PARTIAL REGULARITY OF BRENIER SOLUTIONS OF THE MONGE-AMPÈRE EQUATION

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Abstract. Given $\Omega, \Lambda \subset \mathbb{R}^n$ two bounded open sets, and f and g two probability densities concentrated on Ω and Λ respectively, we investigate the regularity of the optimal map $\nabla\varphi$ (the optimality referring to the Euclidean quadratic cost) sending f onto g. We show that if f and g are both bounded away from zero and infinity, we can find two open sets $\Omega' \subset \Omega$ and $\Lambda' \subset \Lambda$ such that f and g are concentrated on Ω' and Λ' respectively, and $\nabla\varphi : \Omega' \to \Lambda'$ is a (bi-Hölder) homeomorphism. This generalizes the 2-dimensional partial regularity result of [8].

1. Introduction

Let $\Omega, \Lambda \subset \mathbb{R}^n$ be two bounded open sets, and f and g two probability densities concentrated on Ω and Λ respectively. According to Brenier's Theorem [1, 2] there exists a globally Lipschitz convex function $\varphi : \mathbb{R}^n \to \mathbb{R}$ such that $\nabla\varphi_{\#} f = g$ and $\nabla\varphi(x) \in \overline{\Lambda}$ for a.e. $x \in \mathbb{R}^n$. Assuming the existence of a constant $\lambda > 0$ such that $\lambda \le f, g \le 1/\lambda$ inside Ω and Λ respectively, then φ solves the Monge-Ampère equation

$$\lambda^2 \chi_\Omega \le \det(D^2\varphi) \le \frac{1}{\lambda^2} \chi_\Omega \qquad \text{in } \mathbb{R}^n \tag{1.1}$$

in a weak sense (which we will call *Brenier sense*). Moreover, as shown in [6], if Λ is convex then φ solves (1.1) also in the *Alexandrov sense* (see Section 2 for the definition of Brenier and Alexandrov solutions), and this is the starting point to develop a satisfactory regularity theory [3, 4, 5, 6]. On the other hand, if Λ is not convex, $\nabla\varphi$ may not be continuous even if f and g are smooth [6].

The aim of this paper is to generalize the 2-dimensional result of [8] by showing that there exist two open sets $\Omega' \subset \Omega$ and $\Lambda' \subset \Lambda$, with $|\Omega \setminus \Omega'| = |\Lambda \setminus \Lambda'| = 0$, such that φ is $C^{1,\alpha}$ and strictly convex inside Ω', and $\nabla\varphi$ is a bi-Hölder homeomorphism between Ω' and Λ'. In particular φ is an Alexandrov solution of (1.1) inside Ω', so that higher regularity of φ follows when f and g are smooth. Note that by general results on (semi-)convex functions, it is known that the set where φ is not differentiable consists of a countable union of Lipschitz hypersurfaces. Our result prevents such singular set makes a pathological picture: for example, such set cannot be dense inside Ω. Let us however point out that in this higher dimensional case we are not able to find a precise geometric description of the singular set as in [8] (see also [7, 11]).

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The structure of the paper is as follows: in Section 2 we introduce some notation, we recall useful facts about convex functions, and we introduce the notion of Brenier and Alexandrov solutions to (1.1). In Section 3 we first recall how *Alexandrov estimates* play a key role in showing regularity for Alexandrov solutions of (1.1), and then we prove the partial regularity result outlined above (see Theorem 3.5).

It is a pleasure for us to dedicate this paper to Louis Nirenberg on the occasion of his 85th birthday.

2. Notation and preliminary results

Throughout the paper, we use the notation $C(a_1,\ldots,a_n)$ to denote a constant which depends on the quantities $a_1,\ldots,a_n$ only. Given a set $E\subset\mathbb{R}^n$, $|E|$ denotes its Lebesgue measure. Moreover the concept of "almost everywhere" will always be considered with respect to the Lebesgue measure.

