Sharp stability of the Brunn-Minkowski inequality via optimal mass transportation

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Abstract

The Brunn-Minkowski inequality, applicable to bounded measurable sets A and B in $\mathbb{R}^d$, states that $|A+B|^{1/d} \geq |A|^{1/d} + |B|^{1/d}$. Equality is achieved if and only if A and B are convex and homothetic sets in $\mathbb{R}^d$. The concept of stability in this context concerns how, when approaching equality, sets A and B are close to homothetic convex sets. In a recent breakthrough [FvHT23], the authors of this paper proved the following folklore conjectures on the sharp stability for the Brunn-Minkowski inequality:
(1) A linear stability result concerning the distance from A and B to their respective convex hulls.
(2) A quadratic stability result concerning the distance from A and B to their common convex hull.
As announced in [FvHT23], in the present paper, we leverage (1) in conjunction with a novel optimal transportation approach to offer an alternative proof for (2).

1 Introduction

Given measurable sets $X, Y \subset \mathbb{R}^n$ with positive measure, the Brunn-Minkowski inequality says

$$|X+Y|^{\frac{1}{n}} \geq |X|^{\frac{1}{n}} + |Y|^{\frac{1}{n}}.$$

More naturally, for equal sized measurable sets $A, B \subset \mathbb{R}^n$ and a parameter $t \in (0,1)$ this is equivalent to

$$|tA + (1-t)B| \geq |A|,$$

with equality for equal convex sets A and B (less a measure zero set). Here, $A+B = \{a+b \mid a \in A, \text{ and } b \in B\}$ is the *Minkowski sum*, $tA := \{ta : a \in A\}$, and $|\cdot|$ refers to the outer Lebesgue measure. The Brunn-Minkowski inequality is a fundamental tool in analysis and geometry going back to the 19th century, the importance of which is expertly documented in [Gar02].

The Brunn-Minkowski inequality is part of a vast body of geometric inequalities, such as the isoperimetric inequality, the Sobolev inequality, the Prékopa-Leindler inequality, and the Borell-Brascamb-Lieb inequality (e.g. Figure 1 in [Gar02]). The famous isoperimetric inequality states that, for a given volume, the body minimizing its perimeter is the ball. The isoperimetric inequality follows from Brunn-Minkowski by taking A a ball and letting t tend to zero. The Prékopa-Leindler inequality asserts that for $t \in (0,1)$ and functions $f, g, h\colon \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ with the property that $h(tx+(1-t)y) \geq f^t(x)g^{1-t}(y)$ for all $x, y \in \mathbb{R}^n$ and $\int f = \int g$, we have $\int h \geq \int f$ with equality if and only if $f(x) = ag(x-x_0)$ is a log-concave function for some $a \in \mathbb{R}_{>0}$ and $x_0 \in \mathbb{R}^n$. Prékopa-Leindler implies Brunn-Minkowski by taking f and g to be the indicator functions of A and B. The Prékopa-Leindler inequality in turn is subsumed by the Borell-Brascamb-Lieb inequality. Studying these inequalities and their stabilities has sparked a fruitful field of research in recent years.

The stability of Brunn-Minkowski asks for the structure of sets A and B which are close to attaining equality in Brunn-Minkowski. This study goes back to the work of for instance Diskant [Dis73] and Ruzsa [Ruz97]. Two folklore conjectures concern the stability of Brunn-Minkowski: if we are within a factor of $1+\delta$ from equality, then the distance from the sets A and B to a common convex set is $O_{d,t}(\sqrt{\delta})$, and furthermore, the distance from to their individual convex hulls is $O_{d,t}(\delta)$. These conjectures have received a lot of attention becoming central problems in analysis and convex geometry (see e.g. [FMP09, FMP10, Chr12b, Chr12a, EK14, FJ15, Fig15, FJ17, BJ17, CM17, FJ21, vHST22, vHST23a, vHK23a, vHK23b, vHST23b, FvHT23]). Recently, the present authors resolved these conjectures in [FvHT23] (stated as Theorem 1.5 and Theorem 1.3 below).

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The stability of the isoperimetric inequality was first explored in 1921 by Bonnesen [Bon21] who settled the planar case. The optimal result in higher dimensions was established only in 2008 by Fusco, Maggi, and Pratelli [FMP08]. In a cornerstone paper, Figalli, Maggi, and Pratelli [FMP09, FMP10] used mass transportation techniques to generalize this to a sharp stability of the anisotropic isoperimetric inequality while simultaneously proving the following sharp stability for the Brunn-Minkowski inequality for convex sets.

Theorem 1.1 (Figalli, Maggi, and Pratelli [FMP09, FMP10]). *For all $n \in \mathbb{N}$ and $t \in (0, 1/2]$, there are computable constants $c_n^{1.1} > 0$ such that the following holds. Assume that $A, B \subset \mathbb{R}^n$, are convex sets with equal volume so that*

$$|tA + (1-t)B| \leq (1+\delta)|A|.$$

Then, up to translation[1],

$$|A \triangle B| \leq c_n^{1.1} \sqrt{\frac{\delta}{t}} |A|.$$

The aim of this paper is to develop a different mass transportation approach on the stability of the Brunn-Minkowski problem in order to strengthen the above result to non-convex sets.

Theorem 1.2. *For all $n \in \mathbb{N}$ and $t \in (0, 1/2]$, there are computable constants $c_n^{1.2}, d_{n,t}^{1.2}, g_{n,t}^{1.2} > 0$ such that the following holds. Assume $\delta \in [0, d_{n,t}^{1.2})$, $\gamma \in [0, g_{n,t}^{1.2})$, and assume that $A, B \subset \mathbb{R}^n$, are measurable sets with equal volume so that*

$$|tA + (1-t)B| \leq (1+\delta)|A| \qquad \text{and} \qquad |\operatorname{co}(A) \setminus A| + |\operatorname{co}(B) \setminus B| \leq \gamma |A|.$$

Then, up to translation,

$$|A \triangle B| \leq c_n^{1.2} \sqrt{\frac{\delta + \gamma}{t}} |A|.$$

In recent work of the current authors [FvHT23], the following linear stability result to the convex hull of A and B was established, solving one of the aforementioned conjectures.

Theorem 1.3 ([FvHT23]). *For $n \in \mathbb{N}$ and $t \in (0, 1/2]$, there are constants $c_{n,t}^{1.3}, d_{n,t}^{1.3} > 0$ such that the following holds. Assume $\delta \in [0, d_{n,t}^{1.3})$, and assume $A, B \subset \mathbb{R}^n$ are measurable sets of equal volume so that $|tA + (1-t)B| \leq (1+\delta)|A|$, then*

$$|\operatorname{co}(A) \setminus A| + |\operatorname{co}(B) \setminus B| \leq c_{n,t}^{1.3} \delta |A|.$$

A notable application of Theorem 1.2 is that, in combination with Theorem 1.3, it gives the following result.

Corollary 1.4. *For all $n \in \mathbb{N}$ and $t \in (0, 1/2]$, there are computable constants $c_{n,t}^{1.4}, d_{n,t}^{1.4} > 0$ such that the following holds. Assume $\delta \in [0, d_{n,t}^{1.4})$ and assume that $A, B \subset \mathbb{R}^n$, are measurable sets with equal volume so that $|tA + (1-t)B| \leq (1+\delta)|A|$. Then, up to translation,*

$$|A \triangle B| \leq c_{n,t}^{1.4} \sqrt{\delta} |A|.$$

This corollary is a weaker instance of the following quadratic stability recently proved by the current authors.

Theorem 1.5 ([FvHT23]). *For all $n \in \mathbb{N}, n \geq 2$ and $t \in (0, 1/2]$, there are computable constants $c_n^{1.5}, d_{n,t}^{1.5} > 0$ such that the following holds. Assume $\delta \in [0, d_{n,t}^{1.5})$ and let $A, B \subset \mathbb{R}^n$ be measurable sets with equal volume satisfying*

$$|tA + (1-t)B| = (1+\delta)|A|.$$

Then, up to translation[2]*, there is a convex set $K \supset A \cup B$ such that*

$$|K \setminus A| = |K \setminus B| \leq c_n^{1.5} \sqrt{\frac{\delta}{t}} |A|.$$

Note that $\frac{|\operatorname{co}(A \cup B) \setminus A|}{|A \triangle B|} \geq 1/2$, but a priori we don't have any lower bound in terms of n. However, as a consequence of [FvHT23, Theorem 1.7] these two measures are actually equivalent for near-convex sets A, B, i.e., with $|\operatorname{co}(A) \setminus A| + |\operatorname{co}(B) \setminus B| = O_{n,t}(\delta)|A|$. Hence, the main difference between Theorem 1.5 and Corollary 1.4 is in the t dependence of the stability constant. Actually, even combining Theorem 1.2 with the optimal version of Theorem 1.3 (see Conjecture 14.1 in [FvHT23]) would still not obtain the optimal t dependence provided by Theorem 1.5.

[1] That is, there exists $x \in \mathbb{R}^n$ so that $|(A + x) \triangle B| \leq c_n^{1.1} \sqrt{\frac{\delta}{t}} |A|$.

[2] That is, there exist $x, y \in \mathbb{R}^n$ so that $x + A, y + B \subset K$ and $|K \setminus (x + A)| + |K \setminus (y + B)| \leq t^{-c^{1.5} n^8} \delta^{\frac{1}{2}} |A|$.

The first contribution to the study of sumset stability was made by Freiman [Fre59] in dimension $n = 1$. Freiman's celebrated $3k-4$ Theorem [Fre59, LS95, Sta96] from additive combinatorics, implies a strong version of Theorem 1.3 in dimension 1. If $t \in (0, 1/2]$ and $A, B \subset \mathbb{R}$ are measurable sets with equal volume such that $|tA + (1-t)B| \le (1+\delta)|A|$ with $\delta < t$, then $|\operatorname{co}(A) \setminus A| \le t^{-1}\delta|A|$ and $|\operatorname{co}(B) \setminus B| \le (1-t)^{-1}\delta|B|$, which is optimal.

Stability in higher dimensions is considerably more difficult; in [Chr12b, Chr12a] Christ showed a qualitative result: if $n \in \mathbb{N}$, $t, \varepsilon \in (0, 1/2]$ and $A, B \subset \mathbb{R}^n$ are measurable sets with equal volume such that $|tA+(1-t)B| \le (1+\delta)|A|$ with δ sufficiently small in terms of t, n, ε, then there exists a convex set K such that, up to translation, $K \supset A, B$ and $|K \setminus A| = |K \setminus B| \le \varepsilon|A|$. In a cornerstone result, Figalli and Jerison [FJ17] obtained the first quantitative bounds: $|K \setminus A| = |K \setminus B| \le \delta^{(t/|\log(t)|)^{\exp(O(n))}}|A|$. A similar result for the Prékopa-Leindler inequality was recently established by Böröcky, Figalli, and Ramos [BFRar].

Until recently, the only instance of Theorem 1.5 for arbitrary sets was known in two dimensions due to van Hintum, Spink, and Tiba [vHST23b]. In an independent direction, van Hintum and Keevash [vHK23b] determined the optimal value $d_{n,t} = t^n$ for all values $n \in \mathbb{N}$ and $t \in (0, 1/2]$ with the same bound on the distance to a common convex set as in the result of Figalli and Jerison.

Even partial results towards Theorem 1.5 for restricted classes of sets A and B have received much attention. Recall that Figalli, Maggi, and Pratelli [FMP09, FMP10] dealt with the case when A and B are convex. Figalli, Maggi, and Mooney [FMM18] settled the case when A is a ball and B is arbitrary. Barchiesi and Julin [BJ17] extended the previous results to A convex and B arbitrary. Despite all these results supporting Theorem 1.5, a conclusive proof remained wide open and outside the scope of the available techniques for a long time.

The particular case of equal sets $A = B$ in Theorem 1.3 has been thoroughly investigated. Indeed, after establishing in [FJ15] some quantitative bounds for Theorem 1.3 for $A = B$ in all dimensions, Figalli and Jerison [FJ21] proved Theorem 1.3 for $A = B$ in dimensions $n = 1, 2, 3$. Van Hintum, Spink, and Tiba [vHST22] proved Theorem 1.3 for $A = B$ in all dimensions. Moreover, they determined the optimal dependency on t. Furthermore, van Hintum, Spink, and Tiba [vHST23a, Theorem 1.1] established the optimal dependency on d in dimensions $d \le 4$ when $A = B$ is a hypograph of a function over a convex domain. Another closely related result by van Hintum and Keevash [vHK23a] is that if $A \subset \mathbb{R}^n$ with $|\frac{A+A}{2}| \le (1+\delta)|A|$ with $\delta < 1$, then there exists an $A' \subset A$ with $|A'| \ge (1-\delta)|A|$ and $|\operatorname{co}(A')| = O_{n,1-\delta}(|A'|)$.

For distinct sets A and B, showing Theorem 1.3 has proved much more difficult. Van Hintum, Spink, and Tiba in [vHST23a, Theorem 1.5], proved Theorem 1.3, when A and B are hypograph of functions over the same convex domain. The only instance of Theorem 1.3 for arbitrary sets was established by van Hintum, Spink and Tiba [vHST23b, Section 12] in two dimensions. In spite of these determined efforts, for arbitrary sets in higher dimensions a proof of Theorem 1.3 was only recently found by the present authors in [FvHT23].

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1.1 Notation and conventions.

Before starting our proofs, it is convenient to briefly explain the notation that we will use throughout the paper. With $c > 0$, we shall denote a universal constant independent of the dimension, while $c_n > 0$ (and analogous notations) denote dimensional constants. Saying that the quantity a is controlled by $O_n(b)$ means that $|a| \le c_n b$, while notation $a = \Omega_n(b)$ means that $a \ge c_n|b|$. When a constant also depends on t, we write $c_{n,t}$. To distinguish the constants that appear in the different statements, $c^{\ell.m}$ means that the constant c is the one of Theorem $\ell.m$.

Throughout the paper, we fix $n \in \mathbb{N}$ and either $t \in (0, 1/2]$. We use $|\cdot|$ to denote the outer Lebesgue measure in $\mathbb{R}^n$.

Given $s \in \mathbb{R}$ and sets X and Y in $\mathbb{R}^n$, we define $sX = \{sx \colon x \in X\}$ and $X + Y = \{x + y \colon x \in X, y \in Y\}$. A set X in $\mathbb{R}^n$ is convex if for all $t \in [0, 1]$ we have $tX + (1-t)X \subset X$. The convex hull $\operatorname{co}(X)$ of a set X in $\mathbb{R}^n$ is the intersection of all convex sets containing X. In particular, $\operatorname{co}(X)$ is a convex set. Two sets X and Y of $\mathbb{R}^n$ are homothetic if there exist a point z in $\mathbb{R}^n$ and a scalar $s > 0$ such that $X = sY + z$.

Given a bounded convex set X in $\mathbb{R}^n$, we define $\overline{X}$ as the closure of X, which is also a convex set. The vertices of X, denoted by $V(X)$, represent the set $V(X) = \{x \in \overline{X} \colon \operatorname{co}(\overline{X} \setminus \{x\}) \ne \operatorname{co}(\overline{X})\}$. It follows that $\overline{X} = \operatorname{co}(V(X))$.

Measureable sets $X_1, \ldots, X_k$ in $\mathbb{R}^n$ are said to form an essential partition of $\mathbb{R}^n$ if $|\cap_i X_i^c| = 0$ and for $j_1 \ne j_2$, we have $|X_{j_1} \cap X_{j_2}| = 0$. By a basis $e_1, \ldots, e_n$ in $\mathbb{R}^n$, we mean an orthogonal set of vectors with unit length. In light of Proposition 2.8, we can assume that the sets A and B (as well as all parts into which we subdivide A and B) are compact.

