STABILITY RESULTS ON THE SMOOTHNESS OF OPTIMAL TRANSPORT MAPS WITH GENERAL COSTS

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Abstract. We prove some stability results concerning the smoothness of optimal transport maps with general cost functions. In particular, we show that the smoothness of optimal transport maps is an open condition with respect to the cost function and the densities. As a consequence, we obtain regularity for a large class of transport problems where the cost does not necessarily satisfy the MTW condition.

1. Introduction

Given a source domain $X \subset \mathbb{R}^n$ associated with density $f : X \to \mathbb{R}^+$, a target domain $Y \subset \mathbb{R}^n$ associated with density $g : Y \to \mathbb{R}^+$, and a cost function $c : X \times Y \to \mathbb{R}$, the optimal transport problem consists in finding, among all transport maps (i.e., all maps $T : X \to Y$ such that $T_\sharp f = g$), a transport map which minimizes the total transportation cost

$$\int_X c(x, T(x))\, f(x)\, dx.$$

It is by now well-known that, under some rather general assumptions on the cost c, there exists a unique transport map T (see Section 2 for more details). Then, a very natural and important question becomes the following:

If the data f, g, X, Y, c are smooth, is T smooth as well?

While this question is well understood when c is the squared distance function in $\mathbb{R}^n$ (see for instance [15, Chapter 4.5]), the regularity of optimal transport maps with general cost has been for long time a fundamental open problem in the theory of optimal transportation. In 2005, Ma, Trudinger, and Wang [33] found the following fourth order condition (now called MTW condition after their names) on the cost function:

$$\sum_{i,j,k,l,p,q,r,s} c^{p,q}(c_{ij,p}c_{q,rs} - c_{ij,rs})c^{r,k}c^{s,l}\xi_i\xi_j\eta_k\eta_l \geq 0 \qquad \text{in } X \times Y \tag{1.1}$$

for all $\xi, \eta \in \mathbb{R}^n$ satisfying $\xi \perp \eta$, where lower indices before (resp. after) the comma indicate derivatives with respect to x (resp. y) (so for instance $c_{i,j} = \frac{\partial^2 c}{\partial x_i \partial y_j}$), $(c^{i,j})$ is the inverse of $(c_{i,j})$, and all derivatives are evaluated at $(x, y) \in X \times Y$. Under the above condition, they proved in [33] that if the densities are positive and smooth and the domains satisfy some suitable convexity assumptions, then the optimal map is smooth (see also [35, 36]). Later Loeper [30] showed that the MTW condition is actually *necessary* for the smoothness of optimal transport maps. More precisely, if the cost function does not satisfy the MTW

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condition, Loeper constructed two smooth positive densities, supported on smooth domains satisfying the "right" convexity assumptions, for which the optimal map was not even continuous. After these two important works, many experts have contributed to develop a complete regularity theory of optimal transport problem under the MTW condition, to cite a few see [19, 28, 35, 36, 20, 31, 32, 29, 21, 27, 23, 22, 17, 18].

Unfortunately, several interesting costs do not satisfy the MTW condition, for instance $c(x,y)=\frac{1}{p}|x-y|^p$ does not satisfy MTW condition when $p\in(1,2)\cup(2,\infty)$, and actually the class of costs satisfying the MTW condition is very restricted. Recently, De Philippis and Figalli [11] obtained a partial regularity result for optimal transport problem with general cost without assuming neither the MTW condition nor any convexity on the domains. They managed to show that optimal maps are always smooth outside a closed set of measure zero. In a related direction, Caffarelli, Gonzáles, and Nguyen [7] obtained an interior $C^{2,\alpha}$ regularity result of optimal transport problem when the densities are C^α and the cost function is of the form $c(x,y)=\frac{1}{p}|x-y|^p$ with $2<p<2+\epsilon$ for some $\epsilon\ll 1$. This interior regularity result was later extended by us to a global one, and generalized to a larger class of cost functions [9].

Motivated by the recent results and techniques developed in [11, 9], in this paper we show the following stability statement: consider the optimal transport problem from (X,f) to (Y,g) with cost c, and suppose that the optimal maps T_u and $T^*_{u^c}$ sending respectively f to g and g to f have some (suitable) degree of smoothness. Then, if we perturb the problem sightly, regularity persists. More precisely, consider the optimal transport problem from $(X,\tilde f)$ to $(Y,\tilde g)$ with cost $\tilde c$, and assume that $\tilde f$ and $\tilde g$ are close to f and g in C^0 norm respectively, and $\tilde c$ is close to c in C^2 norm. Then the corresponding optimal transport maps $T_{\tilde u}$ and $T^*_{\tilde u^c}$ enjoy the same smoothness as T_u and $T^*_{u^c}$. This is particularly interesting since, even if c satisfies the MTW condition, $\tilde c$ may not satisfy it.

The paper is organized as follows. In section 2 we introduce some notation and state our main results. Then, in Section 3, we collect all the new ingredients that we need to apply the arguments in [11, 9], and finally in the last section we prove our main results.

2. Preliminaries and main results

We begin by introducing some conditions that should be satisfied by the cost. Here and in the following, X and Y are two bounded open subsets of $\mathbb{R}^n$.

(**C0**) The cost function c is of class C^3 with $\|c\|_{C^3(X\times Y)}<\infty$.
(**C1**) For any $x\in X$, the map $Y\ni y\mapsto D_xc(x,y)\in\mathbb{R}^n$ is injective.
(**C2**) For any $y\in Y$, the map $X\ni x\mapsto D_yc(x,y)\in\mathbb{R}^n$ is injective.
(**C3**) $\det(D_{xy}c)(x,y)\neq 0$ for all $(x,y)\in X\times Y$.

A function $u:X\to\mathbb{R}$ is said *c-convex* if it can be written as

$$u(x)=\sup_{y\in Y}\{-c(x,y)+\lambda_y\} \tag{2.1}$$

for some family of constants $\{\lambda_y\}_{y\in Y}\subset\mathbb{R}$. Note that (**C0**) and (2.1) imply that a c-convex function is semiconvex, namely, there exists some constant K depending only on $\|c\|_{C^2(X\times Y)}$ such that $u+K|x|^2$ is convex. One immediate consequence of the semiconvexity is that u is twice differentiable almost everywhere.

