REGULARITY PROPERTIES OF OPTIMAL MAPS BETWEEN NONCONVEX DOMAINS IN THE PLANE

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Abstract. Given two bounded open subsets $\Omega, \Lambda \subset \mathbb{R}^2$, and two densities f and g concentrated on Ω and Λ respectively, we investigate the regularity of the optimal map $\nabla\varphi$ sending f onto g. We show that, if f and g are both bounded away from zero and infinity, then we can find two open sets $\Omega' \subset \Omega$ and $\Lambda' \subset \Lambda$ such that f and g are concentrated on Ω' and Λ' respectively, and $\nabla\varphi : \Omega' \to \Lambda'$ is a homeomorphism. Moreover, if f and g are smooth, then $\nabla\varphi$ is a smooth diffeomorphism between Ω' and Λ'. Finally, we give a quite precise description of the singular set of φ, showing that it is a 1-dimensional manifold of class C^1 out of a countable set.

1. Introduction

Let Ω and Λ be two bounded open sets in the plane, and let $f : \mathbb{R}^2 \to \mathbb{R}$ and $g : \mathbb{R}^2 \to \mathbb{R}$ be two nonnegative functions such that $f = 0$ in $\mathbb{R}^2 \setminus \Omega$, $g = 0$ in $\mathbb{R}^2 \setminus \Lambda$, and

$$\int_\Omega f = \int_\Lambda g = 1.$$

According to Brenier's Theorem [4, 5], there exists a globally Lipschitz convex function $\varphi : \mathbb{R}^2 \to \mathbb{R}$ such that $\nabla\varphi_\# f = g$ and $\nabla\varphi(x) \in \overline{\Lambda}$ for $\mathscr{L}^2$-a.e. $x \in \mathbb{R}^2$. This is a weak way to say that φ solves the Monge-Ampère equation

$$\det(D^2\varphi) = \frac{f}{g \circ \nabla\varphi} \qquad \text{in } \mathbb{R}^2, \tag{1.1}$$

together with the "boundary condition" $\nabla\varphi(\mathbb{R}^2) \subset \overline{\Lambda}$, and we call φ a *Brenier solution.* Assuming that there exists some $\lambda > 0$ such that $\lambda \le f, g \le 1/\lambda$ inside Ω and Λ respectively, (1.1) gives

$$\lambda^2 \le \det(D^2\varphi) \le \frac{1}{\lambda^2} \qquad \text{in } \Omega \tag{1.2}$$

in a weak sense. As shown by Caffarelli [9], if Λ is convex, then φ is strictly convex, and it solves the Monge-Ampère equation (1.2) in the Alexandrov sense (see Section 2 for the definition of Alexandrov solution). Exploiting this fact, one can develop a satisfactory regularity theory with goes as follows (see [6, 7, 8, 9]):

(a) If $\lambda \le f, g \le 1/\lambda$ for some $\lambda > 0$, then $\varphi \in C^{1,\alpha(\lambda)}_{loc}(\Omega)$.
(b) If $|f-1|, |g-1| \le \varepsilon = \varepsilon(p)$, then $\varphi \in W^{2,p}_{loc}(\Omega)$ $(p \in [1,\infty))$.
(c) If $f \in C^{k,\alpha}_{loc}(\Omega)$ and $g \in C^{k,\alpha}_{loc}(\Lambda)$, with $f, g > 0$, then $\varphi \in C^{k+2,\alpha}_{loc}(\Omega)$ $(k \ge 0$, $\alpha \in (0,1))$.

However, if Λ is not convex, one cannot expect for a regularity theory: there exist f and g smooth such that $\varphi \notin C^1(\Omega)$ (see [9]). The aim of this paper is to understand what can be said about the regularity of φ in this case, and to study the set of singularities of φ in terms of the geometry of Λ. This problem has been already investigated in [14] and [11, Section 5] under the assumption that Ω is convex, and under stronger hypotheses on Λ. Let us remark that the assumption that Ω is convex considerably simplifies the situation: indeed, in that case, one can apply Caffarelli's regularity theory to $\nabla\varphi^*$ (φ^* being the Legendre transform of φ) to deduce that $\nabla\varphi^*(\Lambda)$ is an open set of full measure inside Ω, and that φ is C^1 and strictly convex inside $\nabla\varphi^*(\Lambda)$. Moreover the geometric hypotheses on Λ assumed by the authors allow to exclude some of the possible structures which could appear in the singular set (see Figure 3.3).

Here, under some weak assumptions on Ω and Λ, we can prove that there exist two open sets $\Omega' \subset \Omega$ and $\Lambda' \subset \Lambda$, with $\mathscr{L}^2(\Omega \setminus \Omega') = \mathscr{L}^2(\Lambda \setminus \Lambda') = 0$, such that φ is C^1 and strictly convex inside Ω', and $\nabla\varphi$ is a homeomorphism between Ω' and Λ'. In particular φ is an Alexandrov solution of (1.2) inside Ω', and (a)-(b)-(c) above hold with Ω' in place of Ω (see Theorem 3.1).

We then study the geometry of the set *Sing* $\subset \Omega$ where φ is not differentiable. Although it is well-known that the singular set of a convex function is 1-rectifiable, we can prove a more refined result, showing that each connected component is a C^1-manifold out of a countable set (Theorem 3.4), and giving a bound on the numbers of its connected components which do not touch $\partial\Omega$ in terms of the geometry of Λ (Proposition 3.5). Moreover, as a corollary of the regularity of φ inside Ω', we also have $\mathscr{L}^2\left(\overline{Sing} \cap \Omega\right) = 0$. These results generalize and improve the ones in [14] and [11, Section 5].

The structure of the paper is the following: first, in Section 2 we introduce some notation, and we recall some useful facts about convex functions. Then in Section 3 we prove the results described above.

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2. Notation and preliminaries

For any real number $s \in (0, n]$, we denote by $\mathscr{H}^s(B)$ the *s-dimensional Hausdorff measure* of a Borel set $B \subset \mathbb{R}^n$, defined by

$$\mathscr{H}^s(B) := \sup_{\delta>0} \inf \left\{ \sum_{i=1}^{\infty} \frac{[\mathrm{diam}(B_i)]^s}{2^s} \mid B \subset \bigcup_{i=1}^{\infty} B_i,\ \mathrm{diam}(B_i) \leq \delta \right\}.$$

With this definition, $\mathscr{H}^n$ coincides up to a constant factor with the Lebesgue measure $\mathscr{L}^n$.

Given two points $y_0, y_1 \in \mathbb{R}^n$, we will denote by $[y_0, y_1]$ the *segment* joining them, i.e. the set of points of the form $ty_0 + (1-t)y_1$, $t \in [0,1]$.