1.2 Overview of the proof

We now turn to Theorem 1.2. The starting point is the optimal transport approach used in [FMP09] to prove a sharp stability result for the Brunn-Minkowski inequality on convex sets. In our case, the sets A and B are only L^1-close to being convex, and we want to obtain a final estimate where the gap in volume (i.e., γ) appears in the stability estimate with the same power as δ. Because the optimal transport between arbitrary sets can behave very badly in terms of regularity, we consider the optimal transport map sending $\text{co}(A)$ to $\text{co}(B)$. This makes the first part of our argument (the first three steps in the outline below) very similar to the one in [FMP09], but then we immediately face a series of challenges. The key issue is that the optimal transport proof of Brunn-Minkowski provides a control on the transport map only inside the set A (although this map is defined in the whole convex hull), while for us it is crucial to obtain some bounds also in the remaining region $\text{co}(A) \setminus A$. By a series of delicate arguments exploiting the monotonicity of the optimal map (we recall that the optimal map is the gradient of a convex function) and some interior regularity estimates, we obtain a radial control on the transport map along all rays emanating from the origin and contained inside $\text{co}(A)$. This estimate by itself would be too weak. Still, the key observation is that we can repeat our argument by replacing the origin with an arbitrary point o' inside $(1-\varepsilon)C_A$, and replacing our sets A and B with new sets $Q(A)$ and $Q(B)$, where Q varies among all affine transformations with $||Q||_{op}, ||Q^{-1}||_{op} \le \theta$ for some fixed large constant θ. Averaging our radial bound over o' and Q allows us to find a sharp control on $|\text{co}(A) \triangle \text{co}(B)|$, from which the final result follows. We summarize the steps of the proof in the next subsection.

1.3 Outline of the proof of Theorem 1.2

The proof of Theorem 1.2 follows the following steps.

0. Reduce to the case that A and B are sandwiched between two balls of comparable sizes, and look like cones centered at the same vertex

1. Let $C_A \supset A$ and $C_B \supset B$ be convex sets of size $|C_A| = |C_B| = (1+\gamma)|A|$, and let $T : C_A \to C_B$ be the optimal transport map between them.

2. Note that if we let $E := T^{-1}(B) \cap A$, then

$$(\delta + 2\gamma)|A| \ge |tA + (1-t)B| - |E| \ge \int_E \Big(\det D\,(tId + (1-t)T) - 1\Big)dx,$$

   where D is the Jacobian.

3. Analysing the eigenvalues of $D(T)$ (cf Lemma 3.11 akin to the methods in [FMP09]), we find that this implies

$$\int_E ||D(T - Id)||_{op} dx \le O_n\left(\sqrt{\frac{\delta+\gamma}{t}}\right)|A|.$$

4. By an elliptic regularity argument (cf Lemma 3.13) this implies $||D(T-Id)(x)||_{op} \le O_n\left(\sqrt{\frac{\delta+\gamma}{t}}\right)$ for points $x \in (1-\varepsilon)C_A$ and in particular in some small ball around the origin.

5. Next, we note that $C_A \setminus E$ is small, so when integrating a bounded function, we find

$$\int_{C_A} \frac{x^T}{||x||_2}(D(Id-T)(x))\frac{x}{||x||_2}dx \le O_n(\gamma)|A| + \int_E ||D(Id-T)(x)||_{op}dx \le O_n\left(\sqrt{\frac{\delta+\gamma}{t}}\right)|A|.$$

   (Here, we crucially use that DT is nonnegative definite; in particular, we only control the integral on the left-hand side from above.)

6. Combining the two previous steps, we find

$$\int_{C_A} \frac{\frac{x^T}{||x||_2}(D(Id-T)(x))\frac{x}{||x||_2}}{||x||_2^{n-1}}dx \le O_n\left(\sqrt{\frac{\delta+\gamma}{t}}\right)|A|.$$

7. This allows us to integrate radially, giving

$$\int_{\partial C_A} \left\langle (x - T(x)) - (o - T(o)), \frac{x}{||x||_2} \right\rangle dx \le O_n\left(\sqrt{\frac{\delta+\gamma}{t}}\right)|A|.$$

8. Up to this point, we only used that $B(o, \Omega(n)) \in (1-\varepsilon)C_A$. So, in fact, for all $o' \in (1-2\varepsilon)C_A$ we get

$$\int_{\partial C_A} \left\langle (x - T(x)) - (o' - T(o')), \frac{x-o'}{||x-o'||_2} \right\rangle dx \le O_n\left(\sqrt{\frac{\delta+\gamma}{t}}\right)|A|.$$

9. Using the fact that A and B look like cones at the same vertex, we find that $|o' - T(o')| = O_n\left(\sqrt{\frac{\delta+\gamma}{t}}\right)$ (see Lemma 3.6).

10. Hence, we find

$$\int_{\partial C_A} \left\langle x - T(x), \frac{x-o'}{||x-o'||_2} \right\rangle dx \le O_n\left(\sqrt{\frac{\delta+\gamma}{t}}\right)|A|.$$

    This is the conclusion of Proposition 3.1.

11. We find the same result (cf Corollary 3.2) if we first apply an affine transformation Q

$$\int_{x\in\partial C_A} \left\langle Q(x) - T_Q(Q(x)), \frac{Q(x)-Q(o')}{||Q(x)-Q(o')||_2} \right\rangle dx \le O_n\left(\sqrt{\frac{\delta+\gamma}{t}}\right)|A|,$$

12. Proposition 3.4 shows that, considering an appropriately distributed random affine transformation and random point $o' \in (1-\varepsilon)C_A$, then

$$\mathbb{E}_{Q,o'}\left[\left\langle Q(x) - T_Q(Q(x)), \frac{Q(x)-Q(o')}{||Q(x)-Q(o')||_2} \right\rangle\right] \ge \Omega_n(d(x, C_B)).$$

13. Proposition 3.5 shows that

$$|C_A \triangle C_B| \le O_n\left(\int_{\partial C_A} d(x, C_B)dx\right).$$

14. Combining the last three steps gives the desired estimate

$$|A\triangle B| \le |C_A \triangle C_B| + 2\gamma|A| \le O_n\left(\sqrt{\frac{\delta+\gamma}{t}}\right)|A|.$$

2 Initial reduction

We start with a simple reduction (Proposition 2.8) to allow us to assume that A and B are sandwiched between two balls of comparable sizes, and look like cones centered at the same vertex (cf Definition 2.7). Much of this section follows the lines of section 2 in [FvHT23].

2.1 Setup

Definition 2.1. *A convex set $C \subset \mathbb{R}^n$ is called a* cone *if there exists a hyperplane H not containing the origin and a bounded convex set $P \subset H$ such that*

$$C = \bigsqcup_{t\ge0} tP.$$

Definition 2.2. *We write $S^{v_0,\dots,v_n}$ for the simplex with vertices $v_0, \dots, v_n \in \mathbb{R}^n$. Assuming that $S^{v_0,\dots,v_n}$ contains the origin in the interior, construct the family of cones $\mathfrak{C}^{v_0,\dots,v_n} := \{C_i : 0 \le i \le n\}$, where*

$$C_i = \bigsqcup_{t\ge0} t\,\mathrm{co}(v_0, \dots v_{i-1}, v_{i+1}, \dots, v_n).$$

Note that the cones in $\mathfrak{C}^{v_0,\dots,v_n}$ form an essential partition of $\mathbb{R}^n$.

Definition 2.3. *Fix vectors $e_0,\dots,e_n \in \mathbb{R}^n$ such that $S^{e_0,\dots,e_n}$ is a regular unit volume simplex centered at the origin. Denote $S = S^{e_0,\dots,e_n}$ and $\mathfrak{C} = \mathfrak{C}^{e_0,\dots,e_n}$.*

Definition 2.4. *A pair of sets $X, Y \subset \mathbb{R}^n$ is* λ-bounded *if there exists an $r > 0$ so that*

$$rS \subset X, Y \subset \lambda r S.$$

Definition 2.5. *Given a cone $F \subset C' \in \mathfrak{C}$, a pair of sets $X, Y \subset F$ is* (λ, F)-bounded *if there exists an $r > 0$ so that*

$$r(F \cap S) \subset X, Y \subset \lambda r(F \cap S).$$

Definition 2.6. *A pair of sets $X, Y \subset \mathbb{R}^n$ is called a η-sandwich if there exists a convex set P such that $o \in P \subset X, Y \subset (1+\eta)P$.*

Note that given a cone F and a λ-bounded η-sandwich $X, Y \subset \mathbb{R}^n$, the pair $X \cap F, Y \cap F$ is also a (λ, F)-bounded η-sandwich.

Definition 2.7. *Say sets $A, B \subset \mathbb{R}^n$ are* $(\gamma, \ell, \lambda, \mu)$ conelike *if there exist convex sets $C_A \supset A, C_B \supset B$ with $|C_A| = |C_B| = (1+\gamma)|A| = (1+\gamma)|B|$, a convex set K, and a set S'' obtained by intersecting a cone with a half space with the following properties*

1. *$B(o, 1/\ell) \subset C_A, C_B \subset B(o, \ell)$,*
2. *$S'' \subset A - z, C_A - z, B - z, C_B - z \subset \lambda S''$, for some $z \in \mathbb{R}^n$, and*
3. *$K \subset A + x, C_A + x, B + y, C_B + y \subset (1+\mu)K$, for some $x, y \in \mathbb{R}^n$.*

2.2 Proposition

Proposition 2.8. *Assume that for sets $A, B \subset \mathbb{R}^n$ satisfying the conditions of Theorem 1.2 that are $(\gamma, \ell, \lambda, \mu)$ conelike (for μ sufficiently small in terms of n, t, ℓ, and λ), we have $|A \triangle B| \le c_{n,\ell,\lambda}\sqrt{\frac{\delta+\gamma}{t}}|A|$. Then Theorem 1.2 is true for all set $A, B \subset \mathbb{R}^n$.*

2.3 Lemmas

We recall the following result by Michael Christ.

Theorem 2.9 (Christ 2012, [Chr12a]). *For all $n \in \mathbb{N}$, $t \in (0,1)$ and $\eta > 0$, there exist constants $d^{2.9} > 0$, so that for all measurable $X, Y \subset \mathbb{R}^n$ of equal volume with the property that $|tX + (1-t)Y| < (1 + d^{2.9})|X|$, then*

$$\min_{v \in \mathbb{R}^n} |\operatorname{co}(X \cup (v + Y))| \le (1+\eta)|X|.$$

We also use the following three lemmas from [FvHT23]

Lemma 2.10 (Proposition 5.4 in [FvHT23]). *Let $v_0, \dots, v_n \subset \mathbb{R}^n$ be vectors not contained in a halfspace and let $A, B \subset \mathbb{R}^n$ be measurable sets with equal volume. Then there exists a vector $v \in \mathbb{R}^n$ such that for every cone $C \in \mathfrak{C}^{v_0,\dots,v_n}$ we have*

$$|A \cap C| = |(B + v) \cap C|.$$

Moreover, for every $\eta, \lambda > 0$, there is a computable constant $\eta'^{2.10} > 0$ such that the following holds. If $\{v_0, \dots, v_n\} = \{e_0, \dots, e_n\}$ (as in Definition 2.3) and if $A, B \subset \mathbb{R}^n$ is a λ-bounded $\eta'^{2.10}$-sandwich, then $A, B + v$ is a 2λ-bounded η-sandwich.

We won't use Theorem 2.9 directly, but only through Lemma 2.11.

Lemma 2.11 (Lemma 2.11 in [FvHT23]). *For $n \in \mathbb{N}$, $t \in (0, 1/2]$ and $\eta > 0$, there exist constants $c^{2.11}$ and $d^{2.11}_{n,t}(\eta) > 0$ so that the following holds. If $X, Y \subset \mathbb{R}^n$ are measurable sets with $|X| = |Y|$ and $|tX + (1-t)Y| = (1+\delta)|X|$ with $\delta \in [0, d^{2.11}_{n,t}(\eta))$, then, up to translation, there exist measurable sets $X', Y' \subset \mathbb{R}^n$ so that*

1. *X', Y' is an η-sandwich,*
2. *$|X'| = |Y'| = |X|$,*

3. $\operatorname{co}(X') = \operatorname{co}(X)$ *and* $\operatorname{co}(Y') = \operatorname{co}(Y)$,

4. $|X'\triangle X| + |Y'\triangle Y| \leq c^{2.11}t^{-1}\delta|X|$,

5. $|tX' + (1-t)Y'| \leq (1+\delta)|X|$.

Moreover, if $X \subset Y$, we additionally find $X' \subset Y'$.

Lemma 2.12 (Lemma 2.12 in [FvHT23])**.** *For $n \in \mathbb{N}$, and $\eta > 0$ the following holds. If $X, Y \subset \mathbb{R}^n$ is an η-sandwich, then there exists $v \in \mathbb{R}^n$ and there exists a linear transformation $\theta\colon \mathbb{R}^n \to \mathbb{R}^n$ such that $\theta(v+X), \theta(v+Y)$ is a $(n^2 + n^3\eta)$-bounded $n\eta$-sandwich.*

2.4 Proof of Proposition 2.8

Proof of Proposition 2.8. First note that we may assume $|A| = |B| = 1$. Let $\lambda = \lambda_n := 16n^6$. Let ℓ'_n be minimal, so that a translate of $B(0, 1/\ell'_n)$ is contained in $\frac{1}{4n^3}S \cap C_0$, where S and C_0 are defined in Definition 2.3. Let ℓ''_n be minimal, so that $4n^3 S \cap C_0$ is contained in some translate of $B(o, \ell''_n)$. Let $\ell_n := 2\max\{\ell'_n, \ell''_n\}$. Find $\mu = \mu_{n,t} := \mu_{n,t,\ell_n,\lambda_n}$ sufficiently small as required by the assumption. Let $\eta = \eta_{n,t} := \mu$ for notational consistency. Choose η' to be sufficiently small in terms of η and n, so that the second part of Lemma 2.10 applies with parameters $(\eta'_n)^{2.10} = \eta'$, $\eta_n^{2.10} = \eta$ and $\lambda_n^{2.10} = 2n^3$. Choose $d_{n,t}$ smaller than the constant $d_{n,t}(\eta'_{n,t})$ in Lemma 2.11. Let $c_n := (n+1)c_{n,\ell_n,\lambda_n}\sqrt{w_n} + c^{2.11}$, where $w_n = (n+1)(4n^3)^n$, c_{n,ℓ_n,λ_n} is the constant from the assumption and $c^{2.11}$ is the constant from Lemma 2.11.

First, use Lemma 2.11 with parameter η' to find A^1, B^1 which form an η'-sandwich. Note that

$$|A\triangle B| \leq \left|A\triangle A^1\right| + \left|B\triangle B^1\right| + \left|A^1\triangle B^1\right| \leq \left|A^1\triangle B^1\right| + c^{2.11}t^{-1}\delta|A|,$$

so it suffices to show

$$\left|A^1\triangle B^1\right| \leq c_n\sqrt{\frac{\delta+\gamma}{t}}|A^1|.$$

Now apply Lemma 2.12 to A^1, B^1, to find A^2, B^2 an $n^2 + n^3\eta'$-bounded η' sandwich. A^2, B^2 are just a linear transformation and a translation away from A^1, B^1. Note that $n^2 + n^3\eta' \leq 2n^3$, so, in particular, A^2, B^2 is a $2n^3$-bounded η' sandwich.

We then apply Lemma 2.10 with vectors $e_0, \ldots, e_n$ and cones $\mathfrak{C}$ from Definition 2.3. Let $A^3 = A^2$ and B^3 be the translation of B^2 given by the lemma. Note that by definition of η', we find that A^3, B^3 is a $4n^3$-bounded η-sandwich with the property that $|A^3 \cap C| = |B^3 \cap C|$ for all $C \in \mathfrak{C}$.

Fix a $C \in \mathfrak{C}$, and let

$$A' := A^3 \cap C \text{ and } B' := B^3 \cap C.$$

We will show that A' and B' are of the correct form to bound their symmetric difference.

Note that $t(A^3\cap C') + (1-t)(B^3\cap C') \subset (tA^3 + (1-t)B^3)\cap C'$ so that these are disjoint for different $C' \in \mathfrak{C}$. Hence, by Brunn-Minkowski we find that

$$(1+\delta)|A| \geq |tA^3 + (1-t)B^3| \geq \sum_{C'\in\mathfrak{C}} |t(A^3\cap C') + (1-t)(B^3\cap C')| \geq |tA' + (1-t)B'| + \sum_{C\neq C'\in\mathfrak{C}} |A\cap C'|.$$

In particular, we find $|tA' + (1-t)B'| \leq |A'| + \delta|A|$. Since A^3, B^3 is $4n^3$-bounded, there exists some $r > 0$ so that $rS \subset A^3, B^3 \subset 4n^3 rS$. Given that $|A^3| = |A| = 1$ and $|S| = 1$, this implies $1/4n^3 \leq r \leq 1$, and thus $\frac{1}{4n^3}S \subset A^3, B^3 \subset 4n^3 S$. Since A^3, B^3 is a η-sandwich, there exists a convex set $K \subset A^3, B^3 \subset (1+\eta)K$.

