, f, g, 3)$ is the constant of Corollary 4.4 relative to $R = 3$ and if $m \le \varepsilon_0\, C(n, f, g, 3)^{-n}$, then E is convex (by Theorem 3.6) and, provided f is crystalline, E is a convex polygon with sides parallel to that of K (by Theorem 3.7). The proof of Theorem 1.1 is then completed by setting $m_c = \min\{m_0, \varepsilon_0\, C^{-n}\}$. □

4.2. **A general confinement result.** Here, we consider the situation for optimal shapes above the critical mass m_0 of Theorem 4.2. It is convenient to introduce a notation for the infimum in (1.1)

$$\mathfrak{e}(m) = \inf\big\{\mathcal{F}(E) + \mathcal{G}(E) : |E| = m\big\} , \qquad m > 0 ,$$

and for the smallest radius R such that K_R contains every minimizer of mass m

$$R(m) = \inf\big\{R > 0 \,:\, E \subset K_R \text{ for every } E \in \operatorname{argmin} \mathfrak{e}(m)\big\} \in (0, +\infty] .$$

Under a very mild growth condition on g expressed in (4.9) below, we can prove that $R(m)$ is locally bounded as a function of m. We notice that (4.9) is trivially satisfied if g

has locally bounded gradient (for example, if $g \in C^1(\mathbb{R}^n)$). We also recall that g satisfies (4.1) and (4.2).

Theorem 4.5 (A uniform bound with the sub-level sets of g)**.** *Assume that g is continuous and that for every $R > 0$ there exist two constants $\alpha_0 = \alpha_0(R) > 0$ and $\lambda_0 = \lambda_0(R) > 0$, such that*

$$g((1+\lambda)x) \le (1+\alpha_0\,\lambda)\,g(x) + \alpha_0\,\lambda\,, \tag{4.9}$$

whenever $|x| < 2R$ and $0 \le \lambda < \lambda_0(R)$. Then for every $\bar m > 0$ there exists $t(\bar m) > 0$ (depending also on n, f, and g), such that

$$E \subset \{g \le t(\bar m)\}\,,$$

for every minimizer E in (1.1) *with $|E| = m \le \bar m$.*

Proof. Step one. Let E be a minimizer in (1.1) with $|E| = m$ and let R_E be defined as

$$R_E = \inf\{R > 0 : E \subset K_R\} \in (0, +\infty]\,.$$

We now prove that, if $\sigma_0 = \sigma_0(n, f) > 0$ is a suitably small constant which will be fixed later, and $R_0 > 0$ has the property that

$$|E \setminus K_{R_0}| \le \sigma_0 m\,, \tag{4.10}$$

then, either $R_E \le 2R_0$, or

$$\Phi_g(R_E) \le C(n, f, g, R_0)\left(1 + \frac{\mathfrak{e}(m)}{m}\right)\,. \tag{4.11}$$

To this end, we may directly assume that $R_E \in (2\,R_0, +\infty]$, so that

$$\delta_0 = \frac{|E \setminus K_{2\,R_0}|}{m} > 0\,.$$

Hence, for every $\delta \in (0, \delta_0)$, we can find $R_\delta \in (2\,R_0, R_E)$ such that

$$|E \cap K_{R_\delta}| = (1-\delta)m\,. \tag{4.12}$$

Let us consider the Lipschitz decreasing function $\varphi : [0, +\infty) \to [0, 1]$ defined by

$$\varphi(R) = 1\,, \qquad \text{for } R \in (0, R_0)\,, \tag{4.13}$$

$$\varphi(R) = 0\,, \qquad \text{for } R \in (2\,R_0, +\infty)\,, \tag{4.14}$$

$$\varphi'(R) = -\frac{1}{R_0}\,, \qquad \text{for } R \in (R_0, 2R_0)\,, \tag{4.15}$$

and correspondingly define a family of Lipschitz maps $T_\lambda : \mathbb{R}^n \to \mathbb{R}^n$ $(0 < \lambda < 1)$, by setting

$$T_\lambda(x) = (1 + \lambda\,\varphi(f_*(x)))x\,, \qquad x \in \mathbb{R}^n\,.$$

Notice that $T_\lambda(x) = (1 + \lambda\varphi(R))x$ for every $x \in \partial K_R = \{f_* = R\}$, so that $T_\lambda(\partial K_R) = \{f_* = R(1 + \lambda\,\varphi(R))\} = \partial K_{R(1+\lambda\,\varphi(R))}$. If $\lambda \in (0, 1/2)$ then $R \mapsto R(1 + \lambda\,\varphi(R))$ is strictly increasing. Hence, for every $\lambda \in (0, 1/2)$, we have that T_λ is injective on $\mathbb{R}^n$. Also, taking into account that

$$T_\lambda(x) = (1+\lambda)x\,, \qquad \text{if } x \in K_{R_0}\,, \tag{4.16}$$

$$T_\lambda(x) = x\,, \qquad \text{if } x \in \mathbb{R}^n \setminus K_{2R_0}\,, \tag{4.17}$$

by the area formula, we find

$$\begin{aligned} |T_\lambda(E\cap R_\delta)| = \int_{E\cap R_\delta} JT_\lambda(x)dx &= (1+\lambda)^n|E\cap K_{R_0}| + |E\cap(K_{R_\delta}\setminus K_{2R_0})| \\ &\quad + \int_{E\cap(K_{2R_0}\setminus K_{R_0})} JT_\lambda(x)dx\,. \end{aligned} \tag{4.18}$$

We now observe that, by (4.16), (4.17), (4.10), and (4.12),

$$\begin{aligned} & (1+\lambda)^n|E\cap K_{R_0}| + |E\cap(K_{R_\delta}\setminus K_{2R_0})| \\ \geq\ & (1+n\lambda)(m-|E\setminus K_{R_0}|) + |E\cap(K_{R_\delta}\setminus K_{2R_0})| \\ =\ & (1+n\lambda)m - n\lambda|E\setminus K_{R_0}| + (|E\cap(K_{R_\delta}\setminus K_{2R_0})| - |E\setminus K_{R_0}|) \\ \geq\ & (1+n(1-\sigma_0)\lambda)\,m - (|E\setminus K_{R_\delta}| + |E\cap(K_{2R_0}\setminus K_{R_0})|) \\ =\ & (1+n(1-\sigma_0)\lambda-\delta)\,m - |E\cap(K_{2R_0}\setminus K_{R_0})|\,, \end{aligned}$$

that, combined with (4.18), leads to

$$|T_\lambda(E\cap R_\delta)| \geq (1+n(1-\sigma_0)\lambda-\delta)\,m + \int_{E\cap(K_{2R_0}\setminus K_{R_0})} (JT_\lambda(x)-1)\,dx\,. \tag{4.19}$$

Now, for every $x\in\mathbb{R}^n$, we have

$$\nabla T_\lambda(x) = (1+\lambda\,\varphi(f_*(x)))\mathrm{Id}_{\mathbb{R}^n} + \lambda\varphi'(f_*(x))x\otimes\nabla f_*(x)\,.$$

Moreover, (2.6) and (2.7) imply

$$\left|\frac{x}{f_*(x)}\right|\,|\nabla f_*(x)| \leq \frac{\alpha_2}{\alpha_1}\,.$$

Hence, using (4.13), (4.14), and (4.15) we get

$$|\nabla T_\lambda(x) - \mathrm{Id}| \leq \lambda\left\{|\mathrm{Id}_{\mathbb{R}^n}| - \frac{\alpha_2}{\alpha_1}\,\varphi'(f_*(x))f_*(x)\right\} \leq C(n,f)\lambda\,, \tag{4.20}$$

and so, in particular,

$$JT_\lambda - 1 \geq -C(n,f)\lambda\,,$$

for all $\lambda\in(0,1/2)$. Hence, by (4.19) and (4.10) we find

$$\begin{aligned} |T_\lambda(E\cap R_\delta)| &\geq (1+n(1-\sigma_0)\lambda-\delta)\,m - C(n,f)\lambda|E\cap(K_{2R_0}\setminus K_{R_0})| \\ &\geq \Big(1+\big[n-(n+C(n,f))\sigma_0\big]\lambda-\delta\Big)m = \Big(1+\frac{n}{2}\lambda-\delta\Big)\,m\,, \end{aligned}$$

provided we set $\sigma_0 = \sigma_0(n,f) = n/(2n+2C(n,f))$, with $C(n,f)$ as above. So,

$$\lambda > \frac{4\,\delta}{n} \quad\Rightarrow\quad |T_\lambda(E\cap K_{R_\delta})| > m\,,$$

while on the other hand

$$\lim_{\lambda\to 0^+} |T_\lambda(E\cap K_{R_\delta})| = (1-\delta)m\,.$$

By continuity, there exist $\lambda\in(0,4\delta/n)$ such that if we set

$$F = T_\lambda(E\cap K_{R_\delta})$$

then $|F| = m$. We now estimate the free energy of F. By (4.16) and $R_\delta > 2R_0$ we have

$$\mathcal{F}(F; K_{R_0}) = (1+\lambda)^{n-1}\mathcal{F}(E; K_{R_0})\,, \tag{4.21}$$

while by (4.17) and again by $R_\delta > 2R_0$, we also have

$$\mathcal{F}(F; \mathbb{R}^n \setminus K_{2R_0}) = \mathcal{F}(E \cap K_{R_\delta}; \mathbb{R}^n \setminus K_{2R_0}) \le \mathcal{F}(E; \mathbb{R}^n \setminus K_{2R_0})\,. \tag{4.22}$$

Moreover, by (4.20) and since $R_\delta > 2R_0$,

$$\mathcal{F}(F; K_{2R_0} \setminus K_{R_0}) \le (1 + C(n,f)\lambda)\mathcal{F}(E; K_{2R_0} \setminus K_{R_0})\,. \tag{4.23}$$

Correspondingly, on adding up (4.21), (4.22), and (4.23) we conclude that

$$\mathcal{F}(F) \le (1 + C(n,f)\,\lambda)\,\mathcal{F}(E)\,, \tag{4.24}$$

for all $\lambda \in (0, 1/2)$. Concerning the potential energy of F, we first notice that

$$\int_F g = \int_{E\cap R_\delta} (g \circ T_\lambda)\, JT_\lambda\,. \tag{4.25}$$

Since $\lambda < 4\delta/n$, by choosing $\delta > 0$ sufficiently small we can ensure that $\lambda < \lambda_0(R_0)$. Hence by (4.16) and by (4.9) we find

$$\begin{aligned}
\int_{E\cap K_{R_0}} (g\circ T_\lambda)\, JT_\lambda &= (1+\lambda)^n \int_{E\cap K_{R_0}} g((1+\lambda)x)\, dx \\
&\le (1 + C(n)\lambda) \int_{E\cap K_{R_0}} [(1+\alpha_0\lambda)g(x) + \alpha_0\lambda]\, dx \\
&\le (1 + C(n,g,R_0)\lambda) \int_{E\cap K_{R_0}} g(x)\, dx \\
&\qquad + C(n,g,R_0)\lambda\, |E \cap K_{R_0}|\,,
\end{aligned} \tag{4.26}$$

where $\alpha_0 = \alpha_0(R_0)$. By (4.17) and by (4.12) we find that

$$\begin{aligned}
\int_{E\cap(K_{R_\delta}\setminus K_{2R_0})} (g\circ T_\lambda)\, JT_\lambda &= \int_{E\setminus K_{2R_0}} g - \int_{E\setminus K_{R_\delta}} g \\
&\le \int_{E\setminus K_{2R_0}} g(x)\, dx - \delta\Phi_g(R_\delta)\, m\,.
\end{aligned} \tag{4.27}$$

Eventually, by (4.20) and by (4.9), taking also into account that $0 \le \varphi \le 1$, we conclude that

$$\begin{aligned}
\int_{E\cap(K_{2R_0}\setminus K_{R_0})} (g\circ T_\lambda)\, JT_\lambda &\le (1 + C(n,f)\lambda) \int_{E\cap(K_{2R_0}\setminus K_{R_0})} [(1+\alpha_0\lambda)g(x) + \alpha_0\lambda]\, dx \\
&\le (1 + C(n,f,g,R_0)\lambda) \int_{E\cap(K_{2R_0}\setminus K_{R_0})} g(x)\, dx \\
&\qquad + C(n,f,g,R_0)\lambda |E \cap (K_{2R_0} \setminus K_{R_0})|\,.
\end{aligned} \tag{4.28}$$

From (4.25), on adding up (4.26), (4.27), and (4.28), we find

$$\mathcal{G}(F) \le (1 + C(n,f,g,R_0)\lambda)\mathcal{G}(E) + C(n,f,g,R_0)\lambda\, m - \delta\Phi_g(R_\delta)m\,. \tag{4.29}$$

Since E is a minimizer in (1.1), by (4.24) and by (4.29) we get

$$\mathfrak{e}(m) \le \mathcal{F}(F) + \mathcal{G}(F) \le \mathfrak{e}(m) - m\,\delta\,\Phi_g(R_\delta) + C(n,f,g,R_0)\lambda\,(\mathfrak{e}(m)+m)\,,$$

and noticing that $\lambda < 4\delta/n$ we finally obtain

$$\Phi_g(R_\delta) \le C(n,f,g,R_0)\left(1+\frac{\mathfrak{e}(m)}{m}\right). \tag{4.30}$$

By (4.30) we deduce that R_δ is bounded as $\delta \to 0^+$, and since $R_\delta \to R_E$ as $\delta \to 0^+$ we conclude that $R_E < +\infty$. Moreover, since g is continuous, the function Φ_g is also continuous, which proves (4.11).