Let $\varphi : \mathbb{R}^n \to \mathbb{R}$ be a convex function. Its *subdifferential* at a point x is defined by

$$\partial\varphi(x) := \left\{ y \in \mathbb{R}^n \mid \varphi(z) \geq \varphi(x) + y \cdot (z - x) \quad \forall\, z \in \mathbb{R}^n \right\}.$$

It is well-known that the map $x \mapsto \partial\varphi(x)$ is upper semicontinuous, i.e.

$$x_k \to x, \quad y_k \to y, \quad y_k \in \partial\varphi(x_k) \quad \Rightarrow \quad y \in \partial\varphi(x)$$

(see [2, Proposition 2.1]). This implies in particular that $x \mapsto \nabla\varphi(x)$ is continuous on the set where φ is differentiable. Moreover, φ is differentiable at a point x if and only if $\partial\varphi(x)$ is a singleton. Hence, one can decompose the set of non-differentiability points according to the dimension of the singular set:

$$\Sigma_k(\varphi) := \big\{x \in \mathbb{R}^n \mid \dim(\partial\varphi(x)) = k\big\}, \qquad k = 0, \dots, n.$$

For any $k = 0, \dots, n$, the set $\Sigma_k(\varphi)$ is $(n-k)$-rectifiable, i.e. can be covered by a countable union of Lipschitz submanifolds of dimension $n-k$ (see [2, Theorem 4.1]). One also defines the set of *reachable subgradients* at x as

$$\nabla_*\varphi(x) := \Big\{\lim_{k\to+\infty} \nabla\varphi(x_k) \mid x_k \in \Sigma_0,\, x_k \to x\Big\}.$$

It is known (see for instance [12]) that $\mathrm{co}(\nabla_*\varphi(x))$, the convex hull of $\nabla_*\varphi(x)$, coincides with $\partial\varphi(x)$. One key property that we will use to study of the structure of the singular set of Brenier solutions of the Monge-Ampère equation is that, whenever the set $\nabla_*\varphi(x)$ is strictly contained inside the boundary of $\partial\varphi(x)$, then singularities propagate (see Theorem 3.4 below).

We finally recall the definition of *Alexandrov solution* to the Monge-Ampère equation: a convex function $\varphi : \Omega \subset \mathbb{R}^n \to \mathbb{R}$ solves the Monge-Ampère equation (1.2) in the Alexandrov sense if, for any Borel set $B \subset \Omega$,

$$\lambda^2 \mathscr{L}^n(B) \le \mathscr{L}^n\big(\partial\varphi(B)\big) \le \frac{1}{\lambda^2}\mathscr{L}^n(B),$$

where $\partial\varphi(B) := \cup_{x\in B}\partial\varphi(x)$.

3. Regularity results and structure of the singular set

Let $\varphi : \mathbb{R}^2 \to \mathbb{R}$ be a Brenier solution of the Monge-Ampère equation (1.1). We want to study the regularity of φ and the structure of its singular set.

3.1. Regularity of φ.

To prove the regularity of φ out of a set of measure zero, we will assume that $\partial\Omega$ and $\partial\Lambda$ are continuous. This last condition means that, for any $x \in \partial\Omega$ (resp. $y \in \partial\Lambda$), there exists $r > 0$ such that, up to a change of coordinates, $\Omega \cap B_r(x)$ (resp. $\Lambda \cap B_r(y)$) coincides with the epigraph of a continuous function. Observe that we do not even supposed that $\mathscr{L}^2(\partial\Omega) = \mathscr{L}^2(\partial\Lambda) = 0$ (but we assumed that f and g have no mass outside Ω and Λ respectively, so that in particular $\int_{\partial\Omega} f = \int_{\partial\Lambda} g = 0$).

Theorem 3.1. *Assume that there exists $\lambda > 0$ such that*

$$\lambda \le f \le \frac{1}{\lambda} \quad \textit{in } \Omega\,, \qquad \lambda \le g \le \frac{1}{\lambda} \quad \textit{in } \Lambda,$$

and that $\partial\Omega$ and $\partial\Lambda$ are continuous. Then φ is strictly convex inside Ω. Moreover there exist two open sets $\Omega' \subset \Omega$ and $\Lambda' \subset \Lambda$, with $\mathscr{L}^2(\Omega\setminus\Omega') = \mathscr{L}^2(\Lambda\setminus\Lambda') = 0$, such that $\varphi \in C^1(\Omega')$, $\nabla\varphi$ is a homeomorphism between Ω' and Λ', and φ is an Alexandrov solution of (1.2) *inside Ω'. In particular, Caffarelli's regularity theory for strictly convex Alexandrov solutions of the Monge-Ampère equations applies to φ inside Ω'.*

The proof of the above theorem is divided into some preliminary results. First of all we define the set of *regular points* as

$$Reg := \Sigma_0(\varphi) \cup \big\{x \in \mathbb{R}^2 \mid \partial\varphi(x) \cap \overline{\Lambda} \text{ contains a segment}\big\}.$$

The first goal is to prove that φ is C^1 at regular points. Let us remark that, for the proof of this result, we only need an upper bound on f and a lower bound on g, and we do not assume that f vanishes outside Ω:

Proposition 3.2. *Assume that there exists $\lambda > 0$ such that*

$$f \le \frac{1}{\lambda} \quad \text{in } \mathbb{R}^2\,, \qquad \lambda \le g \quad \text{in } \Lambda,$$

and that $\partial\Lambda$ is continuous. If $x_0 \in \mathrm{Reg}$, then $\partial\varphi(x_0)$ is a singleton.

Proof. Suppose that the statement is false. Then, as $x_0 \in Reg$, we can find $y_0, y_1 \in \partial\varphi(x_0)$, with $y_0 \ne y_1$, such that $[y_0, y_1] \subset \overline{\Lambda}$. We will show that in this case the graph of φ contains a suitably chosen half-line, and this will be impossible.

Since $[y_0, y_1] \subset \overline{\Lambda}$ and $\partial\Lambda$ is continuous, there exist $v \in \mathbb{R}^2$ a unit vector orthogonal to $y_1 - y_0$, and $\varepsilon > 0$ small, such that $ty_0 + (1-t)y_1 + \varepsilon v \in \Lambda$ for all $t \in [1/4, 3/4]$. Up to an affine change of coordinates, and by subtracting from φ an affine function, we can assume that $x_0 = (0,0)$, $\varphi(0,0) = 0$, $y_0 = -e_1$, $y_1 = e_1$, and $v = e_2$. Hence

$$\varphi(x_1, x_2) \ge |x_1|, \quad \varphi(0,0) = 0,$$
$$te_1 + \varepsilon e_2 \in \Lambda \quad \text{for all } t \in [-1/2, 1/2], \text{ for some } \varepsilon > 0. \tag{3.1}$$

We remark that, since $\nabla\varphi(x) \in \overline{\Lambda}$ for $\mathscr{L}^2$-a.e. x and $\overline{\Lambda}$ is bounded, there exists $R > 0$ such that φ is R-Lipschitz. We will show that $\varphi(0, x_2) = 0$ for all $x_2 \ge 0$. For this, we assume by contradiction that there exist $h \in (0,1]$ and $\delta > 0$ such that $\varphi(0,\delta) = h$. Then, thanks to [13, Lemma 2.3], we have

$$S_{h,\delta} \subset \partial\varphi(R_{h,\delta}), \tag{3.2}$$

where

$$S_{h,\delta} := [-1/2, 1/2] \times \left[0, \frac{h}{2\delta(1+R)}\right], \quad R_{h,\delta} := [-h, h] \times [0, (1+R)\delta].$$

Let $\alpha := \frac{1}{3}\min\left\{\varepsilon, \frac{h}{2\delta(1+R)}\right\}$, and define

$$T_{h,\delta} := [-1/4, 1/4] \times [\alpha, 2\alpha] \subset \Lambda.$$

By (3.1), (3.2) and an easy geometric argument exploiting the convexity of φ, it is easily seen that $(\nabla\varphi)^{-1}(y) \subset R_{h,\delta}$ for any $y \in T_{h,\delta}$, or equivalently all the mass sent by $\nabla\varphi$ inside $T_{h,\delta}$ comes from $R_{h,\delta}$. However, since $g \ge \lambda$ inside Λ, and $f \le 1/\lambda$ on $\mathbb{R}^2$, we get

$$\frac{\lambda\alpha}{2} = \lambda\mathscr{L}^2(T_{h,\delta}) \le \frac{\mathscr{L}^2(R_{h,\delta})}{\lambda} = \frac{2(1+R)h\delta}{\lambda},$$

which is impossible for $\delta \le \delta_0 := \min\left\{\frac{\lambda^2\varepsilon}{13(1+R)}, \frac{\lambda}{5(1+R)}\right\}$. This proves that $\varphi(0, x_2) = 0$ for all $x_2 \in [0, \delta_0]$, and iterating this argument we deduce that $\varphi(0, x_2) = 0$ for all $x_2 \ge 0$.