Let $\psi:\mathbb{R}^n\to\mathbb{R}$ be a convex function. Its *subdifferential* at a point x is defined as

$$\partial^-\psi(x):=\bigl\{y\in\mathbb{R}^n \mid \psi(z)\geq\psi(x)+y\cdot(z-x)\quad\forall\, z\in\mathbb{R}^n\bigr\}.$$

From the above definition it is easily seen that the map $x\mapsto\partial^-\psi(x)$ is upper semicontinuous, i.e.

$$x_k\to x,\quad y_k\to y,\quad y_k\in\partial^-\psi(x_k)\quad\Rightarrow\quad y\in\partial^-\psi(x). \tag{2.1}$$

This implies in particular that $x\mapsto\nabla\psi(x)$ is continuous on the set where ψ is differentiable. Moreover, ψ is differentiable at a point x if and only if $\partial^-\psi(x)$ is a singleton. Recall that convex functions are locally Lipschitz (and hence differentiable a.e. by Rademacher's Theorem) inside any open set where they are finite. We will denote by $\mathrm{dom}\nabla\psi$ the domain of $\nabla\psi$, i.e. the set of points where ψ is differentiable.

We denote by $\psi^*:\mathbb{R}^n\to\mathbb{R}$ the *Legendre transform* of ψ:

$$\psi^*(y):=\sup_{x\in\mathbb{R}^n} x\cdot y-\psi(x).$$

Let us recall that the following duality relation holds:

$$y\in\partial^-\psi(x)\quad\Leftrightarrow\quad x\in\partial^-\psi^*(y). \tag{2.2}$$

A (open) *section* Z of ψ is any (non-empty) set of the form

$$Z:=\{x:\psi(x)<\ell(x)\}$$

for some linear function $\ell(x)=a\cdot x+b$. Since every section Z is convex, by John's Lemma [10] there exists an affine map $L:\mathbb{R}^n\to\mathbb{R}^n$ such that

$$B_1\subset L(Z)\subset B_n.$$

If the above inclusions hold, we will say that the convex set $L(Z)$ is *renormalized.*

A convex function $\varphi:\mathbb{R}^n\to\mathbb{R}$ solves the Monge-Ampère equation (1.1) in the *Brenier sense* if

$$\lambda^2\,\chi_\Lambda\leq\nabla\varphi_{\#}\chi_\Omega\leq\frac{1}{\lambda^2}\,\chi_\Lambda,$$

together with the "boundary condition" $\nabla\varphi(\mathbb{R}^d)\subset\overline{\Lambda}$ (i.e. $\nabla\varphi(x)\in\overline{\Lambda}$ a.e.).

A convex function $\varphi:\mathbb{R}^n\to\mathbb{R}$ solves the Monge-Ampère equation (1.1) in the *Alexandrov sense* if, for any Borel set $B\subset\mathbb{R}^n$,

$$\lambda^2\,|B\cap\Omega|\le|\partial^-\varphi|(B):=|\partial^-\varphi(B)|\le\frac{1}{\lambda^2}\,|B\cap\Omega|,$$

where $\partial^-\varphi(B):=\cup_{x\in B}\partial^-\varphi(x)$, and $|\partial^-\varphi|$ denotes the Monge-Ampère measure associated to φ. This is can be written in a concise form as

$$\lambda^2\,\chi_\Omega\le|\partial^-\varphi|\le\frac{1}{\lambda^2}\,\chi_\Omega.$$

Let us remark that while the target set Λ enters in the notion of Brenier solution, it plays no role in the notion of Alexandrov solution.

3. Partial regularity

As shown in [6], if the target set Λ is convex, then any Brenier solution of (1.1) is also an Alexandrov solution. More precisely, without any assumption on Λ any Brenier solution φ satisfies

$$|\partial^-\varphi|\ge\lambda^2\,\chi_\Omega,$$

while the other inequality needs the convexity of Λ to be true (see the proof of [6, Lemma 4]).

In order to prove the partial regularity result for Brenier solutions described in the introduction, we first recall the strategy to show regularity for Alexandrov solutions of (1.1).

3.1. Alexandrov estimates, renormalization, and regularity of Alexandrov solutions.

As shown in [3, 5, 6], the regularity theory of Alexandrov solutions of (1.1) relies crucially on the so-called *Alexandrov estimates*. Let us briefly recall the main points of this theory.

