STRONG STABILITY FOR THE WULFF INEQUALITY WITH A CRYSTALLINE NORM

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ABSTRACT. Let K be a convex polyhedron and $\mathscr{F}$ its Wulff energy, and let $\mathscr{C}(K)$ denote the set of convex polyhedra close to K whose faces are parallel to those of K. We show that, for sufficiently small ϵ, all ϵ-minimizers belong to $\mathscr{C}(K)$.

As a consequence of this result we obtain the following sharp stability inequality for crystalline norms: There exist $\gamma = \gamma(K, n) > 0$ and $\sigma = \sigma(K, n) > 0$ such that, whenever $|E| = |K|$ and $|E\Delta K| \leq \sigma$ then

$$\mathscr{F}(E) - \mathscr{F}(K^{\mathbf{a}}) \geq \gamma |E\Delta K^{\mathbf{a}}| \qquad \text{for some } K^{\mathbf{a}} \in \mathscr{C}(K).$$

In other words, the Wulff energy $\mathscr{F}$ grows very fast (with power 1) away from $\mathscr{C}(K)$. The set $K^{\mathbf{a}} \in \mathscr{C}(K)$ appearing in the formula above can be informally thought as a sort of "projection" of E onto $\mathscr{C}(K)$.

Another corollary of our result is a very strong rigidity result for crystals: For crystalline surface tensions, minimizers of $\mathscr{F}(E)+\int_E g$ with small mass are polyhedra with sides parallel to the ones of K. In other words, for small mass, the potential energy cannot destroy the crystalline structure of minimizers. This extends to arbitrary dimensions a two-dimensional result obtained in [9].

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1. Introduction

The Wulff construction [18] gives a way to determine the equilibrium shape of a droplet or crystal of fixed volume inside a separate phase (usually its saturated solution or vapor). Energy minimization arguments are used to show that certain crystal planes are preferred over others, giving the crystal its shape. The anisotropic surface energy is a natural choice in this circumstance.

Given a convex, positive, 1-homogeneous function $f : \mathbb{R}^n \to [0, +\infty)$, we define the anisotropic surface energy for a set of finite perimeter $E \subset \mathbb{R}^n$ as

$$\mathscr{F}(E) = \int_{\partial^* E} f(\nu_E)\, d\mathcal{H}^{n-1},$$

where ν_E is the measure-theoretic outer unit normal to E, and $\partial^* E$ is its reduced boundary. We call f the surface tension for $\mathscr{F}$. Observe that, when $f(\nu) = |\nu|$, we obtain the notion of classical perimeter.

Every volume-constrained minimizer of the surface energy $\mathscr{F}$ is obtained by translation and dilation of a bounded open convex set K, called the *Wulff shape* of f. When $f(\nu) = |\nu|$, K is exactly the unit ball. Also K can be equivalently characterized by

$$K = \bigcap_{\nu \in \mathbb{S}^{n-1}} \{x \in \mathbb{R}^n \colon x \cdot \nu < f(\nu)\} = \{x \in \mathbb{R}^n \colon f_*(x) < 1\}.$$

Here $f_* \colon \mathbb{R}^n \to [0, +\infty)$ is the dual of f defined as

$$f_*(x) = \sup\{x \cdot y \colon f(y) = 1\}, \quad x \in \mathbb{R}^n.$$

The *Wulff inequality* states that, for any set of finite perimeter $E \subset \mathbb{R}^n$, one has

$$\mathscr{F}(E) \geq n|K|^{\frac{1}{n}} |E|^{\frac{n-1}{n}}, \tag{1.1}$$

see e.g. [17, 15]. In particular,

$$\mathscr{F}(E) \geq \mathscr{F}(K) = n|K|$$

whenever $|E| = |K|$.

In recent years, a lot of attention has been given to the stability of the isoperimetric/Wulff inequality. A quantitative but not sharp form of general Wulff inequality was first given by [6]. Later, the sharp stability of classical isoperimetric inequality was shown by Fusco, Maggi and Pratelli in [13], and then in [10] this sharp result was extended to the general Wulff inequality. More precisely, in [10] the authors proved that, for any set of finite perimeter $E \subset \mathbb{R}^n$ with $|E| = |K|$, one has

$$\mathscr{F}(E) - \mathscr{F}(K) \geq c(n, K) \min_{y \in \mathbb{R}^n} \big\{|E \Delta (y + K)|\big\}^2, \tag{1.2}$$

where

$$E \Delta F = (E \setminus F) \cup (F \setminus E)$$

denotes the symmetric difference between E and F. Finally, Fusco and Julin [12] and Neumayer [16] generalized this inequality to a stronger form via a technique known as the selection principle, first introduced in this class of problems by Cicalese and Leonardi [4]. We recommend the survey papers [8, 11] for more details.