Thus, we find that $|A'| \geq \left|\frac{1}{4n^3}S \cap C\right| = \frac{1}{(n+1)(4n^3)^n}|A|$ for all $C \in \mathfrak{C}$. For notational convenience, let $w_n = (n+1)(4n^3)^n$, so that

$$|tA' + (1-t)B'| \leq (1 + w_n\delta)|A'|.$$

With this bound on $|A'|$ and $|B'|$ in hand, we are ready to define $C_{A'}$ and $C_{B'}$. Note that $\operatorname{co}(A')\setminus A' \subset \operatorname{co}(A^3)\setminus A^3$ and $|\operatorname{co}(A^3)\setminus A^3| = |\operatorname{co}(A)\setminus A|$, so that $|\operatorname{co}(A')| \leq |A'| + \left|\operatorname{co}\left(A^3\right)\setminus A^3\right| = |A'| + \gamma|A^3| \leq (1 + w_n\gamma)|A'|$, and analogously $|\operatorname{co}(B')| \leq (1+w_n\gamma)|B'|$. Find convex sets $C_{A'} \supset \operatorname{co}(A')$ and $C_{B'} \supset \operatorname{co}(B')$ so that

$$C_{A'}, C_{B'} \subset C \cap 4n^3 S \cap (1+\eta)K \text{ and } |C_{A'}| = |C_{B'}| \leq (1+w_n\gamma)|A'| = (1+w_n\gamma)|B'|.$$

With these in place we check that these sets are conelike (cf Definition 2.7). Recall that $\frac{1}{4n^3}S \cap C \subset A', B', C_{A'}, C_{B'} \subset 4n^3 S \cap C$. It's easy to see that $\frac{1}{4n^3}S \cap C$ contains a translate of $B(o, 1/\ell'_n)$ and $4n^3 S \cap C$ is contained in a translate of $B(o, \ell''_n)$. Hence, we find that $A', B', C_{A'}, C_{B'}$ satisfy the first condition in Definition 2.7.

For the second condition note that $S'' := \frac{1}{4n^3} S \cap C$ is indeed a set obtained by intersecting a cone with a halfspace. Hence, if we recall that $\lambda_n = (4n^3)^2$, and let $y = o$, we find that $A', B', C_{A'}, C_{B'}$ satisfy the second condition in Definition 2.7.

For the third condition, note that $K \cap C \subset A' \subset C_{A'} \subset (1+\eta)(K \cap C)$, and analogously for $B', C_{B'}$. Hence, if we let η_n sufficiently small in terms of n, t, ℓ_n and λ_n, set $x = y = o$, we find that $A', B', C_{A'}, C_{B'}$ satisfy the third condition in Definition 2.7.

Hence, A', B' are $(w_n\gamma, \ell_n, \lambda_n, \mu_n)$ conelike, with μ_n sufficiently small in terms of n, t, ℓ_n, and λ_n so that by the assumption in the proposition, we have

$$\left|\left(A^3 \cap C\right) \triangle \left(B^3 \cap C\right)\right| = |A' \triangle B'| \leq c_{n,\ell_n,\lambda_n} \sqrt{\frac{w_n\delta + w_n\gamma}{t}} |A'| \leq \frac{c_n - c^{2.11}}{n+1} \sqrt{\frac{\delta+\gamma}{t}} |A|,$$

where we recall that $c_n := (n+1)c_{n,\ell_n,\lambda_n}\sqrt{w_n} + c^{2.11}$. We conclude by adding up the contributions from the different cones $C \in \mathfrak{C}$.

$$\left|A^1 \triangle B^1\right| = \left|A^3 \triangle B^3\right| = \sum_{C \in \mathfrak{C}} \left|\left(A^3 \triangle B^3\right) \cap C\right| \leq (c_n - c^{2.11}) \sqrt{\frac{\delta+\gamma}{t}} |A|.$$

We conclude with the previous note that

$$|A \triangle B| \leq |A^1 \triangle B^1| + c^{2.11} t^{-1} \delta |A| \leq c_n \sqrt{\frac{\delta+\gamma}{t}} |A|.$$

□

3 Intermediate propositions

The proof of Theorem 1.2 relies on optimal transport. For the purpose of this paper, we only need to know the following classical result: given two bounded sets $A, B \subset \mathbb{R}^n$ of positive volume, there exists a convex function $\varphi : \mathbb{R}^n \to \mathbb{R}$ whose gradient sends the normalized indicator function of A onto that of B. More precisely, if we define $T := \nabla\varphi$, then $T_\sharp\left(\frac{1}{|A|}\mathbf{1}_A\right) = \frac{1}{|B|}\mathbf{1}_B$, where $T_\sharp$ denotes the push-forward through the map T. Furthermore, this map is unique: If φ_1 and φ_2 are two convex functions such that $T_i := \nabla\varphi_i$ sends $\frac{1}{|A|}\mathbf{1}_A$ to $\frac{1}{|B|}\mathbf{1}_B$, then $T_1 = T_2$ a.e. inside A.

From now on, whenever we say that T is the optimal transport from A to B, we mean the (unique) gradient of a convex function that sends $\frac{1}{|A|}\mathbf{1}_A$ to $\frac{1}{|B|}\mathbf{1}_B$. We refer to [Fig17, Chapter 4.6] for a quick introduction to this beautiful theory and more references.

3.1 Propositions

Proposition 3.1. *For every $n \in \mathbb{N}$ and all $t, \varepsilon, \lambda, \ell, \delta, \gamma > 0$ with $\delta+\gamma \leq t^{2n-1}/2$, there exists $c^{3.1}_{n,\varepsilon,\lambda,\ell}, \mu^{3.1}_{n,t,\varepsilon,\ell} > 0$ such that the following holds. Assume that $A, B \subset \mathbb{R}^n$ are $(\gamma, \ell, \lambda, \mu^{3.1}_{n,t,\varepsilon,\ell})$ conelike. Moreover, assume that $|tA + (1-t)B| \leq (1+\delta)|A|$. If $T : C_A \to C_B$ is the optimal transport map from C_A to C_B, then*

$$\int_{x \in \partial C_A} \max\left\{\left\langle x - T(x), \frac{x - o'}{||x - o'||_2}\right\rangle, 0\right\} dx \leq c^{3.1}_{n,\varepsilon,\lambda,\ell} \sqrt{\frac{\delta+\gamma}{t}} |A|,$$

for any $o' \in (1-\varepsilon)C_A$.

Corollary 3.2. *In addition, for all $\theta > 0$ there exists $c_{n,\varepsilon,\lambda,\ell,\theta}$ such that the following holds. Let $Q : \mathbb{R}^n \to \mathbb{R}^n$ be an an affine transformation with $||Q||_{op}, ||Q^{-1}||_{op} \leq \theta$. If $T_Q : Q(C_A) \to Q(C_B)$ is the optimal transport map from $Q(C_A)$ to $Q(C_B)$, then*

$$\int_{x \in \partial C_A} \max\left\{\left\langle Q(x) - T_Q(Q(x)), \frac{Q(x) - Q(o')}{||Q(x) - Q(o')||_2}\right\rangle, 0\right\} dx \leq c_{n,\varepsilon,\lambda,\ell,\theta} \sqrt{\frac{\delta+\gamma}{t}} |Q(A)|,$$

for all $o' \in (1-\varepsilon)C_A$. □

Definition 3.3. *Given a parameter θ, let a* random scaling *be the random affine transformation $Q \sim \mathcal{Q}_\theta$ generated as follows. Sample a uniformly random orthonormal basis $e_1, \ldots, e_n \in \mathbb{R}^n$ and sample $\theta_1, \ldots, \theta_n \in [\theta^{-1}, \theta]$ i.i.d. uniformly. Then, in this basis, let Q be the random transformation given by the diagonal matrix with entries θ_i.*

Proposition 3.4. *For every $n \in \mathbb{N}$, $\ell > 1$, there exists constants $\theta = \theta_{n,\ell}, c_{n,\ell} > 0$ such that if*

- *$B(o, 1/\ell) \subset \frac{1}{2}C_A, \frac{1}{2}C_B \subset B(o, \ell)$ where C_A and C_B are convex,*
- *for every affine tranformation $Q\colon \mathbb{R}^n \to \mathbb{R}^n$, T_Q is a map with $T_Q(Q(C_A)) \subset Q(C_B)$,*
- *$Q \sim \mathcal{Q}_\theta$ is a random scaling, and*
- *o' is chosen uniformly random from $B(o, 1/\ell)$,*

then, for all $x \in \partial C_A$,

$$\mathbb{E}_{Q,o'}\left[\max\left\{\left\langle Q(x) - T_Q(Q(x)), \frac{Q(x) - Q(o')}{||Q(x) - Q(o')||_2}\right\rangle, 0\right\}\right] \geq c_{n,\ell} d(x, C_B).$$

Proposition 3.5. *For all $n \in \mathbb{N}, \ell \geq 1$, there exists constants $c_{n,\ell}$, so that given two convex sets $X, Y \subset \mathbb{R}^n$ of equal volume with $B(o, 1/\ell) \subset X, Y \subset B(o, \ell)$ we have*

$$|X \triangle Y| \leq c_{n,\ell} \int_{\partial X} d(x, Y) dx.$$

3.2 Auxiliary Lemmas

Lemma 3.6. *For every $n \in \mathbb{N}$ and $\lambda, \ell > 0$, there exists $\varepsilon^{3.6}_{n,\lambda,\ell}, m^{3.6}_{n,\lambda,\ell}, r^{3.6}_{n,\lambda,\ell}, \sigma^{3.6}_{n,\lambda,\ell} > 0$ such that the following holds. Say sets $A, B \subset \mathbb{R}^n$ are $(\gamma, \ell, \lambda, \mu)$ conelike. Then for every non-zero $y_2 \in \mathbb{R}^n$, there exists $s \in \{\pm 1\}$ such that for every map $M\colon \partial C_A \cap \partial C_B \to \partial C_A \cup \partial C_B$ the following holds. There exist faces F_A of C_A and F_B of C_B with the same supporting hyperplane H, and there exists $w_0 \in H$, such that*

$$B^n(w_0, 1/r^{3.6}_{n,\lambda,\ell}) \cap H \subset F_A \cap F_B.$$

Moreover, for every $w \in B^n(w_0, 1/r^{3.6}_{n,\lambda,\ell}) \cap H$ there exists a ball $X_w \subset \mathbb{R}^n$ such that with $y_1 = M(w) - w$ we have

1. *$X_w \subset (1 - \varepsilon^{3.6}_{n,\lambda,\ell})(C_A \cap C_B)$*
2. *$|X_w| \geq m^{3.6}_{n,\lambda,\ell}$*
3. *$d(w, X_w) \geq 1/(4r^{3.6}_{n\lambda,\ell})$*
4. *$\mathbb{P}_{x \in X_w}\left(\frac{\langle y_1, x-w\rangle}{|y_1||x-w|} \geq 0\right) \geq 1/2$*
5. *$\mathbb{P}_{x \in X_w}\left(\frac{\langle sy_2, x-w\rangle}{|y_2||x-w|} \geq \sigma^{3.6}_{n,\lambda,\ell}\right) = 1.$*

Lemma 3.7. *For every $n \in \mathbb{N}$ and $\lambda, \ell > 0$, there exists $r^{3.7}_{n,\lambda,\ell} > \ell$ such that the following holds. Say sets $A, B \subset \mathbb{R}^n$ are $(\gamma, \ell, \lambda, \mu)$ conelike. Construct simplex $S' = S + z$ with a vertex at z. Let $F_0, F_1, \ldots F_n$ be the faces of S' where F_0 is the face opposite vertex z. Then*

1. *$S' \subset C_A \cap C_B$*
2. *$F_1 \cup \cdots \cup F_n \subset \partial C_A \cap \partial C_B$.*
3. *$B(u, 1/r^{3.7}) \subset S' \subset B(u, r^{3.7})$ for some $u \in \mathbb{R}^n$.*

Lemma 3.8. *For every $n \in \mathbb{N}$ and $r, \sigma > 0$, there exists $k^{3.8}_{n,r,\sigma} > 0$ such that the following holds. Let H be a hyperplane and let H^+ and H^- be the two half-spaces determined by H. Let $w \in H$ with $|w| \leq r$. Let f be the normal vector of H pointing to H^+. Let y_1, y_2 be two vectors such that $\frac{\langle y_1, f\rangle}{|y_1||f|} \geq 0$ and $\frac{\langle y_2, f\rangle}{|y_2||f|} \geq \sigma$. Then the ball*

$$X = B^n(w + f/(2r), k),$$

has the following properties:

1. *$X \subset (1 - 1/(8r^2))(B^n(w, 1/r) \cap H^+)$*
2. *$\mathbb{P}_{x \in X}\left(\frac{\langle y_1, x-w\rangle}{|y_1||x-w|} \geq 0\right) \geq 1/2$*

3. For all $x \in X$ we have $\frac{\langle y_2, x-w\rangle}{|y_2||x-w|} \geq \sigma/4$

4. For all $x \in X$ we have $|x-w| \geq 1/4r$.

Lemma 3.9. *For every $n \in \mathbb{N}$ and $r > 1$, the following holds. Let S' be a simplex such that $B^n(u, 1/r) \subset S' \subset B^n(u, r)$ for some $u \in \mathbb{R}^n$. Let F be a facet of S' and let H be the supporting hyperplane of F. Finally, let H^+ and H^- be the two half spaces determined by H, such that H^+ contains S' and H^- is disjoint from the interior of S'. Then there exists $v \in F$ such that*

$$B^n(v, 1/r) \cap H^+ \subset S'$$

and

$$B^n(v, 1/r) \cap H \subset F \subset B^n(v, 2r) \cap H$$

Lemma 3.10. *For every $n \in \mathbb{N}$ and $r > 0$, there exists $\sigma^{3.10} = \sigma^{3.10}_{n,r} > 0$ such that the following holds. Let S' be a simplex such that $B^n(u, 1/r) \subset S' \subset B^n(u, r)$ for some $u \in \mathbb{R}^n$. Let $f_0, f_1, \dots, f_n$ be the inward normal vectors to the faces of S'. Then for every unit vector $v \in \mathbb{R}^n$ there exists $1 \leq i \leq n$ such that*

$$|\langle f_i, v\rangle| \geq \sigma^{3.10}.$$

Lemma 3.11. *For every $n \in \mathbb{N}$, there exists a constant $c^{3.11}_n > 0$ such that the following holds. If $0 < t \leq 1/2$ and $\lambda_1, \dots, \lambda_n > 0$ and $\lambda_1 \cdots \lambda_n = 1$, then*

$$\sqrt{\sum_i (\lambda_i - 1)^2} \leq c^{3.11}_n t^{-n} \left(\prod_i (t + (1-t)\lambda_i) - 1 \right) + c^{3.11}_n t^{-\frac{1}{2}} \sqrt{\prod_i (t + (1-t)\lambda_i) - 1}.$$

Lemma 3.12. *For every $n \in \mathbb{N}$ and for all $t, \varepsilon, \ell > 0$, there exists $\mu = \mu_{n,t,\varepsilon,\ell} > 0$ such that the following holds. Assume that $A, B \subset \mathbb{R}^n$ are $(\gamma, \ell, \lambda, \mu)$ conelike. Then*

$$tA + (1-t)B \supset t(1 - \varepsilon/4)C_A + (1-t)C_B.$$

Lemma 3.13. *Let C_A, C_B be two convex sets in $\mathbb{R}^n$ with equal volume 1 and satisfying*

$$B^n(o, 1/R) \subset C_A, C_B \subset B^n(o, R) \qquad \text{for some constant } R > 1. \tag{3.1}$$

Let $T = \nabla\varphi$ denote the optimal transport map from C_A to C_B. Then, for every $\varepsilon \in (0, 1)$,

$$\|D(T - \mathrm{Id})\|_{L^\infty((1-\varepsilon)C_A)} \leq C_{n,R,\varepsilon} \|D(T - \mathrm{Id})\|_{L^1((1-\varepsilon/2)C_A)}.$$

3.3 Proofs of propositions

3.3.1 Proof of Proposition 3.1

Proof of Proposition 3.1. We first observe that, by Caffarelli's regularity theory [Caf92b, Caf92a], we can write $T = \nabla\varphi$, where the function $\varphi\colon C_A \to \mathbb{R}^n$ is a smooth strictly convex solution of $\det D^2\varphi = 1$.