First, let $\psi:\mathbb{R}^n\to\mathbb{R}$ be a convex function, and let Z be a renormalized section of ψ such that $\psi=0$ on ∂Z. Then the following estimates hold (recall that since Z is renormalized we have $0<c(n)\le|Z|\le C(n)$):

(LA) *Lower Alexandrov estimate:* for all $\alpha\in(0,1)$

$$|\partial^-\psi(\alpha Z)|\le C(n,\alpha)\,\big|\inf_Z\psi\big|^n.$$

(UA) *Upper Alexandrov estimate:* let C_x denote the cone generated by $(x,\psi(x))$ and $(\partial Z,\psi)$, i.e. $C_x:\mathbb{R}^n\to\mathbb{R}$ satisfies $C_x(x)=\psi(x)$, $C_x=\psi(=0)$ on ∂Z, and $z\mapsto C_x(z-x)$ is 1-homogeneous. Then

$$\big|\psi(x)\big|^n\le C(n)\,|\partial^-C_x(\{x\})|\,\mathrm{dist}(x,\partial Z).$$

Now, let φ be an Alexandrov solution of (1.1). Fix a point $x\in\mathbb{R}^n$, and consider a section $Z:=\{\varphi<\ell\}$ such that $x\in Z$. By John's Lemma there exists an affine map L such $L(Z)$ is renormalized. Set $\varphi_L(z):=|\det L|^{2/n}[\varphi-\ell](L^{-1}z)$. Then φ_L is an Alexandrov solution of

$$\lambda^2\,\chi_{L(\Omega)}\le|\partial^-\varphi_L|\le\frac{1}{\lambda^2}\,\chi_{L(\Omega)},$$

and $L(Z)=\{\varphi_L<0\}$ is renormalized. These facts together with (LA) give

$$\lambda^2\,\big|\big(\alpha L(Z)\big)\cap L(\Omega)\big|\le|\partial^-\varphi_L(\alpha L(Z))|\le C(n,\alpha)\,\big|\inf_{L(Z)}\varphi_L\big|^n,$$

so that, since $c(n)|\det L|^{-1} \leq |Z| \leq C(n)|\det L|^{-1}$ with $C(n), c(n) > 0$, we get

$$\big|\inf_Z \varphi\big|^n \geq c(n,\alpha,\lambda)\,|(\alpha Z)\cap\Omega|\,|Z| > 0. \tag{3.1}$$

On the other hand, let $C_{L,x}$ denote the cone generated by $(L(x), \varphi_L(x))$ and $(\partial(L(Z)), \varphi_L|_{\partial(L(Z))})$. Then (UA) implies

$$|\varphi_L(L(x))|^n \leq C(n)\,|\partial^- C_{L,x}(\{L(x)\})|\ \mathrm{dist}\big(L(x), \partial(L(Z))\big),$$

that is

$$\begin{aligned}\big|\varphi(x)\big|^n &\leq C(n)\,|\partial^- C_x(\{x\})|\,|Z|\ \mathrm{dist}\big(L(x),\partial(L(Z))\big)\\ &\leq C(n)\,|\partial^-\varphi(Z)|\,|Z|\ \mathrm{dist}\big(L(x),\partial(L(Z))\big)\\ &\leq C(n,\lambda)\,|Z\cap\Omega|\,|Z|\ \mathrm{dist}\big(L(x),\partial(L(Z))\big),\end{aligned} \tag{3.2}$$

where the second inequality follows from the inclusion $\partial^- C_x(\{x\}) \subset \partial^-\varphi(Z)$, which can be easily proven by moving down any supporting plane of C_x at x and lifting it up until it touches the graph of φ inside Z.

Using (3.1) and (3.2), the regularity theory for Alexandrov solutions to (1.1) goes as follows:

• *Strict convexity.* Let $y \in \partial^-\varphi(\Omega)$, and set $S_y := \{z : y \in \partial^-\varphi(z)\} = \partial^-\varphi^*(y)$. (Here φ^* denotes the Legendre transform of φ.) We want to prove that S_y is a singleton, which is equivalent to the strict convexity of φ.

Assume that is not the case. Since S_y is convex, there are two possibilities:
(1) S_y contains an infinite line $(\mathbb{R}v) + w$ ($v, w \in \mathbb{R}^n$, $v \neq 0$).
(2) There exists x_0 an exposed point for S_y, namely, there is an affine function $A : \mathbb{R}^n \to \mathbb{R}$ such that $A(x_0) = 0$ and $A < 0$ on $S_y \setminus \{x_0\}$.