It is well known (see for instance [38, Chapter 10]) that (**C0**) and (**C1**) ensure that there exists a unique optimal transport map, and there exists a c-convex function u (also called *potential function*) such that the optimal map is a.e. uniquely characterized in terms of u (and for this reason we denote it by T_u) by the relation

$$-D_x c(x, T_u(x)) = \nabla u(x) \qquad \text{for a.e. } x. \tag{2.2}$$

As explained for instance in [11, Section 2] (see also [12]), the transport condition $(T_u)_\# f = g$ implies that u solves at almost every point the Monge-Ampère type equation

$$\det\Big(D^2u(x)+D_{xx}c\big(x,\text{c-exp}_x(\nabla u(x))\big)\Big) = \big|\det\big(D_{xy}c\big(x,\text{c-exp}_x(\nabla u(x))\big)\big)\big|\,\frac{f(x)}{g(\text{c-exp}_x(\nabla u(x)))}, \tag{2.3}$$

where c-exp denotes the *c-exponential map* defined as

$$\text{for any } x \in X,\ y \in Y,\ p \in \mathbb{R}^n, \qquad \text{c-exp}_x(p) = y \quad \Leftrightarrow \quad p = -D_x c(x, y). \tag{2.4}$$

Notice that, with this notation, $T_u(x) = \text{c-exp}_x(\nabla u(x))$.

For a c-convex function, analogous to the subdifferential for convex function, we can talk about its c-subdifferential: If $u : X \to \mathbb{R}$ is a c-convex function as above, the *c-subdifferential* of u at x is the (nonempty) set

$$\partial_c u(x) := \big\{ y \in \overline{Y} : u(z) \geq -c(z, y) + c(x, y) + u(x) \qquad \forall\, z \in X \big\}.$$

We also define *Frechet subdifferential* of u at x as

$$\partial^- u(x) := \{ p \in \mathbb{R}^n : u(z) \geq u(x) + p \cdot (z - x) + o(|z - x|) \}.$$

It is easy to check that

$$y \in \partial_c u(x) \quad \Longrightarrow \quad -D_x c(x, y) \in \partial^- u(x).$$

Also, it is well known that the optimal transport T_u satisfies

$$T_u(x) \in \partial_c u(x) \qquad \text{for a.e. } x \tag{2.5}$$

Let $c^*(y, x) := c(x, y)$. By exchanging the role of X and Y, we can also study the optimal transport problem from (Y, g) to (X, f). In this case, the corresponding optimal transport map can be obtained from a c^*-convex function φ. It is well known that

$$\varphi(y) = u^c(y) := \sup_{x \in X} \{ -c(x, y) - u(x) \}, \tag{2.6}$$

hence it is natural to denote by $T^*_{u^c} = \text{c}^*\text{-exp}(\nabla u^c)$ the optimal transport from g to f. Also, as a consequence of the identity

$$u = (u^c)^{c^*}, \tag{2.7}$$

one can show that $T^*_{u^c} = (T_u)^{-1}$, that is, $T_u : X \to Y$ and $T^*_{u^c} : Y \to X$ are inverse to each other (see for instance [38]).

Below we will call a constant *universal* if it depends only on the dimension n and the upper bound of $\log f$ and $\log g$. In the following we denote $X^\sigma := \{x \in X : \text{dist}(x, \partial X) > \sigma\}$. We shall state all the results for the potential $\tilde{u}$ below, but of course the same results hold also for $\tilde{u}^c$.

Theorem 2.1. *Let u be the potential function for the optimal transport problem from (X, f) to (Y, g) with cost c satisfying (**C0**)-(**C3**). Suppose f, g are positive continuous densities, and that $u \in C^{1,\alpha}(\bar{X})$ and $u^c \in C^{1,\alpha}(\bar{Y})$ for any $\alpha \in (0,1)$. Then, for any $\sigma > 0$, there exists $\bar{\delta} > 0$ such that such that the following holds: assume that $\tilde{c}$ satisfies (**C0**)-(**C3**), $\tilde{f} : X \to \mathbb{R}$ and $\tilde{g} : Y \to \mathbb{R}$ are positive continuous densities, and*

$$\|\tilde{c} - c\|_{C^2} \le \bar{\delta}, \quad \|f - \tilde{f}\|_{C^0} + \|g - \tilde{g}\|_{C^0} \le \bar{\delta}.$$

Then the potential function $\tilde{u}$ for the optimal transport problem from $(X, \tilde{f})$ to $(Y, \tilde{g})$ with cost $\tilde{c}$ belongs to $C^{1,\beta}(X^\sigma)$ for any $\beta \in (0,1)$.

This result has the following interesting consequence on Riemannian manifolds.

Corollary 2.2. *Let $(M, \mathcal{G})$ be a smooth closed non-focal Riemannian manifold, and denote by d the Riemannian distance induced by $\mathcal{G}$. Let f, g be two positive continuous densities, and let T (resp. T^*) be the optimal transport map for the cost $c = \frac{d^2}{2}$ sending f (resp. g) to g (resp. f). Suppose T and T^* belong to C^α for any $\alpha \in (0,1)$. Then, there exists $\bar{\delta} > 0$ such that if*

$$\|\tilde{\mathcal{G}} - \mathcal{G}\|_{C^2} \le \bar{\delta}, \quad \|f - \tilde{f}\|_{C^0} + \|g - \tilde{g}\|_{C^0} \le \bar{\delta}, \quad \tilde{f}, \tilde{g} \text{ are positive and continuous},$$

then the optimal map $\tilde{T}$ (resp. $\tilde{T}^*$) for the cost $c = \frac{\tilde{d}^2}{2}$ (induced by the metric $\tilde{\mathcal{G}}$ on M) sending $\tilde{f}$ (resp. $\tilde{g}$) to $\tilde{g}$ (resp. $\tilde{f}$) is of class C^β on M for any $\beta \in (0,1)$.

The proof of Theorem 2.1 also gives the following interesting result, which implies that if the potentials are $C^{1,\beta}$ with β sufficiently close to 1 (say $\beta > \frac{4}{5}$), then higher regularity follows provided the densities are smooth enough, even without the MTW condition.

Theorem 2.3. *Let u be the potential function for the optimal transport problem from (X, f) to (Y, g) with cost c. Suppose $u \in C^{1,\beta}(\bar{X})$ and $u^c \in C^{1,\beta}(\bar{Y})$, where $\beta > \frac{4}{5}$. Then*

i) if f and g are positive continuous densities, then $u \in C^{1,\gamma}(X)$ and $u^c \in C^{1,\gamma}(Y)$ for any $\gamma \in (0,1)$.

ii) if $f \in C^\alpha(X)$ and $g \in C^\alpha(Y)$ are positive Hölder continuous densities for some $\alpha \in (0,1)$, then $u \in C^{2,\alpha}(X)$ and $u^c \in C^{2,\alpha}(Y)$.