In this paper we focus on the case when $\mathscr{F}$ is crystalline, i.e., there exists a finite set $\{\hat{x}_\ell\}_{\ell=1}^L \subset \mathbb{R}^n \setminus \{0\}$, with $L \in \mathbb{N}$, such that

$$f(\nu) = \max_{1\le\ell\le L} \hat{x}_\ell \cdot \nu \qquad \forall\, \nu \in \mathbb{S}^{n-1}.$$

Then the corresponding Wulff shape K is a convex polyhedron, and the dual f_* is of the form

$$f_*(x) = \sup_{1\le i\le N} \sigma_i \cdot x \qquad \forall\, x \in \mathbb{R}^n, \tag{1.3}$$

for some $N \in \mathbb{N}$. Here σ_i is a vector parallel to the normal ν_i of the face $\partial K \cap V_i$ of K, where V_i denotes the convex cone

$$V_i = \{x \in \mathbb{R}^n \colon f_*(x) = \sigma_i \cdot x\}.$$

We assume that the set of vectors σ_i is "minimal", i.e.

$$|V_i \cap K| > 0 \text{ for any } i = 1, \dots, N.^1 \tag{1.4}$$

Define $\mathscr{C}(K)$ as the collection of bounded open convex sets close to K, whose faces are parallel to those of K, and whose volume is $|K|$. To be specific, given $\mathbf{a} = (a_1, \cdots, a_N) \in \mathbb{R}^N$ with $|\mathbf{a}| < 1$, let

$$f_*^{\mathbf{a}}(x) = \sup_{1\le i\le N} \frac{\sigma_i \cdot x}{1 + a_i} \quad \text{and} \quad K^{\mathbf{a}} = \{f_*^{\mathbf{a}}(x) \le 1\}. \tag{1.5}$$

Note that $f_* = f_*^{\mathbf{0}}$. Then we define

$$\mathscr{C}(K) = \big\{K^{\mathbf{a}} \ :\ \mathbf{a} \in \mathbb{R}^N,\ |\mathbf{a}| < 1,\ |K^{\mathbf{a}}| = |K|\big\}.$$

Following [9], given a set A and $R > 0$, we denote by $\mathcal{N}_{R,K}(A)$ the R-neighborhood of A with respect to K, namely

$$\mathcal{N}_{R,K}(A) = \{x \in \mathbb{R}^n \ :\ \mathrm{dist}_K(x, A) \le R\}, \qquad \text{where} \quad \mathrm{dist}_K(x, A) = \inf_{y\in A} f_*(x - y).$$

Then we say that a set E is an (ϵ, R)-minimizer[2] for the surface energy $\mathscr{F}$ if, for every set of finite perimeter $G \subset \mathbb{R}^n$ satisfying $|E| = |G|$ and $G \subset \mathcal{N}_{R,K}(E)$, one has

$$\mathscr{F}(E) \le \mathscr{F}(G) + \epsilon \left(\frac{|K|}{|E|}\right)^{\frac{1}{n}} |E\Delta G|.$$

We shall also say that E is an ϵ-minimizer if E is an (ϵ, R)-minimizer with $R = +\infty$. It is clear from the definition that an (ϵ_1, R_1)-minimizer is an (ϵ_2, R_2)-minimizer whenever $\epsilon_1 \le \epsilon_2$ and $R_2 \ge R_1$.

The following result is the main theorem of the paper.

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[1] Note that one could always artificially add some extra vectors σ in the definition of f_* by simply choosing σ small enough so that $f_*(x) > \sigma \cdot x$ for any $x \ne 0$. In this way, f_* is unchanged by adding σ to the set of vectors $\{\sigma_i\}_{1\le i\le N}$. Thus, asking that $|V_i \cap K| > 0$ for any i guarantees that all the vectors σ_i play an active role in the definition of f_*.

[2] Our notion of (ϵ, R)-minimizer differs from the one usually appearing in the literature, since we also impose a volume constraint on the competitors.

Theorem 1.1. *Let $\mathscr{F}$ be crystalline, K be its Wulff shape, and E be a set of finite perimeter with $|E| = |K|$. Then there exists a constant $\epsilon_0 = \epsilon_0(n, K) > 0$ such that, for any $0 < \epsilon \le \epsilon_0$ and $R \ge n+1$, if E is an (ϵ, R)-minimizer of the surface energy associated to f then, up to a translation, $E = K^{\mathbf{a}}$ for some $K^{\mathbf{a}} \in \mathscr{C}(K)$.*

Remark 1.2. The restriction $R \ge n+1$ in Theorem 1.1 comes from the use of Lemma 2.1 below. We do not believe that this assumption can be considerably relaxed, since it encodes a global information on E (see also the comments before the proof of [9, Lemma 5]).

Remark 1.3. In the planar case $n = 2$, the structure of the elements in $\mathscr{C}(K)$ is rather elementary, and a consequence of Theorem 1.1 is that (ϵ, R)-minimizers are bi-Lipschitz equivalent to the Wulff shape (this two-dimensional fact was already proved in [9, Theorem 7]). This fact is not true in general in the higher dimensional case since, depending on the shape of K, the boundary of $K^{\mathbf{a}}$ might have a different number of k-simplices for $k \le n-2$ (see the discussion at the beginning of Section 3).

In [9] the following variational problem was considered:

$$\min \left\{ \mathscr{F}(E) + \int_E g \, dx \colon E \subset \mathbb{R}^n \text{ is a set of finite perimeter} \right\}, \tag{1.6}$$

where $g \colon \mathbb{R}^n \to [0, \infty)$ is a locally bounded Borel function with $g(x) \to \infty$ as $|x| \to \infty$.

In this model, $\mathscr{F}$ represents the surface energy of a droplet/crystal, while g is a (confining) potential term. Hence, the minimization problem aims to understand the equilibrium shapes of droplets/crystals under the action of an external potential.