Case (1) is excluded by the Monge-Ampère equation, since this would imply that $\partial^-\varphi(\mathbb{R}^n)$ is contained inside the $(n-1)$-dimensional subspace $v^\perp$, so that $|\partial^-\varphi|(\mathbb{R}^n) = 0$, a contradiction to (1.1). (See [6] or [9, Lemma 2.4] for more details.)

For the remaining case (2), with no loss of generality we can assume $y = 0$, $x_0 = 0$ and $A(z) = z_n = z \cdot e_n$.

First of all, one shows that $x_0 \notin \Omega$. In fact, if this was the case, we could apply the Alexandrov estimates (3.1) and (3.2) to the sections $Z_\varepsilon := \{\varphi(z) < \varepsilon(z_n - 1)\} \ni 0$. Since 0 is almost a minimum point for $\varphi - \varepsilon(z_n - 1)$ and his image after renormalization is very close to the boundary of the renormalized section, as shown in [3, Theorem 1] this leads to a contradiction to (3.1) and (3.2) as $\varepsilon \to 0$.

Analogously one proves that $x_0 \notin \partial\Omega$. Indeed, one considers the sections $Z_\varepsilon := \{\varphi(z) < \varepsilon(z_n - 2R_0)\}$, where R_0 is such that $\Omega \subset B_{R_0}$. This ensures that after renormalization the measure $|\partial^-\varphi_{L_\varepsilon}|$ has some positive mass inside $L_\varepsilon(Z_\varepsilon)$ (uniformly as $\varepsilon \to 0$), and this allows to conclude as above [6, Lemma 3].

Finally, since $|\partial^-\varphi| = 0$ outside Ω but $|\partial^-\varphi|$ has always some positive mass inside a section Z_ε as the ones above, it is easily seen that x_0 cannot belong to $\mathbb{R}^n \setminus \overline{\Omega}$ (see [6, Lemma 3]). This together with above implies that there is no exposed point, yielding a contradiction. This concludes the proof of the strict convexity of φ.

• *Differentiability and $C^{1,\alpha}$-regularity.* As shown in [3, Corollary 1], (3.1) and (3.2) allow also to show the differentiability of φ at any point of strict convexity. This implies that φ is strictly convex and C^1 inside Ω. Finally, using again the Alexandrov estimates, the $C^{1,\alpha}$-regularity of φ follows by a compactness argument [5, Lemma 2 and Theorem 2].

3.2. **Partial regularity of Brenier solutions.** In this subsection we show the partial regularity result for Brenier solutions described in the introduction. As we already observed at the beginning of this section, if φ is a Brenier solutions of (1.1) then $|\partial^-\varphi| \geq \lambda^2 \chi_\Omega$. In particular (3.1) still holds true. The problem in this case is that we cannot deduce (3.2), since now no upper bound on $|\partial^-\varphi|$ is available [6].

To bypass this problem, we will use the following two preliminary results:

Lemma 3.1. *Let $C \subset B_R$ be a convex set such that $0 \in C$, and fix $\delta \in (0, R)$. Then*
$$|C \cap B_\delta| \geq C(\delta, R)\,|C|.$$

Proof. Since $|C| = |C \cap B_\delta| + |C \setminus B_\delta|$, it suffices to bound $|C \setminus B_\delta|$.

Set $D := C \cap \partial B_\delta$, $E := \mathbb{R}^+ D = \{tx : t > 0,\, x \in D\}$, $E_\delta := E \cap B_\delta$, and $E_R := E \cap B_R$. Since $0 \in C$, by the convexity of C it is easily seen that the following inclusions hold:
$$C \setminus B_\delta \subset E_R, \qquad E_\delta \subset C \cap B_\delta.$$
Hence
$$|C \setminus B_\delta| \leq |E_R| \leq C(\delta, R)\,|E_\delta| \leq C(\delta, R)\,|C \cap B_\delta|,$$
as desired. □

Lemma 3.2. *Let φ be a Brenier solution of (1.1), and let Z be a section of φ. Then*
$$|\partial^-\varphi(Z) \cap \Lambda| \leq \frac{1}{\lambda^2}\,|Z \cap \Omega|.$$