Remark 2.4. In [11], it was proved that for the optimal transport problem from (X, f) to (Y, g) with cost c satisfying (**C0**)-(**C3**), for any given $\beta \in (0,1)$ there exists a closed set Σ_X of measure 0 such that u is of class $C^{1,\beta}$ in $X \setminus \Sigma_X$. However, in that paper, the set Σ_X may a priori depend on β (as can be checked by inspecting the proof there). Our Theorem 2.3 can be used to show that actually there exists a "universal" singular set Σ_X, namely, we can choose Σ_X so that u is $C^{1,\beta}$ in $X \setminus \Sigma_X$ for any $\beta \in (0,1)$.

To show this, one argues as follows: suppose x_0 and $y_0 = T_u(x_0)$ are points where u and u^c are respectively twice differentiable there. Then it follows from [11, Theorem 4.3] (see also the proof of [11, Theorem 1.3]) that there exists a small positive radius ρ such that u (resp. u^c) is of class $C^{1,5/6}$ in $B_\rho(x_0)$ (resp. $T_u(B_\rho)$). Then, by Theorem 2.3 we have that u is $C^{1,\gamma}$ in $B_\rho(x_0)$ for any $\gamma \in (0,1)$, and one concludes by the very same argument as in the proof of [11, Theorem 1.3].

We can also prove that the previous results generalize to higher regularity.

Theorem 2.5. *Let u be the potential function for the optimal transport problem from (X, f) to (Y, g) with cost c satisfying (**C0**)-(**C3**). Suppose that $f \in C^\alpha(X)$ and $g \in C^\alpha(Y)$ are positive Hölder continuous densities, and that $u \in C^{1,1}(X)$. Then for any $\sigma > 0$, there exists $\bar\delta > 0$ such that the following holds: assume that $\tilde c$ satisfies (**C0**)-(**C3**), $\tilde f : X \to \mathbb{R}$ and $\tilde g : Y \to \mathbb{R}$ are positive densities of class C^α, and*

$$\|\tilde c - c\|_{C^2} \le \bar\delta, \quad \|f - \tilde f\|_{C^0} + \|g - \tilde g\|_{C^0} \le \bar\delta.$$

Then the potential function $\tilde u$ for the optimal transport problem from $(X, \tilde f)$ to $(Y, \tilde g)$ with cost $\tilde c$ belongs to $C^{2,\alpha}(X^\sigma)$.

Corollary 2.6. *Let $(M, \mathcal{G})$ be a smooth closed Riemannian manifold, and denote by d the Riemannian distance induced by $\mathcal{G}$. Let $f \in C^\alpha(M)$ and $g \in C^\alpha(M)$ be two positive Hölder continuous densities, and let T be the optimal transport map for the cost $c = \frac{d^2}{2}$ sending f to g. Suppose T is Lipschitz continuous. Then there exists $\bar\delta > 0$ such that if*

$$\|\tilde{\mathcal{G}} - \mathcal{G}\|_{C^2} \le \bar\delta, \quad \|f - \tilde f\|_{C^0} + \|g - \tilde g\|_{C^0} \le \bar\delta, \ \tilde f, \tilde g \text{ are positive and of class } C^\alpha$$

then the optimal map $\tilde T$ for the cost $c = \frac{\tilde d^2}{2}$ sending $\tilde f$ to $\tilde g$ is of class $C^{1,\alpha}$ on M.

Finally, if we assume that the boundaries of X and Y are smooth, then we can also obtain regularity up to the boundary.

Theorem 2.7. *Let u be the potential function for the optimal transport problem from (X, f) to (Y, g) with cost c. Assume that $f \in C^\alpha(\bar X)$ and $g \in C^\alpha(\bar Y)$ are positive Hölder continuous densities, and that ∂X and ∂Y are of class $C^{2,\alpha}$. Suppose $u \in C^{2,\alpha}(\bar X)$ and $u^c \in C^{2,\alpha}(\bar Y)$. Then there exists $\bar\delta > 0$ such that if*

$$\|\tilde c - c\|_{C^2} \le \bar\delta, \quad \|f - \tilde f\|_{C^0} + \|g - \tilde g\|_{C^0} \le \bar\delta, \ \tilde f, \tilde g \text{ are positive and } C^\alpha \tag{2.8}$$

then the potential function $\tilde u$ for the optimal transport problem from $(X, \tilde f)$ to $(Y, \tilde g)$ with cost $\tilde c$ belongs to $C^{2,\alpha'}(\bar X)$ for some $0 < \alpha' < \alpha$.

3. Localizing the problem

In the following, $C, \{C_i\}_{i=1,\dots,5}$ will always denote positive constants, which may change from line to line, but without specifically mentioned they depend only on the local $C^{1,\alpha}$ norms of u and u^c, and the C^2 norm of c. Although we could have decided to use only C everywhere, we believe that the introduction of the additional constants $C_1, \dots, C_5$ should help the reader.

From now on we set $\delta := \|\tilde c - c\|_{C^2}$.

Given $x_0 \in X^\sigma$ and $y_0 := T_u(x_0)$, as in [11] we perform the transform

$$c_1(x, y) := c(x, y) - c(x, y_0) - c(x_0, y) + c(x_0, y_0), \tag{3.1}$$

$$\tilde c_1(x, y) := \tilde c(x, y) - \tilde c(x, y_0) - \tilde c(x_0, y) + \tilde c(x_0, y_0), \tag{3.2}$$

$$u_1 := u(x) - u(x_0) + c(x, y_0) - c(x_0, y_0), \tag{3.3}$$

$$\tilde u_1 := \tilde u(x) - \tilde u(x_0) + \tilde c(x, y_0) - \tilde c(x_0, y_0). \tag{3.4}$$

It is easy to see that

$$\|\tilde c_1 - c_1\|_{C^2} \le 4\delta. \tag{3.5}$$

Up to a change of coordinates we can assume that $x_0 = y_0 = 0$ and that $D_{xy}c(0,0) = -\mathrm{Id}$, so that a Taylor expansion of c_1 around $(x_0, y_0) = (0,0)$ yields

$$c_1(x,y) = -x\cdot y + O(|x|^2|y| + |y|^2|x|). \tag{3.6}$$

In the next result, we study the geometry of the sub-level sets $\{u_1 < h\}$ when h is small enough.

Lemma 3.1. *Under the assumptions of Theorem 2.1, for h sufficiently small we have that*

$$\frac{1}{C}B_{h^{\frac{1}{2}+\epsilon}} \subset \{u_1 < h\} \subset CB_{h^{\frac{1}{2}-\epsilon}}, \tag{3.7}$$

where $\epsilon := \frac{1}{1+\alpha} - \frac{1}{2} \to 0$ *as* $\alpha \to 1$.