Step two. We conclude the proof of the theorem. By Theorem 4.3 we can directly assume that $\overline{m} > m_0$, and limit ourselves to consider minimizers E such that

$$m_0 \le m = |E| < \overline{m}\,. \tag{4.31}$$

If we set $s = s(m) = (m/|K|)^{1/n}$ as in Theorem 4.2, then for every such E we find that

$$\mathfrak{e}(m) = \mathcal{F}(E) + \mathcal{G}(E) \le \mathcal{F}(K_s) + \mathcal{G}(K_s) \le n|K|^{1/n}m^{1/n'} + \Psi_g(s)\,m\,. \tag{4.32}$$

In particular, for every $R > 0$, we have that

$$|E \setminus K_R| \le \frac{\mathcal{G}(E)}{\Phi_g(R)} \le \frac{n|K|^{1/n}m^{1/n'} + \Psi_g(s)\,m}{\Phi_g(R)}\,,$$

so that we can define an increasing function $R_0 = R_0(m,n,f,g)$ (depending on n, f, and g, but independent from E), such that

$$|E \setminus K_{R_0}| \le \delta_0\,m\,, \tag{4.33}$$

where $\delta_0 = \delta_0(n,f)$ was introduced in (4.10) above. By step one (see, in particular, (4.11)), defining $\tilde{\Phi}_g$ as in the proof of Theorem 4.3, we obtain

$$R(m) \le \min\left\{2\,R_0(m,n,f,g)\,,\tilde{\Phi}_g\left(C(n,f,g,R_0(m,n,f,g))\left(\frac{\mathfrak{e}(m)}{m}+1\right)\right)\right\}. \tag{4.34}$$

Moreover, (4.31) and (4.32) give

$$\frac{\mathfrak{e}(m)}{m} \le \frac{n|K|^{1/n}m^{1/n'} + \Psi_g(s)\,m}{m} \le \frac{n|K|^{1/n}}{m_0^{1/n}} + \Psi_g(s(\bar{m}))\,,$$

which combined with (4.34) implies

$$E \subset K_{R(\bar{m})}\,,$$

with $R(\bar{m}) \le C(n,f,g,m_0,\bar{m})$ as wanted. □

4.3. **Smoothness and convexity in the uniformly elliptic case.** In this section we consider minimizers for uniformly elliptic surface tensions and address their smoothness and convexity in the small regime, eventually proving Theorem 1.3. On combining Lemma 4.4 with the results from Section 3.4 we now come to the following theorem.

Theorem 4.6 ($C^{2,\alpha}$-regularity of minimizers)**.** *If $\alpha \in (0,1)$, f is λ-elliptic, $f \in C^{2,\alpha}(\mathbb{R}^n \setminus \{0\})$, $g \in C^{0,\alpha}_{loc}(\mathbb{R}^n)$, then there exist positive constants $m_0 = m_0(n,f,g)$, $\eta_0 = \eta_0(f)$, $r_0 = r_0(n,f)$, $N = N(n,f) \in \mathbb{N}$ and $C = C(n,f,g,\alpha)$ with the following property:*

If E is a minimizer in (1.1) *with $|E| = m \le m_0$, then there exist $Q_i : \mathbb{R}^n \to \mathbb{R}^n$ and $u_i : \mathbb{R}^{n-1} \to \mathbb{R}$, $1 \le i \le N$, such that*

(i) *each Q_i is an isometry of $\mathbb{R}^n$;*
(ii) *each u_i belongs to $C^{1,\alpha}(\mathbf{D}(r_0)) \cap C^{2,\alpha}(\mathbf{D}(r_0/2))$, with*
$$\|\nabla' u_i\|_{C^{1,\alpha}(\mathbf{D}(r_0/2))} \le \frac{C(n,f,g,\alpha)}{m^{1/n}}\,;$$
(iii) *we have*
$$\partial E = \bigcup_{i=1}^N Q_i\big(\mathrm{graph}(u_i) \cap \mathbf{C}(r_0,\eta_0)\big) = \bigcup_{i=1}^N Q_i\big(\mathrm{graph}(u_i) \cap \mathbf{C}(r_0/2,\eta_0)\big)\,.$$

Remark 4.7. We can write the result of Theorem 4.6 in the more suggestive form
$$\|\partial E\|_{C^{2,\alpha}} \le \frac{C(n,f,g,\alpha)}{m^{1/n}}\,. \tag{4.35}$$
Hence, if we set $F = (|K|/m^{1/n})E$ so that $|F| = |K|$, then
$$\|\partial F\|_{C^{2,\alpha}} \le C(n,f,g,\alpha)\,. \tag{4.36}$$

Proof of Theorem 4.6. Let us apply Theorem 3.8 with $\bar\alpha = 1/2$, and let $\eta_0 = \eta_0(f)$, $\varepsilon = \varepsilon(n,f,1/2)$, $r_0 = r_0(n,f,1/2)$, $L = L(n,f,1/2)$, and $N = N(n,f,1/2) \in \mathbb{N}$ be the constants provided by that theorem. By Corollary 4.4 applied with $R = n+1$, provided $m_0 = m_0(n,f,g)$ is small enough, we see that a minimizer E in (1.1) with $|E| = m \le m_0$ is, in fact, a $(\varepsilon, n+1)$-minimizer for $\mathcal{F}$. In particular, by Theorem 3.8, there exist $Q_i : \mathbb{R}^n \to \mathbb{R}^n$ and $u_i : \mathbb{R}^{n-1} \to \mathbb{R}$, $1 \le i \le N$, such that properties (i) and (iii) hold true, and moreover
$$\|\nabla' u\|_{C^{0,1/2}(\mathbf{D}(r_0))} \le L\,.$$
As seen in Appendix A.2, Schauder theory gives
$$\|\nabla' u_i\|_{C^{1,\alpha}(\mathbf{D}(r_0/2))} \le C(n,r_0,L,\alpha)\|h_i\|_{C^{0,\alpha}(\mathbf{D}(r_0))}\,, \tag{4.37}$$
where $h_i(z) = g(z,u_i(z)) - \mu$, $z \in \mathbb{R}^{n-1}$, and μ satisfies
$$\mu = \frac{(n-1)\mathcal{F}(E) + \int_{\partial^* E} g\, x\cdot\nu_E\, d\mathcal{H}^{n-1}}{n|E|}\,, \tag{4.38}$$
see (A.3). To deduce (ii) from (4.37) we have to provide a bound on μ. To this end we exploit the minimality of E to see that
$$\mathcal{F}(E) + \mathcal{G}(E) \le \mathcal{F}(E+tv) + \mathcal{G}(E+t\,v) \qquad \forall\, t \in \mathbb{R},\, v \in \mathbb{S}^{n-1}\,.$$

Since $\mathcal{F}(E)=\mathcal{F}(E+tv)$, letting $t\to 0$ we obtain that

$$\int_{\partial E} g\,\nu_E\cdot v\,d\mathcal{H}^{n-1}=0\qquad\forall\ v\in\mathbb{S}^{n-1}\,,$$

i.e.,

$$\int_{\partial E} g\,\nu_E\,d\mathcal{H}^{n-1}=0\,.$$

We combine this condition with (4.38) to find that

$$\mu=\frac{(n-1)\mathcal{F}(E)+\int_{\partial E} g\,(x-x_0)\cdot\nu_E\,d\mathcal{H}^{n-1}}{n|E|}\,,\qquad\forall\ x_0\in\mathbb{R}^n\,. \tag{4.39}$$

Provided m_0 is small enough we can choose x_0 as in Theorem 4.3, and thanks to the Hölder continuity of g we get $t(m)\le C\,m^{\alpha/n}$, so that

$$\left|\int_{\partial E} g\,(x-x_0)\cdot\nu_E\,d\mathcal{H}^{n-1}\right|\le\Big(\max_{\partial E} g\Big)\,2\int_{\partial K_{s(m)}}|x|\,d\mathcal{H}^{n-1}\le C\,m^{1+\alpha/n}$$

where $s(m)=(m/|K|)^{1/n}$. Moreover, Lemma 3.3 gives

$$\mathcal{F}(E)=n\,|K|^{1/n}m^{1/n'}\big(1+O(m^{2/n})\big),$$

so that

$$\mu=\frac{n(n-1)\,|K|^{1/n}m^{1/n'}\big(1+O(m^{2/n})\big)+O(m^{1+\alpha/n})}{n\,m}=\frac{(n-1)\,|K|^{1/n}}{m^{1/n}}+O(m^{\alpha/n})\,.$$

Combining all together we conclude that

$$\max_{1\le i\le N}\|\nabla' u_i\|_{C^{1,\alpha}(\mathbf{D}(r_0/2))}\le\frac{C(n,f,g,\alpha)}{m^{1/n}}\,,$$

that is (ii), as required. □

4.3.1. *Second variation and convexity.* As recalled in Appendix A.1, under the assumptions of Theorem 4.6, a minimizer E in (1.1) satisfies the stationarity condition

$$H_f+g=\mu\qquad\text{on }\partial E\,, \tag{4.40}$$

where μ is characterized as in (4.38) and where H_f is the anisotropic mean curvature of ∂E, i.e.,

$$H_f=\mathrm{tr}(\mathrm{Hess}f(\nu_E)A_E)\,.$$

Here, and in the following, we let "grad" and "Hess" denote, respectively, the first and second tangential derivatives with respect to ∂E of a function and let A_E be the second fundamental form of ∂E, and compute the trace operator on $T_x\partial E$ (see Appendix A.1 and [13] for more details). To avoid confusion, we remark that $\mathrm{Hess}f(\nu_E)=\mathrm{Hess}f(\nu_E(x))$ denotes the Hessian of f restricted to tangent space $T_x\partial E$ evaluated at $\nu_E(x)$. When it is further assumed that $g\in C^1_{\mathrm{loc}}(\mathbb{R}^n)$ one can exploit the non-negativity of the second variation of the free energy with respect to normal variations to deduce the validity of the minimality condition

$$\int_{\partial E}\mathrm{grad}\zeta\cdot(\mathrm{Hess}f(\nu_E)\mathrm{grad}\zeta)-\zeta^2\left[\mathrm{tr}(\mathrm{Hess}f(\nu_E)A_E^2)-(\nabla g\cdot\nu_E)\right]\,d\mathcal{H}^{n-1}\ge 0\,, \tag{4.41}$$

on every $\zeta \in C^\infty_c(\mathbb{R}^n)$ satisfying the constraint

$$\int_{\partial E} \zeta \, d\mathcal{H}^{n-1} = 0\,. \tag{4.42}$$

In this section we exploit (4.41) to prove a quantitative bound on the L^2-distance of the second fundamental form of ∂E from that of ∂K. Once this is established, the uniform $C^{2,\alpha}$-bound on ∂E of Theorem 4.6 combined with standard interpolation inequalities allows us to deduce the C^0-proximity of the second fundamental form of ∂E to that of ∂K, implying in particular the convexity of E at small mass.