As shown in [9, Corollary 2], for $|E| \ll 1$ any minimizer of (1.6) is an $(\epsilon, n+1)$-minimizer of $\mathscr{F}$. Also, it was noted in [9, Theorem 7] that, when $n = 2$, minimizers with sufficiently small mass are polyhedra with sides parallel to K. It was then asked in [9, Remark 1] whether this result would hold in every dimension. Our result gives a positive answer to this question, by simply applying Theorem 1.1 to the rescaled set $\left(\frac{|K|}{|E|}\right)^{\frac{1}{n}} E$. We summarize this in the following:

Corollary 1.4. *Let $\mathscr{F}$ be crystalline, K be its Wulff shape, and $g \colon \mathbb{R}^n \to [0, \infty)$ a locally bounded Borel function such that $g(x) \to \infty$ as $|x| \to \infty$. Let E be a minimizer of (1.6). There exists $m_0 = m_0(n, K, g) > 0$ small enough such that if $|E| \le m_0$ then $\left(\frac{|K|}{|E|}\right)^{\frac{1}{n}} E \in \mathscr{C}(K)$.*

This result is particularly interesting from a "numerical" viewpoint: since the space $\mathscr{C}(K)$ is finite dimensional, minimizers of (1.6) can be found explicitly by minimizing the energy functional over a small neighborhood of K in this finite dimensional space. In addition, in some explicit cases, it can be used to analytically find the exact minimizer. For instance, in [3] the authors used the two-dimensional analogue of Corollary 1.4 to explicitly find the minimizers of some variational problems coming from plasma physics. Thanks to Corollary 1.4 one can now perform the same kind of analysis also in the physical dimension $n = 3$.

While Theorem 1.1 proves that all (ϵ, R)-minimizers with $R \geq n+1$ are in $\mathscr{C}(K)$, one may wonder whether all elements of $\mathscr{C}(K)$ are (ϵ, R)-minimizers for $R \geq n+1$. This is our second result, which gives a full characterization of ϵ-minimizers for $\mathscr{F}$, when ϵ and $|\mathbf{a}|$ are sufficiently small.

Theorem 1.5. *Let $\mathscr{F}$ be crystalline, K be its Wulff shape, and ϵ_0 be as in Theorem 1.1. Then, for any $\epsilon \leq \epsilon_0$ there exists $a_0 = a_0(n, K, \epsilon) > 0$ such that, for $0 < |\mathbf{a}| \leq a_0$, any set $K^{\mathbf{a}}$ is an ϵ-minimizer (and so also an (ϵ, R)-minimizer for all $R \geq n+1$) for $\mathscr{F}$.*

Finally, with the help of Theorems 1.1 and 1.5, we show the following stability inequality.

Theorem 1.6. *Let $\mathscr{F}$ be crystalline, and let K be its Wulff shape. Then there exists $\sigma_0 = \sigma_0(n, K) > 0$ such that the following holds: For any set of finite perimeter E satisfying $|E \Delta K| \leq \sigma_0$, there exists $K^{\mathbf{a}} \in \mathscr{C}(K)$ such that*

$$\mathscr{F}(E) - \mathscr{F}(K^{\mathbf{a}}) \geq \gamma |E \Delta K^{\mathbf{a}}|.$$

The set $K^{\mathbf{a}}$ is given via Lemma 2.2 below, and can be informally thought as a sort of "projection" of E onto $\mathscr{C}(K)$. An immediate corollary of Theorem 1.6 is that, under the constraint

$$|E \cap V_i^{\mathbf{a}}| = |K^{\mathbf{a}} \cap V_i^{\mathbf{a}}| \qquad \forall\, i = 1, \ldots, N,$$

$K^{\mathbf{a}}$ is the unique minimizer for $\mathscr{F}$, where

$$V_i^{\mathbf{a}} = \left\{ x \in \mathbb{R}^n \colon f_*^{\mathbf{a}}(x) = \frac{\sigma_i \cdot x}{1 + a_i} \right\}.$$

The stability inequality provided by Theorem 1.6 is different from the classical one in (1.2). Indeed, first of all elements in $\mathscr{C}(K)$ might not be a minimizer of $\mathscr{F}$ (only K and its translates are minimizers). In addition, thanks to the non-smoothness of f, we are able to obtain a stability result for the isoperimetric inequality with a linear control (power 1) on $|E \Delta K^{\mathbf{a}}|$.

One could understand this result geometrically in the following way:
If we plot the "graph" of $\mathscr{F}$ in a neighborhood of K, $\mathscr{F}$ is constant on the translates of K, while it increases smoothly on the space $\mathscr{C}(K)$ near K; see Lemma 2.3 below. On the other hand, given a set E close to K but not in $\mathscr{C}(K)$, the value of $\mathscr{F}$ grows very fast (linearly) from the value of $\mathscr{F}$ on the "best approximation of E in $\mathscr{C}(K)$" provided by Lemma 2.2. In other words, the energy $\mathscr{F}$ varies smoothly on $\mathscr{C}(K)$, while it has a Lipschitz singularity when moving away from $\mathscr{C}(K)$.