Proof. The first inclusion follows from the $C^{1,\alpha}$ regularity of u_1.

To prove the second inclusion we fix r small. We first notice that, as a consequence of (2.5) and (2.7), the supremum in the relation

$$u_1(x) = \sup_y -u_1^{c_1}(y) - c_1(x,y)$$

is attained for $y = T_{u_1}(x)$. Now, for any fixed $x \in B_r$, let $r_1 := |x|^{\frac{1}{\alpha}-\frac{1}{2}}$. Since u_1 is $C^{1,\alpha}$, we have $|T_{u_1}(x)| \le C|x|^\alpha \le |x|^{\frac{1}{\alpha}-\frac{1}{2}}$ for $x \in B_r$ and $\alpha > 1/2$. Hence, recalling that also $u_1^{c_1} \in C^{1,\alpha}$, for $x \in B_r$ we get

$$\begin{aligned}
u_1(x) &= \sup_{y\in B_{r_1}} -u_1^{c_1}(y) - c_1(x,y)\\
&= \sup_{y\in B_{r_1}} -u_1^{c_1}(y) + x\cdot y + O(|x|^2|y| + |y|^2|x|)\\
&\ge \sup_{y\in B_{r_1}} -C|y|^{1+\alpha} + x\cdot y + O(|x|^2|y| + |y|^2|x|)\\
&\ge \sup_{y\in B_{r_1}} -(C+1)|y|^{1+\alpha} + x\cdot y - C|x|^2\\
&\ge C_1|x|^{1+\frac{1}{\alpha}},
\end{aligned}$$

where we used that the standard Legendre transform of $|y|^{1+\alpha}$ is $|x|^{1+\frac{1}{\alpha}}$. Thanks to the above estimate, also the second inclusion follows. □

Now, we exploit the "almost convex" property implied by the Taylor expansion of c_1 (see (3.6)). More precisely, although in our situation the set $\{u_1 \le h\}$ is not convex in general, we are able to show that this set is so close to its convex envelope that we can still use an affine transformation to normalize it. In the following we will use $[E]$ to denote the convex envelope of the set E, and we use $\mathrm{dist}(A,B)$ to denote the Hausdorff distance between two sets A and B.

Lemma 3.2. *Let ϵ be as in Lemma 3.1. Then, for h small enough,*

$$\mathrm{dist}(\{u_1 \le h\}, [\{u_1 \le h\}]) \le Ch^{1-6\epsilon}.$$

Proof. Denote $S := \{u_1 \le h\}$ and notice that, by the argument above, we have

$$S \subset \bigcap_{x\in\partial S} \{z \in B_{r_1} : -c_1(z, T_{u_1}(x)) + c_1(x, T_{u_1}(x)) \le 0\}, \tag{3.8}$$

where $r_1 = Ch^{\frac{1}{2}-\epsilon}$.

Now, given any point $\bar{x} \in [\{u_1 \le h\}] - \{u_1 \le h\}$, we can write $\bar{x} = x + \xi$, where $x \in \partial\{u_1 \le h\}$ and ξ is perpendicular to $\partial\{u_1 \le h\}$. Since the function

$$z \mapsto -c_1(z, T_{u_1}(x)) - u_1^{c_1}(T_{u_1}(x))$$

touches u_1 from below at the point x (this follows from (2.5)), we deduce that ξ is parallel to the the vector $-D_x c_1(x,y)$, where $y := T_{u_1}(x)$. Moreover, by (3.6) we have that $-D_x c_1(x,y)$ is approximately equal to y, in particular

$$\xi \cdot y \ge \frac{1}{2}|\xi|\,|y|. \tag{3.9}$$

Note also that, by the $C^{1,\alpha}$ regularity of u_1, we have $|y| \le C|x|^\alpha \le Ch^{(\frac{1}{2}-\epsilon)\alpha}$.

Now, consider the linear function

$$L(z) := -z \cdot y + (x+\xi)\cdot y.$$

We shall show that if $|\xi|$ is not small enough then $L(z) < 0$ for any $z \in S$, which will contradict the fact that $x + \xi = \bar{x} \in [S]$. In fact, for any $z \in S$, by (3.6) and (3.9) we have

$$\begin{aligned}
&L(z) + (c_1(z,y) - c_1(x,y)) \\
&= -z\cdot y + (x+\xi)\cdot y - \big(-z\cdot y + O(|z|^2|y| + |y|^2|z|) + x\cdot y + O(|x|^2|y| + |y|^2|x|)\big) \\
&\le -\xi\cdot y + C(|z|^2 + |y||z| + |x|^2 + |x||y|)|y| \\
&\le \left(-\frac{1}{2}|\xi| + C(|z|^2 + |y||z| + |x|^2 + |x||y|)\right)|y| \\
&\le \left(-\frac{1}{2}|\xi| + C(h^{1-2\epsilon} + h^{(1-2\epsilon)\alpha})\right)|y| \\
&\le \left(-\frac{1}{2}|\xi| + C_1 h^{1-6\epsilon}\right)|y|,
\end{aligned}$$

where we used that $(1-2\epsilon)\alpha \ge 1 - 6\epsilon$ for all $\alpha \in (0,1)$. Hence

$$L(z) < c_1(x,y) - c_1(z,y) \le -u_1(x) + u_1(z) \le 0 \qquad \forall\, z \in S$$

when $|\xi| > \frac{C_1}{2}h^{1-6\epsilon}$, a contradiction.

This proves that

$$|\xi| \le \frac{C_1}{2}h^{1-6\epsilon},$$

as desired. □

Let $A : \mathbb{R}^n \to \mathbb{R}^n$ be a unitary linear transformation normalizing the convex set $[S]$, that is

$$\det A = 1, \qquad B_r(x_0) \subset A[S] \subset B_{nr}(x_0)$$

for some $r > 0$ and $x_0 \in \mathbb{R}^n$ (the existence of this map follows from John's Lemma [26]). Now, we perform the family of transformations:

$$X \to X_2 := \frac{1}{\sqrt{t}}AX;$$

$$Y \to Y_2 := \frac{1}{\sqrt{t}}A'^{-1}Y;$$

$$u_1(x) \to \frac{1}{t}u_1(\sqrt{t}A^{-1}x) := u_2(x);$$

$$\tilde{u}_1(x) \to \frac{1}{t}\tilde{u}_1(\sqrt{t}A^{-1}x) := \tilde{u}_2(x);$$

$$c_1(x,y) \to \frac{1}{t}c_1(\sqrt{t}A^{-1}x, \sqrt{t}A'y) := c_2(x,y);$$

$$\tilde{c}_1(x,y) \to \frac{1}{t}\tilde{c}_1(\sqrt{t}A^{-1}x, \sqrt{t}A'y) := \tilde{c}_2(x,y);$$

$$f(x),\ g(y) \to f_2(x) := f(\sqrt{t}A^{-1}x),\ g_2(y) := g(\sqrt{t}A'y);$$

$$\tilde{f}(x),\ \tilde{g}(y) \to \tilde{f}_2(x) := \tilde{f}(\sqrt{t}A^{-1}x),\ \tilde{g}_2(y) := \tilde{g}(\sqrt{t}A'y);$$

$$S \to \tilde{S} := \frac{1}{\sqrt{t}}AS,$$

where $t := r^2$ is chosen so that

$$B_1(\tilde{x}_0) \subset [\tilde{S}] \subset B_n(\tilde{x}_0), \qquad \tilde{x}_0 := \frac{x_0}{\sqrt{t}}. \tag{3.10}$$