Therefore, for $x \in C_A$, $DT(x) = D^2\varphi(x)$ is a positive definite symmetric matrix with determinant 1 and its eigenvalues $\lambda_1(x), \lambda_2(x), \dots, \lambda_n(x)$ satisfy

$$\lambda_1(x), \lambda_2(x), \dots, \lambda_n(x) > 0 \text{ and } \lambda_1(x)\lambda_2(x)\cdots\lambda_n(x) = 1. \tag{3.2}$$

Note that we can write $tId + (1-t)T = \nabla(\frac{t}{2}||x||_2^2 + (1-t)\varphi)$ and that the function $\frac{t}{2}||x||_2^2 + (1-t)\varphi$ is also strictly convex.

Therefore, for $x \in C_A$,

$$D(tId + (1-t)T)(x) = D^2\left(\frac{t}{2}||x||_2^2 + (1-t)\varphi\right)(x)$$

is a positive definite symmetric matrix with eigenvalues

$$t + (1-t)\lambda_1(x), t + (1-t)\lambda_2(x), \dots, t + (1-t)\lambda_n(x).$$

In particular, the function $tId + (1-t)T\colon C_A \to \mathbb{R}^n$ is injective.

The above discussion shows that for every compact subset $E \subset C_A$ we have

$$\left| \bigcup_{x\in E} tx + (1-t)T(x) \right| = \int_E \det D\left(tId + (1-t)T\right) dx = \int_E \left(\prod_i t + (1-t)\lambda_i \right) dx$$

Construct the set $E := (T^{-1}(B) \cap A) \cup (1-\varepsilon/4)C_A$. By Lemma 3.12, it follows that $tA + (1-t)B \supset \bigcup_{x\in E} tx + (1-t)T(x)$. By hypothesis, we have $|tA + (1-t)B| \leq (1+\delta)|A|$.

Combining the last three lines, we get

$$\int_E \left(\prod_i t + (1-t)\lambda_i \right) dx \leq (1+\delta)|A|$$

By hypothesis, we also have

$$|C_A \setminus A| = |C_B \setminus B| = \gamma|A|, \tag{3.3}$$

and because T is bijective and measure preserving, we get

$$|E| \geq |T^{-1}(B) \cap A| = |A \setminus T^{-1}(C_B \setminus B)| \geq |A| - |C_B \setminus B| \geq (1-\gamma)|A|. \tag{3.4}$$

Combining the last two inequalities, we get

$$\int_E \left[\prod_i (t + (1-t)\lambda_i) - 1 \right] dx \leq (\delta+\gamma)|A| \tag{3.5}$$

Also, Lemma 3.11 together with (3.2) imply that

$$\sqrt{\sum_i (\lambda_i - 1)^2} \leq c_n^{3.11} t^{-n} \left(\prod_i (t + (1-t)\lambda_i) - 1 \right) + c_n^{3.11} t^{-\frac{1}{2}} \sqrt{\prod_i (t + (1-t)\lambda_i) - 1}. \tag{3.6}$$

Therefore, we get

$$\begin{aligned}
\int_E ||D(T - Id)||_{op} dx &\leq \int_E \sqrt{\sum_i (\lambda_i - 1)^2} dx \\
&\leq \int_E \left(c_n^{3.11} t^{-n} \left(\prod_i (t + (1-t)\lambda_i) - 1 \right) + c_n^{3.11} t^{-\frac{1}{2}} \sqrt{\prod_i (t + (1-t)\lambda_i) - 1} \right) dx \\
&\leq c_n^{3.11} t^{-n} (\delta+\gamma)|A| + c_n^{3.11} t^{-\frac{1}{2}} \int_E \sqrt{\prod_i (t + (1-t)\lambda_i) - 1}\, dx \\
&\leq c_n^{3.11} t^{-n} (\delta+\gamma)|A| + c_n^{3.11} t^{-\frac{1}{2}} \sqrt{|E|} \sqrt{\int_E \left(\prod_i (t + (1-t)\lambda_i) - 1 \right) dx} \\
&\leq c_n^{3.11} t^{-n} (\delta+\gamma)|A| + c_n^{3.11} t^{-\frac{1}{2}} \sqrt{\delta+\gamma} \sqrt{|E|} \sqrt{|A|} \\
&\leq c_n^{3.11} t^{-n} (\delta+\gamma)|A| + 2c_n^{3.11} t^{-\frac{1}{2}} \sqrt{\delta+\gamma} |A| \leq 3c_n^{3.11} \sqrt{\frac{\delta+\gamma}{t}} |A|.
\end{aligned} \tag{3.7}$$

Here, the first inequality follows from the fact that the operator norm is upper bounded by the Hilbert-Schmidt norm. The second inequality follows from (3.6). The third inequality follows from (3.5). The fourth inequality follows from the concavity of the function $x^{\frac{1}{2}}$. The fifth inequality follows again from (3.5). The sixth inequality follows from the hypothesis $|E| \leq |C_A| \leq 2|A|$ and the final inequality follows from the hypothesis $\delta + \gamma \leq t^{2n-1}/2$.

Thus, Lemma 3.13, together with (3.7) and the fact that $E \supset (1-\varepsilon/4)C_A$, implies that for $x \in (1-\varepsilon/2)C_A$

$$||D(T - Id)(x)||_{op} \leq c_{n,\varepsilon,\ell}^{3.13} \sqrt{\frac{\delta+\gamma}{t}} \tag{3.8}$$

Now, fix $o' \in (1-\varepsilon)C_A$ and set $P = B(o', \varepsilon/2\ell)$. Note that as $C_A \supset B(o, \ell^{-1})$, it follows that $P \subset (1-\varepsilon/2)C_A \subset E$. Combining (3.7) and (3.8), we deduce that

$$\begin{aligned}
\int_E \frac{||D(T-Id)||_{op}}{||x-o'||_2^{n-1}}dx &\le \int_P \frac{||D(T-Id)||_{op}}{||x-o'||_2^{n-1}}dx + \int_{E\setminus P} \frac{||D(T-Id)||_{op}}{||x-o'||_2^{n-1}}dx \\
&\le c_{n,\varepsilon,\ell}^{3.13}\sqrt{\frac{\delta+\gamma}{t}}\int_P \frac{1}{||x-o'||_2^{n-1}}dx + \int_{E\setminus P} \frac{||D(T-Id)||_{op}}{||x-o'||_2^{n-1}}dx \\
&\le c_{n,\varepsilon,\ell}^{3.13}\sqrt{\frac{\delta+\gamma}{t}}\int_{B(o,\varepsilon/2\ell)} \frac{1}{||x||_2^{n-1}}dx + (2\ell\varepsilon^{-1})^{n-1}\int_{E\setminus P}||D(T-Id)||_{op}dx \qquad (3.9)\\
&\le c_{n,\varepsilon,\ell}^{3.13}\sqrt{\frac{\delta+\gamma}{t}}\frac{\varepsilon}{2\ell}|S^{n-1}(o,1)| + (2\ell\varepsilon^{-1})^{n-1}\int_E ||D(T-Id)||_{op}dx \\
&\le c_{n,\varepsilon,\ell}^{3.13}\sqrt{\frac{\delta+\gamma}{t}}\frac{\varepsilon}{2\ell}|S^{n-1}(o,1)| + (2\ell\varepsilon^{-1})^{n-1}3c_n^{3.11}\sqrt{\frac{\delta+\gamma}{t}}|A| \le c_{n,\varepsilon,\ell}^{3.9}\sqrt{\frac{\delta+\gamma}{t}}|A|.
\end{aligned}$$

Here, the first inequality is immediate. The second inequality follows from (3.8) and the fact that $P \subset (1-\varepsilon/2)C_A$. The third inequality follows from the trivial bound that for $x \notin P$ we have $||x-o'||_2^{-1} \le 2\ell\varepsilon^{-1}$. The fourth inequality follows from a simple change of variables. The fifth inequality follows from (3.7). The final inequality follows from the hypothesis $|A| \ge 2^{-1}|B(o,\ell^{-1})|$.

In particular, by definition of the operator norm,

$$\int_E \max\left\{\frac{\frac{(x-o')^T}{||x-o'||_2}(D(Id-T)(x))\frac{x-o'}{||x-o'||_2}}{||x-o'||_2^{n-1}}, 0\right\}dx \le \int_E \frac{||D(T-Id)||_{op}}{||x-o'||_2^{n-1}}dx \le c_{n,\varepsilon,\ell}^{3.9}\sqrt{\frac{\delta+\gamma}{t}}|A|.$$

Note now that, for $x \in C_A$, as the eigenvalues of $D(T)(x)$ are all positive by (3.2), it follows that the eigenvalues of $D(Id-T)(x)$ are at most 1, which implies that

$$\frac{(x-o')^T}{||x-o'||_2}(D(Id-T)(x))\frac{x-o'}{||x-o'||_2} \le 1.$$

As $P \subset (1-\varepsilon/2)C_A \subset E$, it follows that for $x \in C_A \setminus E$ we have $||x-o'||_2 \ge \varepsilon/2\ell$, which implies that $\frac{1}{||x-o'||_2^{n-1}} \le (2\ell\varepsilon^{-1})^{n-1}$.

Combining the last three inequalities with (3.3) and (3.4), we deduce

$$\begin{aligned}
\int_{C_A} \max\left\{\frac{\frac{(x-o')^T}{||x-o'||_2}(D(Id-T)(x))\frac{x-o'}{||x-o'||_2}}{||x-o'||_2^{n-1}}, 0\right\}dx &\le (2\ell\varepsilon^{-1})^{n-1}|C_A \setminus E| + c_{n,\varepsilon,\ell}^{3.9}\sqrt{\frac{\delta+\gamma}{t}}|A| \\
&\le (2\ell\varepsilon^{-1})^{n-1}2\gamma|A| + c_{n,\varepsilon,\ell}^{3.9}\sqrt{\frac{\delta+\gamma}{t}}|A| \le c_{n,\varepsilon,\ell}^{3.10}\sqrt{\frac{\delta+\gamma}{t}}|A|. \qquad (3.10)
\end{aligned}$$

Now, for a unit vector $y \in S^{n-1}(o,1)$ define $s_y := \max\{s\colon o'+sy \in C_A\}$. Define the function $f_y(s)\colon [0,s_y] \to \mathbb{R}$, by $f_y(s) := \langle (o'+sy) - T(o'+sy), y\rangle$. It is easy to check that $\frac{d}{ds}(f_y)(s) = y^T\ D(Id-T)(o'+sy)\ y$. Thus, by performing the change of variable $x(s,y)\colon \mathbb{R}\times S^{n-1}(o,1)\to\mathbb{R}^n, x(s,y)=o'+sy$, we get

$$\begin{aligned}
&\int_{C_A} \max\left\{\frac{\frac{(x-o')^T}{||x-o'||_2}(D(Id-T)(x))\frac{x-o'}{||x-o'||_2}}{||x-o'||_2^{n-1}},0\right\}dx\\
&\quad=\int_{S^{n-1}(o,1)}\int_{0\le s\le s_y}\max\left\{y^T\ D(Id-T)(o'+sy)\ y,0\right\}dsdy\\
&\quad\ge\int_{S^{n-1}(o,1)}\max\left\{\int_{0\le s\le s_y}y^T\ D(Id-T)(o'+sy)\ y\ ds,0\right\}dy\\
&\quad=\int_{S^{n-1}(o,1)}\max\left\{\int_{0\le s\le s_y}\frac{d}{ds}(f_y)(s)ds,0\right\}dy\\
&\quad=\int_{S^{n-1}(o,1)}\max\left\{f_y(s_y)-f_y(0),0\right\}dy\\
&\quad=\int_{S^{n-1}(o,1)}\max\left\{\langle(o'+s_yy)-T(o'+s_yy),y\rangle-\langle o'-T(o'),y\rangle,0\right\}dy
\end{aligned}\tag{3.11}$$

Combining (3.10) and (3.11), it follows that

$$\int_{S^{n-1}(o,1)}\max\left\{\langle(o'+s_yy)-T(o'+s_yy),y\rangle-\langle o'-T(o'),y\rangle,0\right\}dy\le c^{3.10}_{n,\varepsilon,\ell}\sqrt{\frac{\delta+\gamma}{t}}|A|.\tag{3.12}$$

Integrating (3.8) between o and o', and using that $o'\in B(o,2\ell)$ we find

$$\begin{aligned}
|(T(o)-o)-(T(o')-o')|&=\left|\int_0^{|o-o'|}\left[D(T-Id)\left(o+t\frac{o'-o}{|o'-o|}\right)\right]\frac{o'-o}{|o'-o|}dt\right|\\
&\le\int_0^{|o'-o|}\left|D(T-Id)\left(o+t\frac{o'-o}{|o'-o|}\right)\right|_{op}dt\\
&\le|o-o'|c^{3.13}_{n,\varepsilon,\ell}\sqrt{\frac{\delta+\gamma}{t}}\le 2\ell c^{3.13}_{n,\varepsilon,\ell}\sqrt{\frac{\delta+\gamma}{t}}
\end{aligned}\tag{3.13}$$

Integrating this further over the unit sphere, we find

$$\int_{S^{n-1}(o,1)}\max\left\{\langle(T(o)-o)-(T(o')-o'),y\rangle,0\right\}dy\le|S^{n-1}(o,1)|2\ell c^{3.13}_{n,\varepsilon,\ell}\sqrt{\frac{\delta+\gamma}{t}}\le c^{3.14}_{n,\varepsilon,\ell}\sqrt{\frac{\delta+\gamma}{t}}|A|.\tag{3.14}$$

Hence, we can adjust (3.12) to give

$$\begin{aligned}
&\int_{S^{n-1}(o,1)}\max\left\{\langle(o'+s_yy)-T(o'+s_yy),y\rangle-\langle o-T(o),y\rangle,0\right\}dy\\
&\quad\le\int_{S^{n-1}(o,1)}\max\left\{\langle(o'+s_yy)-T(o'+s_yy),y\rangle-\langle o'-T(o'),y\rangle,0\right\}dy\\
&\quad+\int_{S^{n-1}(o,1)}\max\left\{\langle(T(o)-o)-(T(o')-o'),y\rangle,0\right\}dy\\
&\quad\le c^{3.10}_{n,\varepsilon,\ell}\sqrt{\frac{\delta+\gamma}{t}}|A|+c^{3.14}_{n,\varepsilon,\ell}\sqrt{\frac{\delta+\gamma}{t}}|A|=c^{3.15}_{n,\varepsilon,\ell}\sqrt{\frac{\delta+\gamma}{t}}|A|
\end{aligned}\tag{3.15}$$

We aim to evaluate this as an integral over the boundary ∂C_A rather than the unit sphere $S^{n-1}(o,1)$. Recall that $o'\in(1-\varepsilon)C_A$ so that $(1-\varepsilon)o'+\varepsilon C_A\subset C_A$. In particular, as $B(o,1/\ell)\subset C_A\subset B(o,\ell)$, we have $B(o',\varepsilon/\ell)\subset C_A\subset B(o',2\ell)$. Considering the map $z\colon S^{n-1}(o,1)\to\partial C_A; y\mapsto o'+s_yy$, so that $y=\frac{z-o'}{||z-o'||}$, then we find that the Jacobian of this map has determinant bounded by some constant, say $k^{3.16}_{n,\varepsilon,\ell}$, depending only on ε, ℓ, and n. Hence, changing variables, we find

$$\begin{aligned}
&\int_{\partial C_A} \max\left\{\left\langle z - T(z), \frac{z-o'}{|z-o'|}\right\rangle - \left\langle o - T(o), \frac{z-o'}{|z-o'|}\right\rangle, 0\right\} dz \\
&\quad \le k^{3.16}_{n,\varepsilon,\ell} \int_{S^{n-1}(o,1)} \max\left\{\langle (o' + s_y y) - T(o' + s_y y), y\rangle - \langle o - T(o), y\rangle, 0\right\} dy \\
&\quad \le k^{3.16}_{n,\varepsilon,\ell} c^{3.15}_{n,\varepsilon,\ell} \sqrt{\frac{\delta+\gamma}{t}} |A| \le c^{3.16}_{n,\varepsilon,\ell} \sqrt{\frac{\delta+\gamma}{t}} |A|.
\end{aligned} \tag{3.16}$$

Note that $T^{-1}\colon C_B \to C_A$ is also an optimal transport map. By repeating the entire argument above, we get that for $o^\star \in (1-\varepsilon)C_B$

$$\int_{\partial C_B} \max\left\{\left\langle w - T^{-1}(w), \frac{w-o^\star}{|w-o^\star|}\right\rangle - \left\langle o - T^{-1}(o), \frac{w-o^\star}{|w-o^\star|}\right\rangle, 0\right\} dw \le c^{3.16}_{n,\varepsilon,\ell} \sqrt{\frac{\delta+\gamma}{t}} |A|.$$

We now observe that $T^{-1}(o)$ belongs to $(1-\varepsilon)C_A$. Indeed, Lemma 3.13 applied to T^{-1} implies that T^{-1} is uniformly close to the affine map $x + T^{-1}(o)$ inside $(1-\varepsilon)C_B$. Since $T^{-1}((1-\varepsilon)C_B) \subset C_A$, this implies that $T^{-1}(o)$ remains at some uniform positive distance from ∂C_A.