Claim: $\overline{\Lambda} \subset \{(y_1, y_2) \mid y_2 \le 0\}$.

We observe that the claim contradicts (3.1), and this concludes the proof of the proposition. Thus we are left with proving the claim.

Let $y \in \varphi(x)$ for some $x \in \mathbb{R}^2$. We have

$$0 = \varphi(te_2) \geq \varphi(x) + y \cdot (te_2 - x) \qquad \forall\, t \geq 0.$$

Letting $t \to +\infty$ we deduce $y_2 = y \cdot e_2 \leq 0$, so that $\partial\varphi(\mathbb{R}^2) \subset \{(y_1, y_2) \mid y_2 \leq 0\}$. On the other hand, since $\mathscr{L}^2$-a.e. $y \in \Lambda$ belong to the image of $\nabla\varphi$, we have $\partial\varphi(\mathbb{R}^2) \supset \Lambda$, and the claim follows. □

As a corollary, we immediately deduce a well-known result on the strict convexity of solutions of (1.2), see for instance [10]:

Corollary 3.3. *Assume that there exists $\lambda > 0$ such that*

$$\lambda \leq f \quad \text{in } \Omega\,, \qquad g \leq \frac{1}{\lambda} \quad \text{in } \mathbb{R}^2,$$

and that $\partial\Omega$ is continuous. Then φ is strictly convex inside Ω.

Proof. Suppose by contradiction that there exist $x_0, x_1 \in \Omega$ such that $y_0 \in \partial\varphi(x_0) \cap \partial\varphi(x_1)$, and let us consider the function φ^* given by

$$\varphi^*(y) := \sup_{x \in \Omega} \big\{x \cdot y - \varphi(x)\big\}, \qquad \forall\, y \in \mathbb{R}^2. \tag{3.3}$$

Then $\varphi^* : \mathbb{R}^2 \to \mathbb{R}$ is a globally Lipschitz convex function such that $(\nabla\varphi^*)_\# g = f$, and $\nabla\varphi^*(y) \in \overline{\Omega}$ for $\mathscr{L}^2$-a.e. $y \in \mathbb{R}^2$ (see for instance [15, Chapter 2]). Since $x_0, x_1 \in \partial\varphi^*(y_0)$, by applying Proposition 3.2 with φ^* in place of φ, we deduce that $\partial\varphi^*(y)$ is a singleton for every $y \in \mathit{Reg}^*$, where

$$\mathit{Reg}^* := \Sigma_0(\varphi^*) \cup \big\{y \in \mathbb{R}^2 \mid \partial\varphi^*(y) \cap \overline{\Omega} \text{ contains a segment}\big\}.$$

As $x_0, x_1 \in \partial\varphi^*(y_0) \cap \Omega$, $\partial\varphi^*(y_0) \cap \overline{\Omega}$ contains a segment, and therefore $y_0 \in \mathit{Reg}^*$, absurd. □

Proof of Theorem 3.1. Let us define the open sets

$$\Omega' := \{x \in \Omega \mid \partial\varphi(x) \subset \Lambda\}, \qquad \Lambda' := \{x \in \Lambda \mid \partial\varphi^*(x) \subset \Omega\}. \tag{3.4}$$

Since all the mass of g is contained inside Λ, $\nabla\varphi(x)$ exists and belongs to Λ for $\mathscr{L}^2$-a.e. $x \in \Omega$. This implies that $\mathscr{L}^2(\Omega \setminus \Omega') = 0$, and analogously $\mathscr{L}^2(\Lambda \setminus \Lambda') = 0$. Moreover, since $\Omega' \subset \mathit{Reg}$ and $\Lambda' \subset \mathit{Reg}^*$, by Proposition 3.2 applied to both φ and φ^* we get $\varphi \in C^1(\Omega')$ and $\varphi^* \in C^1(\Lambda')$. Recalling the identities

$$\nabla\varphi^*(\nabla\varphi(x)) = x \;\; \mathscr{L}^2\text{-a.e. in } \Omega \quad \text{and} \quad \nabla\varphi(\nabla\varphi^*(y)) = y \;\; \mathscr{L}^2\text{-a.e. in } \Lambda$$

and the fact that φ and φ^* are differentiable inside Reg and Reg^* respectively, we immediately obtain that $\nabla\varphi(x) \in \Lambda'$ for any $x \in \Omega'$, $\nabla\varphi^*(y) \in \Omega'$ for any $y \in \Lambda'$, and $\nabla\varphi : \Omega' \to \Lambda'$ is a homeomorphism. Finally, the fact that φ is C^1 and strictly convex inside Ω' (cfr. Corollary 3.3) implies easily that φ is an Alexandrov solution of (1.2) in Ω'. Indeed, by the definition of push-forward, we know that

$$\int_A g(y)\, dy = \int_{(\nabla\varphi)^{-1}(A)} f(x)\, dx \qquad \text{for all } A \subset \Lambda' \text{ Borel},$$

which is equivalent to

$$\int_{\nabla\varphi(B)} g(y)\, dy = \int_B f(x)\, dx \qquad \text{for all } B \subset \Omega' \text{ Borel}.$$

Hence, being differentiable inside Ω', φ solves (1.2) in the Alexandrov sense, as desired. □