Notice that, as a consequence of Lemma 3.1,

$$\frac{1}{C}h^{1+2\epsilon} \leq t \leq Ch^{1-2\epsilon}. \tag{3.11}$$

Moreover, Lemmas 3.2 and 3.1 imply that

$$\|A\|, \|A^{-1}\| \leq Ch^{-2\epsilon}. \tag{3.12}$$

In addition, Lemma 3.2 combined with (3.11) and (3.12) yields

$$\mathrm{dist}([\tilde{S}], \tilde{S}) \leq Ch^{\frac{1}{2}-9\epsilon}. \tag{3.13}$$

Hence, if α is sufficiently close to 1 so that $\epsilon < 1/18$, it follows by (3.10) that

$$B_{\frac{1}{C_1}}(\tilde{x}_0) \subset \tilde{S} \subset B_{C_1}(\tilde{x}_0) \tag{3.14}$$

for some dimensional constant C_1.

Now by (3.6), (3.11), (3.12), and the above definition of c_2, for all $R \geq 1$ we have

$$\begin{aligned} \|c_2 + x \cdot y\|_{C^2(B_R \times B_R)} &\leq C\frac{1}{t}t^{\frac{3}{2}}(\|A\|^3 + \|A^{-1}\|^3)R^3 \\ &\leq Ch^{\frac{1}{2}-7\epsilon}R^3 \leq Ch^{\epsilon} \end{aligned} \tag{3.15}$$

provided we choose $R := h^{\frac{8}{3}\epsilon - \frac{1}{6}} \geq 1$, which is true under the assumption $\epsilon < 1/18$.

Remark 3.3. Recalling that $\epsilon = \frac{1}{1+\alpha} - \frac{1}{2}$, we see that the assumption $\epsilon < 1/18$ is equivalent to $\alpha > 4/5$.

We now improve (3.11) showing that actually t and h are comparable.

Lemma 3.4. *Assume that $\epsilon < 1/18$ and that h is small enough. Then there exists a universal constant C_2 such that $\frac{1}{C_2}h \leq t \leq C_2 h$.*

Proof. By subtracting $\frac{h}{t}$, we can assume $u_2 = 0$ on $\partial\tilde{S}$. All we need to show is that $\frac{1}{C_1} \le |\inf u_2| \le C_1$.

Suppose first that $|\inf u_2| > 1$, and let us prove a universal upper bound on it. Noticing that $|\inf u_2| = -u_2(0)$ and that $\tilde{S} \subset B_{2C_1}(0)$ (this follows from (3.14) and the fact that $0 \in \tilde{S}$), it follows by (3.15) that, for all unit vectors ξ,

$$\begin{aligned} u_2(0) - c_2\left(x, \frac{|\inf u_2|}{4C_1}\xi\right) &\le u_2(0) + \frac{|\inf u_2|}{4C_1}\xi \cdot x + Ch^\epsilon \\ &\le -\frac{|\inf u_2|}{2} + Ch^\epsilon \\ &< 0 \qquad \text{on } \partial\tilde{S}, \end{aligned}$$

provided h is small and $\frac{|\inf u_2|}{4C}\xi \in B_R$. Hence $\frac{|\inf u_2|}{4C}B_1 \cap B_R \subset T_{u_2}(\tilde{S})$. Since

$$\mathrm{vol}\left(T_{u_2}(\tilde{S})\right) \approx \mathrm{vol}(\tilde{S}) \le C$$

(this follows by the transport condition and the fact that f and g are bounded away from zero and infinity) and $R = h^{\frac{8}{3}\epsilon - \frac{1}{6}}$ can be made arbitrarily large by choosing h small enough, we deduce that $|\inf u_2| \le C_1$, as desired.

To prove the converse bound, we assume that $|\inf u_2| \le \frac{1}{C_2}$ for some universal constant C_2, and we prove that C_2 is universal bounded from above. Since $\mathrm{dist}(\partial\tilde{S}, \frac{1}{2}\tilde{S}) \ge \frac{1}{C_3}$, we deduce that

$$u(x) + a\xi \cdot (z - x) \ge \frac{a}{C_3} - |\inf u_2| \ge \frac{1}{C_2},$$

whenever $a > \frac{2C_3}{C_2}$, $x \in \frac{1}{2}\tilde{S}$, ξ is a unit vector, $z \in \partial\tilde{S}$, and $z - x$ is parallel to ξ. Then by (3.15), we have that

$$u(x) - (c_2(z, a\xi) - c_2(x, a\xi)) \ge \frac{1}{C_2} - Ch^\epsilon \ge \frac{1}{2C_2},$$

which implies $a\xi \notin T_{u_2}(\frac{1}{2}\tilde{S})$. This yields $T_{u_2}(\frac{1}{2}\tilde{S}) \subset B_{\frac{2C_3}{C_2}}(0)$, and since

$$\mathrm{vol}\left(T_{u_2}\left(\frac{1}{2}\tilde{S}\right)\right) \approx \mathrm{vol}\left(\frac{1}{2}\tilde{S}\right) = \frac{1}{2^n}\mathrm{vol}(\tilde{S}) \ge \frac{1}{C}$$

(recall (3.14)), we obtain

$$C_4\left(\frac{2C_3}{C_2}\right)^n \ge 1.$$

This proves that C_2 is universally bounded from above, concluding the proof. □

Remark 3.5. By (3.5) and the above definitions of c_2 and $\tilde{c}_2$ we see that

$$\|\tilde{c}_2 - c_2\|_{C^2(X_2 \times Y_2)} = \tilde{\delta}, \tag{3.16}$$

where $\tilde{\delta}$ can be as small as we want provided we take δ sufficiently small (the smallness depends also on h).

We now show that $\partial\tilde{S}$ cannot be too close to the origin.