This argument should be clarified by an explanation of its origins. In the fundamental case that g is constant and f is isotropic, following Barbosa and do Carmo [7] (see also [52]), one tests (4.41) by means of

$$\zeta(x) = 1 - \beta(x - x_0)\cdot \nu_E(x)\,, \qquad \forall\, x \in \partial E\,, \tag{4.43}$$

(where β is determined by (4.42) and x_0 is arbitrary), to discover that the principal curvatures of ∂E have to be all equal to each other and constant. In particular, ∂E is forced to be an Euclidean sphere. If we still keep g constant, but now allow f to be anisotropic (with the smoothness required by Theorem 4.6), then, following Winklmann [53] (see also [37]), one sees that the same method applies to prove that E is a Wulff shape. In this case one has to modify the test function (4.43), and choose instead

$$\zeta(x) = f(\nu_E(x)) - \beta\,(x - x_0)\cdot \nu_E(x)\,, \qquad \forall\, x \in \partial E\,, \tag{4.44}$$

with β and x_0 as before. This time (4.41) shall force H_f to be a constant multiple of the identity, and hence E to be a Wulff shape.

In our situation, due to the small mass regime, we can consider the term $\nabla g \cdot \nu_E$ in (4.41) as a small perturbation and try to gain some information by using (4.41) with the test function used by Winklmann. Let us observe that, since now g is not constant, we have to work out the subsequent computations by replacing the constant anisotropic mean curvature condition $H_f = \mu$ with the stationarity condition (4.40). In this way we will prove that the quantity

$$\frac{1}{P(E)}\int_{\partial E} \|\mathrm{Hess} f(\nu_E) A_E - \mu\, \mathrm{Id}_{T_x\partial E}\|^2\, d\mathcal{H}^{n-1}\,,$$

is bounded in terms of n, f, and g only. As we will see, after a proper rescaling, this kind of bound implies the L^2-proximity of the second fundamental form of ∂E to the one of ∂K (see (4.63) below).

We now present the details of this argument. We denote by Δ_f the f-Laplace-Beltrami operator on ∂E, i.e.,

$$\Delta_f \zeta = \mathrm{Div}\,\big(\mathrm{Hess} f(\nu_E)\mathrm{grad}\zeta\big)\,, \qquad \zeta \in C^\infty_c(\partial E)\,,$$

where Div denotes the tangential divergence on ∂E, see also [13, Equation (1.13)]. It is also convenient to set

$$\begin{aligned} S_f &= \mathrm{Hess} f(\nu_E) A_E \qquad (\text{so that } H_f = \mathrm{tr}(S_f))\,,\\ \kappa_f &= \mathrm{tr}(\mathrm{Hess} f(\nu_E) A_E^2)\,. \end{aligned}$$

With this notation, (4.41) takes the form

$$\int_{\partial E} -\zeta\Delta_f\zeta - \zeta^2\left[\kappa_f - (\nabla g\cdot\nu_E)\right]\, d\mathcal{H}^{n-1} \geq 0\,. \tag{4.45}$$

We now prove a lemma that allows computing $\Delta_f\zeta$ when ζ is given by (4.44). The analogous formulas for the case that g is constant appears, of course, in [53, Theorem 3.1] (we notice however that we use a different convention for the sign of H_f).

Lemma 4.8. *Given $x_0 \in \mathbb{R}^n$ let*

$$h(x) = (x-x_0)\cdot\nu_E(x)\,, \qquad x\in\partial E\,. \tag{4.46}$$

Then the following identities hold true,

$$\begin{aligned} \Delta_f\nu_E + \kappa_f\,\nu_E &= -\mathrm{grad} g\,, && (4.47)\\ \Delta_f h + \kappa_f\, h &= -\mathrm{grad} g\cdot(x-x_0) + H_f\,, && (4.48)\\ \Delta_f\big(f(\nu_E)\big) + \kappa_f\, f(\nu_E) &= \mathrm{tr}\big(S_f^2\big) - \mathrm{grad} g\cdot\nabla f(\nu_E)\,. && (4.49)\end{aligned}$$

Proof. We observe that [13, Equation (1.19)], written with our notation, reads as

$$\Delta_f\nu_E + \big(\kappa_f - (\nabla g\cdot\nu_E)\big)\nu_E = -\nabla g.$$

By taking into account that

$$\nabla g = \mathrm{grad} g + (\nabla g\cdot\nu_E)\nu_E\,,$$

we immediately deduce (4.47). To prove (4.48), we denote by ∇_i the i-th tangential derivative on ∂E, and notice that $\nu_E(x)\cdot\nabla_i x = 0$ for every $x\in\partial E$. Thus, adopting Einstein's summation convention,

$$\begin{aligned} \Delta_f h &= \nabla_i\big(\nabla^2_{ij}f(\nu_E)(x-x_0)\cdot\nabla_j\nu_E\big)\\ &= \nabla^2_{ij}f(\nu_E)\nabla_i x\cdot\nabla_j\nu_E + (x-x_0)\cdot\nabla_i\big(\nabla^2_{ij}f(\nu_E)\nabla_j\nu_E\big)\\ &= \nabla^2_{ij}f(\nu_E)(A_E)_{ij} + (x-x_0)\cdot\Delta_f\nu_E\\ &= H_f - \kappa_f\, h - \mathrm{grad} g\cdot(x-x_0), \end{aligned}$$

where in the last step we have applied (4.48). We similarly prove (4.49), as

$$\begin{aligned} \Delta_f\big(f(\nu_E)\big) &= \nabla_i\big(\nabla^2_{ij}f(\nu_E)\nabla f(\nu_E)\cdot\nabla_j\nu_E\big)\\ &= \nabla^2_{ij}f(\nu_E)\nabla_i[\nabla f(\nu_E)]\cdot\nabla_j\nu_E + \nabla f(\nu_E)\cdot\Delta_f\nu_E\\ &= \nabla^2_{ij}f(\nu_E)\nabla^2_{kl}f(\nu_E)(A_E)_{ik}(A_E)_{jl} + \nabla f(\nu_E)\cdot\Delta_f\nu_E\\ &= \mathrm{tr}\big(S_f^2\big) - \kappa_f\,\nabla f(\nu_E)\cdot\nu_E - \mathrm{grad} g\cdot\nabla f(\nu_E)\,, \end{aligned}$$

and $\nabla f(\nu_E)\cdot\nu_E = f(\nu_E)$ by 1-homogeneity of f. □

Theorem 4.9. *If $\alpha\in(0,1)$, f is λ-elliptic, $f\in C^{2,\alpha}(\mathbb{R}^n\setminus\{0\})$, $g\in C^1_{loc}(\mathbb{R}^n)$, then there exist positive constants $m_0 = m_0(n,f,g)$ and $C = C(n,f,g,\alpha)$ with the following property: If E is a minimizer in (1.1) with $|E| = m\leq m_0$, then*

$$\frac{1}{P(E)}\int_{\partial E}\left\|\mathrm{Hess} f(\nu_E)A_E - \frac{\mathrm{Id}_{T_x\partial E}}{s(m)}\right\|^2 d\mathcal{H}^{n-1} \leq C\,, \tag{4.50}$$

where $s(m) = (m/|K|)^{1/n}$.

Proof. The constant m_0 is chosen so that Theorem 4.2 and Theorem 4.6 apply. In particular, we let x_0 be the point provided by Theorem 4.2, and correspondingly introduce the test function $\zeta = f(\nu_E) - \beta\, h$, where β is defined as

$$\beta = \frac{\mathcal{F}(E)}{n|E|}\,, \tag{4.51}$$

and where h is as in (4.46). Let us notice that by Corollary 4.4 and since $\mathcal{F}(K_{s(m)}) = n|K|^{1/n} m^{1/n'}$, we clearly have

$$\beta = \frac{\mathcal{F}(K_{s(m)})(1 + O(m^{2/n}))}{nm} = \frac{1}{s(m)} + O(m^{1/n})\,. \tag{4.52}$$

Moreover, by the divergence theorem,

$$\int_{\partial E} \zeta\, d\mathcal{H}^{n-1} = \int_{\partial E} f(\nu_E)\, d\mathcal{H}^{n-1} - \beta \int_{\partial E} (x - x_0)\cdot \nu_E\, d\mathcal{H}^{n-1} = \mathcal{F}(E) - \beta\, n|E| = 0\,,$$

so that that ζ satisfies (4.42). In particular, ζ is an admissible test function for (4.45). Thus,

$$\int_{\partial E} \zeta \Delta_f \zeta + \kappa_f\, \zeta^2\, d\mathcal{H}^{n-1} \le \int_{\partial E} (\nabla g \cdot \nu_E)\zeta^2\, d\mathcal{H}^{n-1}\,. \tag{4.53}$$

We now set for simplicity

$$f_E = f(\nu_E)\,,$$

and apply (4.48) and (4.49) to prove that

$$\begin{aligned} \Delta_f \zeta &= \Delta_f f_E - \beta \Delta_f h \\ &= -\kappa_f\, \zeta + \mathrm{tr}(S_f^2) - \mathrm{grad} g \cdot \nabla f_E + \beta\, \mathrm{grad} g \cdot (x - x_0) - \beta H_f\,. \end{aligned}$$

Therefore

$$\zeta \Delta_f \zeta + \kappa_f \zeta^2 = \zeta\Big(\mathrm{tr}\big(S_f^2\big) - \beta\, H_f - \mathrm{grad} g \cdot \nabla f_E + \beta\, \mathrm{grad} g \cdot (x - x_0)\Big)\,,$$

and we may deduce from (4.53) that

$$\begin{aligned} \int_{\partial E} (f_E - \beta h)\big(\mathrm{tr}\big(S_f^2\big) - \beta\, H_f\big)\, d\mathcal{H}^{n-1} \;\le\; & \int_{\partial E} (f_E - \beta h)^2 |\nabla g|\, d\mathcal{H}^{n-1} \\ & + \int_{\partial E} |f_E - \beta h||\nabla g||\nabla f_E|\, d\mathcal{H}^{n-1} \\ & + \beta \int_{\partial E} |f_E - \beta h||\nabla g||x - x_0|\, d\mathcal{H}^{n-1}. \end{aligned} \tag{4.54}$$

We now divide the proof of (4.50) in two steps.

Step one. We prove that the right hand side of (4.54) is controlled by $C\,P(E)$, where $C = C(n, f, g, \alpha)$, so that

$$\int_{\partial E} (f_E - \beta h)\big(\mathrm{tr}\big(S_f^2\big) - \beta\, H_f\big)\, d\mathcal{H}^{n-1} \le C\, P(E)\,. \tag{4.55}$$

To this end, we first notice that by our choice of m_0, Theorem 4.2, and by Theorem 4.3 we have the confinement estimates

$$E \subset x_0 + K_{s(m)}\,, \qquad\qquad s(m) = \left(\frac{m}{|K|}\right)^{1/n}\,, \tag{4.56}$$

and

$$E \subset \{g < t(m)\}\,, \tag{4.57}$$

where t as a locally bounded and increasing function of m depends on n, f, g, and α. By (4.57) and thanks to the local boundedness of ∇g, we find that

$$\sup_E |\nabla g| \le C\,. \tag{4.58}$$

By (4.8) we have $\mathcal{F}(E) \le C\,\mathcal{F}(K_{s(m)}) \le Cm^{1/n'}$. Hence, by (4.52) and by (4.57),

$$\beta \max_{x\in\partial E} |x - x_0| \le \frac{C}{m^{1/n}} \max_{x\in K_{s(m)}} |x| \le C\,. \tag{4.59}$$

On combining (4.58) and (4.59) we easily deduce (4.55) from (4.54), as desired.