The paper is organized as follows. In Section 2, we introduce a method to associate an element of $\mathscr{C}(K)$ to any set of finite perimeter E with $|E| = |K|$ and $|E \Delta K| \ll 1$, and we study the behaviour of $\mathscr{F}$ on $\mathscr{C}(K)$. Then we prove Theorem 1.1 in Section 3. Finally, the proofs of Theorem 1.5 and Theorem 1.6 are given in the last section. Some useful technical results are collected in the appendix.

2. Associate $K^{\mathbf{a}}$ to a set of finite perimeter E

We begin by setting the notation used in this paper. We denote by $C = C(\cdot)$ and $c = c(\cdot)$ positive constant, with the parentheses including all the parameters on which the constants depend. The constants $C(\cdot)$ and $c(\cdot)$ may vary between appearances, even within a chain of inequalities. We denote by $|A|$ the Lebesgue measure of A, and by $\mathcal{H}^\alpha$ the α-dimensional Hausdorff measure. The reduced boundary of a set of finite perimeter E is denoted by $\partial^* E$. By $\mathrm{dist}(A, B)$ we denote the Euclidean distance between A and B.

As in the introduction, K is the Wulff shape associated to f_*. We denote by rK the dilation of K by a factor r with respect to the origin $O \in \mathbb{R}^n$. Also, given $\mathbf{a} \in \mathbb{R}^N$ with $|\mathbf{a}| < 1$, we define

$$V_i^{\mathbf{a}} = \left\{ x \in \mathbb{R}^n \colon f_*^{\mathbf{a}}(x) = \frac{\sigma_i \cdot x}{1 + a_i} \right\},$$

where $\{\sigma_i\}_{1 \le i \le N}$ are the vectors defining f_* (see (1.3)).

For a vector $x \in \mathbb{R}^n$, we denote by x' its first $(n-1)$-coordinates, and by x_n its last coordinate, so that $x = (x', x_n)$. We denote by ν_E the measure-theoretic outer unit normal to a set of finite perimeter E, and omit the subindex if the set E in question is clear from the context.

We first recall the following result.

Lemma 2.1. [9, Lemma 5 and Theorem 5] *If E is an $(\epsilon, n+1)$-minimizer of $\mathscr{F}$ with $|E| = |K|$, then there exists $x \in \mathbb{R}^n$ so that*

$$|(x + E) \Delta K| \le C(n)|K|\epsilon.$$

In addition, up to a translation, ∂E is uniformly close to ∂K, namely

$$(1 - r)K \subset x + E \subset (1 + r)K, \qquad r \le C(n, K)\epsilon^{\frac{1}{n}}.$$

The lemma above allows us to apply the following result to $(\epsilon, n+1)$-minimizers, when ϵ is sufficiently small.

Lemma 2.2. *Let $E \subset \mathbb{R}^n$ be a set of finite perimeter and $|E| = |K|$. There exist $\eta = \eta(n, K) > 0$ and $C = C(n, K) > 0$ such that the following holds: If*

$$(1 - \eta)K \subset E \subset (1 + \eta)K$$

then there exists a unique $\mathbf{a} = \mathbf{a}(K, |E \Delta K|) \in \mathbb{R}^N$, with $|\mathbf{a}| \le C|E \Delta K|$, such that

$$|E \cap V_i^{\mathbf{a}}| = |K^{\mathbf{a}} \cap V_i^{\mathbf{a}}| \qquad \forall\, i = 1, \dots, N,$$

for some $K^{\mathbf{a}} \in \mathscr{C}(K)$.

Proof. Note that, by the definition of V_i and K,

$$\partial K \cap V_i \subset \{\sigma_i \cdot x = 1\} \qquad \forall\, i = 1, \dots, N.$$

Given $t > 0$, set $H_i^t := \{x \in \mathbb{R}^n : \sigma_i \cdot x < t\}$. Then it follows by our assumption on E that, for any $\mathbf{a}$ small enough,

$$E \cap V_i^{\mathbf{a}} \supset H_i^{1-2\eta} \cap V_i^{\mathbf{a}} \qquad \forall\, i = 1, \dots, N. \tag{2.1}$$

Similarly, for $\mathbf{a}$ small,

$$K^{\mathbf{a}} \cap V_i^{\mathbf{a}} \supset H_i^{1-2\eta} \cap V_i^{\mathbf{a}} \qquad \forall\, i = 1, \dots, N. \tag{2.2}$$

Consider the map

$$\Phi \colon \mathbf{a} \mapsto (|K^{\mathbf{a}} \cap V_1^{\mathbf{a}}| - |E \cap V_1^{\mathbf{a}}|, \dots, |K^{\mathbf{a}} \cap V_N^{\mathbf{a}}| - |E \cap V_N^{\mathbf{a}}|) = (\Phi_1(\mathbf{a}), \dots, \Phi_N(\mathbf{a})).$$

Our goal is to show that $\Phi(\mathbf{a}) = \mathbf{0}$ for a unique vector $\mathbf{a}$ satisfying $|\mathbf{a}| \le C|E\Delta K|$.