Proof. Let φ^* denote the Legendre transform of φ. Since $\nabla\varphi_\# \chi_\Omega \geq \lambda^2 \chi_\Lambda$, for a.e. $y \in \Lambda$ there exists $x \in \Omega$ such that $x \in \partial^-\varphi^*(y)$. Thanks to (2.1), this implies that $\partial^-\varphi^*(y) \cap \overline{\Omega} \neq \emptyset$ for all $y \in \Lambda$, so by the boundedness of Ω we deduce that φ^* is Lipschitz, and hence differentiable a.e., inside Λ. This gives
$$|\partial^-\varphi(Z) \cap \Lambda| = |\partial^-\varphi(Z) \cap \Lambda \cap (\mathrm{dom}\nabla\varphi^*)|.$$
Now, thanks to the duality relation (2.2), if $y \in \partial^-\varphi(x_1) \cap \partial^-\varphi(x_2)$ and $y \in \mathrm{dom}\nabla\varphi^*$ then $x_1 = x_2$. This implies
$$(\nabla\varphi)^{-1}\big(\partial^-\varphi(Z) \cap (\mathrm{dom}\nabla\varphi^*)\big) \subset Z.$$
Combining all together and using the definition of Brenier solution we finally obtain
$$\begin{aligned}|\partial^-\varphi(Z) \cap \Lambda| &= |\partial^-\varphi(Z) \cap \Lambda \cap (\mathrm{dom}\nabla\varphi^*)| \\ &\leq \frac{1}{\lambda^2}\,\big|(\nabla\varphi)^{-1}\big(\partial^-\varphi(Z) \cap (\mathrm{dom}\nabla\varphi^*)\big) \cap \Omega\big| \leq \frac{1}{\lambda^2}\,|Z \cap \Omega|.\end{aligned}$$
□

Thanks to the two results above, we can show the following key estimate, which will play the role of (3.2) in our situation:

Proposition 3.3. *Fix $\delta > 0$, let φ be a Brenier solution of* (1.1)*, and Z be a section of φ. Let $L : \mathbb{R}^n \to \mathbb{R}^n$ an affine map which renormalize Z, and let $x \in Z$ be a point such that $\partial^-\varphi(x) \cap \{y \in \Lambda : \mathrm{dist}(y, \partial\Lambda) > \delta\} \neq \emptyset$. Then*

$$\big|\varphi(x)\big|^n \leq C(n, \lambda, \delta, R)\,|Z \cap \Omega|\,|Z|\ \mathrm{dist}\big(L(x), \partial(L(Z))\big), \tag{3.3}$$

where $R := \mathrm{diam}(\Lambda)$.

Proof. Let $y \in \partial^-\varphi(x) \cap \{y \in \Lambda : \mathrm{dist}(y, \partial\Lambda) > \delta\}$. Then $B_\delta(y) \subset \Lambda$, and moreover $\Lambda \subset B_R(y)$.

Let C_x be the cone generated by $(x, \varphi(x))$ and $(\partial Z, \varphi|_{\partial Z})$. Then $\partial^- C_x(\{x\}) - y$ is a convex set satisfying the assumptions of Lemma 3.1. Indeed $y \in \partial^- C_x(\{x\})$ and $\partial^- C_x(\{x\}) \subset \partial^-\varphi(Z) \subset \mathrm{co}(\overline{\Lambda}) \subset B_R(y)$. (Here $\mathrm{co}(\overline{\Lambda})$ denotes the convex envelope of $\overline{\Lambda}$, and the inclusion $\partial^- C_x(\{x\}) \subset \partial^-\varphi(Z)$ follows as in (3.2).) Hence, thanks to Lemma 3.1 and recalling that $B_\delta(y) \subset \Lambda$ we obtain

$$\begin{aligned}
|\partial^- C_x(\{x\})| = |\partial^- C_x(\{x\}) - y| &\leq C(\delta, R)\,|(\partial^- C_x(\{x\}) - y) \cap B_\delta| \\
&= C(\delta, R)\,|\partial^- C_x(\{x\}) \cap B_\delta(y)| \\
&\leq C(\delta, R)\,|\partial^- C_x(\{x\}) \cap \Lambda| \leq C(\delta, R)\,|\partial^-\varphi(Z) \cap \Lambda|.
\end{aligned}$$

Combining the above estimate with Lemma 3.2 and the first inequality in (3.2) (which is a consequence of (UA)) we get the result. □

We can now prove the strict convexity and differentiability on a open subset of Ω of full measure. Fix $y \in \Lambda$. We first show that the contact set S_y still reduces to a singleton.