Now, integrating (3.8) between o and $T^{-1}(o)$ (both of which are in $(1-\varepsilon)C_A$), we get

$$\begin{aligned}
\left|(T(o) - o) - (o - T^{-1}(o))\right| &= \left|(T(o) - o) - (T(T^{-1}(o)) - T^{-1}(o))\right| \\
&= \left|\int_0^{|o-T^{-1}(o)|} \left[D(T - Id)\left(o + s\frac{T^{-1}(o) - o}{|T^{-1}(o) - o|}\right)\right] \frac{T^{-1}(o) - o}{|T^{-1}(o) - o|} ds\right| \\
&\le \int_0^{|o-T^{-1}(o)|} \left| D(T - Id)\left(o + s\frac{T^{-1}(o) - o}{|T^{-1}(o) - o|}\right)\right|_{op} ds \\
&\le |o - T^{-1}(o)| c^{3.13}_{n,\varepsilon,\ell} \sqrt{\frac{\delta+\gamma}{t}} \le 2\ell c^{3.13}_{n,\varepsilon,\ell} \sqrt{\frac{\delta+\gamma}{t}}
\end{aligned}$$

Combining the last two equations and using the fact that $|\partial C_B| \le |S^{n-1}(o,\ell)|$, we get that

$$\begin{aligned}
&\int_{\partial C_B} \max\left\{\left\langle w - T^{-1}(w), \frac{w-o^\star}{|w-o^\star|}\right\rangle - \left\langle T(o) - o, \frac{w-o^\star}{|w-o^\star|}\right\rangle, 0\right\} dw \\
&\quad \le c^{3.16}_{n,\varepsilon,\ell} \sqrt{\frac{\delta+\gamma}{t}} |A| + |S^{n-1}(o,\ell)| 2\ell c^{3.13}_{n,\varepsilon,\ell} \sqrt{\frac{\delta+\gamma}{t}} \le c^{3.17}_{n,\varepsilon,\ell} \sqrt{\frac{\delta+\gamma}{t}} |A|.
\end{aligned} \tag{3.17}$$

We apply Lemma 3.6 to the $(\gamma, \ell, \lambda, \mu)$ conelike sets A, B, together with the vector $y_2 = o - T(o)$ and the map $M = T$ in the case $s = 1$ and the map $M = T^{-1}$ in the case $s = -1$ (restricted to $\partial C_A \cap \partial C_B$). Thus, we find faces F_A of C_A and F_B of C_B with the same supporting hyperplane H, and we find $w_0 \in H$ such that $B^n(w_0, 1/r^{3.6}_{n,\lambda,\ell}) \cap H \subset F_A \cap F_B$. Moreover, for every $w \in B^n(w_0, 1/r^{3.6}_{n,\lambda,\ell}) \cap H$ there exists a ball $X_w \subset \mathbb{R}^n$ such that with $y_1 = M(w) - w$ we have

1. $X_w \subset (1 - \varepsilon^{3.6}_{n,\lambda,\ell})(C_A \cap C_B)$
2. $|X_w| \ge m^{3.6}_{n,\lambda,\ell}$
3. $d(w, X_w) \ge 1/(4r^{3.6}_{n\lambda,\ell})$
4. $\mathbb{P}_{x\in X_w}\left(\frac{\langle y_1, x-w\rangle}{|y_1||x-w|} \ge 0\right) \ge 1/2$
5. $\mathbb{P}_{x\in X_w}\left(\frac{\langle sy_2, x-w\rangle}{|y_2||x-w|} \ge \sigma^{3.6}_{n,\lambda,\ell}\right) = 1.$

Consider the case $s = 1$ and $M = T$ (the other case is analogous). By averaging (3.16) over $o' \in (1-\varepsilon)C_A$, we get

$$\mathbb{E}_{o'\in(1-\varepsilon)C_A} \int_{\partial C_A} \max\left\{\left\langle z - T(z), \frac{z-o'}{|z-o'|}\right\rangle - \left\langle o - T(o), \frac{z-o'}{|z-o'|}\right\rangle, 0\right\} dz \le c^{3.16}_{n,\varepsilon,\ell} \sqrt{\frac{\delta+\gamma}{t}} |A|,$$

which, by interchanging the integration and the average, is equivalent to

$$\int_{\partial C_A} \mathbb{E}_{o'\in(1-\varepsilon)C_A} \max\left\{\left\langle z-T(z), \frac{z-o'}{|z-o'|}\right\rangle - \left\langle o-T(o), \frac{z-o'}{|z-o'|}\right\rangle, 0\right\} dz \leq c^{3.16}_{n,\varepsilon,\ell}\sqrt{\frac{\delta+\gamma}{t}}|A|.$$

By restricting our attention to a certain part of the boundary, namely $B^n(w_0, 1/r^{3.6}_{n,\lambda,\ell}) \cap H \subset \partial C_A$, we deduce

$$\int_{B^n\left(w_0,1/r^{3.6}_{n,\lambda,\ell}\right)\cap H} \mathbb{E}_{o'\in(1-\varepsilon)C_A} \max\left\{\left\langle z-T(z), \frac{z-o'}{|z-o'|}\right\rangle - \left\langle o-T(o), \frac{z-o'}{|z-o'|}\right\rangle, 0\right\} dz \leq c^{3.16}_{n,\varepsilon,\ell}\sqrt{\frac{\delta+\gamma}{t}}|A|.$$

For each $z \in B^n(w_0, 1/r^{3.6}_{n,\lambda,\ell}) \cap H$, by conditioning on the event $o' \in X_z$ and using the first two properties of X_z, namely that $X_z \subset (1-\varepsilon^{3.6}_{n,\lambda,\ell})C_A \subset (1-\varepsilon)C_A$, and that $|X_z| \geq m^{3.6}_{n,\lambda,\ell}$ we get

$$\int_{B^n(w_0,1/r^{3.6}_{n,\lambda,\ell})\cap H} \mathbb{E}_{o'\in X_z} \max\left\{\left\langle z-T(z), \frac{z-o'}{|z-o'|}\right\rangle - \left\langle o-T(o), \frac{z-o'}{|z-o'|}\right\rangle, 0\right\} dz \leq (m^{3.6}_{n,\lambda,\ell})^{-1} c^{3.16}_{n,\varepsilon,\ell}\sqrt{\frac{\delta+\gamma}{t}}|A|.$$

Now for each $z \in B^n(w_0, 1/r^{3.6}_{n,\lambda,\ell}) \cap H$, by conditioning on the event $o' \in E_z$, where

$$E_z := \left\{o' \in X_z \colon \frac{\langle y_1, o'-z\rangle}{|y_1||o'-z|} \geq 0 \text{ and } \frac{\langle y_2, o'-z\rangle}{|y_2||o'-z|} \geq \sigma^{3.6}_{n,\lambda,\ell}\right\}$$

and using the last two properties of X_z which imply $|E_z| \geq \frac{1}{2}|X_z|$, we get

$$\int_{B^n\left(w_0,1/r^{3.6}_{n,\lambda,\ell}\right)\cap H} \mathbb{E}_{o'\in E_z} \max\left\{\left\langle z-T(z), \frac{z-o'}{|z-o'|}\right\rangle - \left\langle o-T(o), \frac{z-o'}{|z-o'|}\right\rangle, 0\right\} dz \leq 2(m^{3.6}_{n,\lambda,\ell})^{-1} c^{3.16}_{n,\varepsilon,\ell}\sqrt{\frac{\delta+\gamma}{t}}|A|.$$

For each $z \in B^n(w_0, 1/r^{3.6}_{n,\lambda,\ell}) \cap H$ and each $o' \in E_z$, by the definition of E_z, y_1 and y_2, we have $\left\langle z-T(z), \frac{z-o'}{|z-o'|}\right\rangle \geq 0$ and $-\left\langle o-T(o), \frac{z-o'}{|z-o'|}\right\rangle \geq \sigma^{3.6}_{n,\lambda,\ell}|o-T(o)|$.

By combining the last three inequalities, we obtain

$$\left|B^n(w_0, 1/r^{3.6}_{n,\lambda,\ell}) \cap H\right| \sigma^{3.6}_{n,\lambda,\ell}|o-T(o)| \leq 2(m^{3.6}_{n,\lambda,\ell})^{-1} c^{3.16}_{n,\varepsilon,\ell}\sqrt{\frac{\delta+\gamma}{t}}|A|,$$

hence

$$|T(o)-o| \leq |B^n(w_0, 1/r^{3.6}_{n,\lambda,\ell}) \cap H|^{-1}(\sigma^{3.6}_{n,\lambda,\ell})^{-1} 2(m^{3.6}_{n,\lambda,\ell})^{-1} c^{3.16}_{n,\varepsilon,\ell}\sqrt{\frac{\delta+\gamma}{t}}|A| \leq c^{3.18}_{n,\lambda,\varepsilon,\ell}\sqrt{\frac{\delta+\gamma}{t}}. \qquad (3.18)$$

By combining (3.18) with (3.16), we conclude

$$\begin{aligned}
&\int_{\partial C_A} \max\left\{\left\langle z-T(z), \frac{z-o'}{|z-o'|}\right\rangle, 0\right\} dz \\
&\leq \int_{\partial C_A} \max\left\{\left\langle z-T(z), \frac{z-o'}{|z-o'|}\right\rangle - c^{3.18}_{n,\lambda,\varepsilon,\ell}\sqrt{\frac{\delta+\gamma}{t}}, 0\right\} + c^{3.18}_{n,\lambda,\varepsilon,\ell}\sqrt{\frac{\delta+\gamma}{t}} dz \\
&\leq |\partial C_A| c^{3.18}_{n,\lambda,\varepsilon,\ell}\sqrt{\frac{\delta+\gamma}{t}} + \int_{\partial C_A} \max\left\{\left\langle z-T(z), \frac{z-o'}{|z-o'|}\right\rangle - \left\langle o-T(o), \frac{z-o'}{|z-o'|}\right\rangle, 0\right\} dz \\
&\leq |\partial C_A| c^{3.18}_{n,\lambda,\varepsilon,\ell}\sqrt{\frac{\delta+\gamma}{t}} + c^{3.16}_{n,\varepsilon,\ell}\sqrt{\frac{\delta+\gamma}{t}}|A| \leq c_{n,\lambda,\varepsilon,\ell}\sqrt{\frac{\delta+\gamma}{t}}|A|.
\end{aligned}$$

□

3.3.2 Proof of Proposition 3.4

Proof of Proposition 3.4. First note that if $x \in C_B$, then $d(x, C_B) = 0$, so the inequality trivially holds. Henceforth assume $x \notin C_B$. Define

$$\psi := 0.1, \quad \phi = \frac{1}{4\ell}, \quad \xi := \min\left\{\frac{1}{12}\phi\ell^{-1}, \frac{1}{2}\psi\ell^{-1}\right\}, \quad \theta := 2\xi^{-2}\ell^2, \quad \zeta := \frac{1}{24}\psi\theta^{-2}\ell^{-2},$$

$$\alpha := \min\left\{\frac{1}{4}\xi^2\theta^{-2}\ell^{-2}(n-1)^{-1}, \frac{1}{48^2}\psi^2 n\theta^{-6}\ell^{-6}, \frac{1}{2}\right\}, \text{ and } \eta := \min\left\{\frac{1}{3}\psi\theta^{-1}\ell^{-1}, \frac{1}{2}\phi\right\}$$

Write $e_1, \ldots, e_n \in \mathbb{R}^n$ and $\theta_1, \ldots, \theta_n \in [\theta^{-1}, \theta]$ for the random parameters corresponding to transformation $Q \sim \mathcal{Q}_\theta$.

We first restrict our attention to a controlled set of transformations Q. We condition on the event that $\theta_1 \leq \theta^{-1} \min_{i>1}\{\theta_i\}$ and the event that e_1 points roughly in the direction x, viz $\left\langle \frac{x}{|x|}, e_1 \right\rangle \geq 1 - \alpha$. As these events are independent, there exists a constant $c_n^{3.19}$ so that

$$\mathbb{P}\left(\theta_1 \leq \theta^{-1} \min_{i>1}\{\theta_i\} \text{ and } \left\langle \frac{x}{|x|}, e_1 \right\rangle \geq 1 - \alpha\right) \geq c_n^{3.19}. \tag{3.19}$$

Henceforth, we condition on these events. We will show that, for these Q, the stated inequality holds. For notational convenience, rescale by a factor θ^{-1}/θ_1, so that we may assume that $\theta_1 = \theta^{-1}$ and $\theta_2, \ldots, \theta_n \in [1, \theta]$.

First, note that as $\langle x, e_1 \rangle \geq (1-\alpha)|x|$, we have $\langle x, e_i \rangle \leq \sqrt{1-(1-\alpha)^2}|x| \leq \sqrt{2\alpha}\ell$, which implies that

$$|Q(x)| = \sqrt{\sum_i \langle Q(x), e_i \rangle^2} \leq \sqrt{\theta^{-1}\langle x, e_1 \rangle^2 + \sum_{i>1} \theta^2 \langle x, e_i \rangle^2} \leq \sqrt{\theta^{-1}\ell^2 + (n-1)2\alpha\theta^2\ell^2} \leq \xi, \tag{3.20}$$

$$|Q(x)| \geq \langle Q(x), e_1 \rangle = \theta^{-1}\langle x, e_1 \rangle > 0.9\theta^{-1}\ell^{-1}. \tag{3.21}$$

Let $u := \frac{Q(x) - T_Q(Q(x))}{|Q(x) - T_Q(Q(x))|}$. We show that $\langle u, e_1 \rangle$ is not very negative.

Claim 3.14. *$\langle u, e_1 \rangle \geq -\psi$.*

Proof of claim. Assume for a contradiction $\langle u, e_1 \rangle < -\psi$. Let p be the point where the line through $Q(x)$ and $T_Q(Q(x))$ intersects the plane spanned by $e_2, \ldots, e_n$. Write $p - Q(x) = su$ for some $s \in \mathbb{R}$. Note $s > 0$ as $\langle Q(x), e_1 \rangle > 0$ and $\langle u, e_1 \rangle < 0$. Since $\langle Q(x), e_1 \rangle \leq |Q(x)| \leq \xi$ and $\langle u, e_1 \rangle < -\psi$, we find that $s \leq \xi/\psi$. By the triangle inequality, this implies $|p| \leq |p - Q(x)| + |Q(x)| \leq \xi + \xi/\psi < 1/\ell$. Since, $\theta_i \geq 1$ for all $i > 1$, we have $B^n(o, 1/\ell) \cap \operatorname{span}(e_2, \ldots, e_n) \subset Q(B^n(o, 1/\ell)) \cap \operatorname{span}(e_2, \ldots, e_n)$, so that $|p| \leq 1/\ell$ implies $p \in Q(B^n(o, 1/\ell))$. Moreover, $Q(B^n(o, 1/\ell)) \subset Q(C_B)$, so $p \in Q(C_B)$. However, this implies $Q(x)$ lies on the line segment between p and $T_Q(Q(x))$, both of which are in $Q(C_B)$. Since affine transformations preserve convexity, this implies $Q(x) \in Q(C_B)$, i.e., $x \in C_B$, a contradiction. □

Let us return to the inner product $\left\langle Q(x) - T_Q(Q(x)), \frac{Q(x) - Q(o')}{||Q(x) - Q(o')||_2} \right\rangle = |Q(x) - T_Q(Q(x))| \left\langle u, \frac{Q(x) - o'}{||Q(x) - o'||_2} \right\rangle$, for some $o' \in Q(B(o, 1/\ell))$. Write

$$\mathcal{O} := \left\{ o' \in Q(B(o, 1/\ell)) : \left\langle u, \frac{Q(x) - o'}{||Q(x) - o'||_2} \right\rangle \geq \eta \right\}.$$

We shall argue $|\mathcal{O}| \geq c_n^{3.22}|Q(B(o, 1/\ell))|$. Write π for the projection onto the plane spanned by $e_2, \ldots, e_n$, thus

$$\left\langle u, \frac{Q(x) - o'}{||Q(x) - o'||_2} \right\rangle = \left\langle \pi(u), \pi\left(\frac{Q(x) - o'}{||Q(x) - o'||_2}\right) \right\rangle + \left(\langle u, e_1 \rangle \cdot \left\langle \frac{Q(x) - o'}{||Q(x) - o'||_2}, e_1 \right\rangle \right),$$

and distinguish two cases; either $\langle u, e_1 \rangle \geq \psi$ or $\langle u, e_1 \rangle \in [-\psi, \psi]$.