It is easy to check that Φ is Lipschitz, and we now want to compute its differential. Note that, thanks to (2.1) and (2.2),

$$\Phi_i(\mathbf{a}) = |(K^{\mathbf{a}} \cap V_i^{\mathbf{a}}) \setminus H_i^{1-2\eta}| - |(E \cap V_i^{\mathbf{a}}) \setminus H_i^{1-2\eta}| =: \Phi_{i,1}(\mathbf{a}) + \Phi_{i,2}(\mathbf{a}).$$

Let $d_i^{\mathbf{a}}$ be the distance from the origin to $\partial K^{\mathbf{a}} \cap V_i^{\mathbf{a}}$. Then, by (1.5) we have

$$d_i^{\mathbf{a}} = \frac{1+a_i}{|\sigma_i|},$$

hence

$$\begin{aligned}|(K^{\mathbf{a}} \cap V_i^{\mathbf{a}}) \setminus H_i^{1-2\eta}| &= |K^{\mathbf{a}} \cap V_i^{\mathbf{a}}| - |H_i^{1-2\eta} \cap V_i^{\mathbf{a}}| \\ &= \frac{1}{n}\mathcal{H}^{n-1}(\partial K^{\mathbf{a}} \cap V_i^{\mathbf{a}})\big(d_i^{\mathbf{a}} - (1-2\eta)d_i\big) \\ &= \frac{1}{n}\mathcal{H}^{n-1}(\partial K^{\mathbf{a}} \cap V_i^{\mathbf{a}})\frac{2\eta + a_i}{|\sigma_i|},\end{aligned}$$

and therefore

$$\begin{aligned}\Phi_{i,1}(\mathbf{a}') - \Phi_{i,1}(\mathbf{a}) - \frac{1}{n}\frac{\mathcal{H}^{n-1}(\partial K^{\mathbf{a}} \cap V_i^{\mathbf{a}})}{|\sigma_i|}(a_i' - a_i)& \\ = \frac{2\eta + a_i'}{n|\sigma_i|}\Big(\mathcal{H}^{n-1}(\partial K^{\mathbf{a}'} \cap V_i^{\mathbf{a}'}) - \mathcal{H}^{n-1}(\partial K^{\mathbf{a}} \cap V_i^{\mathbf{a}})\Big)&.\end{aligned}$$

Thus, defining

$$v_i := |K \cap V_i| = \frac{1}{n}\frac{\mathcal{H}^{n-1}(\partial K \cap V_i)}{|\sigma_i|},$$

since the map $\mathbf{a} \mapsto \mathcal{H}^{n-1}(\partial K^{\mathbf{a}} \cap V_i^{\mathbf{a}})$ is Lipschitz (although not needed here, in Lemma 2.3 we also show how compute its differential) it follows that

$$|\Phi_{i,1}(\mathbf{a}') - \Phi_{i,1}(\mathbf{a}) - v_i(a_i' - a_i)| \le C(n,K)\,\big(\eta + |\mathbf{a}'| + |\mathbf{a}|\big)|\mathbf{a}' - \mathbf{a}|.$$

On the other hand, since $E \subset (1+\eta)K$ and $\cap_{i=1}^N H_i^{1-2\eta} = (1-2\eta)K$, we can bound

$$|\Phi_{i,2}(\mathbf{a}') - \Phi_{i,2}(\mathbf{a})| \le \big|(V_i^{\mathbf{a}'}\Delta V_i^{\mathbf{a}}) \cap \big((1+\eta)K \setminus (1-2\eta)K\big)\big| \le C(n,K)\,\eta\,|\mathbf{a}' - \mathbf{a}|.$$

Thus, if we consider the linear map $A : \mathbb{R}^N \to \mathbb{R}^N$ given by

$$A := \begin{pmatrix} v_1 & 0 & \cdots & 0 \\ 0 & v_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & v_N \end{pmatrix},$$

we proved that

$$|\Phi(\mathbf{a}') - \Phi(\mathbf{a}) - A(\mathbf{a}' - \mathbf{a})| \le C(n,K)\,\big(\eta + |\mathbf{a}'| + |\mathbf{a}|\big)|\mathbf{a}' - \mathbf{a}|.$$

In particular, at every differentiability point of Φ we have

$$|D\Phi(\mathbf{a}) - A| \le C(n,K)\,\big(\eta + |\mathbf{a}|\big). \tag{2.3}$$

Since Φ is differentiable a.e. (by Rademacher Theorem), for every $\mathbf{a}$ we can define $\partial_C\Phi(\mathbf{a})$ as the convex hull of all the limit of gradients, namely

$$\partial_C\Phi(\mathbf{a}) := \mathrm{co}\Big\{\lim_{j\to\infty} D\Phi(a_j)\,:\,\mathbf{a}_j\to\mathbf{a},\,\Phi\text{ is differentiable at }\mathbf{a}_j\Big\},$$

and it follows by (2.3) that

$$|A_{\mathbf{a}} - A| \le C(n,K)\big(\eta + |\mathbf{a}|\big) \qquad \forall\, A_{\mathbf{a}} \in \partial_C\Phi(\mathbf{a}).$$

Since A has rank N (recall that $v_i > 0$ for any i, see (1.4)), there exists $r_0 > 0$ such that $A_{\mathbf{a}}$ is invertible for every $A_{\mathbf{a}} \in \partial_C\Phi(\mathbf{a})$, provided $\eta \le r_0$ and $|\mathbf{a}| \le r_0$. This allows us to apply the inverse function theorem for Lipschitz mappings [5] to deduce that

$$\Phi : B_{r_0}(\mathbf{0}) \to \Phi(B_{r_0}(\mathbf{0}))$$

is a bi-Lipschitz homeomorphism, with bi-Lipschitz constant depending only on n and K. In particular, since $|\Phi(\mathbf{0})| \le C(n,K)|E\Delta K| \le C(n,K)\eta$ (by our assumption on E), if $\eta = \eta(n,K)$ is sufficiently small with respect to r_0, then $\Phi(B_{r_0}(\mathbf{0}))$ contains the origin.