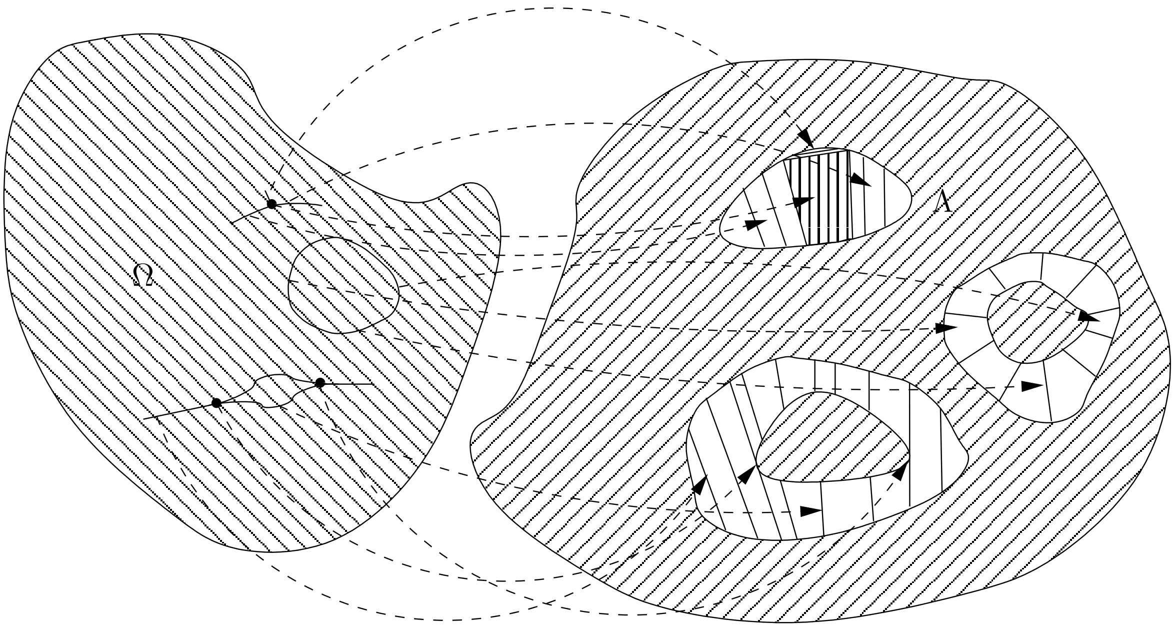

FIGURE 3.1. With our definition, also an annulus inside Λ is considered as a hole. Observe that if a connected component of the singular set does not touch $\partial\Omega$, then its image through $\partial\varphi$ has to fill a hole (indeed, if not, we could continue to propagate the singularity, see Proposition 3.5). Moreover, the number of connected components of Λ bound the number of closed injective curves in *Sing*. In the above figure, the connected components of Λ are 3, and the number of closed injective curves is $2 = 3 - 1$. Observe also that the images of the two curve through $\partial\varphi$ fill the annuli in different ways (in one case the subdifferentials are almost parallel, while in the other case they turn around along the annulus).

3.2. **The structure of *Sing*.** We now want to study the singular set of φ in Ω, i.e. the set of points $x \in \Omega$ where φ is not differentiable, in terms of the geometry of Λ. In all this paragraph, we assume that the hypotheses of Theorem 3.1 hold.

Let us define a *hole* in Λ as a connected open set $\mathcal{O}$ such that $\mathcal{O} \cap \Lambda = \emptyset$ and $\partial\mathcal{O} \subset \partial\Lambda$. Thanks to Proposition 3.2, it is clear that the singular set of φ, which we denote by *Sing*, coincides with $\Omega \setminus$ *Reg*. We will show that all but at most m connected components of *Sing* touch $\partial\Omega$, where $m \geq 0$ is the number of holes in Λ (see Figure 3.1). Moreover we will give a quite precise description of the connected components of *Sing*, showing that they are C^1-manifolds outside a countable set.

First of all we observe that *Sing* is a 1-rectifiable set, and in particular it has σ-finite $\mathscr{H}^1$-measure. Moreover, thanks to Theorem 3.1, we also have $\mathscr{L}^2\left(\overline{Sing} \cap \Omega\right) = 0$ (a property which is false for a general convex function, see also Remark 3.6). Since $\nabla\varphi(x) \in \overline{\Lambda}$ for $\mathscr{L}^2$-a.e. $x \in \mathbb{R}^2$ and $x \mapsto \nabla\varphi(x)$ is continuous on its domain of definition, we get that $\nabla\varphi(x) \in \overline{\Lambda}$ at every point where φ is differentiable. Hence, by the definition of *Reg* and the identity $\mathrm{co}(\nabla_*\varphi(x)) = \partial\varphi(x)$, we easily obtain the following characterization:

$$Sing = \{x \in \Omega \mid \partial\varphi(x) \cap \Lambda = \emptyset,\ \nabla_*\varphi(x) \subset \partial\Lambda,\ \partial\varphi(x) \not\subset \partial\Lambda\}. \tag{3.5}$$

Let us write it as the disjoint union of its connected components:

$$Sing := \bigcup_i S_i.$$

We remark that, thanks to Corollary 3.3, we have

$$\partial\varphi(S_i) \cap \partial\varphi(S_\ell) = \emptyset \qquad \text{if } i \neq \ell. \tag{3.6}$$

The following structure theorem for the singular set generalizes the ones in [14, Theorem 1.1], [11, Theorem 5.1]:

Theorem 3.4. *The number of connected components of* Sing *is at most countable. Moreover:*

(1) *either S_i coincides with an isolated point $\{x_i\}$ for some $x_i \in \Omega$, and in this case the boundary of $\partial\varphi(x_i)$ is enterely contained inside $\partial\Lambda$ (so that $\partial\varphi(x_i)$ completely fills a hole in Λ, see Figure 3.2);*

(2) *or S_i can be written as a disjoint union as follows:*

$$S_i = \bigcup_{j \in \mathbb{N}} \gamma_{ij}, \tag{3.7}$$

where $\gamma_{ij} : I_{ij} \to$ Sing are embedded Lipschitz curves parameterized by arc-length, with $I_{ij} = [0, t_{ij})$ or $I_{ij} = (0, t_{ij})$, depending whether they are periodic curves or not (see Figure 3.2).

Furthermore, in case (2), if $\{t_k^{ij}\}_k \subset I_{ij}$ is the (at most countable) set of times such that $\partial\varphi\big(\gamma_{ij}(t_k^{ij})\big) \in \Sigma_2(\varphi)$, and we define $J_{ij} := I_{ij} \setminus \big(\cup_k \{t_k^{ij}\}\big)$, then:

(2,a) *$\gamma_{ij}(t) \in \Sigma_1(\varphi)$ for every $t \in J_{ij}$, and there exist two injective curves $J_{ij} \ni t \mapsto y_0^{ij}(t), y_1^{ij}(t) \subset \partial\Lambda$ such that $\partial\varphi(\gamma_{ij}(t)) = [y_0^{ij}(t), y_1^{ij}(t)]$ for any $t \in J_{ij}$, and*

$$J_{ij} \ni t \mapsto \frac{y_1^{ij}(t) - y_0^{ij}(t)}{|y_1^{ij}(t) - y_0^{ij}(t)|}$$

is continuous.

(2,b) *γ_{ij} is right and left differentiable at every $t \in J_{ij}$, and the right and left derivatives $\dot\gamma_{ij}^\pm$ coincide up to a countable sets of times $\{\bar t_\ell^{ij}\}_\ell$ where $\dot\gamma_{ij}^+(\bar t_\ell^{ij}) = -\dot\gamma_{ij}^-(\bar t_\ell^{ij})$. The set of times when these discontinuities in the derivative may happen, can be characterized as the set of $t \in J_{ij}$ where the segment $[y_0^{ij}(t), y_1^{ij}(t)] = \partial\varphi(\gamma_{ij}(t))$ intersects $\partial\Lambda$ in at least three points (see Figure 3.3). Moreover, the map $J_{ij} \setminus \{\bar t_\ell^{ij}\}_\ell \ni t \mapsto \dot\gamma_{ij}(t)$ is continuous,*

$$\dot\gamma_{ij}^\pm(t) \cdot (y_1^{ij}(t) - y_0^{ij}(t)) = 0 \qquad \forall\, t \in J_{ij}.$$

(2,c) *The curves γ_{ij} can be chosen so that, as $t \to 0^+$ and as $t \to t_{ij}^-$, one of the following happens:*

- $\mathrm{dist}(\gamma_{ij}(t), \partial\Omega) \to 0$*;*
- $|y_0^{ij}(t) - y_1^{ij}(t)| \to 0$*;*
- *there exist $\ell \in \mathbb{N}$ and $t_0 \in I_{i\ell}$ such that $\gamma_{ij}(t) \to \gamma_{i\ell}(t_0)$ (that is, at the point $\gamma_{i\ell}(t_0)$ there is a bifurcation, see Figure 3.2).*