In the former case, consider the set

$$\mathcal{O}' := \{o' \in Q(B(o, 1/\ell)) : \langle o', e_1 \rangle \leq 0, ||o'|| \leq \zeta\}.$$

Note that as $\zeta < \theta^{-1}\ell^{-1}$, we have that $\{o' \in Q(B(o, 1/\ell)) : ||o'|| \leq \zeta\} = B(o, \zeta)$, so that using symmetry in the plane spanned by $e_2, \ldots, e_n$ we have $|\mathcal{O}'| = \frac{1}{2}|B(o, \zeta)| \geq \frac{\ell^n \zeta^n}{2\theta^{n-1}}|Q(B(o, 1/\ell))|$.

For points $o' \in \mathcal{O}'$, using Equation (3.21) and a version of Equation (3.20), we get

$$\left| \left\langle \pi(u), \pi\left(\frac{Q(x) - o'}{||Q(x) - o'||_2}\right) \right\rangle \right| \leq \left| \pi\left(\frac{Q(x) - o'}{||Q(x) - o'||_2}\right) \right| \leq \frac{|\pi(Q(x))| + |\pi(o')|}{0.9||Q(x)||_2} \leq \frac{2\sqrt{(n-1)2\alpha}\theta\ell}{\theta^{-1}\ell^{-1}} + \frac{2\zeta}{\theta^{-1}\ell^{-1}}.$$

On the other hand, because $\langle o', e_1 \rangle \leq 0$ we have

$$\begin{aligned} \langle u, e_1 \rangle \cdot \left\langle \frac{Q(x) - o'}{||Q(x) - o'||_2}, e_1 \right\rangle &\geq \psi \left\langle \frac{Q(x)}{||Q(x) - o'||_2}, e_1 \right\rangle \geq \psi\theta^{-1} \left\langle \frac{x}{||Q(x)|| + ||o'||}, e_1 \right\rangle \\ &\geq \frac{\psi\theta^{-1}}{\xi + \zeta} \langle x, e_1 \rangle \geq \frac{\psi\theta^{-1}}{\xi + \zeta}(1-\alpha)\ell^{-1} \end{aligned}$$

Combining these two bounds we find

$$\left\langle u, \frac{Q(x)-o'}{||Q(x)-o'||_2}\right\rangle = \left(\langle u, e_1\rangle \cdot \left\langle \frac{Q(x)-o'}{||Q(x)-o'||_2}, e_1\right\rangle\right) + \left\langle \pi(u), \pi\left(\frac{Q(x)-o'}{||Q(x)-o'||_2}\right)\right\rangle$$
$$\geq \frac{\psi\theta^{-1}}{\xi+\zeta}(1-\alpha)\ell^{-1} - \frac{2\sqrt{(n-1)2\alpha}\theta\ell}{\theta^{-1}\ell^{-1}} - \frac{2\zeta}{\theta^{-1}\ell^{-1}}$$
$$= \theta^{-1}\ell^{-1}\left(\frac{\psi}{2} - 4\sqrt{n\alpha}\theta^3\ell^3 - 2\zeta\theta^2\ell^2\right) \geq \frac{\psi}{3\theta\ell} \geq \eta.$$

Hence, we find $\mathcal{O}' \subset \mathcal{O}$, so that $|\mathcal{O}| \geq |\mathcal{O}'| \geq \frac{1}{2}|B(o,\zeta)| \geq \frac{\ell^n\zeta^n}{2\theta^{n-1}}|Q(B(o,1/\ell))|$.

Now consider the other case, i.e., $\langle u, e_1\rangle \in [-\psi, \psi]$. This implies that $|\pi(u)| \geq \sqrt{1-\psi^2} \geq \frac{1}{2}$. Write $u' := \pi(u)/|\pi(u)|$. We consider the set

$$\mathcal{O}'' := \{o' \in Q(B(o,1/\ell)) : \langle o', e_1\rangle \geq 0, ||\pi(o')|| \in (1/2\ell, 1/\ell), \langle \pi(o'), u'\rangle < -\phi\}.$$

By symmetry in the plane spanned by $e_2, \ldots, e_n$, we have that

$$|\mathcal{O}''| = \frac{1}{2}\left|\{o' \in Q(B(o,1/\ell)) : ||\pi(o')|| \in (1/2\ell, 1/\ell), \langle \pi(o'), u'\rangle < -\phi\}\right|$$

Consider the transformation $Q' \sim \mathcal{Q}_\theta$ with parameters $e_1, \ldots, e_n \in \mathbb{R}^n$ (same as Q) and also $\theta^{-1}, 1, \ldots, 1$. As $\theta_1 = \theta^{-1}$, $\theta_2, \ldots, \theta_n \in [1, \theta]$, we get $Q'(B(o,1/\ell)) \subset Q(B(o,1/\ell))$ and $\frac{|Q'(B(o,1/\ell))|}{|Q(B(o,1/\ell))|} \geq \theta^{-n+1}$. From this containment and the rotational symmetry of $Q'(B(o,1/\ell))$ around the e_1 axis, we deduce

$$|O''| \geq \frac{1}{2}\left|\{o' \in Q'(B(o,1/\ell)) : ||\pi(o')|| \in (1/2\ell, 1/\ell), \langle \pi(o'), u'\rangle < -\phi\}\right|$$
$$\geq \frac{1}{2}\left|\{o' \in Q'(B(o,1/\ell)) : ||\pi(o')|| \in (1/2\ell, 1/\ell), \langle \frac{\pi(o')}{||\pi(o')||}, u'\rangle < -2\ell\phi\}\right|$$
$$\geq \frac{1}{2}\frac{\cos^{(-1)}(2\ell\phi)}{2\pi}\left|\{o' \in Q'(B(o,1/\ell)) : ||\pi(o')|| \in (1/2\ell, 1/\ell)\}\right|$$
$$\geq \frac{1}{6}\left|\{o' \in Q'(B(o,1/\ell)) : ||\pi(o')|| \in (1/2\ell, 1/\ell)\}\right|$$
$$\geq \frac{1}{12}\left|\{o' \in Q'(B(o,1/\ell))\}\right| \geq \frac{1}{12\theta^{n-1}}\left|\{o' \in Q(B(o,1/\ell))\}\right|.$$

Assume that $o' \in \mathcal{O}''$. We have

$$\left|\langle u, e_1\rangle \cdot \left\langle \frac{Q(x)-o'}{||Q(x)-o'||_2}, e_1\right\rangle\right| \leq \psi\left|\left\langle \frac{Q(x)}{||Q(x)-o'||_2}, e_1\right\rangle\right| \leq \psi\frac{||Q(x)||}{||o'||-|Q(x)|} \leq 3\ell\psi\xi,$$

where the first inequality follows from $\langle o', e_1\rangle \geq 0$, the second inequality follows from the triangle inequality and $|e_1| = 1$ and the final inequality follows from $|Q(x)| \leq \xi$ and $||o'|| - |Q(x)| \geq 1/2\ell - \xi \geq 1/3\ell$.

For the other term, we use $|Q(x)| \leq \xi$ and $|o'| \leq \sqrt{|\pi(o')|^2 + \langle o', e_1\rangle^2} \leq \sqrt{1/\ell^2 + 1/\ell^2} \leq 2/\ell$ to find that

$$\left\langle \pi(u), \pi\left(\frac{Q(x)-o'}{||Q(x)-o'||_2}\right)\right\rangle \geq \frac{|\pi(u)|}{||o'||_2 + |Q(x)|}\left(\langle u', \pi(-o')\rangle - |\langle \pi(Q(x)), u'\rangle|\right) \geq \frac{1/2}{2/\ell+\xi}\left(\langle u', \pi(-o')\rangle - \xi\right) \geq \phi - \xi.$$

Combining these two inequalities, we find

$$\left\langle u, \frac{Q(x)-o'}{||Q(x)-o'||_2}\right\rangle = \left(\langle u, e_1\rangle \cdot \left\langle \frac{Q(x)-o'}{||Q(x)-o'||_2}, e_1\right\rangle\right) + \left\langle \pi(u), \pi\left(\frac{Q(x)-o'}{||Q(x)-o'||_2}\right)\right\rangle \geq \phi - \xi - 3\ell\psi\xi \geq \phi/2 \geq \eta$$

This proves that $\mathcal{O}'' \subset \mathcal{O}$, hence $|\mathcal{O}''| \leq |\mathcal{O}|$.

Returning to the two cases $\langle u, e_1\rangle \geq \psi$ and $\langle u, e_1\rangle \in [-\psi, \psi]$, we now find that in both cases

$$|\mathcal{O}| \geq \min\{|\mathcal{O}'|, |\mathcal{O}''|\} \geq c_n^{3.22}|Q(B(o,1/\ell))|, \tag{3.22}$$

where $c_n^{3.22} > 0$ can be found in terms of ζ, θ, ℓ, and n. Note that if $o' \in \mathcal{O}$, then

$$\left\langle Q(x) - T_Q(Q(x)), \frac{Q(x)-o'}{||Q(x)-o'||_2}\right\rangle = |Q(x) - T_Q(Q(x))|\left\langle u, \frac{Q(x)-o'}{||Q(x)-o'||_2}\right\rangle \geq \eta|Q(x) - T_Q(Q(x))|$$
$$\geq \eta\theta^{-1}|x - Q^{-1}(T_Q(Q(x)))| \geq \eta\theta^{-1}d(x, C_B),$$

where the first inequality follows from the definition of $\mathcal{O}$, the second inequality follows from $|Q^{-1}|_{op} \leq \theta$ and the last inequality follows from the fact that $Q^{-1}(T_Q(Q(x))) \in C_B$.

Now we are ready to conclude using the following Markov bound on the expectation we are trying to control:

$$\begin{aligned}
&\mathbb{E}_{Q,o'}\left[\max\left\{\left\langle Q(x) - T_Q(Q(x)), \frac{Q(x) - Q(o')}{||Q(x) - Q(o')||_2}\right\rangle, 0\right\}\right] \\
&\geq \mathbb{P}\left(\theta_1 \leq \theta^{-1}\min_{i>1}\{\theta_i\} \text{ and } \left\langle \frac{x}{|x|}, e_1\right\rangle \geq 1-\alpha\right)\mathbb{P}\left(Q(o') \in \mathcal{O}|Q\right)\eta\theta^{-1}d(x, C_B) \\
&\geq c_n^{3.19}c_n^{3.22}\eta\theta^{-1}d(x, C_B) \geq c_n^{3.4}d(x, C_B).
\end{aligned}$$

Here we used that if o' is chosen uniformly from $B(o, 1/\ell)$, then $Q(o')$ is chosen uniformly from $Q(B(o, 1/\ell))$. This concludes the lemma. □

3.3.3 Proof of Proposition 3.5

Proof of Proposition 3.5. First note that $|X \triangle Y| = 2|X \setminus Y|$, so it suffices to bound $|X \setminus Y|$.

Given $x \in (\partial X) \setminus Y$, let y_x be the intersection between the line segment ox and ∂Y. We'll show that $|x - y_x| = O_\ell(d(x, Y))$ and integrate $|x - y_x|$ over x to find the lemma.

Claim 3.15. *$|x - y_x| \leq \ell^2 d(x, Y)$.*

Proof of claim. Let p_x be the projection of x onto ∂Y, so that $d(x, Y) = |x - p_x|$. Note that as x, y_x, and o are colinear, x, y_x, p_x and o are coplanar. Restrict attention to this plane, and let L be the ray (line) through p_x tangent to $B(o, 1/\ell)$ so that L intersects the line segment ox. Write y'_x for that intersection. Note that because $p_x \in Y$ and $B(o, 1/\ell) \subset Y$, we have $|x - y'_x| \geq |x - y_x|$, so it suffices to upper bound $|x - y'_x|$. We show that the angle $\angle L, ox$ is lower bounded away from 0 in terms of ℓ.

Let t be the tangent point of L to $B(0, 1/\ell)$, so that $\angle L, ox = \angle ty'_x o$. Using the sin rule in the triangle $ty'_x o$, we find $\frac{\sin(\angle ty'_x o)}{|t-o|} = \frac{\sin(\angle y'_x to)}{|y'_x - o|} = \frac{1}{|y'_x - o|}$, so that using $|y'_x - o| \leq \ell$ and $|t - o| = 1/\ell$, we find $\sin(\angle ty'_x o) \geq \ell^{-2}$. Considering the triangle $y'_x p_x x$, we find $\angle p_x y'_x x = \angle ty'_x o$, so that applying the sin rule again, we find $|y'_x - x| = \frac{\sin(\angle x p_x y'_x)}{\sin(\angle p_x y'_x x)}|x - p_x| \leq \ell^2|x - p_x|$. We conclude

$$|x - y_x| \leq |y'_x - x| \leq \ell^2|x - p_x| = \ell^2 d(x, Y).$$

□

Using this claim, we find

$$\int_{\partial X}|x - y_x|dx \leq \ell^2\int_{\partial X}d(x, Y)dx.$$

Note that $\bigcup_{x\in\partial X}[x, y_x] = X \setminus Y$. Let $z\colon S^{n-1}(o, \ell) \to \partial X$ be the map taking a direction $v \in S^{n-1}(o, \ell)$ to the intersection between $\mathbb{R}^+ v$ and ∂X. Note that

$$\begin{aligned}
\left|\bigcup_{x\in\partial X}[x, y_x]\right| &\leq \left|\bigcup_{v\in\partial S^{n-1}(o,\ell)}[v - (z(v) - y_{z(v)}), v]\right| = \int_{0\leq s\leq\ell}\left|S^{n-1}(o, s)\bigcap\bigcup_{v\in\partial S^{n-1}(o,\ell)}[v - (z(v) - y_{z(v)}), v]\right|ds \\
&= \int_{0\leq s\leq\ell}\frac{|S^{n-1}(o, s)|}{|S^{n-1}(o, \ell)|}\left|\{v \in S^{n-1}(o, \ell)\colon |z(v) - y_{z(v)}| \geq \ell - s\}\right|ds \\
&\leq \int_{0\leq s\leq\ell}\left|\{v \in S^{n-1}(o, \ell)\colon |z(v) - y_{z(v)}| \geq \ell - s\}\right|ds = \int_{v\in\partial S^{n-1}(o,\ell)}|z(v) - y_{z(v)}|dv
\end{aligned}$$

The first inequality is immediate from the fact that we compress segments inside $B^n(o, \ell)$ radially outwards onto the sphere $S^{n-1}(o, \ell)$.

As $B(o, 1/\ell) \subset X \subset B(o, \ell)$, we find that the Jacobian of the map z has determinant bounded by some constant, say $k_{n,\ell}$, depending only on ℓ, and n. Hence, we find

$$|X \setminus Y| \leq \int_{v\in\partial S^{n-1}(o,\ell)}|z(v) - y_{z(v)}|dv \leq k_{n,\ell}\int_{\partial X}|x - y_x|dx \leq k_{n,\ell}\ell^2\int_{\partial X}d(x, Y)dx,$$

which concludes the proof. □

3.4 Proofs of Lemmas

3.4.1 Proof of Lemma 3.6

Proof of Lemma 3.6. Fix $r^{3.6}_{n\lambda,\ell} = 2r^{3.7}_{n\lambda,\ell}$, $\sigma^{3.6}_{n,\lambda,\ell} = \sigma^{3.10}_{n,r^{3.7}_{n\lambda,\ell}}/4$, $m^{3.6}_{n,\lambda,\ell} = \left(k^{3.8}_{n,2r^{3.7}_{n\lambda,\ell},\sigma^{3.10}_{n,r^{3.7}_{n\lambda,\ell}}}\right)^n |B^n(o,1)|$, $\varepsilon^{3.6}_{n,\lambda,\ell} = 1/(32(r^{3.7}_{n\lambda,\ell})^2)$.