This means exactly that there exists a unique $\mathbf{a} \in B_{r_0}(\mathbf{0})$ such that $\Phi(\mathbf{a}) = \mathbf{0}$. In addition, again by bi-Lipschitz regularity of Φ, the inequality

$$|\Phi(\mathbf{a}) - \Phi(\mathbf{0})| = |\Phi(\mathbf{0})| \le C(n,K)|E\Delta K|$$

implies that $|\mathbf{a}| \le C(n,K)|E\Delta K|$, as desired. □

We conclude this section with a simple result on the behaviour of $\mathscr{F}$ on $\mathscr{C}(K)$, showing that $\mathscr{F}|_{\mathscr{C}(K)}$ is C^1 at K, with zero gradient.

Lemma 2.3. *There exists a modulus of continuity $\omega : \mathbb{R}^+ \to \mathbb{R}^+$ such that the following holds: if $|\mathbf{a}| + |\mathbf{a}'| < 1$ and $|K^{\mathbf{a}}| = |K^{\mathbf{a}'}|$, then*

$$\big|\mathscr{F}(K^{\mathbf{a}}) - \mathscr{F}(K^{\mathbf{a}'})\big| \le \omega(|\mathbf{a}| + |\mathbf{a}'|)|\mathbf{a} - \mathbf{a}'|.$$

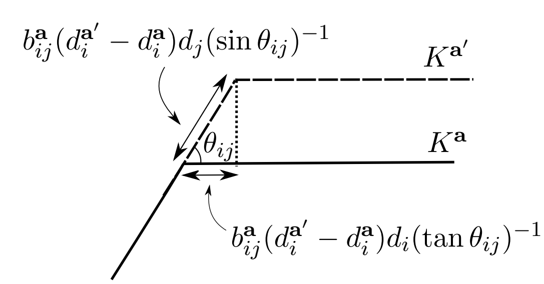

FIGURE 1. We show how moving the faces changes the surface tension.

Proof. Obviously it suffices to prove the result when $|\mathbf{a}'-\mathbf{a}|$ is small. In this proof we denote by $o(1)$ a quantity that goes to 0 as $|\mathbf{a}|+|\mathbf{a}'|\to 0$.

By writing $b_{ij}^{\mathbf{a}} = \mathcal{H}^{n-2}(\partial K^{\mathbf{a}}\cap\partial V_i^{\mathbf{a}}\cap\partial V_j^{\mathbf{a}})$ (note that $b_{ij}^{\mathbf{a}}=0$ when the two sets $V_i^{\mathbf{a}}$ and $V_j^{\mathbf{a}}$ are not adjacent) and choosing $\theta_{ij}\in(0,\pi)$ so that $\cos\theta_{ij}=\nu_i\cdot\nu_j$, a simple geometric construction (see Figure 1) shows that, at first order in $\mathbf{a}'-\mathbf{a}$, the surface energy of $K^{\mathbf{a}'}$ inside $V_i^{\mathbf{a}}$ has an extra term $b_{ij}^{\mathbf{a}}(d_i^{\mathbf{a}'}-d_i^{\mathbf{a}})f(\nu_j)(\sin\theta_{ij})^{-1}$ with respect to the surface energy of $K^{\mathbf{a}}$, but also a negative term $-b_{ij}^{\mathbf{a}}(d_i^{\mathbf{a}'}-d_i^{\mathbf{a}})f(\nu_i)(\tan\theta_{ij})^{-1}$.

Since $f(\nu_i)=d_i$ and $f(\nu_j)=d_j$ (this follows by the relation between f and K), this gives

$$\begin{aligned}
&\mathscr{F}(K^{\mathbf{a}'})-\mathscr{F}(K^{\mathbf{a}})\\
&=\sum_{1\le i,\,j\le N,\,i\ne j} b_{ij}^{\mathbf{a}}(d_i^{\mathbf{a}'}-d_i^{\mathbf{a}})d_j(\sin\theta_{ij})^{-1}-b_{ij}^{\mathbf{a}}(d_i^{\mathbf{a}'}-d_i^{\mathbf{a}})d_i(\tan\theta_{ij})^{-1}+o(1)|\mathbf{a}-\mathbf{a}'|\\
&=\sum_{1\le i,\,j\le N,\,i\ne j} b_{ij}^{\mathbf{a}}(d_i^{\mathbf{a}'}-d_i^{\mathbf{a}})\left[d_j(\sin\theta_{ij})^{-1}-d_i(\tan\theta_{ij})^{-1}\right]+o(1)|\mathbf{a}-\mathbf{a}'|.
\end{aligned}\tag{2.4}$$