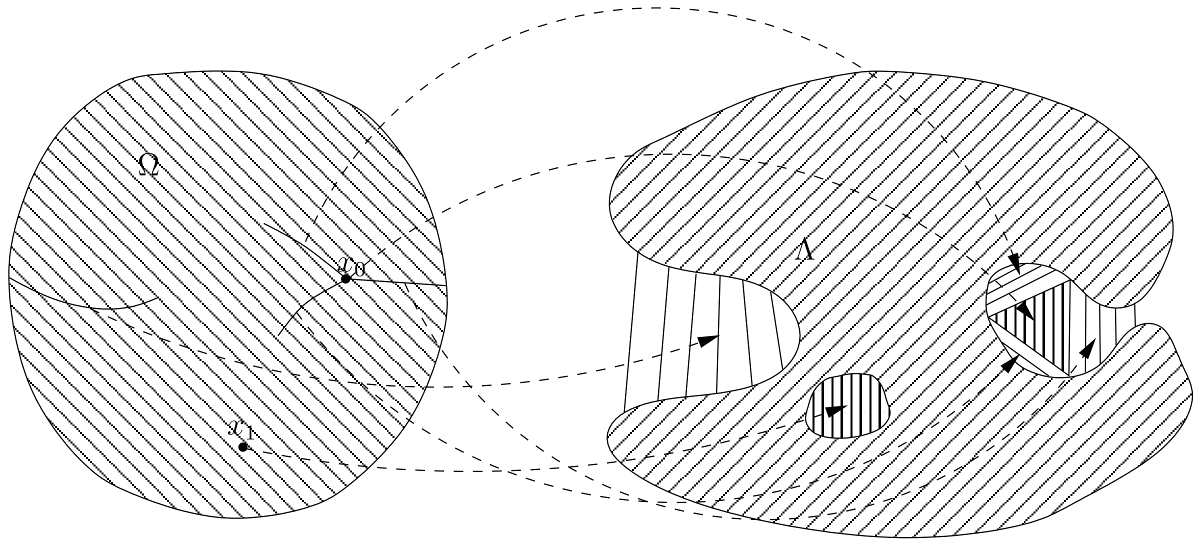

FIGURE 3.2. The subdfferential of φ at the point x_0 is two-dimensional. This generates a Lipschitz bifurcation in *Sing* at x_0. At the point x_1, $\partial\varphi(x_1)$ completely fills a hole in Λ, and x_1 is an isolated singularity.

(2,d) for every $t \in \{t_k^{ij}\}_k$, for any $y_0^{ik}, y_1^{ik} \in \nabla_*\varphi(\gamma_{ij}(t_k^{ij}))$ such that $[y_0^{ik}, y_1^{ik}] \cap \partial\Lambda = \{y_0^{ik}, y_1^{ik}\}$, we have

$$\partial\varphi(\gamma_{ij}(t)) \to [y_0^{ik}, y_1^{ik}] \quad \text{and} \quad \dot\gamma_{ij}^{\pm}(t)\cdot(y_1^{ik} - y_0^{ik}) \to 0$$

either as $t \to (t_k^{ij})^+$ or as $t \to (t_k^{ij})^-$.

Finally, if Λ has n connected components, then the number of closed injective curves inside Sing *is bounded by $n-1$ (see Figure 3.1).*

Proof. Since *Sing* has σ-finite $\mathscr{H}^1$-measure, the fact that the number of connected components of *Sing* is countable follows easily from points (1) and (2). Let us prove them.

PROOF OF (1) AND (2). Let $x \in S_i$. Then (3.5) implies that $\partial\varphi(x)$ is a convex set contained in $\overline{\mathrm{co}(\Lambda)} \setminus \Lambda$ such that $\nabla_*\varphi(x) \subset \partial\Lambda$. This gives

$$\partial\varphi(x) \setminus \nabla_*\varphi(x) = \partial\varphi(x) \cap \left(\overline{\mathrm{co}(\Lambda)} \setminus \overline{\Lambda}\right) \neq \emptyset.$$

We now distinguish two cases: $\dim(\partial\varphi(x)) = 2$, or $\dim(\partial\varphi(x)) = 1$.

In the first case, two possibilities arise:

(i) $\nabla_*\varphi(x)$ coincides with the boundary of $\partial\varphi(x)$.

(ii) $\nabla_*\varphi(x)$ is strictly contained in the boundary of $\partial\varphi(x)$.

In case (i), since $\partial\varphi(x) \subset \overline{\mathrm{co}(\Lambda)} \setminus \Lambda$ and $\nabla_*\varphi(x) \subset \partial\Lambda$, the boundary of $\partial\varphi(x)$ is contained inside $\partial\Lambda$, and so $\partial\varphi(x)$ coincides with a hole inside Λ. Moreover, the upper semicontinuity of $\partial\varphi$ implies that $\partial\varphi(z) \cap \Lambda \neq \emptyset$ for z near x, which means that any point near x belongs to *Reg*. Hence $S_i = \{x\}$ (see Figure 3.2).

In case (ii), let us consider any couple of vectors $y_0, y_1 \in \nabla_*\varphi(x)$ such that $[y_0, y_1] \cap \partial\Lambda = \{y_0, y_1\}$,

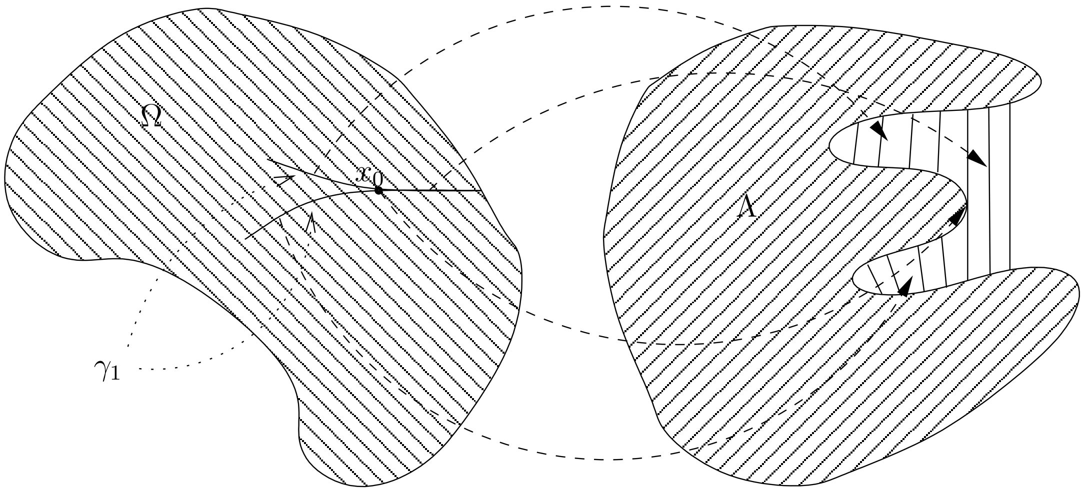

FIGURE 3.3. The subdifferential of φ at x_0 touches $\partial\Lambda$ at three different points. This generates a C^1-bifurcation in *Sing*, and it may be possible that the curve γ_1 that we have selected in the partition of *Sing* consists of the two arcs to the left of x_0, so its derivative change direction at x_0 (of course, if the number of bifurcations as the one above is finite, then we can always choose the curves γ_i is such a way to avoid such discontinuities, so that they are all of class C^1).

and $[y_0, y_1]$ is contained in the boundary of $\partial\varphi(x)$ (recall that $\partial\varphi(x) = \mathrm{co}(\nabla_*\varphi(x))$). Then, thanks to [1, Theorem 4.2], there exists a Lipschitz curve $\gamma : [0, \rho] \to \Omega$ with $\gamma(0) = x$, a vector $v \neq 0$ orthogonal to $y_1 - y_0$, and a positive number $\delta > 0$, such that

$$\lim_{t\to 0^+} \frac{\gamma(t) - \gamma(0)}{t} = v, \qquad \mathrm{diam}\big(\partial\varphi(\gamma(t))\big) \geq \delta \quad \forall\, t \in [0, \rho].$$

This implies that there is a curve of singular points leaving from x.