Step two. We deduce (4.50) from (4.55). We first observe that, if we multiply (4.48) by f_E and (4.49) by h, then we find

$$\begin{aligned} f_E\,\Delta_f h &= -\kappa_f f_E h - f_E\,\mathrm{grad} g\cdot(x-x_0) + f_E\,H_f\,,\\ h\,\Delta_f f_E &= -\kappa_f f_E h + h\,\mathrm{tr}(S_f^2) - h(\mathrm{grad} g\cdot\nabla f_E)\,. \end{aligned}$$

Thus an integration by parts leads to the identity

$$-\int_{\partial E} f_E\,\mathrm{grad} g\cdot(x-x_0) - f_E\,H_f\,d\mathcal{H}^{n-1} = \int_{\partial E} h\,\mathrm{tr}\big(S_f^2\big) - h\,\mathrm{grad} g\cdot\nabla f_E\,d\mathcal{H}^{n-1}\,,$$

that by the same argument used in Step one leads to

$$\beta\int_{\partial E} f_E\,H_f - h\,\mathrm{tr}\big(S_f^2\big)\,d\mathcal{H}^{n-1} \le C\,P(E)\,.$$

We combine this estimate with (4.55) to find

$$\int_{\partial E} f_E\,\mathrm{tr}\big(S_f^2\big) - 2\beta\,f_E\,H_f + \beta^2 h\,H_f\,d\mathcal{H}^{n-1} \le C\,P(E)\,.$$

We rearrange the terms in the above expression to get

$$\begin{aligned} \int_{\partial E} f_E\,\mathrm{tr}\big(S_f^2\big) - 2\beta\,f_E\,H_f + \beta^2 h\,H_f\,d\mathcal{H}^{n-1} &= \int_{\partial E} f_E\left(\mathrm{tr}\big(S_f^2\big) - \frac{H_f^2}{n-1}\right)d\mathcal{H}^{n-1}\\ &\quad + \int_{\partial E} f_E\left(\frac{H_f}{\sqrt{n-1}} - \beta\sqrt{n-1}\right)^2 d\mathcal{H}^{n-1} + \beta^2\int_{\partial E}\big(h\,H_f - (n-1)\,f_E\big)\,d\mathcal{H}^{n-1}\,. \end{aligned}$$

The last term in the expression above vanishes. Indeed, since $H_f + g = \mu$ we have

$$\begin{aligned} &\int_{\partial E}\big(h\,H_f - (n-1)\,f_E\big)\,d\mathcal{H}^{n-1}\\ &= \int_{\partial E} h(\mu - g)\,d\mathcal{H}^{n-1} - (n-1)\mathcal{F}(E) \end{aligned}$$

$$= \quad n|E|\,\mu-(n-1)\mathcal{F}(E)-\int_{\partial E} g\,(x-x_0)\cdot\nu_E\,d\mathcal{H}^{n-1}=0\,.$$

where in the last equation we have used the identity (4.39). Taking into account that $f_E\ge\alpha_1$ (see (2.4)), we have thus proved that

$$\int_{\partial E}\left(\mathrm{tr}\big(S_f^2\big)-\frac{H_f^2}{n-1}\right)+\left(\frac{H_f}{\sqrt{n-1}}-\beta\sqrt{n-1}\right)^2 d\mathcal{H}^{n-1}\le C\,P(E)\,. \tag{4.60}$$

Now, as in the proof of [53, Theorem 1.1], we choose at every point $x\in\partial E$ an orthonormal basis $\{e_i(x)\}_{i=1,\dots,n-1}$ for $T_x\partial E$ which diagonalize $\mathrm{Hess}f_E(x)$, so that $\mathrm{Hess}f_E\,e_i=\gamma_i e_i$. In this way we find $S_f(e_i)=\sum_j s_{ij}\gamma_j e_j$, where $s_{ij}=e_j\cdot(A_E e_i)$. We now observe the following two algebraic identities: for any $N\in\mathbb{N}$, $\{\lambda_k\}_{k=1}^N\subset\mathbb{R}$, $b\in\mathbb{R}$, we have

$$\frac{1}{N}\left(\sum_k\lambda_k\right)^2 \;=\; \sum_k\lambda_k^2-\frac{1}{N}\sum_{k<h}(\lambda_k-\lambda_h)^2\,, \tag{4.61}$$

$$\frac{1}{N}\sum_{k<h}(\lambda_k-\lambda_h)^2 \;=\; \sum_k(\lambda_k-b)^2-\frac{1}{N}\left(\sum_k\lambda_k-Nb\right)^2\,. \tag{4.62}$$

By (4.61) with $N=n-1$ and $\lambda_k=\gamma_k s_{kk}$ we get

$$\begin{aligned}
\mathrm{tr}\big(S_f^2\big)-\frac{H_f^2}{n-1} &= \sum_{i,j}\gamma_i\gamma_j s_{ij}^2-\frac{1}{(n-1)}\left(\sum_i\gamma_i s_{ii}\right)^2\\
&= \sum_{i,j}\gamma_i\gamma_j s_{ij}^2-\sum_i(\gamma_i s_{ii})^2+\frac{1}{(n-1)}\sum_{i<j}(\gamma_i s_{ii}-\gamma_j s_{jj})^2\\
&= \sum_{i\ne j}\gamma_i\gamma_j s_{ij}^2+\frac{1}{(n-1)}\sum_{i<j}(\gamma_i s_{ii}-\gamma_j s_{jj})^2\,.
\end{aligned}$$

Applying now (4.62) with $b=\beta$ we find

$$\begin{aligned}
\mathrm{tr}\big(S_f^2\big)-\frac{H_f^2}{n-1} &= \sum_{i\ne j}\gamma_i\gamma_j s_{ij}^2+\sum_i(\gamma_i s_{ii}-\beta)^2-\frac{1}{n-1}\left(\sum_i\gamma_i s_{ii}-(n-1)\beta\right)^2\\
&= \|\mathrm{Hess}f_E A_E-\beta\,\mathrm{Id}_{T_x\partial E}\|^2-\left(\frac{H_f}{\sqrt{n-1}}-\beta\sqrt{n-1}\right)^2
\end{aligned}$$

Thanks to (4.60), we finally conclude

$$\begin{aligned}
&\int_{\partial E}\|\mathrm{Hess}f_E A_E-\beta\,\mathrm{Id}_{T_x\partial E}\|^2\,d\mathcal{H}^{n-1}\\
&= \int_{\partial E}\left(\mathrm{tr}\big(S_f^2\big)-\frac{H_f^2}{n-1}\right)+\left(\frac{H_f}{\sqrt{n-1}}-\beta\sqrt{n-1}\right)^2 d\mathcal{H}^{n-1}\\
&\le C(n,f,g,\alpha)P(E)\,,
\end{aligned}$$

and we get (4.50) by means of (4.52). $\square$

Proof of Theorem 1.3: We consider m_0 as in Theorem 4.9 and let $F = (|K|/m)^{1/n}E = s(m)^{-1}E$. Since $\mathrm{Hess} f(\nu_E)A_E = \nabla^2 f(\nu_E)\nabla\nu_E$, $\nu_F(y) = \nu_E(s(m)y)$ and $T_y\partial F = T_{s(m)x}\partial E$ for every $y \in \partial F$, by the change of variable $x = s(m)y$, we deduce that

$$\begin{aligned} & \frac{1}{P(E)}\int_{\partial E}\left\|\mathrm{Hess} f(\nu_E)A_E - \frac{\mathrm{Id}_{T_x\partial E}}{s(m)}\right\|^2 d\mathcal{H}^{n-1} \\ = & \frac{1}{P(F)}\int_{\partial F}\left\|\nabla^2 f(\nu_F)\frac{\nabla\nu_F}{s(m)} - \frac{\mathrm{Id}_{T_x\partial E}}{s(m)}\right\|^2 d\mathcal{H}^{n-1}. \end{aligned}$$

Hence, by (4.50),

$$\frac{1}{P(F)}\int_{\partial F}\left\|\nabla^2 f(\nu_F)\nabla\nu_F - \mathrm{Id}_{T_x\partial F}\right\|^2 d\mathcal{H}^{n-1} \le C\, m^{2/n}. \tag{4.63}$$

On the other hand by (4.36) we know that

$$\|\nabla\nu_F\|_{C^{0,\alpha}(\partial F)} \le C,$$

so that, in fact,

$$[\nabla^2 f(\nu_F)\nabla\nu_F]_{C^{0,\alpha}(\partial F)} \le C.$$

Now, thanks to the uniform bound (4.36), we can cover ∂F with $M = M(n, f, g, \alpha)$ balls of radius $r_0 = r_0(n, f, g, \alpha) > 0$ and apply Lemma 4.10 below (up to a diffeomorphism with uniform bi-Lipschitz norm) on each of these balls to get

$$\sup_{\partial F}\left\|\nabla^2 f(\nu_F)\nabla\nu_F - \mathrm{Id}_{T_x\partial F}\right\| \le C m^{2\alpha/(n+2\alpha)}.$$

Provided m is small enough with respect to n, we find that $\nabla^2 f(\nu_F)\nabla\nu_F$ is positive definite. In particular, $\nabla\nu_F$ is positive definite, so that F is convex. □

Lemma 4.10. *Let $u \in C^{0,\alpha}(B_1) \cap L^2(B_1)$. Then*

$$\sup_{B_{1/2}}|u| \le C(n)\,\left([u]_{C^{0,\alpha}(B_1)}^{n/(n+2\alpha)}\|u\|_{L^2(B_1)}^{2\alpha/(n+2\alpha)} + \|u\|_{L^2(B_1)}\right).$$

Proof. Given $x \in B_{1/2}$ and $y \in B_{1/2}(x) \subset B_1$, we write

$$|u(x)|^2 \le 2|u(y) - u(x)|^2 + 2|u(y)|^2.$$

We now consider $r \in (0, 1/2]$ (to be chosen later) and we integrate the above inequality with respect to y inside $B_r(x)$. Then we get

$$\begin{aligned} |u(x)|^2 & \le \frac{2}{\omega_n r^n}\int_{B_r(x)}|u(y) - u(x)|^2\,dy + \frac{2}{\omega_n r^n}\int_{B_r(x)}|u(y)|^2\,dy \\ & \le 2\,[u]_{C^{0,\alpha}(B_1)}^2\, r^{2\alpha} + \frac{2\,\|u\|_{L^2(B_1)}^2}{\omega_n r^n}. \end{aligned}$$

Now two cases arise, depending on whether or not $\|u\|_{L^2(B_1)} \le [u]_{C^{0,\alpha}(B_1)}$. In the first case we set

$$r = \frac{1}{2}\left(\frac{\|u\|_{L^2(B_1)}}{[u]_{C^{0,\alpha}(B_1)}}\right)^{2/(n+2\alpha)} \in (0, 1/2],$$

while in the second case we take $r = 1/2$. Taking the supremum over $x \in B_{1/2}$ finally leads to desired estimate. □

4.4. **A rigidity result for planar minimizers in the crystalline case.** In Theorem 3.7 we proved that if the surface tension f is crystalline and E is an $(\varepsilon, 3)$-minimizer with ε sufficiently small, then E is a convex polygon with normals coinciding with the ones of K. In lieu of Corollary 4.4, in the small mass regime the same happens to minimizers in (1.1). In this section we prove that, even outside of the small mass regime, planar crystals have a remarkably rigid structure. More precisely, assuming the continuity of g we are going to show that if f is crystalline then ∂E consists of two pieces, one of which is included in some level set $\{g = \ell\}$ and the other which is polygonal, with normal directions chosen among the normal directions to ∂K. We notice that, in this case, E may well be disconnected. For instance, on constructing an "ad hoc" potential g, we easily see that E may consist of the disjoint union of two Wulff shapes.

Theorem 4.11. *Let $n = 2$, let f be a crystalline surface tension, so that the Wulff shape K is a convex polygon with outer unit normals $\{\nu_i\}_{i=1}^N$, and let g be continuous. If E is a minimizer in (1.1), then there exists a constant $\ell > 0$ such that $\partial E = \Gamma_1 \cup \Gamma_2$, where*

$$\Gamma_1 \subset \{g = \ell\}, \qquad \nu(x) \in \{\nu_i\}_{i=1}^N \quad \text{at } \mathcal{H}^1\text{-a.e. } x \in \Gamma_2 \,.$$

Proof. We use the same notation as in the proof of Theorem 3.7. Let us recall that by Theorem 3.1, ∂E is differentiable at every point of $\partial^* E$.