Analogously, the difference of the volumes of $K^{\mathbf{a}'}$ and $K^{\mathbf{a}}$ inside $V_i^{\mathbf{a}}$ is given, at first order in $\mathbf{a}'-\mathbf{a}$, by $(d_i^{\mathbf{a}'}-d_i^{\mathbf{a}})\mathcal{H}^{n-1}(\partial K^{\mathbf{a}}\cap V_i^{\mathbf{a}})$, therefore

$$|K^{\mathbf{a}'}|-|K^{\mathbf{a}}|=\sum_{1\le i\le N}(d_i^{\mathbf{a}'}-d_i^{\mathbf{a}})\mathcal{H}^{n-1}(\partial K^{\mathbf{a}}\cap V_i^{\mathbf{a}})+o(1)|\mathbf{a}-\mathbf{a}'|.$$

Let us now denote by $\pi_i^{\mathbf{a}}(O)$ the orthogonal projection of the origin onto the hyperplane $H_i^{\mathbf{a}}$ containing $\partial K^{\mathbf{a}}\cap V_i^{\mathbf{a}}$, let $H_{ij}^{\mathbf{a}}\subset H_i^{\mathbf{a}}$ denote the half-hyperplane such that

$$\partial H_{ij}^{\mathbf{a}}\supset\partial V_i^{\mathbf{a}}\cap\partial V_j^{\mathbf{a}}\qquad\text{and}\qquad H_{ij}^{\mathbf{a}}\supset\partial K^{\mathbf{a}}\cap V_i^{\mathbf{a}},$$

and let us define $\mathrm{dist}_{\mathrm{s}}(\pi_i^{\mathbf{a}}(O),\,\partial H_{ij}^{\mathbf{a}})$ the "signed distance of $\pi_i^{\mathbf{a}}(O)$ from $\partial H_{ij}^{\mathbf{a}}$", namely

$$\mathrm{dist}_{\mathrm{s}}(\pi_i^{\mathbf{a}}(O),\,\partial H_{ij}^{\mathbf{a}})=\begin{cases}\mathrm{dist}(\pi_i^{\mathbf{a}}(O),\,\partial H_{ij}^{\mathbf{a}}) & \text{if } \pi_i^{\mathbf{a}}(O)\in H_{ij}^{\mathbf{a}},\\ -\mathrm{dist}(\pi_i^{\mathbf{a}}(O),\,\partial H_{ij}^{\mathbf{a}}) & \text{if } \pi_i^{\mathbf{a}}(O)\notin H_{ij}^{\mathbf{a}}.\end{cases}$$

Then, with this definition, using the formula for the volume of a cone we get

$$\mathcal{H}^{n-1}(\partial K^{\mathbf{a}}\cap V_i^{\mathbf{a}})=\frac{1}{n}\sum_{j\ne i}b_{ij}^{\mathbf{a}}\,\mathrm{dist}_{\mathrm{s}}(\pi_i^{\mathbf{a}}(O),\,\partial H_{ij}^{\mathbf{a}}),$$

from which we deduce that

$$|K^{\mathbf{a}'}|-|K^{\mathbf{a}}|=\frac{1}{n}\sum_{1\le i,\,j\le N,\,i\ne j}b_{ij}^{\mathbf{a}}(d_i^{\mathbf{a}'}-d_i^{\mathbf{a}})\,\mathrm{dist}_{\mathrm{s}}(\pi_i^{\mathbf{a}}(O),\,\partial H_{ij}^{\mathbf{a}})+o(1)|\mathbf{a}-\mathbf{a}'|.$$

Since by assumption $|K^{\mathbf{a}'}|=|K^{\mathbf{a}}|$, this proves that

$$\sum_{1\le i,\,j\le N,\,i\ne j}b_{ij}^{\mathbf{a}}(d_i^{\mathbf{a}'}-d_i^{\mathbf{a}})\,\mathrm{dist}_{\mathrm{s}}(\pi_i^{\mathbf{a}}(O),\,\partial H_{ij}^{\mathbf{a}})=o(1)|\mathbf{a}-\mathbf{a}'|.\tag{2.5}$$

Using π_i (resp. ∂H_{ij}) to denote $\pi_i^{\mathbf{0}}$ (resp. $\partial H_{ij}^{\mathbf{0}}$), we see that

$$\mathrm{dist}_{\mathrm{s}}(\pi_i^{\mathbf{a}}(O),\,\partial H_{ij}^{\mathbf{a}})=\mathrm{dist}_{\mathrm{s}}(\pi_i(O),\,\partial H_{ij})+o(1).$$

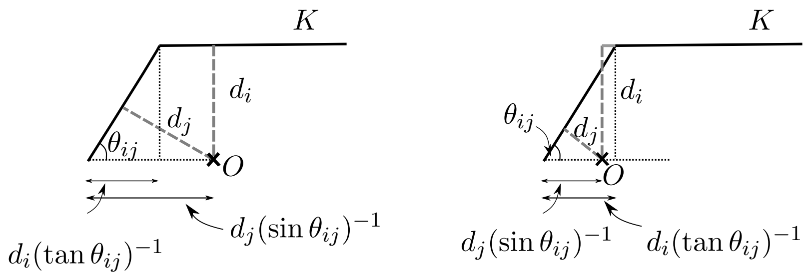

FIGURE 2. We illustrate why formula (2.7) holds. On the left-hand side, we consider the case when the origin O projects inside the i-th fact of ∂K, while on the right-hand side the case when the origin O projects outside the i-th fact of ∂K (so in this case $\mathrm{dist_s}(\pi(O), \partial H_{ij})$ is negative).