In the case $\dim(\partial\varphi(x)) = 1$, we can write $\partial\varphi(x) = [y_0, y_1]$ for some $y_0, y_1 \in \partial\Lambda$. Then, applying again [1, Theorem 4.2] (and in particular its proof, see also the proof of Proposition 3.5 below), we deduce that there exists a Lipschitz curve $\gamma : [-\rho, \rho] \to \Omega$ with $\gamma(0) = x$, a vector $v \neq 0$ orthogonal to $y_1 - y_0$, and a positive number $\delta > 0$, such that

$$\lim_{t\to 0} \frac{\gamma(t) - \gamma(0)}{t} = v, \qquad \mathrm{diam}\big(\partial\varphi(\gamma(t))\big) \geq \delta \quad \forall\, t \in [-\rho, \rho]$$

(see also [11, Theorem 4.2]). Hence, if S_i is not an isolated point, to decompose it as in (3.7) we proceed as follows: we start from any point in S_i and we use the results above to propagate our singularity as long as we can, that is either up to the moment when the diameter of the subdifferential goes to 0, or the singular curve γ_{i1} hit $\partial\Omega$, or the curve closes onto itself (observe that in general there could be more than one possibility to propagate the singularity, and we just choose one of them). Then we remove the curve γ_{i1} constructed in this way from S_i, and we iterate the procedure, but now we stop also in case γ_{i2} hit γ_{i1}. Going on in this way, and reparameterizing all the curves γ_{ij} by arc-length, we finally get (3.7). (The procedure necessarily ends after countably many iterations, as every S_i has σ-finite $\mathscr{H}^1$-measure.) This completes the proof of (1) and (2).

Proof of $(2,a)$ and $(2,b)$. To study the differentiability of γ_{ij}, we observe that for any $t \in J_{ij}$ there exist $y_0^{ij}(t), y_1^{ij}(t) \in \partial\Lambda$ such that $\partial\varphi(\gamma_{ij}(t)) = [y_0^{ij}(t), y_1^{ij}(t)]$. Thanks to the upper semicontinuity of the subdifferential and the fact that $\partial\Lambda$ is (uniformly) continuous, we see that if $t_n \in J_{ij}$ and $t_n \to t_\infty$, then both $y_0^{ij}(t_n)$ and $y_1^{ij}(t_n)$ converge, and

$$[\lim_n y_0^{ij}(t_n), \lim_n y_1^{ij}(t_n)] \subset [y_0^{ij}(t_\infty), y_1^{ij}(t_\infty)]. \tag{3.8}$$

Let us remark that, if the above inclusion is strict, then there exists a point $y \neq y_0^{ij}(t_\infty), y_1^{ij}(t_\infty)$ such that $y \in [y_0^{ij}(t_\infty), y_1^{ij}(t_\infty)] \cap \partial\Lambda$ (see Figure 3.3).

Thanks to (3.8) and the fact that $\partial\Lambda$ is (uniformly) continuous, we deduce that we can always exchange $y_0(t)$ with $y_1(t)$ for $t \in J_{ij}$, so that the map $J_{ij} \ni t \mapsto \frac{y_1^{ij}(t)-y_0^{ij}(t)}{|y_1^{ij}(t)-y_0^{ij}(t)|}$ is continuous[1]. Let us recall that, if $u : [0,1] \to \mathbb{R}$ is a function which admits at every point both left and right limit, then at any discontinuity point it can only jumps, and the number of its jumps is countable. Combining this fact with (3.8), we get that the function

$$J_{ij} \ni t \mapsto |y_1^{ij}(t) - y_1^{ij}(t)|$$

is continuous up to a countable sets of times $\{\bar t_\ell^{ij}\}_\ell$ where it may jump, and at any $t \in \{\bar t_\ell^{ij}\}_\ell$ it always admits a limit from the left and one from the right.

This fact, together with the strict convexity of φ (cfr. Corollary 3.3), implies that the curves

$$J_{ij} \ni t \mapsto y_0^{ij}(t) \subset \partial\Lambda, \qquad J_{ij} \ni t \mapsto y_1^{ij}(t) \subset \partial\Lambda,$$

are injective, they are continuous up to a countable number of times $\{\bar t_\ell^{ij}\}_\ell$, and at any time $t \in \{\bar t_\ell^{ij}\}_\ell$ they both admit a left and a right limit.

We now apply [2, Proposition 2.2] and [3, Theorem 2.3] to obtain that, for any $t_0 \in J_{ij}$, if $v^\pm(t_0)$ is a limit point of $\frac{\gamma_{ij}(t)-\gamma_{ij}(t_0)}{|\gamma_{ij}(t)-\gamma_{ij}(t_0)|}$ as $t \to t_0^\pm$, then

$$v^\pm(t_0) \cdot (y_1^{ij}(t_0) - y_0^{ij}(t_0)) = 0. \tag{3.9}$$

We distinguish two cases, depending whether t_0 belongs to $\{\bar t_\ell^{ij}\}_\ell$ or not.

If $t_0 \notin \{\bar t_\ell^{ij}\}_\ell$, we know that the (injective) curves $J_{ij} \ni t \mapsto y_0^{ij}(t), y_1^{ij}(t)$ are continuous at t_0. Let $w(t) := [y_1^{ij}(t) - y_0^{ij}(t)]^\perp$, where $[y_1^{ij}(t) - y_0^{ij}(t)]^\perp$ denotes the clockwise rotation of $\pi/2$. Then, thanks to the continuity of

$$J_{ij} \ni t \mapsto [y_0^{ij}(t), y_1^{ij}(t)]$$

[1]Indeed, for every $\bar t \in J_{ij}$, thanks to (3.8) and the fact that $y_0(\bar t) \neq y_1(\bar t)$ we can find a small open interval $J_{\bar t} \subset J_{ij}$ containing $\bar t$ where $\mathrm{diam}(\partial\varphi(\gamma_{ij}(t)))$ is bounded away from zero. Then, since $\partial\Lambda$ is continuous, using again (3.8) we can define two continuous functions $y_0^{\bar t}, y_1^{\bar t} : J_{\bar t} \to \partial\Lambda$ such that $\partial\varphi(\gamma_{ij}(t)) = [y_0^{\bar t}(t), y_1^{\bar t}(t)]$ on $J_{\bar t}$. We observe that for every $t \in J_{\bar t_1} \cap J_{\bar t_2}$, either $y_0^{\bar t_1}(t) = y_0^{\bar t_2}(t)$ and $y_1^{\bar t_1}(t) = y_1^{\bar t_2}(t)$, or $y_0^{\bar t_1}(t) = y_1^{\bar t_2}(t)$ and $y_1^{\bar t_1}(t) = y_0^{\bar t_2}(t)$. Hence, thanks to the local compactness of J_{ij}, we can find a locally finite covering of J_{ij} made by intervals of the form $J_{\bar t_n}$, and on any of these intervals we can define $y_0, y_1 : J_{ij} \to \Lambda$ in a coherent way by setting either $y_0 := y_0^{\bar t_n}$ and $y_1 := y_1^{\bar t_n}$, or $y_0 := y_1^{\bar t_n}$ and $y_1 := y_0^{\bar t_n}$, in such a way that y_0 and y_1 are both continuous on the whole J_{ij}.