Lemma 3.6. *Assume that $\epsilon < 1/18$ and that h is small enough. Then there exists a universal constant C_3 such that* $\mathrm{dist}(0, \partial\tilde{S}) \ge \frac{1}{C_3}$.

Proof. Thanks to Lemma 3.4, without loss of generality we can take $t = h$ in the normalization. Hence $u_2 = 1$ on $\partial\tilde{S}$. Suppose $\mathrm{dist}(0, \partial\tilde{S}) \leq \frac{1}{C_3}$ for some C_3 large. By (3.13), we can assume $\mathrm{dist}(0, \partial[\tilde{S}]) \leq \frac{2}{C_3}$ by taking h sufficiently small. Let $\mathcal{C}$ be the convex cone with vertex $(0, 0) \in \mathbb{R}^n \times \mathbb{R}$ and base $[\tilde{S}] \times \{\frac{1}{2}\}$, and denote by v the convex function whose graph is $\mathcal{C}$. It follows by the classical Alexandrov's estimates (see for instance [15, Theorem 2.2.4]) that

$$\mathrm{vol}\big(\partial^- v(\mathcal{C})\big) \geq \frac{C_3}{C_4}, \tag{3.17}$$

for some universal constant $C_4 > 0$.

Now, for any $\eta \in \partial^- v(\mathcal{C}) \cap B_R$ we have $\eta \cdot x \leq v$, so it follows by (3.15) that

$$\begin{aligned} -c_2(x, \eta) &\leq \eta \cdot x + Ch^\epsilon \\ &\leq v + Ch^\epsilon \\ &\leq \frac{1}{2} + Ch^\epsilon < 1 \end{aligned}$$

for $x \in \partial\tilde{S}$ provided h is small. This implies that, for any $\eta \in \partial^- v(\mathcal{C}) \cap B_R$,

$$-D_x c(x, \eta) \text{ belongs to } \partial^- u_2(x) \text{ for some } x \in \tilde{S}$$

(it suffices to lower the graph of $-c_2(\cdot, \eta)$ and then lift it until it touches the graph go u_2 at some point $x \in \tilde{S}$). Since $u_2 \in C^{1,\alpha}$, this implies that $\eta \in \partial^{c_2} u_2(x) = T_{u_2}(x)$ (see [11, Remark 4.4]), hence

$$\partial^- v(\mathcal{C}) \cap B_R \subset T_{u_2}(\tilde{S}).$$

Therefore

$$C_5 \geq \mathrm{vol}\big(\tilde{S}\big) \approx \mathrm{vol}(T_{u_2}(\tilde{S})) \geq \mathrm{vol}\left(\partial^- v(\mathcal{C})) \cap B_R\right).$$

Recalling that $R = h^{\frac{8}{3}\epsilon - \frac{1}{6}}$ can be made arbitrarily large by choosing h small enough, combining the estimate above with (3.17) we deduce that

$$C_3 \leq \frac{2C_5}{C_4},$$

as desired. □

Remark 3.7. Lemma 3.6 implies that, up to slightly changing the constant in (3.10) and (3.14), we can assume the center of the balls to be the origin.

From now on, to simplify the notation, B_r will denote the ball of radius r centered around the origin.

Lemma 3.8. *Assume that $\epsilon < 1/18$ and that h is small enough, and set $S_{1/2} := \{u_2 \leq 1/2\}$. There exists a universal constant C_4 such that*

$$B_{\frac{1}{C_4}} \subset [S_{1/2}] \subset B_{C_4}, \tag{3.18}$$

$$B_{\frac{1}{C_4}} \subset T_{u_2}([S_{1/2}]) \subset B_{C_4}. \tag{3.19}$$

Proof. The second inclusion in (3.18) follows immediately from (3.10) and Remark 3.7. The first inclusion in (3.18) follows by repeating the proof of Lemma 3.6 with $S_{1/2}$ in place of S.

To prove (3.19) we notice that, for any fixed $y \in B_{1/4C_4}$, (3.18) implies that

$$x \cdot y < 1/4$$

for $x \in \partial[S_{1/2}]$. Hence it follows by (3.15) that

$$-c_2(x, y) \le x \cdot y + Ch^\epsilon < 1/2 \le u_2(x)$$

for $x \in \partial[S_{1/2}]$ provided h is small enough, and as in the proof of Lemma 3.6 we deduce that $y \in T_{u_2}([S_{1/2}])$. Since $y \in B_{1/4C_4}$ is arbitrary, we have that

$$B_{\frac{1}{4C_4}} \subset T_{u_2}([S_{1/2}]),$$

which proves the first inclusion in (3.19).

For the second inclusion, the same argument as that in the proof of Lemma 3.6 gives also

$$\operatorname{dist}(\partial\tilde{S}, S_{1/2}) \ge \frac{1}{C_5} \tag{3.20}$$

for some large constant C_5. Hence, by (3.13) applied to $S_{1/2}$ we obtain

$$\operatorname{dist}(\partial\tilde{S}, [S_{1/2}]) \ge \frac{1}{2C_5}. \tag{3.21}$$

Now, for any fixed $x \in [S_{1/2}]$, let $y := a\xi$, where ξ is a unit vector and $4C_5 < a < R$. Then

$$u(x) + y \cdot (z - x) \ge a\frac{1}{2C_5} > 2$$

where $z \in \partial\tilde{S}$ is chosen so that $z - x$ is parallel to ξ. Hence

$$u(x) - c_2(z, y) + c_2(x, y) \ge u(x) + y \cdot (z - x) - Ch^\epsilon > 1 \qquad \forall\, x \in [S_{1/2}],$$

which implies that $y \notin T_{u_2}([S_{1/2}])$ and proves that

$$T_{u_2}([S_{1/2}]) \subset B_{4C_5}.$$

□

All the estimates above hold for our solution u. We now want to extend these bounds to $\tilde{u}$. For this, we begin begin stating a simple lemma which follows by a standard compactness argument (see for instance [11]). Here and in the sequel, $\tilde{\delta}$ is as in Remark 3.5.

Lemma 3.9. *With the same notation as before, it holds*

$$\|\tilde{u}_2 - u_2\|_{L^\infty(X_2)} \le \omega(\tilde{\delta}), \qquad \|\tilde{u}_2^{\tilde{c}_2} - u_2^{c_2}\|_{L^\infty(Y_2)} \le \omega(\tilde{\delta}),$$

where the nondecreasing function $\omega : \mathbb{R}^+ \to \mathbb{R}^+$ satisfies $\omega(s) \to 0$ as $s \to 0$.