Assume this is not the case. As in the previous subsection, we can exclude that S_y contains an infinite line. Moreover, if S_y as an exposed point, then it has to belong to $\overline{\Omega}$.

Since Λ is open, there exists a small ball $B_\delta(y) \subset \Lambda$. As above we can assume $y = 0$. Suppose there exists an exposed point, say $x \in S_y \cap \Omega$ (the case $x \in S_y \cap \partial\Omega$ is analogous). Then we can consider a section Z_ε around x as in the proof of the strict convexity outlined above, and by combining (3.1) and (3.3) the proof of the strict convexity goes through as in the case of Alexandrov solutions, and we deduce that S_y is singleton. This shows that for any $x \in \Omega$ with $\partial^-\varphi(x) \cap \Lambda \neq \emptyset$, φ is strictly convex at x. Moreover, at such a point x, (3.1) and (3.3) allow to prove the differentiability of φ exactly as in the classical case [3, Corollary 1].

All in all, we have proved:

Proposition 3.4. *Let φ be a Brenier solution of* (1.1)*, fix $x \in \Omega$, and assume that $\partial^-\varphi(x) \cap \Lambda \neq \emptyset$. Then φ is strictly convex and differentiable at x.*

Let us now define the set

$$\Omega' := \{x \in \Omega : \partial^-\varphi(x) \cap \Lambda \neq \emptyset\}.$$

Observe that Proposition 3.4 implies that $\Omega' = \{x \in \Omega : x \in \mathrm{dom}\nabla\varphi,\ \nabla\varphi(x) \in \Lambda\}$. In particular, since $\nabla\varphi(x) \in \Lambda$ for a.e. $x \in \Omega$ (which follows by $\nabla\varphi_{\#}\chi_\Omega \leq \frac{1}{\lambda^2}\chi_\Lambda$), we get $|\Omega \setminus \Omega'| = 0$. We claim that Ω' is open. Indeed, fix $x \in \Omega'$ and assume by contradiction that $x_n \to x$ but $\partial^-\varphi(x_n) \subset \mathbb{R}^n \setminus \Lambda$. Since φ is differentiable at x, (2.1) gives $\partial^-\varphi(x_n) \to \nabla\varphi(x) \in \Lambda$, impossible.

Hence Proposition 3.4 together with the continuity of the subdifferential at every differentiability point (see (2.1)) implies that φ is strictly convex and C^1 inside the open set Ω'.

In particular one can easily deduce that φ is a strictly convex Alexandrov solution of (1.1) in Ω' (see for instance the proof of [8, Theorem 3.1]), so that by [5] we get that φ is $C^{1,\alpha}$ inside Ω' (and it is smooth if $\nabla\varphi_{\#}f = g$, with f and g smooth).

To conclude the proof of the result described in the introduction, it suffices to observe that, since $\nabla\varphi|_{\Omega'}$ is continuous and injective, the set $\Lambda' := \nabla\varphi(\Omega')$ is open. Moreover the Legendre transform φ^* of φ is strictly convex and of class C^1 inside Λ', and it is an Alexandrov solution of (1.1) with Λ' instead of Ω. In particular φ^* is $C^{1,\alpha}$ inside Λ' thanks to [5], and since $\nabla\varphi^*|_{\Lambda'} = (\nabla\varphi|_{\Omega'})^{-1}$ we conclude:

Theorem 3.5. *Let φ be a Brenier solution of* (1.1). *Then there exist two open sets $\Omega' \subset \Omega$ and $\Lambda' \subset \Lambda$, with $|\Omega \setminus \Omega'| = |\Lambda \setminus \Lambda'| = 0$, such that φ is $C^{1,\alpha}$ and strictly convex inside Ω', and $\nabla\varphi : \Omega' \to \Lambda'$ is a bi-Hölder homeomorphism.*

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