Let $\bar{x} \in \partial^* E$, and assume that $\nu_E(\bar{x}) \notin \{\nu_i\}_{i=1}^N$. With no loss of generality, we can assume that $\nu_1 < \nu(\bar{x}) < \nu_2$. Fix $\varepsilon > 0$ small (to be chosen later), and let $r_0 = r_0(\bar{x}, \varepsilon) > 0$ be sufficiently small so that

$$\partial E \cap B(\bar{x}, r_0) \subset \bigl\{x : |(x - \bar{x}) \cdot \nu(\bar{x})| \le \varepsilon r_0\bigr\}$$

(the existence of such r_0 is ensured by the differentiability of ∂E at $\bar{x}$). We now construct two possible "perturbations" of E near $\bar{x}$ as follows:
Perturbation 1 (adding mass): For $r \in (0, r_0)$, let $x_{+,r} \le x'_{+,r}$ be the two points obtained by intersecting $\partial B(\bar{x}, r)$ with the line $\{x : (x - \bar{x}) \cdot \nu(\bar{x}) + \varepsilon r = 0\}$ (the notation $\le$ means that $x_{-,r}$ is to the left of $x'_{-,r}$ once we rotate the coordinates so that $\nu(\bar{x})$ points upward), and consider the rhomb

$$R_+(\bar{x}, r) := \bigl\{x : (x'_{+,r} - x) \cdot \nu_1 \le 0 \le (x_{+,r} - x) \cdot \nu_1,\, (x_{+,r} - x) \cdot \nu_2 \le 0 \le (x'_{+,r} - x) \cdot \nu_2\bigr\}.$$

With these choices it is not difficult to see that $\partial R_+(r) \setminus E$ is contained inside the lines

$$\{x : (x_{+,r} - x) \cdot \nu_1 = 0\} \quad \text{and} \quad \{x :\, (x'_{+,r} - x) \cdot \nu_2 = 0\}.$$

Moreover, if $\varepsilon = \varepsilon(\nu_1, \nu_2) > 0$ is sufficiently small then the set $R_+(\bar{x}, r) \setminus E$ has positive measure for every $r \in (0, r_0)$, and by continuity $|R_+(\bar{x}, r) \setminus E|$ can be any number $\delta \in (0, \delta_0)$, where $\delta_0 = |R_+(\bar{x}, r_0) \setminus E|$. Finally, by arguing as in the proof of Theorem 3.7, it is easily seen that there exists a point x_j such that, if $\nu(x)$ denotes the exterior normal to $\partial(E \cup R_+(\bar{x}, r))$, then $f(\nu(x)) = \nu(x) \cdot x_j$ for all $x \in \partial R_+(\bar{x}, r) \setminus E$ while $f(\nu(x)) \ge \nu(x) \cdot x_j$ for all $x \in \partial E \setminus R_+(r)$, which implies

$$\mathcal{F}(E \cup R_+(\bar{x}, r)) \le \mathcal{F}(E).$$

Perturbation 2 (removing mass): Let $x_{-,r} \le x'_{-,r}$ be the two points obtained by intersecting $\partial B(\bar{x}, r)$ with the line $\{x : (x - \bar{x}) \cdot \nu(\bar{x}) - \varepsilon r = 0\}$, and consider now the rhomb

$$R_{-}(\bar{x}, r) := \left\{x : (x'_{-,r} - x) \cdot \nu_1 \le 0 \le (x_{-,r} - x) \cdot \nu_1,\, (x_{-,r} - x) \cdot \nu_2 \le 0 \le (x'_{-,r} - x) \cdot \nu_2\right\}.$$

Then we can remove $R_{-}(\bar{x}, r)$ from E, and arguing as above we have

$$\mathcal{F}(E \setminus R_{-}(\bar{x}, r)) \le \mathcal{F}(E).$$

Now, fix two points $x_1, x_2 \in \partial^* E$, and assume that $\nu(x_1), \nu(x_2) \notin \{\nu_i\}_{i=1}^N$. We want to show that there is a value λ such that $x_1, x_2 \in \{g = \lambda\}$. By the arbitrariness of x_1, x_2 this will prove the result.

Fix a small constant $\delta > 0$, and choose $r_1 > 0$ such that $R^\delta_{1,+} = R^\delta_{+}(x_1, r_1)$ satisfies $|R^\delta_{1,+} \setminus E| = \delta$. (This can always be done if δ is sufficiently small.) Similarly, choose $r_2 > 0$ such that $R^\delta_{2,-} = R^\delta_{-}(x_2, r_2)$ satisfies $|E \setminus R^\delta_{2,-}| = \delta$. By choosing δ sufficiently small we can also ensure that $R^\delta_{1,+} \cap R^\delta_{2,-} = \emptyset$.

Let us compare E with the set $F := (E \cup R^\delta_{1,+}) \setminus R^\delta_{2,-}$. Since $|F| = |E|$ by the minimality of E, we have

$$\mathcal{F}(E) + \mathcal{G}(E) \le \mathcal{F}(F) + \mathcal{G}(F).$$

Moreover, since both $\mathcal{F}(E \cup R^\delta_{1,+})$ and $\mathcal{F}(E \setminus R^\delta_{2,-})$ are bounded by $\mathcal{F}(E)$, it is easily seen that $\mathcal{F}(F) \le \mathcal{F}(E)$, which implies

$$\mathcal{G}(E) \le \mathcal{G}(F) = \mathcal{G}(E) + \int_{R^\delta_{1,+} \setminus E} g(x)\, dx - \int_{E \setminus R^\delta_{2,-}} g(x)\, dx,$$

that is

$$\int_{R^\delta_{1,+} \setminus E} g(x)\, dx \le \int_{E \setminus R^\delta_{2,-}} g(x)\, dx.$$

Dividing both sides by δ and letting $\delta \to 0$, thanks to the continuity of g we get $g(x_1) \le g(x_2)$. By symmetry we also have $g(x_2) \le g(x_1)$, and the result follows. $\square$

Remark 4.12. The above result can be easily generalized to potentials which may take the value $+\infty$ on a closed set, and are continuous inside $\{g < +\infty\}$. In that case, our result becomes that $\partial E = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3$, where

$$\Gamma_1 \subset \{g = \lambda\}, \qquad \nu(x) \in \{\nu_i\}_{i=1}^N \quad \text{at } \mathcal{H}^1\text{-a.e. } x \in \Gamma_2, \qquad \Gamma_3 \subset \{g = +\infty\}.$$

Appendix A. Variation formulae and higher regularity

A.1. **First and second variation formulae.** In this section, we show how the energy $\mathcal{F} + \mathcal{G}$ changes under infinitesimal transformations. We compute the first variation of $\mathcal{F} + \mathcal{G}$ on a set of locally finite perimeter. Next, we recall from [13] the second variation formula of $\mathcal{F} + \mathcal{G}$ at a stationary set with smooth boundary.

If E is a set of finite perimeter and $\Phi : \mathbb{R}^n \to \mathbb{R}^n$ is a diffeomorphism, then $\Phi(E)$ is a set of finite perimeter, with

$$\nu_{\Phi(E)}\, \mathcal{H}^{n-1} \llcorner \partial^* \Phi(E) = (\Phi)_{\#} \left[J\Phi\, (\nabla \Phi^{-1} \circ \Phi)^* \nu_E\, \mathcal{H}^{n-1} \llcorner \partial^* E \right].$$

Given $T \in C^1_c(\mathbb{R}^n;\mathbb{R}^n)$ we define a one parameter family of diffeomorphisms $\{\Phi_t\}_{|t|<\varepsilon}$ by setting

$$\Phi_t(x) = x + t\,T(x)\,, \qquad x \in \mathbb{R}^n\,,$$

so that

$$\begin{aligned}
\mathcal{F}(\Phi_t(E)) &= \int_{\partial^* E} J\Phi_t\, f((\nabla\Phi_t^{-1}\circ\Phi_t)^*\nu_E)\,d\mathcal{H}^{n-1}\,,\\
\mathcal{G}(\Phi_t(E)) &= \int_E J\Phi_t (g\circ\Phi_t)\,,
\end{aligned}$$

are smooth functions of t in a neighborhood of $t=0$. It is well known that

$$\nabla\Phi_t^{-1}\circ\Phi_t = (\mathrm{Id} + t\nabla T)^{-1} = \mathrm{Id} - t\nabla T + t^2(\nabla T)^2 + O(t^3)\,,$$
$$J\Phi_t = 1 + t\,\mathrm{div}\, T + \frac{t^2}{2}\big((\mathrm{div}\, T)^2 - \mathrm{tr}(\nabla T)^2\big) + O(t^3)$$

(see, for instance, [42, Section 9]). Hence,

$$\begin{aligned}
\mathcal{F}(\Phi_t(E)) &= \int_{\partial^* E} J\Phi_t\, f((\nabla\Phi_t^{-1}\circ\Phi_t)^*\nu_E)\,d\mathcal{H}^{n-1}\\
&= \int_{\partial^* E} (1 + t\,\mathrm{div}\,T)\, f(\nu_E - t(\nabla T)^*\nu_E + O(t^2))\,d\mathcal{H}^{n-1} + O(t^2)\\
&= \int_{\partial^* E} (1 + t\,\mathrm{div}\,T)\, \big(f(\nu_E) - t\nabla f(\nu_E)\cdot(\nabla T)^*\nu_E\big)\,d\mathcal{H}^{n-1} + O(t^2)\\
&= \mathcal{F}(E) + t\int_{\partial^* E} \big(f(\nu_E)\,\mathrm{div}\,T - \nabla f(\nu_E)\cdot(\nabla T)^*\nu_E\big)\,d\mathcal{H}^{n-1} + O(t^2)\,,
\end{aligned}$$

i.e.,

$$\delta\mathcal{F}(E;T) = \int_{\partial^* E} \big(f(\nu_E)\,\mathrm{div}\,T - \nabla f(\nu_E)\cdot(\nabla T)^*\nu_E\big)\,d\mathcal{H}^{n-1}\,.$$

Analogously

$$\delta\mathcal{G}(E;T) = \int_{\partial^* E} g\,T\cdot\nu_E\,d\mathcal{H}^{n-1}\,. \tag{A.1}$$

By a standard argument based on these first order Taylor expansions, for every minimizer E of $\mathcal{F}+\mathcal{G}$ with volume constraint, there exists $\mu\in\mathbb{R}$ such that

$$0 = \delta\mathcal{F}(E;T) + \delta\mathcal{G}(E;T) - \mu\,\delta\mathcal{V}(E;T)\,. \tag{A.2}$$

We now characterize μ: consider the family of transformations given by $t\mapsto\Phi_t(E)$ with $\Phi_t(x) = x + tx$, i.e., $\Phi_t(E) = (1+t)^n E$ and $T = \mathrm{id}$ is the identity map. Then $\mathcal{F}(\Phi_t(E)) = (1+t)^{n-1}\mathcal{F}(E)$, $\mathcal{V}(\Phi_t(E)) = (1+t)^n\mathcal{V}(E)$, and by (A.1) and (A.2) we get

$$\begin{aligned}
0 &= \delta\mathcal{F}(E;\mathrm{id}) + \delta\mathcal{G}(E;\mathrm{id}) - \mu\,\delta\mathcal{V}(E;\mathrm{id})\\
&= (n-1)\mathcal{F}(E) + \int_{\partial^* E} g\,x\cdot\nu_E\,d\mathcal{H}^{n-1} - \mu\,n\,|E|\,,
\end{aligned}$$

that is

$$\mu = \frac{(n-1)\mathcal{F}(E) + \int_{\partial^* E} g\,x\cdot\nu_E\,d\mathcal{H}^{n-1}}{n|E|}\,. \tag{A.3}$$

Let us remark again that thus far we have just assumed to work with a generic set of finite perimeter. Let us now assume that E is an open set with smooth boundary, and let us restrict to consider normal variations T of the form

$$T = \zeta\, N\,,$$

where $\zeta \in C^\infty_c(\mathbb{R}^m)$ and N is the gradient of the signed distance function from ∂E. In particular, N is a smooth extension of ν_E in a neighborhood of ∂E, and $A = \nabla N$ is a field of symmetric tensors that extends the second fundamental form A_E of ∂E to a neighborhood of ∂E, with the property that $AN = 0$ and $N \cdot (Av) = 0$ for every $v \in \mathbb{R}^n$. Correspondingly, the first variation of $\mathcal{F}$ along $\zeta\, N$ takes the form

$$\delta\mathcal{F}(E;\zeta\, N) = \int_{\partial E} \zeta \mathrm{tr}(\mathrm{Hess} f(\nu_E) A_E)\, d\mathcal{H}^{n-1}\,.$$

Hence, if E is stationary for $\mathcal{F}+\mathcal{G}$ under a volume constraint, then (A.2) takes the form

$$\mathrm{tr}(\mathrm{Hess} f(\nu_E) A_E) + g = \mu\,, \qquad \text{on } \partial E, \tag{A.4}$$

where $\mathrm{Hess} f(\nu_E)$ denotes the Hessian of f restricted to the tangent space of ∂E evaluated at ν_E (see [13, Equation (1.10)]). Under (A.4), the second variation is given by

$$\delta^2(\mathcal{F}+\mathcal{G})(E;\zeta\, N) = \int_{\partial E} \mathrm{grad}\zeta \cdot (\mathrm{Hess} f(\nu_E) \mathrm{grad}\zeta) - \zeta^2 \left[\mathrm{tr}(\mathrm{Hess} f(\nu_E) A_E^2) - (\nabla g \cdot \nu_E)\right]\, d\mathcal{H}^{n-1}$$

(see [13, Corollary 4.2]), where $\mathrm{grad}\zeta$ is the tangential gradient on ∂E of the function ζ. Hence we have

$$\delta^2(\mathcal{F}+\mathcal{G})(E;\zeta\, N) \ge 0\,,$$

whenever $\zeta \in C^\infty_c(\mathbb{R}^n)$ is such that $\int_{\partial E} \zeta\, d\mathcal{H}^{n-1} = 0$.