at $t = t_0$, the line

$$s \mapsto \frac{y_0^{ij}(t) + y_0^{ij}(t_0)}{2} + sw(t_0)$$

intersects transversally any of the segments $[y_0^{ij}(t), y_1^{ij}(t)]$ for $t \in J_{ij}$ close to t_0. Denoting by $y(t) \in \partial\varphi(\gamma(t))$ such intersection points, by the monotonicity of the subdifferential we have

$$\frac{\gamma_{ij}(t) - \gamma_{ij}(t_0)}{|\gamma_{ij}(t) - \gamma_{ij}(t_0)} \cdot \frac{w(t_0)}{|w(t_0)|} = \frac{\gamma_{ij}(t) - \gamma_{ij}(t_0)}{|\gamma_{ij}(t) - \gamma_{ij}(t_0)|} \cdot \frac{y(t) - y(t_0)}{|y(t) - y(t_0)|} \geq 0,$$

so that letting $t \to t_0$ we obtain

$$v^{\pm}(t_0) \cdot [y_1^{ij}(t_0) - y_0^{ij}(t_0)]^{\perp} \geq 0. \tag{3.10}$$

Combinining (3.9) and (3.10), we see that the directions of $v^{\pm}(t_0)$ are uniquely determined and vary continuously on $J_{ij} \setminus \{\bar{t}_\ell^{ij}\}_\ell$. Therefore, since $|v^{\pm}(t_0)| = 1$, $v^{\pm}(t_0)$ are unique and they coincide. Hence, since γ_{ij} is parameterized by arc-length, we get that $\dot\gamma_{ij}(t) = v(t)$ exists and it is continuous for every $t \in J_{ij} \setminus \{\bar{t}_\ell^{ij}\}_\ell$ (see also [14, Proposition 2.7] and [11, Theorem 5.1]).

We now have to consider the case $t_0 \in \{\bar{t}_\ell^{ij}\}_\ell$. As we already observed, the (multivalued) map

$$J_{ij} \ni t \mapsto [y_0^{ij}(t), y_1^{ij}(t)],$$

always admits a limit from the left and from the right at every $t \in J_{ij}$. Hence the argument used above shows that

$$v^{\pm}(t_0) \cdot [y_1^{ij}(t_0^{\pm}) - y_0^{ij}(t_0^{\pm})]^{\perp} \geq 0, \tag{3.11}$$

where $y_0^{ij}(t_0^{\pm}), y_1^{ij}(t_0^{\pm})$ denote the limits of $y_0^{ij}(t), y_1^{ij}(t)$ as $t \to t_0^{\pm}$. In this case, since a priori the directions of $y_1^{ij}(t_0^{+}) - y_0^{ij}(t_0^{+})$ and $y_1^{ij}(t_0^{-}) - y_0^{ij}(t_0^{-})$ may be opposite to the other, we obtain that $\dot\gamma_{ij}^{\pm}(t_0)$ both exist, they are continuous at t_0, and they are either equal or opposite to each other (see Figure 3.3). This proves (2,a) and (2,b)

Proof of $(2,c)$ and $(2,d)$. The properties stated in (2,c) are an easy consequence of the way the curves γ_{ij} were constructed. Concerning (2,d), it follows from the result on propagation of singularities described above, and from the argument we used to prove the right and left differentiability of γ_{ij}.

Finally, let us estimate the number of closed injective curves inside *Sing*. If $\gamma \subset Sing$ is a closed injective curve, then by definition it is a Jordan curve, and so there exists an open set O inside Ω such that $\partial O = \gamma$. We claim that $\partial\big(\nabla\varphi(O \cap \Omega')\big) \cap \Lambda = \emptyset$, where Ω' was defined in (3.4).

Indeed, assume by contradiction that there is a point $y \in \partial\big(\nabla\varphi(O \cap \Omega')\big) \cap \Lambda$. Then we can find a sequence $\{x_k\}_{k\in\mathbb{N}} \subset O \cap \Omega'$ such that $\nabla\varphi(x_k) \to y$. Let $x \in \overline{O}$ be any limit point of $\{x_k\}_{k\in\mathbb{N}}$. By the upper semicontinuity of the subdifferential we have $y \in \partial\varphi(x)$, and since $y \in \Lambda$ we get $x \in Reg$. Hence $\partial\varphi(x)$ is a singleton, and $x \in \Omega'$. Combining this with the fact that $\partial O = \gamma \subset Sing$, we obtain $x \in O \cap \Omega'$. Recalling that $\nabla\varphi$ is an homeomorphism between Ω' and its image, we have that the set $\nabla\varphi(O \cap \Omega')$ is open and $y = \nabla\varphi(x)$ belongs to $\nabla\varphi(O \cap \Omega')$, contradiction.

Thanks to the claim and the fact that $\nabla\varphi(O \cap \Omega')$ is open, $\nabla\varphi(O \cap \Omega')$ contains at least one connected component of Λ. This implies that the number of periodic curves is bounded by n.

To get the right estimate (i.e., with $n-1$), let $\gamma_1,\ldots,\gamma_s$, with $s\le n$, be the periodic curves inside *Sing*. For $k=1,\ldots,s$, let O_k denote the open set inside Λ such that $\partial O_k=\gamma_k$, and define $O_{s+1}:=\Lambda\setminus\big(\cup_{k=1}^s O_k\big)$. Then, for any $k=1,\ldots,s+1$, $\nabla\varphi(O_k\cap\Omega')$ contains at least one connected component of Λ, and since the sets $\nabla\varphi(O_k\cap\Omega')$ are disjoint this implies $s\le n-1$. $\square$

We now estimate the number of components S_i which do not touch $\partial\Omega$ in terms of the holes in Λ:

Proposition 3.5. *The number of connected components S_i such that $\overline{S}_i\cap\partial\Omega=\emptyset$ is bounded by the number of holes inside Λ*

Proof. Let S_i be such that $\overline{S}_i\cap\partial\Omega=\emptyset$. We recall that $\partial\varphi(S_i)$ is a connected set, and we want to show that $\partial\varphi(S_i)$ necessarily fills a hole of Λ.

Assume not. Then, since $\partial\varphi(\overline{S}_i)$ is a closed set strictly contained in $\overline{\mathrm{co}(\Lambda)}\setminus\Lambda$, we can find a small open ball $B_r(\bar y)$ outside $\overline{\Lambda}$ such that $B_r(\bar y)\cap\partial\varphi(\overline{S}_i)=\emptyset$, $\overline{B_r}(\bar y)\cap\partial\varphi(\overline{S}_i)\ni y_i$, and $y_i\notin\overline{\Lambda}$. Let $x_i\in\overline{S}_i$ be any point such that $y_i\in\partial\varphi(x_i)$. Then, since $y_i\notin\overline{\Lambda}$, Proposition 3.2 implies that $x_i\in$ *Sing*, so that $x_i\in S_i$.