Recall Definition 2.7 and construct simplex $S' = S'' + z$ with a vertex at z. Let $F_0, F_1, \dots F_n$ be the faces of S' where F_0 is the face opposite vertex z. Then, by Lemma 3.7,

1. $S' \subset C_A \cap C_B$
2. $F_1 \cup \dots \cup F_n \subset \partial C_A \cap \partial C_B$.
3. $B(u, 1/r^{3.7}_{n\lambda,\ell}) \subset S' \subset B(u, r^{3.7}_{n\lambda,\ell})$ for some $u \in \mathbb{R}^n$.

Let $f_0, f_1, \dots, f_n$ be the inward normal vectors to the faces $F_0, \dots, F_n$, respectively. By Lemma 3.10 together with (3), there exists $1 \le i \le n$ such that $\frac{|\langle f_i, y_2\rangle|}{|f_i||y_2|} \ge \sigma^{3.10}_{n,r^{3.7}_{n\lambda,\ell}}$. Hence there exists $s \in \{\pm 1\}$ such that

$$\frac{\langle f_i, sy_2\rangle}{|f_i||sy_2|} \ge \sigma^{3.10}_{n,r^{3.7}_{n\lambda,\ell}}. \tag{3.23}$$

Write $F = F_i$ and $f = f_i$. Let H be the supporting hyperplane of F and let H^+ and H^- be the partition into half-spaces determined by H with H^+ containing S' and H^- disjoint from the interior of S'.

By Lemma 3.9, together with (3), we deduce there exists $w_0 \in F$ such that $B^n(w_0, 1/r^{3.7}_{n,\lambda,\ell}) \cap H^+ \subset S'$ and $B^n(w_0, 1/r^{3.7}_{n,\lambda,\ell}) \cap H \subset F$.

By (1), $S' \subset C_A, C_B$. By (2), there exists faces F_A of C_A and F_B of C_B such that $F \subset F_A \cap F_B$. Clearly faces F, F_A and F_B share the supporting hyperplane H; in particular, F, F_A and F_B share the same inward normal vector f. Therefore, we get $w_0 \in H$ and $B^n(w_0, 1/r^{3.7}_{n\lambda,\ell}) \cap H^+ \subset C_A \cap C_B$. and $B^n(w_0, 1/r^{3.7}_{n\lambda,\ell}) \cap H \subset F_A \cap F_B$. It immediately follows that for every $w \in B^n(w_0, 1/2r^{3.7}_{n\lambda,\ell}) \cap H$, we also have

$$B^n(w, 1/2r^{3.7}_{n\lambda,\ell}) \cap H^+ \subset C_A \cap C_B. \tag{3.24}$$

Fix $w \in B^n(w_0, 1/2r^{3.7}_{n\lambda,\ell}) \cap H \subset F_A \cap F_B$. As $C_A \cup C_B \subset B^n(o, \ell)$, it follows that

$$|w| \le \ell. \tag{3.25}$$

Recall that faces F, F_A and F_B share the same inward normal vector f. Because $w \in F_A \cap F_B$ and $M(w) \in \partial C_A \cup \partial C_B$, by convexity we deduce that $y_1 = M(w) - w$ satisfies

$$\frac{\langle y_1, f\rangle}{|y_1||f|} \ge 0. \tag{3.26}$$

By Lemma 3.8, together with (3.23), (3.25) and (3.26), applied with parameters $n, 2r^{3.7}_{n\lambda,\ell}, \sigma^{3.10}_{n,r^{3.7}_{n\lambda,\ell}}$ (recall $r^{3.7}_{n,\lambda,\ell} > \ell$), the ball $X_w = B^n\left(w + f/(4r^{3.7}_{n\lambda,\ell}), k^{3.8}_{n,2r^{3.7}_{n\lambda,\ell},\sigma^{3.10}_{n,r^{3.7}_{n\lambda,\ell}}}\right)$ has the following properties:

1. $X_w \subset (1 - 1/(32(r^{3.7}_{n\lambda,\ell})^2))(B(w, 1/2r^{3.7}_{n\lambda,\ell}) \cap H^+)$.
2. $\mathbb{P}_{x\in X_w}\left(\frac{\langle y_1, x-w\rangle}{|y_1||x-w|} \ge 0\right) \ge 1/2$
3. For all $x \in X_w$, we have $\frac{\langle sy_2, x-w\rangle}{|sy_2||x-w|} \ge \sigma^{3.10}_{n,r^{3.7}_{n\lambda,\ell}}/4$
4. For all $x \in X_w$, we have $|x - w| \ge 1/8r^{3.7}_{n\lambda,\ell}$.

By construction, $|X_w| = \left(k^{3.8}_{n,2r^{3.7}_{n\lambda,\ell},\sigma^{3.10}_{n,r^{3.7}_{n\lambda,\ell}}}\right)^n |B^n(o,1)| = m^{3.6}_{n,\lambda,\ell}$. By the first property of X_w, together with (3.24), we get $X_w \subset (1 - 1/(32(r^{3.7}_{n\lambda,\ell})^2))(C_A \cap C_B) = (1 - \varepsilon^{3.6}_{n,\lambda,\ell})(C_A \cap C_B)$. The second and third property of X_w exactly give $\mathbb{P}_{x\in X_w}\left(\frac{\langle y_1, x-w\rangle}{|y_1||x-w|} \ge 0\right) \ge 1/2$ and $\mathbb{P}_{x\in X_w}\left(\frac{\langle sy_2, x-w\rangle}{|y_2||x-w|} \ge \sigma^{3.6}_{n,\lambda,\ell}\right) = 1$. The last property of X_w is exactly $d(w, X_w) \ge 1/(4r^{3.6}_{n\lambda,\ell})$. Finally, note that all of these hold for all $w \in B^n(w_0, 1/r^{3.6}_{n\lambda,\ell}) \cap H$, which concludes the proof. □

3.4.2 Proof of Lemma 3.7

Proof of Lemma 3.7. Set $r^{3.7} = 2\ell\lambda$. The first two parts follow immediately from Definition 2.7 (2). For the third part, note that by Definition 2.7 (1) and (2) we have $B(o, 1/\ell) \subset C_A \subset \lambda S'' + z = \lambda S' + (1-\lambda)z$. After rearranging, we conclude $B\left(\frac{\lambda-1}{\lambda}z, \frac{1}{\ell\lambda}\right) \subset S'$. In addition, $z \in S' \subset C_A \subset B(o, \ell)$. After rearranging, we conclude

$$S' \subset B(o,\ell) \subset B\left(\frac{\lambda-1}{\lambda}z, \ell + \frac{\lambda-1}{\lambda}|z|\right) \subset B\left(\frac{\lambda-1}{\lambda}z, 2\ell\right).$$

□

3.4.3 Proof of Lemma 3.8

Proof of Lemma 3.8. Set $k = (4r)^{-1}\sigma$ and $\varepsilon = 1/(8r)^2$. As everything is normalized, without loss of generality we can assume $|y_1| = |y_2| = 1$.

For the second part, consider the half-space $Y = \{x \colon \langle y_1, x - w\rangle \geq 0\}$. We need to show that $|X \cap Y|/|X| \geq 1/2$. Because X is a ball and Y is a half-space, it is enough to show that the center of the ball belongs to the half space. In other words, we need to check $\langle y_1, w + f/(2r) - w\rangle \geq 0$, which follows from the hypothesis $\langle y_1, f\rangle \geq 0$.

For the rest of the proof fix $x \in X = B^n(w + f/(2r), k)$. For the third part, note that we can write $x = w + f/(2r) + \alpha g$ where g is a unit vector and $k \geq \alpha \geq 0$. Thus we have

$$\begin{aligned}
\frac{\langle y_2, x-w\rangle}{|y_2||x-w|} &= \frac{\langle y_2, f/(2r) + \alpha g\rangle}{|y_2||f/(2r)+\alpha g|} = \frac{(2r)^{-1}\langle y_2, f\rangle + \alpha\langle y_2, g\rangle}{|y_2||f/(2r)+\alpha g|} \geq \frac{(2r)^{-1}\sigma + \alpha\langle y_2, g\rangle}{|y_2||f/(2r)+\alpha g|} \\
&\geq \frac{(2r)^{-1}\sigma - \alpha}{|y_2||f/(2r)+\alpha g|} \geq \frac{(4r)^{-1}\sigma}{|y_2||f/(2r)+\alpha g|} \geq \frac{(4r)^{-1}\sigma}{|y_2|(|f/(2r)| + |\alpha g|)} \\
&\geq \frac{(4r)^{-1}\sigma}{1/(2r)+\alpha} \geq \frac{(4r)^{-1}\sigma}{3/(4r)} \geq \sigma/4.
\end{aligned}$$

Here the first inequality follows from the hypothesis $\langle y_2, f\rangle \geq \sigma$. The second inequality follows from the simple fact that for unit vectors y_2, g $\langle y_2, g\rangle \geq -1$. The third inequality follows from the fact that $\alpha \leq k \leq (4r)^{-1}\sigma$. The forth inequality is the triangle inequality. The fifth inequality follows from the fact that y_2, f, g have norm 1. The sixth inequality follows from the fact that $\alpha \leq k \leq (4r)^{-1}$.

For the forth and first parts, we recall that $|x - (w + (f/2r))| \leq k$, $|w| \leq r$ and $|f| = 1$ and apply the triangle inequality.

$$|x - w| \geq |w + f/(2r) - w| - |x - (w + f/(2r))| \geq 1/(2r) - k \geq 1/(4r).$$

Here we used the hypothesis $k \leq 1/(4r)$.

$$|x - (1-\varepsilon)w| \leq |w + f/(2r) - w| + |\varepsilon w| + |x - (w + f/(2r))| \leq 1/(2r) + \varepsilon r + k \geq 7/(8r) \leq (1-\varepsilon)r.$$

Here we used the hypothesis $k \leq 1/(4r)$ and $\varepsilon \leq 1/(8r^2) \leq 1/8$. Finally, we can again write $x = w + f/(2r) + \alpha g$ with g a unit vector and $0 \leq \alpha \leq k$, so that we have

$$\langle f, x\rangle = \langle f, w\rangle + \langle f, f/(2r)\rangle + \langle f, \alpha g\rangle = 0 + 1/(2r) + \alpha\langle f, g\rangle \geq 0.$$

Here we used that $0 \leq \alpha \leq k \leq 1/(2r)$ and $\langle f, g\rangle \geq -1$. Hence, we find that $X \subset H^+$. □

3.4.4 Proof of Lemma 3.9

Proof of Lemma 3.9. Let x be the vertex of S' opposite to F. Let $v = xu \cap F$ be the intersection of the ray xu with the face F. Set $\lambda = |xu|/|xv| \leq 1$. Then it is easy to see that $(1-\lambda)x + \lambda B^n(v, 1/r) = B^n(u, \lambda/r) \subset B^n(u, 1/r)$.

Let $F, F_1, \dots, F_n$ be the faces of F and let $H, H_1, \dots, H_n$ be the supporting hyperplanes, respectively. For each $1 \leq i \leq n$ let H_i^+ and H_i^- be the two half spaces determined by H_i, such that H_i^+ contains S' and H_i^- is disjoint from the interior of S'. Then $S' = H^+ \cap_{i=1}^n H_i^+$.

For the first part, as $B^n(v, 1/r) \cap H^+ \subset H^+$, it is enough to show that for $1 \leq i \leq n$, we have $B^n(v, 1/r) \subset H_i^+$. Assume for the sake of contradiction that there exists $y \in B^n(v, 1/r) \cap H_i^{-\circ}$. As vertex x belongs to all faces except F, we have $x \in F_i \subset H_i \subset H_i^-$. Hence, as H_i^- is convex, we have $(1-\lambda)x + \lambda y \subset H_i^{-\circ}$. However, by the above discussion, we have $(1-\lambda)x + \lambda y \subset B^n(u, 1/r) \subset S' \subset H_i^+$. As H_i^+ and $H_i^{-\circ}$ are disjoint, this gives the desired contradiction. Thus, we conclude the first part.

For the second part, on the one hand we have $B^n(v, 1/r) \cap H = B^n(v, 1/r) \cap H^+ \cap H \subset S' \cap H = F$. On the other hand, $F \subset H$ by definition and $F \subset S' \subset B^n(u, r) \subset B^n(v, 2r)$ by hypothesis. For the last inclusion we just used the fact that $v \in F \subset S' \subset B^n(u, r)$. Thus, we conclude the second part. □

3.4.5 Proof of Lemma 3.10

Proof of Lemma 3.10. For a contradiction assume there is a sequence of simplices S^i and unit vectors v^i so that $\max_{1\le j\le n} |\langle f_j^i, v^i\rangle| \le \sigma_i$, where $\sigma_i \to 0$ as $i \to \infty$. By compactness there exists a converging subsequence so that $v^i \to v$ and $S^i \to S'$ (each of the vertices of S_i converging to the corresponding vertices of S'). S' has the property that $B(u, 1/r) \subset S' \subset \overline{B(u,r)}$ and letting f_i be the inward normal vectors to the faces of S', we have $\langle f_i, v\rangle = 0$ for all $1 \le i \le n$.

Consider the line $u+\mathbb{R}v$ through u. As $B(u, 1/r) \subset S'$, this line goes through the interior of S', so intersects the boundary $\partial S'$ exactly twice, in two distinct faces. In particular, this line intersects some face i with normal f_i with $1 \le i \le n$. However, as $\langle f_i, v\rangle = 0$ it follows that this line is contained inside face i. However, this line goes through the interior of S', contradiction. □

3.4.6 Proof of Lemma 3.11

Proof of Lemma 3.11. The statement is equivalent to the following statement. There exists $0 < \alpha_n < 1$ such that the following holds. If $0 < t \le 1/2$ and $\lambda_1, \dots, \lambda_n > 0$ and $\lambda_1 \dots \lambda_n = 1$, then

$$\alpha_n(\lambda_1 - 1)^2 \le t^{-1}\left(\prod_i (t + (1-t)\lambda_i) - 1\right) + t^{-2n}\left(\prod_i (t + (1-t)\lambda_i) - 1\right)^2.$$

It is easy to check that for fixed $\lambda_1 > 0$, and conditioned on $\lambda_1 \dots \lambda_n = 1$, the right hand side is minimised when $\lambda_2 = \dots = \lambda_n = \lambda_1^{\frac{1}{1-n}}$. This is because for $a, b > 0$ we have $(t+(1-t)a)(t+(1-t)b) \ge (t+(1-t)\sqrt{ab})^2$. Write $\lambda_1 = \lambda^{1-n}$ and $\lambda_2 = \dots = \lambda_n = \lambda$ for some $\lambda > 0$. Then the inequality becomes

$$\alpha_n(\lambda^{1-n} - 1)^2 \le t^{-1}\Big((t + (1-t)\lambda^{1-n})(t + (1-t)\lambda)^{n-1} - 1\Big) + t^{-2n}\Big((t + (1-t)\lambda^{1-n})(t + (1-t)\lambda)^{n-1} - 1\Big)^2$$

We first assume that $0 < \lambda \le 1$ and note that

$$\begin{aligned}(t+&(1-t)\lambda^{1-n})(t + (1-t)\lambda)^{n-1} - 1 \ge \lambda^{(1-t)(1-n)}(t + (1-t)\lambda)^{n-1} - 1 = (t\lambda^{t-1} + (1-t)\lambda^t)^{n-1} - 1 \\ &\ge t\lambda^{t-1} + (1-t)\lambda^t - 1 = t\exp(-\log(\lambda)(1-t)) + (1-t)\exp(t\log(\lambda)) - 1 \\ &\ge t(1 - \log(\lambda)(1-t) + \frac{1}{2}\log^2(\lambda)(1-t)^2) + (1-t)(1 + t\log(\lambda)) - 1 = \frac{t}{2}\log^2(\lambda)(1-t)^2 \ge \frac{t}{8}\log^2(\lambda).\end{aligned}$$