A.2. **Euler-Lagrange equations on graphs and higher regularity.** We decompose $\mathbb{R}^n$ as $\mathbb{R}^{n-1}\times\mathbb{R}$ and let $\mathbf{p} : \mathbb{R}^n \to \mathbb{R}^{n-1}$ and $\mathbf{q} : \mathbb{R}^n \to \mathbb{R}$ denote the coordinate projections. We define $f^\# : \mathbb{R}^{n-1} \to (0,+\infty)$ on setting

$$f^\#(z) = f(-z,1)\,, \quad z \in \mathbb{R}^{n-1}\,.$$

If $u : \mathbb{R}^{n-1} \to \mathbb{R}$ is a Lipschitz function such that, for some $x \in \partial E$, $\eta_0 > 0$ and $r > 0$,

$$\begin{aligned} \mathbf{C}_x(r,\eta_0) \cap \partial^* E &= \{(z,u(z)) : z \in \mathbf{D}(\mathbf{p}x, r)\}\,,\\ \mathbf{C}_x(r,\eta_0) \cap E &= \{(z,t) : z \in \mathbf{D}(\mathbf{p}x, r)\,, -\eta_0 < t < u(z)\}\,, \end{aligned}$$

then (see, for example, [4, Proposition 2.85])

$$\nu_E(z,u(z)) = \frac{(-\nabla' u(z), 1)}{\sqrt{1+|\nabla' u(z)|^2}}\,, \quad z \in \mathbf{D}(\mathbf{p}x, r)\,,$$

where ∇' denotes the gradient operator on $\mathbb{R}^{n-1}$. In particular,

$$\mathcal{F}(E;\mathbf{C}(x,r)) = \int_{\mathbf{D}(\mathbf{p}x,r)} f^\#(\nabla' u(z))dz\,.$$

If now E is a volume-constrained local minimizer for $\mathcal{F}+\mathcal{G}$ in $\mathbf{C}(x,r)$, and if we set $G(z,t)=\int_0^t g(z,s)\,ds$ for $(z,t)\in\mathbb{R}^n$, then it is easily seen that u is a local minimizer for the non-parametric functional

$$\int_{\mathbf{D}(\mathbf{p}x,r)} f^\#(\nabla' v(z)) + G(z,v(z))\,dz\,,$$

among Lipschitz functions $v:\mathbb{R}^{n-1}\to\mathbb{R}$ satisfying the mass constraint

$$\int_{\mathbf{D}(\mathbf{p}x,r)} v(z)\,dz = \int_{\mathbf{D}(\mathbf{p}x,r)} u(z)\,dz\,.$$

As a direct consequence of (A.2), one sees that u satisfies the Euler-Lagrange equation

$$\int_{\mathbf{D}(\mathbf{p}x,r)} \nabla'\varphi(z)\cdot(\nabla' f^\#(\nabla' u(z))) + \varphi(z)\,[g(z,u(z))-\mu]\,dz = 0\,,\quad \forall\,\varphi\in C^\infty_c(\mathbf{D}(\mathbf{p}x,r))\,,$$

with μ given by (A.3). If we now further assume that f is λ-elliptic with $f\in C^{2,\alpha}(\mathbb{R}^n\setminus\{0\})$ for some $\alpha\in(0,1)$, then we readily check that

$$\nabla' f^\#(z) = -\nabla' f(-z,1)\,,\qquad (\nabla')^2 f^\#(z) = (\nabla')^2 f(-z,1)\,.$$

In particular, $f^\#\in C^{2,\alpha}(\mathbb{R}^{n-1})$, with $(\nabla')^2 f^\#(z)$ uniformly elliptic as z lies in a given bounded set. Thus, by a classical argument (based on incremental ratios and on the Caccioppoli inequality) we find that $u\in W^{2,2}_{\rm loc}(\mathbf{D}(\mathbf{p}x,r))$, with

$$\mathrm{div}'\big(\nabla' f^\#(\nabla' u(z))\big) = g(z,u(z))-\mu\,,$$

a.e. in $\mathbf{D}(\mathbf{p}x,r)$. Since

$$\mathrm{div}'\big(\nabla' f^\#(\nabla' u(z))\big) = \sum_{i,j=1}^{n-1} a_{ij}(z)\partial^2_{z_iz_j}u(z)\,,$$

where

$$a_{ij}(z) = [(\nabla')^2 f^\#]_{ij}\circ(\nabla' u(z))\,,$$

on taking into account the boundedness of $\nabla' u$, we have thus proved that u solves a linear second order elliptic equation with bounded measurable coefficients. If, moreover, $u\in C^{1,\bar\alpha}(\mathbf{D}(\mathbf{p}x,r))$, then the coefficients are of class $C^{0,\alpha\bar\alpha}$ (being obtained as the composition of a $C^{0,\alpha}$ function with a $C^{0,\bar\alpha}$ function), and u solves an elliptic partial differential equations in non-divergence form with $(\alpha\bar\alpha)$-Hölder continuous coefficients. By Schauder theory (see for instance [24, Theorem 6.2]) the $C^{2,\alpha\bar\alpha}$-norm of u inside $\mathbf{D}(x,r/2)$ is bounded by a constant times the $C^{0,\alpha}$-norm of $g(z,u(z))-\mu$ inside $\mathbf{D}(x,r)$, where the constant depends on r, $f^\#$, and the $C^{1,\bar\alpha}$-norm of u inside $\mathbf{D}(x,r)$. The C^2-regularity of u shows now that the coefficients a_{ij} are actually $C^{0,\alpha}$, and applying Schauder theory again we find

$$\|u\|_{C^{2,\alpha}(\mathbf{D}(x,r/2))}\le C\big(r,f^\#,\|u\|_{C^{1,\bar\alpha}(\mathbf{D}(x,r))},\alpha\big)\|g(z,u(z))-\mu\|_{C^{0,\alpha}(\mathbf{D}(x,r))}\,. \tag{A.5}$$

Moreover, if f and g are smoother, then higher regularity of u follows easily.

Appendix B. A remark about Question (Q1)

In this section we assume that the convexity of the sub-level sets of the potential energy g, i.e., we assume that the sets $\{g \le t\}$ are convex for all $t \ge 0$. Under this assumption we prove the following result, that may be seen as a first step in the direction of answering the convexity question (Q1) stated in the introduction.

Proposition B.1. *If the sub-level sets of g has convex, then every minimizer E in* (1.1) *satisfies*

$$\mathcal{F}(E) \le \mathcal{F}(E \cup F)\,, \tag{B.1}$$

whenever $|F \cap E| = 0$.

Proof. Let F be such that $|F \cap E| = 0$, $|F| > 0$, and consider the smallest value $t > 0$ such that

$$|E| \le |(E \cup F) \cap \{g \le t\}|\,.$$

Since by definition of t we have

$$|E| \ge |(E \cup F) \cap \{g < t\}|\,,$$

we can find a convex set G such that

$$\{g < t\} \subset G \subset \{g \le t\}\,, \qquad |E| = |(E \cup F) \cap G|\,.$$

(Of course, if $|E| = |(E \cup F) \cap \{g \le t\}|$, then we set $G = \{g \le t\}$.) In particular, we have

$$g \le t \quad \text{on } G\,, \qquad g \ge t \quad \text{on } \mathbb{R}^n \setminus G\,. \tag{B.2}$$

By the convexity of G one has

$$\mathcal{F}((E \cup F) \cap G) \le \mathcal{F}(E \cup F)\,.$$

Moreover,

$$|E \cap (\mathbb{R}^n \setminus G)| = |F \cap G|\,.$$

Hence, by minimality of E and using (B.2) we get

$$\begin{aligned}
\mathcal{F}(E) + \mathcal{G}(E) &\le \mathcal{F}((E \cup F) \cap G) + \mathcal{G}((E \cup F) \cap G) \\
&\le \mathcal{F}(E \cup F) + \int_{E \cap G} g\,dx + \int_{F \cap G} g\,dx \\
&\le \mathcal{F}(E \cup F) + \int_{E \cap G} g\,dx + t\,|F \cap G| \\
&= \mathcal{F}(E \cup F) + \int_{E \cap G} g\,dx + t\,|E \cap (\mathbb{R}^n \setminus G)| \\
&\le \mathcal{F}(E \cup F) + \int_{E \cap G} g\,dx + \int_{E \cap (\mathbb{R}^n \setminus G)} g\,dx \\
&= \mathcal{F}(E \cup F) + \mathcal{G}(E)\,,
\end{aligned}$$

as desired. □

Let us point out that the condition found above implies, in particular, that the surface energy increase if we infinitesimally enlarge E. Hence, by the first variation formula for $\mathcal{F}$ (see appendix A.1) we deduce that the anisotropic mean curvature of ∂E is non-negative. In two dimension this is sufficient to show that every connected component of E is convex. On the other hand, though non-negative mean curvature is not sufficient for convexity, condition (B.1) is global and stronger, and we may expect that it could be useful to prove that E is convex for small masses.

Appendix C. Almost minimal currents and $C^{1,\alpha}$-regularity of $(\varepsilon, n+1)$-minimizers

In this section we discuss the $C^{1,\alpha}$-regularity of (ε, R)-minimizers of $\mathcal{F}$ as a consequence of the standard regularity theory for rectifiable currents that are "almost minimizers" of an elliptic integrand [1, 9, 40, 41]. We refer to [42, 30, 36] for an introduction to the theory of currents. Let us recall that the surface tension f defines a functional $\mathbf{F}$ on the set $\mathcal{R}_{n-1}(\mathbb{R}^n)$ of $(n-1)$-dimensional integer multiplicity rectifiable currents T,

$$T(\omega) = \int_{M_T} \theta_T \, \langle \overrightarrow{T}, \omega \rangle \, d\mathcal{H}^{n-1} , \quad \omega \in \mathcal{D}_{n-1}(\mathbb{R}^n) ,$$

(here, $\mathcal{D}_{n-1}(\mathbb{R}^n)$ is the space of the $(n-1)$-dimensional compactly supported smooth forms on $\mathbb{R}^n$, M_T is a countably $(n-1)$-rectifiable set, $\overrightarrow{T}$ is an orientation of M_T, and θ_T is an integer valued function on M_T) by setting

$$\mathbf{F}(T) = \int_{M_T} \theta_T \, f(*\overrightarrow{T}) \, d\mathcal{H}^{n-1} ,$$

where $*$ denotes the Hodge star-operation. Note that the choice $f(\nu) = |\nu|$ leads to define the mass of the current T,

$$\mathbf{M}(T) = \int_{M_T} \theta_T \, d\mathcal{H}^{n-1} .$$

Thus, if we associate to the set of finite perimeter E the current T_E defined by

$$T_E(\omega) = \int_{\partial^* E} \langle *\nu_E, \omega \rangle \, d\mathcal{H}^{n-1} \tag{C.1}$$

then we find

$$\mathcal{F}(E) = \mathbf{F}(T_E) , \qquad P(E) = \mathcal{H}^{n-1}(\partial^* E) = \mathbf{M}(T_E) .$$

Regularity results for $\mathbf{F}$-minimizing currents are valid when the surface tension f is λ-elliptic in the sense defined in section 1.2, i.e., $f \in C^2(\mathbb{R}^n \setminus \{0\})$ and (1.9) holds. We recall that if f is λ-elliptic then

$$\mathbf{F}(T) - \mathbf{F}(S) \geq \lambda(\mathbf{M}(T) - \mathbf{M}(S))$$

whenever $T , S \in \mathcal{R}_{n-1}(\mathbb{R}^n)$ with $\partial T = \partial S$, and S corresponds to a $(n-1)$-dimensional disk with constant orientation (see [40, Section 1.1]).