Let $v:=\frac{\bar y-y_i}{|\bar y-y_i|}$. We can observe that, since $B_r(\bar y)$ touches the convex set $\partial\varphi(x_i)$ at y_i, v is orthogonal to a segment $[y_0,y_1]\subset\partial\varphi(x_i)$, with $y_0,y_1\in\partial\Lambda$. We now apply [1, Theorem 4.2] (and in particular its proof) to deduce the existence of a Lipschitz curve $\gamma:[0,\rho]\to\Omega$ with $\gamma(0)=x_i$, and a positive number $\delta>0$, such that

$$\lim_{t\to0}\frac{\gamma(t)-\gamma(0)}{t}=v,\qquad \mathrm{diam}\big(\partial\varphi(\gamma(t))\big)\ge\delta\quad\forall\,t\in[-\rho,\rho].$$

Moreover, according to [1, Lemma 4.5, Equations (4.10) and (4.11)], the curve γ can be constructed so that there exists a continuous path $[0,\rho]\ni t\mapsto y(t)\in\partial\varphi(\gamma(t))$ such that $y(0)=y_i$. Hence, exploiting the monotonicity of the subdifferential and the strict convexity of φ, we get

$$\big(y(t)-y\big)\cdot\big(\gamma(t)-\gamma(0)\big)>0\qquad\forall\,y\in[y_0,y_1],\ t>0\text{ small},$$

which combined with $\gamma(t)=\gamma(0)+tv+o(t)$ implies

$$\big(y(t)-y(0)\big)\cdot v>0\qquad\forall\,t>0\text{ small}.$$

Since $y(t)\in\partial\varphi(\gamma(t))$, the curve $t\mapsto y(t)$ is continuous, and $\partial\varphi(\gamma(t))=\mathrm{co}(\nabla_*\varphi(\gamma(t)))$ with $\nabla_*\varphi(\gamma(t))\subset\partial\Lambda$, we easily deduce that $B_r(\bar y)\cap\partial\varphi(\gamma(t))\neq\emptyset$ for some $t>0$ close to 0, a contradiction. $\square$

Remark 3.6. As we already observed before, $\mathscr{L}^2\big(\overline{Sing}\cap\Omega\big)=0$. However, thanks to the description of *Sing* given above, one would be tempted to conjecture that a better result is true, that is $\mathscr{H}^1\big(\overline{Sing}\setminus Sing\big)=0$. A first step in this direction would be to prove that, for any $K\subset\subset\Omega$, there exist only a finite number of connected components S_i such that $S_i\cap K\neq\emptyset$ (indeed, it is well-known that if a set A has locally finite $\mathscr{H}^1$-measure, and the number of its connected components is locally finite, then $\mathscr{H}^1\big(\overline{A}\setminus A\big)=0$). However, we believe that this local bound on the number of connected components is false if $\partial\Lambda$ is merely continuous: one can imagine to construct a boundary with strong oscillations (something like $[0,\varepsilon]\ni s\mapsto\sqrt{\frac{s}{\varepsilon}}\sin\big(\frac{1}{\varepsilon s}\big)$, repeated countably many times, at different places of $\partial\Lambda$, with different values of ε), which can produce a countable number of connected components S_i which intersect a fixed compact set in Ω. On the other hand, thanks to the fact that φ is strictly convex and the description of

Sing given in Theorem 3.4, we do believe that the result should be true if $\partial\Lambda$ is Lipschitz, and actually it is not difficult (although tedious) to prove it when $\partial\Lambda$ is a smooth closed curve whose curvature changes sign only a finite number times.

Nevertheless, also assuming that one is able to bound the number of connected components, it is still not clear whether one can hope that $\mathscr{H}^1\left(\overline{Sing}\setminus Sing\right)=0$. Indeed, let us consider the following example: let $u:\mathbb{R}^2\to\mathbb{R}$ be a compactly supported semi-convex function such that its singular set is given by $[-1,1]\times\{0\}$, and outside this set u is C^∞. Set $u^\perp(x_1,x_2):=u(x_2,x_1)$, so that its singular set is $\{0\}\times[-1,1]$. Define now

$$\varphi(x):=\frac{|x|^2}{2}+\alpha\Bigg[\sum_{i\in\mathbb{N}}\gamma_i\,u\Big(\frac{x-y^i}{\delta_i}\Big)+\sum_{i\in\mathbb{N}}\varepsilon_i\,u^\perp\Big(\frac{x-z^i}{\eta_i}\Big)\Bigg],$$

where $\gamma_i,\delta_i,\varepsilon_i,\eta_i\in(0,1)$ are small numbers such that

$$\sum_{i\in\mathbb{N}}\frac{\gamma_i}{\delta_i^2}+\sum_{i\in\mathbb{N}}\frac{\varepsilon_i}{\eta_i^2}\le 1, \tag{3.12}$$

$\alpha>0$ is sufficiently small so that

$$\alpha\left(\inf_{x\in\mathbb{R}^2}D^2u(x)\right)\ge-\frac{1}{2}\,\mathrm{Id},$$

and $y^i,z^i\in\mathbb{R}^2$ are points chosen in such a way that the singular set of φ,

$$Sing=\bigcup_{i\in\mathbb{N}}\Big[\big([-\delta_i,\delta_i]\times\{0\}+y^i\big)\cup\big(\{0\}\times[-\eta_i,\eta_i]+z^i\big)\Big],$$

is path-connected. Moreover, δ_i,η_i,y^i,z^i can be chosen such that $Sing\subset B_2(0)$.

Hence φ is a uniformly convex function, of class C^2 outside its singular set, and the push-forward of $\det(D^2\varphi)\chi_{B_3(0)}$ under $\nabla\varphi$ is given by the characteristic function of a set Λ with only one hole inside (if desired, by replacing φ with $\varphi+\varepsilon u^\perp(2(\cdot-e_2))$ for some $\varepsilon>0$ small, one can even remove the hole in Λ, so that Λ will be simply connected).

We now observe that $\mathscr{H}^1(Sing)=2\sum_i\big(\delta_i+\eta_i\big)$. Thus, if δ_i,η_i are small enough so that $\mathscr{H}^1(Sing)<+\infty$, since *Sing* is connected one can prove that $\mathscr{H}^1\big(\overline{Sing}\big)<+\infty$ too. On the other hand, it is possible to choose δ_i,η_i,y^i,z^i in such a way that $\mathscr{H}^1(Sing)=+\infty$ and $\overline{Sing}$ has not σ-finite $\mathscr{H}^1$-measure (observe that, after this choice of δ_i,η_i,y^i,z^i is done, one can always choose γ_i and ε_i sufficiently small so that (3.12) holds).

We further remark that δ_i,η_i,y^i,z^i can even be chosen such that *Sing* is dense inside $B_1(0)$, which would give $\mathscr{L}^2\big(\overline{Sing}\cap\Omega\big)>0$. Thanks to Theorem 3.1, we deduce that in this case Λ cannot be an open set with continuous boundary. Therefore we see that the geometric assumptions on Λ allow to prevent *Sing* to be too much nasty. However, it is not clear how to exploit these informations to prevent $\overline{Sing}$ from having zero Lebesgue measure but Hausdorff dimension greater than 1.

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