We now assume that $\lambda \ge 1$ and note that

$$\begin{aligned}(t+&(1-t)\lambda^{1-n})(t + (1-t)\lambda)^{n-1} - 1 \ge (t + (1-t)\lambda^{1-n})\lambda^{(1-t)(n-1)} - 1 = t\lambda^{(1-t)(n-1)} + (1-t)\lambda^{-t(n-1)} - 1 \\ &= t\exp(\log(\lambda)(1-t)(n-1)) + (1-t)\exp(-t\log(\lambda)(n-1)) - 1 \\ &\ge t\Big(1 + \log(\lambda)(1-t)(n-1) + \frac{1}{2}\log^2(\lambda)(1-t)^2(n-1)^2\Big) + (1-t)\Big(1 - t\log(\lambda)(n-1)\Big) - 1 \\ &= \frac{t}{2}\log^2(\lambda)(1-t)^2(n-1)^2 \ge \frac{t}{8}\log^2(\lambda).\end{aligned}$$

In both cases ($0 < \lambda \le 1$ and $\lambda > 1$), the first inequality follows from the AM-GM inequality: $px + (1-p)y \ge x^p y^{1-p}$ for $0 < p < 1$ and $0 < x, y$. Also, the penultimate inequality follows from the inequalities $\exp(x) \ge 1 + x + \frac{x^2}{2}$ and $\exp(-x) \ge 1 - x$, for $x \ge 0$. Combining the two cases, for $\lambda > 0$ we get that

$$(t + (1-t)\lambda^{1-n})(t + (1-t)\lambda)^{n-1} - 1 \ge \frac{t}{8}\log^2(\lambda).$$

Therefore, for $|\lambda - 1| \le 0.25$, using the simple inequality $|\log(\lambda)| \ge \frac{|\lambda-1|}{2}$, we deduce that

$$(t + (1-t)\lambda^{1-n})(t + (1-t)\lambda)^{n-1} - 1 \ge \frac{t}{2^5}|\lambda - 1|^2. \tag{3.27}$$

Moreover, for $|\lambda - 1| \ge 0.25$, using the inequality $|\log(\lambda)| \ge 1/8$, we deduce that $(t + (1-t)\lambda^{1-n})(t + (1-t)\lambda)^{n-1} - 1 \ge \frac{t}{2^9}$. The last inequality implies that for $t/2 < \lambda \le 0.75$, we have

$$t^{-2n}\Big((t + (1-t)\lambda^{1-n})(t + (1-t)\lambda)^{n-1} - 1\Big)^2 \ge \frac{t^{2-2n}}{2^{18}} \ge \frac{\lambda^{2-2n}}{2^{16+2n}} \ge \frac{(\lambda^{1-n} - 1)^2}{2^{16+2n}}. \tag{3.28}$$

It also implies that for $1.25 \le \lambda$, we have

$$t^{-2n}\Big((t+(1-t)\lambda^{1-n})(t+(1-t)\lambda)^{n-1}-1\Big)^2 \ge \frac{t^{2-2n}}{2^{18}} \ge \frac{1}{2^{18}} \ge \frac{(\lambda^{1-n}-1)^2}{2^{18}}. \tag{3.29}$$

For $0<\lambda\le t/2$, we have the simple bound

$$(t+(1-t)\lambda^{1-n})(t+(1-t)\lambda)^{n-1}-1 \ge (1-t)\lambda^{1-n}t^{n-1}-1 \ge 3^{-1}\lambda^{1-n}t^{n-1},$$

where the last inequality follows from $\lambda^{1-n}t^{n-1}\ge 2^{n-1}$ and $0<t\le 1/2$. Therefore, for $0<\lambda\le t/2$ we infer

$$t^{-2n}\Big((t+(1-t)\lambda^{1-n})(t+(1-t)\lambda)^{n-1}-1\Big)^2 \ge t^{-2n}3^{-2}\lambda^{2-2n}t^{2n-2} \ge 3^{-2}\lambda^{2-2n} \ge 3^{-2}(\lambda^{1-n}-1)^2. \tag{3.30}$$

where the last inequality follows from $\lambda\le 1$.

Combining (3.27), (3.28), (3.29) and (3.30), we conclude

$$\frac{(\lambda^{1-n}-1)^2}{2^{18+2n}} \le t^{-1}\Big((t+(1-t)\lambda^{1-n})(t+(1-t)\lambda)^{n-1}-1\Big) + t^{-2n}\Big((t+(1-t)\lambda^{1-n})(t+(1-t)\lambda)^{n-1}-1\Big)^2.$$

□

3.4.7 Proof of Lemma 3.12

Proof of Lemma 3.12. Choose maximal $1\ge\mu>0$ such that $(\ell^{-2}+1)^{-1}(1+\mu)\le(t\varepsilon/4)^{-1}(t(1+\mu)\varepsilon/4-\mu)$. By hypothesis, we have $K-x\subset A\subset C_A\subset(1+\mu)K-x$ and $K-y\subset B\subset C_B\subset(1+\mu)K-y$.

Therefore, it is enough to show that $t(K-x)+(1-t)(K-y)\supset t(1-\varepsilon/4)((1+\mu)K-x)+(1-t)((1+\mu)K-y)$. After rearranging, this is equivalent to $K-tx-(1-t)y\supset(1-t\varepsilon/4)(1+\mu)K-t(1-\varepsilon/4)x-(1-t)y$. After further rearranging, this is equivalent to $K\supset(1+\mu-t(1+\mu)\varepsilon/4)K+(t\varepsilon/4)\,x$. Therefore, it is enough to show

$$x\in(t\varepsilon/4)^{-1}(t(1+\mu)\varepsilon/4-\mu)K$$

By hypothesis, we know $K\subset(1+\mu)K$ which implies that $o\in K$ (assuming wlog K is compact). By hypothesis, we also know $K-x\subset B(o,\ell)$. Combining the last two inclusions, we get $-x\in B(o,\ell)$ i.e., $x\in B(o,\ell)$. Finally, by hypothesis we have $(1+\mu)K-x\supset B(o,\ell^{-1})$.

Combining the last two inclusions and rearranging, we get

$$x\in(\ell^{-2}+1)^{-1}(1+\mu)K.$$

By the choice of parameters, we have $(\ell^{-2}+1)^{-1}(1+\mu)\le(t\varepsilon/4)^{-1}(t(1+\mu)\varepsilon/4-\mu)$, from which the conclusion follows. □

3.4.8 Proof of Lemma 3.13

Proof of Lemma 3.13. We first observe that, by Caffarelli's regularity theory [Caf92b, Caf92a], the function φ is a strictly convex Alexandrov solution of $\det D^2\varphi=1$. Also, thanks to (3.1), the modulus of strict convexity depends only on R and the dimension. Hence, we can apply the interior regularity theory for Alexandrov solutions (see for instance [Fig17, Theorem 4.42]) to deduce that, for every $\theta,\alpha\in(0,1)$, $D^2\varphi$ is uniformly α-Hölder continuous inside $(1-\theta)C_A$. More precisely, there exists a constant $\hat{C}_{n,R,\theta,\alpha}>0$ such that

$$\|D^2\varphi\|_{C^{0,\alpha}((1-\theta)C_A)} := \|D^2\varphi\|_{L^\infty((1-\theta)C_A)} + \sup_{x,y\in(1-\theta)C_A}\frac{|D^2\varphi(x)-D^2\varphi(y)|}{|x-y|^\alpha} \le \hat{C}_{n,R,\theta,\alpha} \tag{3.31}$$

(here the choice of the norm for $D^2\varphi(x)$ is irrelevant, since all norms are equivalent up to dimensional constants).

Now, given any affine function $\ell(x):=b\cdot x+c$ ($b\in\mathbb{R}^n$, $c\in\mathbb{R}$), consider the second-order polynomial $p_\ell(x):=\frac{|x|^2}{2}+\ell(x)$. Since $\det D^2p_\ell=1$, applying [Fig17, Lemma A.1] we write

$$\begin{aligned} 0=\det D^2\varphi-\det D^2p_\ell &= \int_0^1\frac{d}{dt}\det\big(tD^2\varphi+(1-t)D^2p_\ell\big)\,dt \\ &= \sum_{i,j=1}^n\Big(\int_0^1\operatorname{cof}\big(tD^2\varphi+(1-t)\mathrm{Id}\big)\,dt\Big)_{ij}\partial_{ij}(\varphi-p_\ell), \end{aligned}$$

where, given a symmetric matrix A, $\mathrm{cof}(A)$ denotes its cofactor matrix. In other words, if we define the functions

$$a_{ij}(x) := \left(\int_0^1 \mathrm{cof}\big(tD^2\varphi(x) + (1-t)\mathrm{Id}\big)\,dt\right)_{ij} \qquad i,j \in \{1,\dots,n\}$$

and $\psi_\ell := \varphi - p_\ell$, then ψ_ℓ solves the equation

$$\sum_{i,j=1}^n a_{ij}\partial_{ij}\psi_\ell = 0.$$

Note that, thanks to (3.31), the matrices $(a_{ij}(x))_{i,j=1}^n$ are uniformly positive definite and Hölder continuous. Hence, recalling (3.1), it follows from classical elliptic regularity (see for instance [GT98, Corollary 6.3 and Theorem 9.20]) and a covering argument that

$$\|D^2\psi_\ell\|_{L^\infty((1-2\theta)C_A)} \le C'_{n,R,\theta}\|\psi_\ell\|_{L^1((1-\theta)C_A)}, \tag{3.32}$$

where $C'_{n,R,\theta}$ depends on n, R, and θ only.

Now, set $\psi(x) := \varphi(x) - \frac{|x|^2}{2}$ and fix $\bar\ell(x) = \bar b\cdot x + \bar c$ with

$$\bar b := \frac{1}{|(1-\theta)C_A|}\int_{(1-\theta)C_A}\nabla\psi(x)\,dx, \qquad \bar c := \frac{1}{|(1-\theta)C_A|}\int_{(1-\theta)C_A}(\psi(x) - \bar b\cdot x)\,dx.$$

Then, by applying twice the 1-Poincaré inequality (see [GT98, Equation (7.45)] with $p=1$) and recalling (3.1), we have

$$\|\psi_{\bar\ell}\|_{L^1((1-\theta)C_A)} \le 2^nR^{2n-1}\|\nabla\psi_{\bar\ell}\|_{L^1((1-\theta)C_A)} \le 4^nR^{4n-2}\|D^2\psi_{\bar\ell}\|_{L^1((1-\theta)C_A)}. \tag{3.33}$$

Noticing that $D^2\psi_{\bar\ell} = D^2\psi$, combining (3.32) (with $\ell = \bar\ell$) and (3.33) we conclude that

$$\|D^2\psi\|_{L^\infty((1-2\theta)C_A)} \le 4^nR^{4n-2}C'_{n,R,\theta}\|D^2\psi\|_{L^1((1-\theta)C_A)}.$$

Choosing $\theta = \varepsilon/2$, this proves the desired estimate with $C_{n,R,\varepsilon} = 4^nR^{4n-2}C'_{n,R,\varepsilon/2}$. □

4 Putting it all together: Proof of Theorem 1.2

Proof of Theorem 1.2. Consider any n,t,ℓ, and λ. Choose $g_{n,t} = d_{n,t} := t^{2n-1}/4$. Choose $\theta = \theta^{3.4}_{n,\ell/2}$ as given by Proposition 3.4. Choose $\varepsilon = \frac{1}{2}$. Choose $\mu := \mu^{3.1}_{n,t,\varepsilon,\ell}$ as given by Proposition 3.1. Choose $c_{n,\ell,\lambda} := \frac{c^{3.5}_{n,\ell}c^{3.2}_{n,\varepsilon,\lambda,\ell,\theta}}{c^{3.4}_{n,\ell}}\theta^{-n} + 2$, where $c^{3.5}_{n,\ell}, c^{3.2}_{n,\varepsilon,\lambda,\ell,\theta}$,and $c^{3.4}_{n,\ell}$ are the constants from Proposition 3.5, Corollary 3.2 and Proposition 3.4 respectively.

By Proposition 2.8, we may assume that A, B are $(\gamma,\ell,\lambda,\mu)$ conelike with μ sufficiently small in terms of n,t,ℓ and λ.

By Corollary 3.2, we find that for any affine transformation $Q:\mathbb{R}^n\to\mathbb{R}^n$, if $||Q||_{op}, ||Q^{-1}||_{op} \le \theta$ and if $T_Q: Q(C_A)\to Q(C_B)$ is the optimal transport map from $Q(C_A)$ to $Q(C_B)$, then

$$\int_{x\in\partial C_A}\max\left\{\left\langle Q(x) - T_Q(Q(x)), \frac{Q(x)-Q(o')}{||Q(x)-Q(o')||_2}\right\rangle, 0\right\}dx \le c^{3.2}_{n,\varepsilon,\lambda,\ell,\theta}\sqrt{\frac{\delta+\gamma}{t}}|Q(A)|,$$

for all $o' \in (1-\varepsilon)C_A$.

Let $Q\sim\mathcal{Q}_\theta$ be random scaling, choose o' uniformly random from $B(o,1/2\ell)$. Note that $T_Q(Q(C_A))\subset Q(C_B)$ and $B(o,1/2\ell)\subset\frac{1}{2}C_A, \frac{1}{2}C_B\subset B(o,2\ell)$. Hence, by Proposition 3.4, we have

$$\mathbb{E}_{Q,o'}\left[\max\left\{\left\langle Q(x) - T_Q(Q(x)), \frac{Q(x)-Q(o')}{||Q(x)-Q(o')||_2}\right\rangle, 0\right\}\right] \ge c^{3.4}_{n,2\ell}d(x,C_B).$$

Since we have $||Q||_{op}, ||Q^{-1}||_{op}\le\theta$ for every random scaling $Q\sim\mathcal{Q}_\theta$, and $B(o,1/2\ell)\subset(1-\varepsilon)C_A, (1-\varepsilon)C_B$, we can combine these two to find:

$$\begin{aligned}
c^{3.4}_{n,2\ell}\int_{x\in\partial C_A}d(x,C_B)dx &\le \int_{x\in\partial C_A}\mathbb{E}_{Q,o'}\left[\max\left\{\left\langle Q(x)-T_Q(Q(x)), \frac{Q(x)-Q(o')}{||Q(x)-Q(o')||_2}\right\rangle, 0\right\}\right]dx \\
&= \mathbb{E}_{Q,o'}\left[\int_{x\in\partial C_A}\max\left\{\left\langle Q(x)-T_Q(Q(x)), \frac{Q(x)-Q(o')}{||Q(x)-Q(o')||_2}\right\rangle, 0\right\}dx\right] \\
&\le \mathbb{E}_{Q,o'}\left[c^{3.2}_{n,\varepsilon,\lambda,\ell,\theta}\sqrt{\frac{\delta+\gamma}{t}}|Q(A)|\right] \le c^{3.2}_{n,\varepsilon,\lambda,\ell,\theta}\sqrt{\frac{\delta+\gamma}{t}}\mathbb{E}_Q\left[|Q(A)|\right] \le c^{3.2}_{n,\varepsilon,\lambda,\ell,\theta}\sqrt{\frac{\delta+\gamma}{t}}\theta^n|A|,
\end{aligned}$$

where the final inequality follows as every $Q \sim \mathcal{Q}_\theta$ has determinant at most θ^n. Applying Proposition 3.5 we find

$$|C_A \triangle C_B| \leq c^{3.5}_{n,\ell} \int_{x \in \partial C_A} d(x, C_B) dx \leq \frac{c^{3.5}_{n,\ell} c^{3.2}_{n,\varepsilon,\lambda,\ell,\theta}}{c^{3.4}_{n,2\ell}} \sqrt{\frac{\delta + \gamma}{t}} \theta^n |A|.$$

We conclude recalling the definition of C_A, C_B:

$$|A \triangle B| \leq |C_A \triangle C_B| + |C_A \setminus A| + |C_B \setminus B| \leq \frac{c^{3.5}_{n,\ell} c^{3.2}_{n,\varepsilon,\lambda,\ell,\theta}}{c^{3.4}_{n,2\ell}} \sqrt{\frac{\delta + \gamma}{t}} \theta^n |A| + 2\gamma |A| \leq c_{n,\ell,\lambda} \sqrt{\frac{\delta + \gamma}{t}} |A|.$$

This concludes the proof of the theorem. □

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