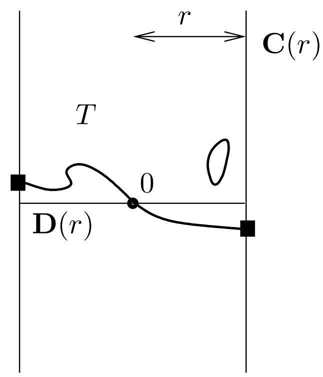

FIGURE 4. The statement of Theorem C.1 is depicted in the figure above: Let T be an almost minimizer of $\mathbf{F}$ in the sense of (C.2) such that: (C.3), T contains 0; (C.4), T is contained in the open cylinder $\mathbf{C}(r)$; (C.5), the boundary of T (depicted by black squares) is contained in $\partial\mathbf{C}(r)$; (C.6), the vertical push-forward of T amounts to the (canonically) oriented integration over the $(n-1)$-dimensional disk $\mathbf{D}(r)$, the cross section of $\mathbf{C}(r)$ (note in particular that T may contain multiple sheets over $\mathbf{D}(r)$ but with opposite orientations, so that they cancel under $\mathbf{p}_\#$); (C.7), the cylindrical excess of T in $\mathbf{C}(r)$, i.e. the deviation from 1 of the ratio between the mass of T and $\mathcal{H}^{n-1}(\mathbf{D}(r))$, is small enough. Then $T\llcorner\mathbf{C}(r/2)$ is representable as the graph of $C^{1,\overline{\alpha}}$-function from $\mathbf{D}(r/2)$ to $\mathbb{R}$.

We now introduce the regularity theorem we are going to apply. Let us recall that $\mathbf{p}:\mathbb{R}^n\to\mathbb{R}^{n-1}$ and $\mathbf{q}:\mathbb{R}^n\to\mathbb{R}$ denote the canonical projections, so that $x=(\mathbf{p}x,\mathbf{q}x)$, and we set

$$\mathbf{C}(r,s)=\{x\in\mathbb{R}^n:|\mathbf{p}x|<r\,,|\mathbf{q}x|<s\}\,,\quad \mathbf{C}(r)=\mathbf{C}(r,\infty)\,,\quad \mathbf{D}(r)=\mathbf{C}(r,0)\,.$$

We define the cylindrical excess of the current T over $\mathbf{C}(r)$,

$$\mathbf{e}(T,\mathbf{C}(r))=\frac{\mathbf{M}(T\llcorner\mathbf{C}(r))-\mathbf{M}(\mathbf{p}_\#(T\llcorner\mathbf{C}(r)))}{\omega_{n-1}r^{n-1}}\,,$$

where $\mathbf{p}_\#(T\llcorner\mathbf{C}(r))$ denotes the push-forward of the current $T\llcorner\mathbf{C}(r)$ through the map $\mathbf{p}$. The following theorem is a particular case of a more general result proved in [16, Lemma 2.2 and Theorem 6.1].

Theorem C.1. *Let f be λ-elliptic, with $\Lambda=\sup_{S^{n-1}}|\nabla^2 f|$. Given $\bar\alpha\in(0,1)$ and $\beta>0$, there exist $\sigma=\sigma(n,\lambda,\Lambda,\bar\alpha,\beta)$ and $L=L(n,\lambda,\Lambda,\bar\alpha,\beta)$ with the following property: If $T\in\mathcal{R}_{n-1}(\mathbb{R}^n)$ is such that*

$$\mathbf{F}(T)\le\mathbf{F}(T+X)+\beta\,r\mathbf{M}(T\llcorner K+X)\,,\tag{C.2}$$

whenever $X\in\mathcal{R}_{n-1}(\mathbb{R}^n)$, $\partial X=0$ and $K=\mathrm{spt}(X)$ is contained in a ball of radius r, and if, moreover,

$$0\in\mathrm{spt}\,T\,,\tag{C.3}$$

$$T\llcorner\mathbf{C}(r)=T\,,\tag{C.4}$$

$$\partial T\llcorner\mathbf{C}(r)=0\,,\tag{C.5}$$

$$\mathbf{p}_{\#}T(\omega) = \int_{\mathbf{D}(r)} \langle \omega, *e_n \rangle \, d\mathcal{H}^{n-1} \,, \quad \forall \, \omega \in \mathcal{D}_{n-1}(\mathbb{R}^n) \,, \tag{C.6}$$

$$\mathbf{e}(T, \mathbf{C}(r)) \le \sigma \,, \tag{C.7}$$

then $T \llcorner \mathbf{C}(r/2)$ *is represented by the graph of a* $C^{1,\bar\alpha}$*-function* $u : \mathbf{D}(r/2) \to \mathbb{R}$ *such that*

$$\begin{aligned} |u(z)| &\le L\,\mathbf{e}(T, \mathbf{C}(r))^{1/(2n)} \,, \\ |\nabla u(z) - \nabla u(z')| &\le L|z - z'|^{\bar\alpha} \,, \end{aligned}$$

for every $z, z' \in \mathbf{D}(r/2)$.

Having in mind the minimality condition (C.2), we now prove that $(\varepsilon, n+1)$-minimizers satisfy a perturbed local $\mathcal{F}$-minimality condition, with no volume constraint.

Lemma C.2. *There exist positive constants* $\varepsilon_0 = \varepsilon_0(n)$, $\beta = \beta(n, |K|)$, *and* $r_1 = r_1(n, |K|)$ *with the following property: If* E *is a* $(\varepsilon, n+1)$*-minimizer for* $\mathcal{F}$ *with* $|E| = |K|$ *and* $\varepsilon < \varepsilon(n)$, *then*

$$\mathcal{F}(E) \le \mathcal{F}(F) + \beta \, r \, P(E\Delta F) \,, \tag{C.8}$$

whenever $E\Delta F$ *is compactly contained in a ball of radius* $r < r_1$.

Remark C.3. It is well-known that volume-constrained minimizers (like our (ε, R)-minimizers) satisfy local minimality conditions like (C.8). This is usually shown by using Lemma 2.3. In the present situation this strategy would present the drawback of producing a value of r_1 depending on E. We can instead use the proximity of E to a Wulff shape (Corollary 3.5) together with Lemma 2.4 to prove (C.8) with $r_1 = r_1(n, K)$. The fact that r_1 does not depend on E played a key role in section 4.3.

Proof of Lemma C.2. By applying Lemma 2.2 we can directly assume that $B \subset K \subset B_n$. Let now F be such that $E\Delta F$ is compactly contained in some ball of radius $r < r_1$. If this ball is disjoint from ∂E then $\mathcal{F}(F) \ge \mathcal{F}(E)$ and (C.8) is trivially valid. If this is not the case then, provided r_1 is small enough we can assume that $F \subset I_n(E)$. Hence, thanks to Corollary 3.5, up to a translation and to suitably choosing the value of ε_0, we can assume that $K_{1/2} \subset E \subset K_2$. Let now $s > 0$ be such that $|s\,E| = |F|$, so that

$$|s - 1| \le |s^n - 1| \le \frac{|E\Delta F|}{|E|} = \frac{|E\Delta F|}{|K|} \,, \tag{C.9}$$

and, in particular,

$$|s - 1| \le \frac{\omega_n r_1^n}{|K|} \,. \tag{C.10}$$

By the isoperimetric inequality and since $E\Delta F$ is contained in a ball of radius r,

$$|E\Delta F| \le |E\Delta F|^{1/n} |E\Delta F|^{1/n'} \le \omega_n^{1/n} \, r \, \frac{P(E\Delta F)}{n\omega_n^{1/n}} = \frac{r}{n} \, P(E\Delta F) \,. \tag{C.11}$$

Thus, on combining (C.11) with (C.9), we have

$$|s - 1| \le \beta(n, |K|) \, r \, P(E\Delta F) \,. \tag{C.12}$$

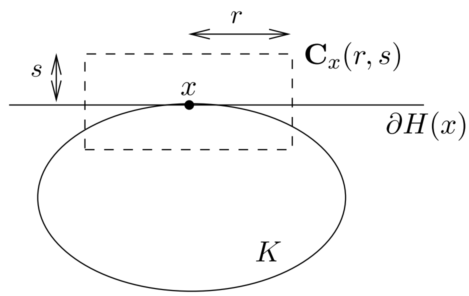

FIGURE 5. A cylinder $\mathbf{C}_x(r,s)$.

We now notice that $s\,E$ is a $(\varepsilon, s(n+1))$-minimizer of $\mathcal{F}$. Since $F \subset I_n(E)$, choosing r_1 sufficiently small by (C.10) we have $F \subset I_{s(n+1)}(s\,E)$. Therefore,

$$\begin{aligned}\mathcal{F}(s\,E) &\leq \mathcal{F}(F) + \varepsilon\,|K|^{1/n}\frac{|(s\,E)\Delta F|}{|s\,E|^{1/n}} \leq \mathcal{F}(F) + \frac{\varepsilon}{s}\,|(s\,E)\Delta F| \\ &\leq \mathcal{F}(F) + \beta(n,|K|)|(s\,E)\Delta F|\,, \end{aligned} \tag{C.13}$$

where in the last step we have applied (C.10). We now notice that by Lemma 2.4, by (C.12), and since $E \subset K_2 \subset B_{2n}$ and $B \subset K$,

$$|(s\,E)\Delta E| \leq C(n)|s-1|P(E) \leq C(n)|s-1|\mathcal{F}(E) \leq \beta(n,|K|)\,r\,P(E\Delta F)\,, \tag{C.14}$$

where we have also taken into account that

$$\mathcal{F}(E) \leq \mathcal{F}(K) + \varepsilon|E\Delta K| = (n + 2\,\varepsilon_0)|K|\,.$$

By a similar argument we have

$$|\mathcal{F}(E) - \mathcal{F}(s\,E)| \leq C(n)|s-1|\mathcal{F}(E) \leq \beta(n,|K|)\,r\,P(E\Delta F)\,. \tag{C.15}$$

On combining (C.14), (C.15), and (C.11) with (C.13) we finally obtain the validity of (C.8). □

We are now in the position to prove the $C^{1,\alpha}$-regularity of $(\varepsilon, n+1)$-minimizers of any $\mathcal{F}$ corresponding to a λ-elliptic surface tension.

Proof of Theorem 3.8. Given $x \in \partial K$ we denote by $\pi(x)$ the tangent plane to ∂K at x, and by $H(x)$ the supporting half-space to K at x that contains K (note that $\partial H(x) = x+\pi(x)$). We denote by $\mathbf{p}_x : \mathbb{R}^n \to \pi(x)$ and by $\mathbf{q}_x : \mathbb{R}^n \to \pi(x)^\perp$ the projections of $\mathbb{R}^n$ onto $\pi(x)$ and the line $\pi(x)^\perp$, respectively. We also consider the cylinders

$$\mathbf{C}_x(r,s) = \big\{y \in \mathbb{R}^n : |\mathbf{p}_x(y-x)| < r\,, \quad |\mathbf{q}_x(y-x)| < s\big\}\,, \quad r,s \in [0,+\infty]\,,$$

so that $\mathbf{C}_x(r) = \mathbf{C}_x(r,\infty)$ denotes a cylinder of infinite height and $(n-1)$-dimensional cross section $\mathbf{D}_x(r) = \{y \in \partial H(x) : |y-x| < r\}$. We let $\eta_0 > 0$ be such that for every $x \in \partial K$,

$$K \cap \mathbf{C}_x(\sqrt{\eta_0},\eta_0) = \big\{y \in \mathbf{C}_x(\sqrt{\eta_0},\eta_0) : \mathbf{q}_x(y-x) < v_x(\mathbf{p}_x(y-x))\big\}\,,$$

for some concave function $v_x : \pi(x) \to (-\infty, 0]$. Then the regularity of K implies the existence of a constant $c = c(f) \in (0,1)$, such that

$$N_\eta(\partial K) \cap \mathbf{C}_x(c\sqrt{\eta}, \eta_0) \subset N_{2\eta}(\partial H(x)) \,, \quad \forall\, \eta \in \left(0, \frac{\eta_0}{2}\right) \,, \tag{C.16}$$

where $N_\mu(A)$ denotes the (Euclidean) open μ-neighborhood of a set A, i.e., $N_\mu(A) = \{y : \mathrm{dist}(y, A) < \mu\}$, see Figure C. Given $\eta \in (0, \eta_0/2)$ we can choose ε so that, by Corollary 3.5,

$$K_{1-\eta} \subset E \subset K_{1+\eta} \,, \qquad \partial E \subset I_\eta(\partial K) \,. \tag{C.17}$$

Moreover, by Lemma C.2 there exists $\beta = \beta(n, |K|) > 0$, such that

$$\mathcal{F}(E) \le \mathcal{F}(F) + \beta\, r\, P(E\Delta F) \tag{C.18}$$

whenever $E\Delta F$ compactly contained into some ball of radius $r < r_1 = r_1(n, |K|)$. We now consider the $(n-1)$-dimensional current T_E associated to E by (C.1), and then show that Theorem C.1 applies (up to a small vertical translation of size η to ensure (C.3)) to the current

$$T = T_E \llcorner \mathbf{C}_x(c\sqrt{\eta}, \eta_0) \,,$$

inside the cylinder $\mathbf{C}_x(c\sqrt{\eta})$, provided we choose $\eta \in (0, \eta_0/2)$ (and, correspondingly, ε) small enough, depending on n, λ, Λ, and α, but independent of the point $x \in \partial K$. Once this fact will be established, the statement of the theorem will follow easily by covering ∂K with cylinders $\{\mathbf{C}_x(c\sqrt{\eta}/4, \eta_0)\}_{x \in \partial K}$ and using the compactness of ∂K. So we only need to show that, given $x \in \partial K$ and defined T as above, the assumptions in Theorem C.1 are satisfied (up to a rigid motion) by T. Observe that by (C.16) and (C.17), the uniform bound

$$\mathrm{spt} T \subset N_{3\eta}(\partial H(x)) \,, \quad \forall\, \eta \in \left(0, \frac{\eta_0}{2}\right) \,, \tag{C.19}$$

holds true.