Set

$$X_2^\sigma := \{x : \operatorname{dist}(x, \partial X_2) > \sigma\}, \quad Y_2^\sigma := \{y : \operatorname{dist}(x, \partial Y_2) > \sigma\},$$

and define

$$K_1 := \max\{\|u_2\|_{C^{1,\alpha}(X_2^\sigma)}, \|c_2\|_{C^2}, \|\tilde{c}_2\|_{C^2}\},$$

$$K_2 = \inf\Big\{\frac{|D_x c_2(x, y) - D_x c_2(x, y')|}{|y - y'|} \;:\; x \in X_2, y, y' \in Y_2\}.$$

Note that $0 < K_1, K_2 < \infty$ (thanks to our assumptions), and that these constants may depend on h.

Lemma 3.10. *Given any $x_0 \in X_2^\sigma$ satisfying $B_{\omega(\tilde\delta)}(x_0) \subset X_2^\sigma$, given $y_0 \in \partial^{\tilde c_2}\tilde u_2(x_0)$, we have that*

$$|y_0 - T_{u_2}(x_0)| \le K\left(\tilde\delta + \omega(\tilde\delta)^{\frac{\alpha}{2}}\right),$$

where K depends only on K_1, K_2.

Proof. Since $\tilde u_2$ is $\tilde c_2$-convex, we have that $\tilde u_2(x) + K_1|x-x_0|^2$ is convex. Hence for any unit vector e we have

$$\tilde u_2\left(x_0 + \omega(\tilde\delta)^{\frac12}e\right) + K_1\omega(\tilde\delta) \ge \tilde u_2(x_0) - \omega(\tilde\delta)^{\frac12}D_x\tilde c_2(x_0,y_0)\cdot e.$$

Noticing that $-D_x\tilde c_2(x_0,y_0) \in \partial^-\tilde u_2(x_0) = \{Du_2(x_0)\}$, by the $C^{1,\alpha}$ regularity of u_2 we have

$$u_2\left(x_0 + \omega(\tilde\delta)^{\frac12}e\right) \le u_2(x_0) - \omega(\tilde\delta)^{\frac12}D_xc_2(x_0,T_{u_2}(x_0))\cdot e + K_1\omega(\tilde\delta)^{\frac{1+\alpha}{2}}.$$

Hence, combining the two estimates above with Lemma 3.9, we obtain

$$(-D_x\tilde c_2(x_0,y_0) + D_xc_2(x_0,T_{u_2}(x_0)))\cdot e \le 2K_1\omega(\tilde\delta)^{\frac{\alpha}{2}} + 2\omega(\tilde\delta)^{\frac12} \le 2(1+K_1)\omega(\tilde\delta)^{\frac{\alpha}{2}}.$$

Since e is an arbitrary unit vector, we see that

$$|D_x\tilde c_2(x_0,y_0) - D_xc_2(x_0,T_{u_2}(x_0))| \le 2(1+K_1)\omega(\tilde\delta)^{\frac{\alpha}{2}}.$$

By Remark 3.5 we also have $|D_x\tilde c_2(x_0,y_0) - D_xc_2(x_0,y_0)| \le \tilde\delta$, thus

$$|D_xc_2(x_0,y_0) - D_xc_2(x_0,T_{u_2}(x_0))| \le \tilde\delta + 2(1+K_1)\omega(\tilde\delta)^{\frac{\alpha}{2}}.$$

Hence, recalling the definition of K_2 we get

$$K_2|y_0 - T_{u_2}(x_0)| \le 2\Big(\tilde\delta + 2(1+K_1)\omega(\tilde\delta)^{\frac{\alpha}{2}}\Big),$$

which proves the desired estimate with $K := \frac{4(1+K_1)}{K_2}$. □

Thanks to Remark 3.5 and Lemma 3.10 we see that by first taking h small enough, and then δ sufficiently small depending on h, it follows by Lemma 3.8 that, up to slightly enlarging the constant C_4, we can ensure that

$$\partial_{\tilde c_2}\tilde u_2([S_{1/2}]) \subset B_{C_4}. \tag{3.22}$$

Since the same argument can be repeated for the dual problem, we also have

$$\partial_{\tilde c_2^*}\tilde u_2^{\tilde c_2}(B_{\frac{1}{C_4}}) \subset [S_{1/2}],$$

thus

$$B_{\frac{1}{C_4}} \subset \partial_{\tilde c_2}\tilde u_2\Big(\partial_{\tilde c_2^*}\tilde u_2^{\tilde c_2}(B_{\frac{1}{C_4}})\Big) \subset \partial_{\tilde c_2}\tilde u_2([S_{1/2}]) \subset B_{C_4}.$$

Therefore, we can now summarize all facts proved above in the following lemma.

Lemma 3.11. *Let $\mathcal{C}_1 := [S_{1/2}]$, $\mathcal{C}_2 := \partial_{\tilde c_2}\tilde u_2([S_{1/2}])$, $\bar f := \tilde f_2\mathbf{1}_{\mathcal{C}_1}$, and $\bar g := \tilde g_2\mathbf{1}_{\mathcal{C}_2}$. Then the following properties hold:*

$$B_{\frac{1}{C_4}} \subset \mathcal{C}_1 \subset B_{C_4};$$

$$B_{\frac{1}{C_4}} \subset \mathcal{C}_2 \subset B_{C_4};$$

$$\|\bar f - \mathbf{1}_{\mathcal{C}_1}\|_{L^\infty(B_{C_4})} = o(1), \quad \|\bar g - \mathbf{1}_{\mathcal{C}_2}\|_{L^\infty(B_{C_4})} = o(1) \to 0 \quad \textit{as } h,\delta \to 0;$$

$$\|\tilde{c}_2 + x \cdot y\|_{C^2(B_{C_4} \times B_{C_4})} = o(1) \to 0 \quad as\ h, \delta \to 0.$$

4. Proof of the results

4.1. **Proof of Theorem 2.1.** The proof of Theorem 2.1 is very similar to that of [11, Theorem 4.3]. More precisely, in [11, Theorem 4.3] the authors have the assumption that the potential function is close to a quadratic function, which is used to find an approximating solution with interior C^3 estimate. Thanks to the convexity of $\mathcal{C}_1$, in our case we also have the interior C^3 estimate for the approximating solution as explained below.