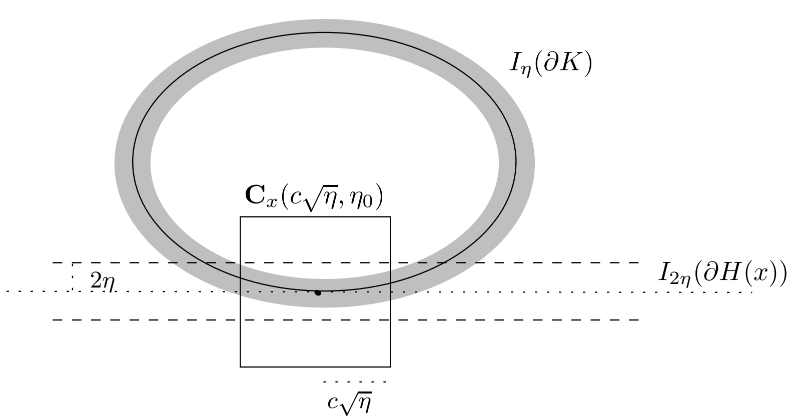

FIGURE 6. Decomposing the boundary of K.

Step one. We prove that for every $\eta \in (0, \eta_0/3)$, the current $T = T_E \llcorner \mathbf{C}_x(c\sqrt{\eta}, \eta_0)$, i.e.,

$$T(\omega) = \int_{\mathbf{C}_x(c\sqrt{\eta},\eta_0)\cap\partial^* E} \langle \omega, *\nu_E\rangle \, d\mathcal{H}^{n-1}\,, \quad \omega \in \mathcal{D}_{n-1}(\mathbb{R}^n)\,,$$

satisfies the following properties:

$$T \llcorner \mathbf{C}_x(c\sqrt{\eta}) = T\,, \tag{C.20}$$

$$\partial T \llcorner \mathbf{C}_x(c\sqrt{\eta}) = 0\,, \tag{C.21}$$

$$(\mathbf{p}_x)_\# T(\omega) = \int_{\mathbf{D}_x(c\sqrt{\eta})} \langle \omega, *\nu_K(x)\rangle d\mathcal{H}^{n-1}\,, \quad \forall\, \omega \in \mathcal{D}_{n-1}(\mathbb{R}^n)\,, \tag{C.22}$$

$$\mathbf{F}(T) \le \mathbf{F}(T+X) + \beta\, r\, \mathbf{M}(T \llcorner K + X)\,, \tag{C.23}$$

whenever $X \in \mathcal{R}_{n-1}(\mathbb{R}^n)$, $\partial X = 0$, and $K = \mathrm{spt} X$ is contained in a ball of radius $r < r_0$. First of all, we observe that if η is such that

$$\mathcal{H}^{n-1}\big(\partial^* E \cap \partial \mathbf{C}_x(c\sqrt{\eta}, \eta_0)\big) = 0\,,$$

then (C.20) holds true. Since $\partial T_E = 0$ we check the validity of (C.21). By (C.19)

$$\partial E \cap \mathbf{C}_x(c\sqrt{\eta}, \eta_0) \subset I_{3\eta}(\partial H(x))\,, \quad \forall\, \eta \in \left(0, \frac{\eta_0}{2}\right)\,,$$

so that by connectedness

$$|\{y \in \mathbf{C}_x(c\sqrt{\eta}, \eta_0) \cap E : \mathbf{q}_x(y-x) > 3\eta\}| = 0\,,$$
$$|\{y \in \mathbf{C}_x(c\sqrt{\eta}, \eta_0) \setminus E : \mathbf{q}_x(y-x) < -3\eta\}| = 0\,.$$

Thus by a standard application of the the divergence theorem we find that for every $\varphi \in C^\infty_c(\pi(x))$,

$$\int_{\mathbf{D}_x(c\sqrt{\eta})} \varphi(y-x)\, d\mathcal{H}^{n-1}(y) = \int_{\mathbf{C}_x(c\sqrt{\eta},\eta_0)\cap\partial^* E} \varphi(\mathbf{p}_x(y))(\nu_E(y)\cdot\nu_K(x))\, d\mathcal{H}^{n-1}(y)\,,$$

that is (C.22). Finally, to check the validity of (C.23), we write $X = \partial U$ for some $U \in \mathcal{R}_n(\mathbb{R}^n)$ (recall that $\partial X = 0$). By [30, Theorem 7.5.5] there exists a disjoint family of sets of finite perimeter $\{A_i\}_{i\in\mathbb{Z}}$ such that $U = \sum_{i\in\mathbb{Z}} i\, \chi_{A_i}$. If we define

$$A_+ = \bigcup_{i>0} A_i \setminus E\,, \qquad A_- = \bigcup_{i<0} A_i \cap E\,,$$

then (C.18) gives

$$\mathcal{F}(E) \le \mathcal{F}((E \cup A_+) \setminus A_-) + \beta\, r\, [P(A_+) + P(A_-)],$$

and it is easily checked that

$$\mathcal{F}((E \cup A_+) \setminus A_-) + \beta\, r\, [P(A_+) + P(A_-)] \le \mathbf{F}(T+X) + \beta\, r\, \mathbf{M}(T \llcorner K + X),$$

which proves (C.23).

Step two. We now show that if $\eta \le \eta_0/3$ is small enough then

$$\mathbf{e}(T, \mathbf{C}_x(c\sqrt{\eta})) = \frac{\mathcal{H}^{n-1}(\mathbf{C}_x(c\sqrt{\eta}, \eta_0) \cap \partial^* E)}{\omega_{n-1}(c\sqrt{\eta})^{n-1}} - 1 \le \sigma\,.$$

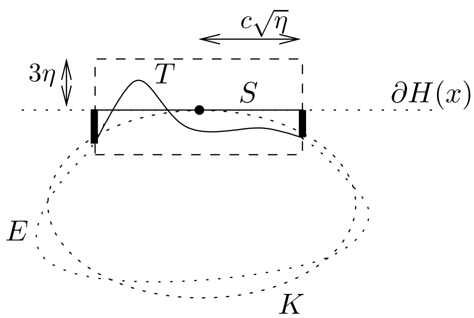

FIGURE 7. The current P matches T and S, so that $\partial(-T+S-P)=0$, and is carried by a subset M_P (represented with bold lines) of $\partial\mathbf{C}_x(c\sqrt{\eta})\cap\partial\mathbf{C}_x(c\sqrt{\eta},3\eta)$.

Let us consider the $(n-1)$-dimensional rectifiable current

$$S(\omega)=\int_{\mathbf{D}_x(c\sqrt{\eta})}\langle *\nu_K(x),\omega\rangle\,d\mathcal{H}^{n-1}\,,$$

corresponding to the integration over the disk $\mathbf{D}_x(c\sqrt{\eta})$ oriented by $\nu_K(x)$, and let P be the $(n-1)$-dimensional rectifiable current with $M_P\subset\partial\mathbf{C}_x(c\sqrt{\eta})\cap\partial\mathbf{C}_x(c\sqrt{\eta},3\eta)$ (see (C.19)) such that

$$\partial(T+P)=\partial S\,.$$

Since $\mathrm{spt}(-T+S-P)$ is contained in ball of radius controlled by $C(n,f)\sqrt{\eta}$ and since

$$\mathcal{H}^{n-1}(M_P)\le C(n,f)\eta^{1+(n-2)/2}\,, \tag{C.24}$$

by (C.23), we find that

$$\mathbf{F}(T)-\mathbf{F}(S-P)\le C(n,f)\,\eta^{1+(n-1)/2}\,.$$

We now notice that

$$\begin{aligned}\mathbf{F}(T)-\mathbf{F}(S-P) &= \int_{\partial^*E\cap\mathbf{C}_x(c\sqrt{\eta},3\eta)}f(\nu_E)\,d\mathcal{H}^{n-1}-f(\nu_K(x))\omega_{n-1}(c\sqrt{\eta})^{n-1}\\ &\quad+\int_{M_P}f(\pm\nu_{\mathbf{C}_x(c\sqrt{\eta},3\eta)})\,d\mathcal{H}^{n-1}\,,\end{aligned}$$

where $\int_{M_P}f(\pm\nu_{\mathbf{C}_x(c\sqrt{\eta},3\eta)})\,d\mathcal{H}^{n-1}\le\alpha_2\mathcal{H}^{n-1}(M_P)$ (see (2.4)), hence

$$\int_{\partial^*E\cap\mathbf{C}_x(c\sqrt{\eta},3\eta)}f(\nu_E)\,d\mathcal{H}^{n-1}-f(\nu_K(x))\omega_{n-1}(c\sqrt{\eta})^{n-1}\le C(n,f)\eta^{1+(n-2)/2}\,. \tag{C.25}$$

On the other hand we have

$$\begin{aligned}\mathbf{F}(T+P)-\mathbf{F}(S) &= \int_{\partial^*E\cap\mathbf{C}_x(c\sqrt{\eta},\eta_0)}f(\nu_E)\,d\mathcal{H}^{n-1}\\ &\quad+\int_{M_P}f(\pm\nu_{\mathbf{C}_x(c\sqrt{\eta},\eta_0)})\,d\mathcal{H}^{n-1}-f(\nu_K(x))\omega_{n-1}(c\sqrt{\eta})^{n-1}\,,\\ \mathbf{M}(T+P)-\mathbf{M}(S) &= \mathcal{H}^{n-1}(\partial^*E\cap\mathbf{C}_x(c\sqrt{\eta},\eta_0))+\mathcal{H}^{n-1}(M_P)-\omega_{n-1}(c\sqrt{\eta})^{n-1}\,,\end{aligned}$$

and since $\mathbf{F}(T+P)-\mathbf{F}(S)\geq\lambda(\mathbf{M}(T+P)-\mathbf{M}(S))$, taking again (C.24) into account, we find that

$$\begin{aligned}&\int_{\partial^*E\cap\mathbf{C}_x(c\sqrt{\eta},\eta_0))}f(\nu_E)\,d\mathcal{H}^{n-1}-f(\nu_K(x))\omega_{n-1}(c\sqrt{\eta})^{n-1} \qquad (C.26)\\&\geq\lambda\left(\mathcal{H}^{n-1}(\partial^*E\cap\mathbf{C}_x(c\sqrt{\eta},\eta_0)))-\omega_{n-1}(c\sqrt{\eta})^{n-1}\right)-\lambda C(n,f)\eta^{1+(n-2)/2}\,.\end{aligned}$$

Combining (C.25) and (C.26) we find that

$$\mathcal{H}^{n-1}(\partial^*E\cap\mathbf{C}_x(c\sqrt{\eta},\eta_0)))-\omega_{n-1}(c\sqrt{\eta})^{n-1}\leq C(n,f)\eta^{1+(n-2)/2}\,,$$

that immediately gives

$$\mathbf{e}(T,\mathbf{C}_x(c\sqrt{\eta}))\leq C(n,f)\sqrt{\eta}\,,$$

and concludes the proof. □

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