Given any δ_1 small, by Lemma 3.11 we can assume that $o(1) < \delta_1$, and that $\tilde{\delta} + Ch^{\epsilon} < \delta_1$ provided h and δ are sufficiently small. Let v^* be the convex function satisfying

$$(Dv^*)_\sharp \mathbf{1}_{\mathcal{C}_2} = \mathbf{1}_{\rho\mathcal{C}_1},$$

where ρ is chosen so that $\mathrm{vol}(\rho\mathcal{C}_1) = \mathcal{C}_2$. By Lemma 3.11 and [11, Lemma 4.1] we have that

$$\|\tilde{u}_2^{\tilde{c}_2} - v^*\|_{L^\infty(B_{1/C_4})} \le \omega(\delta_1), \tag{4.1}$$

and since ρ is close to 1, up to enlarging C_4 slightly we still have

$$B_{1/C_4} \subset \rho\mathcal{C}_1 \subset B_{C_4}.$$

Hence by Caffarelli's interior regularity result (see [4, 2] or [15, Chapter 4.5]) we see that

$$\|v^*\|_{C^3(B_{\frac{1}{2C_4}})} \le C_5,$$

where C_5 depends only on C_4. This estimate allows us to repeat the very same argument in the proof of [11, Theorem 4.3] to show that $\tilde{u}_2^{\tilde{c}_2} \in C^{1,5/6}(B_{\frac{1}{3C_4}})$. Then by a standard covering argument, we have that $\tilde{u}^{\tilde{c}} \in C^{1,5/6}(Y^\sigma)$. By switching X and Y, the same argument shows that $\tilde{u} \in C^{1,5/6}(X^\sigma)$, provided $\bar{\delta}$ is sufficiently small. Then by Theorem 2.3 we have that $\tilde{u} \in C^{1,\beta}(X^\sigma)$ for any $\beta \in (0,1)$, as desired. □

4.2. **Proof of Theorem 2.3.** Let $\mathcal{C}_3 := T_{u_2}([S_{1/2}])$, so that $(\partial_{c_2^*} u_2^{c_2})_\sharp g_2 \mathbf{1}_{\mathcal{C}_3} = f_2 \mathbf{1}_{\mathcal{C}_1}$. For any fixed small positive constant δ_1, by (3.15) and Lemma 3.8 we have that

$$B_{\frac{1}{C_4}} \subset \mathcal{C}_1 \subset B_{C_4},$$

$$B_{\frac{1}{C_4}} \subset T_{u_2}([S_{1/2}]) \subset B_{C_4},$$

$$\|g_2 - 1\|_{L^\infty(\mathcal{C}_3)} + \|f_2 - 1\|_{L^\infty(\mathcal{C}_1)} \le \delta_1,$$

$$\|c_2 + x \cdot y\|_{C^2(B_{C_4} \times B_{C_4})} \le \delta_1,$$

provided h is sufficiently small. Then we can proceed similarly to the proof of [11, Theorems 4.3 and 5.3]. Indeed, as in the proof of Theorem 2.1, the only difference between our conditions and those of [11, Theorems 4.3 and 5.3] is that, instead of $u_2^{c_2}$ being close to a quadratic function, we have that $u_2^{c_2}$ is close to a smooth function v^* which satisfies

$$(Dv^*)_\sharp \mathbf{1}_{\mathcal{C}_3} = \mathbf{1}_{\rho\mathcal{C}_1},$$

where ρ is chosen so that $\mathrm{vol}(\rho\mathcal{C}_1) = \mathcal{C}_3$. Since ρ is close to 1 and $\rho\mathcal{C}_1$ is convex, v^* has C^3 estimate in $B_{\frac{1}{2C_4}}$, which plays the same role of "$u_2^{c_2}$ being close to a quadratic function" in the argument of the proof of [11, Theorems 4.3 and 5.3]. □

4.3. **Proof of Theorem 2.5.** Fix any point $x_0 \in X$, and set $y_0 := T_u(x_0)$. Without loss of generality we may assume $x_0 = y_0 = 0$.

Since $u \in C^{1,1}(X)$, the transport map T_u is continuous, hence we can choose small positive constants r_1, r_2 so that $B_{r_1} \subset X$ and $B_{r_2} \subset T_u(B_{r_1}) \subset Y$. Also, the $C^{1,1}$ regularity of u combined with the Monge-Ampère equation (2.3) implies that u is uniformly c-convex, which implies by duality that $u^c \in C^{1,1}(B_{r_2})$. Hence, we can apply Theorem 2.3 with $X = T^*_{u^c}(B_{r_2})$ and $Y = B_{r_2}$, and $0 < r_3 \ll r_1$ so that $B_{r_3} \subset\subset X$, to deduce that $\tilde{u} \in C^{1,5/6}(B_{r_3})$ and $\tilde{u}^{\tilde{c}} \in C^{1,5/6}(T_u(B_{r_3}))$, provided $\bar{\delta}$ is sufficiently small. Then by Theorem 2.3 we have that $\tilde{u} \in C^{2,\alpha}(B_{r_3/2})$, and we conclude the proof by a standard covering argument. □

4.4. **Proof of Corollary 2.2 and 2.6.** For Corollary 2.2, the condition that $(M, \mathcal{G})$ is non-focal combined with the property $u \in C^1$ ensure that, for any given $x_0 \in M$, the point $T_u(x_0)$ stays at some positive distance away from the cut locus of x_0 (see for instance [32, 23]). Hence we can localize the problem by using coordinate charts around x_0 and $T_u(x_0)$ so that the cost satisfies (**C0**)-(**C3**) in a neighborhood of $(x_0, T_u(x_0))$. Then we can finish the proof of Corollary 2.2 by applying Theorem 2.1 to this localized problem and using a standard covering argument.

The proof of Corollary 2.6 is similar. Note that the Lipschitz continuity of T_u is equivalent to $u \in C^{1,1}$, which implies that the "stay away from the cut locus" property holds even without the nonfocality assumption (see for instance [10]). □

4.5. **Proof of Theorem 2.7.** As in [9], it is enough to understand the regularity of $\tilde{u}$ at the boundary. Given $x_0 \in \partial X$, consider $y_0 = T_u(x_0) \in \partial Y$, and assume without loss of generality that $x_0 = y_0 = 0$.

We first perform a transformation as that of (3.1), and up to another transformation with uniformly bounded norm, we can assume that $u_1 = \frac{1}{2}|x|^2 + O(|x|^{2+\alpha})$ and that $\{x^n = 0\}$ is the tangent plane to ∂X at 0. Then it follows by (3.6) that $u_1^{c_1} = \frac{1}{2}|y|^2 + O(|y|^{2+\alpha})$ and that $\{y^n = 0\}$ is the tangent plane to ∂Y at 0. This allows us to follow the same argument of the proof of [9, Corollaries 2.3 and 2.4] to conclude that $\tilde{u} \in C^{2,\alpha'}(\bar{X})$ for some $\alpha' < \alpha$, provided $\bar{\delta}$ is sufficiently small. □

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