Therefore, since $|d_i^{\mathbf{a}'} - d_i^{\mathbf{a}}| \le C|\mathbf{a}' - \mathbf{a}|$, it follows by (2.5) that

$$\sum_{1\le i,\, j\le N,\, i\ne j} b_{ij}^{\mathbf{a}}(d_i^{\mathbf{a}'} - d_i^{\mathbf{a}})\,\mathrm{dist_s}(\pi_i(O),\, \partial H_{ij}) = o(1)|\mathbf{a} - \mathbf{a}'|. \tag{2.6}$$

Noticing now that

$$d_j(\sin\theta_{ij})^{-1} - d_i(\tan\theta_{ij})^{-1} = \mathrm{dist_s}(\pi_i(O),\, \partial H_{ij}) \tag{2.7}$$

(see Figure 2), combining (2.4), (2.6), and (2.7) we obtain that

$$\mathscr{F}(K^{\mathbf{a}'}) - \mathscr{F}(K^{\mathbf{a}}) = o(1)|\mathbf{a} - \mathbf{a}'|,$$

as desired. □

3. Proof of Theorem 1.1

Let $E \subset \mathbb{R}^n$ be an (ϵ, R)-minimizer with $|E| = |K|$ and $R \ge n+1$. Thanks to Lemma 2.1, we know that for ϵ small enough E is close to a translation of K. In particular, up to a translation, we can apply Lemma 2.2 to E. We denote by $K^{\mathbf{a}} \in \mathscr{C}(K)$ the set provided by such lemma.

3.1. **An almost identity map.** Recall that $\partial K^{\mathbf{a}} \cap V_i^{\mathbf{a}}$ is parallel to $\partial K \cap V_i$ for all $1 \le i \le N$. Moreover, by the construction of $K^{\mathbf{a}}$, when $|\mathbf{a}|$ is much smaller than the diameter of any edges of K, the number of vertices of each face of $K^{\mathbf{a}}$ is not less than that of K.

More precisely, let $\{\hat{x}_\ell\}_{\ell=1}^L$ denote the vertices of K. Any vertex of K corresponds to an intersection of at least n transversal hyperplanes (the hyperplanes H_i, in the notation in the proof of Lemma 2.3). When the number of hyperplanes meeting at a vertex $\hat{x}_\ell$ is exactly n, then (by transversality of the hyperplanes) any small perturbation of them will still meet at a single point $\hat{x}_\ell^{\mathbf{a}}$ (the new vertex of $\partial K^{\mathbf{a}}$). On the other hand, if the number of hyperplanes meeting at $\hat{x}_\ell$ is larger than n, again by transversality it follows that the intersection points of the translated hyperplanes (equivalently, the number of vertices of $\partial K^{\mathbf{a}}$ close to $\hat{x}_\ell$) may

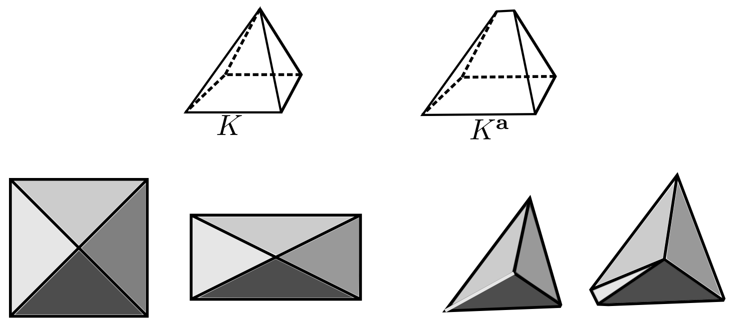

FIGURE 3. The sets K and $K^{\mathbf{a}}$ do not necessarily have the same number of vertices. For example, when K is a pyramid and $K^{\mathbf{a}}$ is obtained by moving one of the lateral sides of K, the number of vertices may increase.
If one looks at one face of ∂K and the corresponding one in $\partial K^{\mathbf{a}}$, whenever the number of vertices of that face did not increase (as in the square depicted above) then one can find a nice transformation between the two faces. Instead, whenever the number of vertices increases (as in the case of the triangle), then there is a well defined map from the face of $\partial K^{\mathbf{a}}$ to the one of ∂K which collapses the small triangle onto a segment.

increase but never become zero. So, in every case, the vertex $\hat{x}_\ell$ generates a finite family $\{\hat{x}^{\mathbf{a}}_{\ell,m}\}_{m=1}^{M_\ell}$, with $M_\ell \geq 1$.

Now, since the number of vertices of each face of $K^{\mathbf{a}}$ is not less than that of K, there is a natural map between the vertices of $\partial K^{\mathbf{a}}$ to those of ∂K: this map sends each vertex of $\partial K^{\mathbf{a}}$ to the closest one on ∂K within a distance proportional to $|\mathbf{a}|$ (in the notation above, this map sends the family $\{\hat{x}^{\mathbf{a}}_{\ell,m}\}_{m=1}^{M_\ell}$ onto $\hat{x}_\ell$). We note that this map is not necessarily one to one, see Figure 3.

Starting from this map, we construct a map $\phi\colon \partial K^{\mathbf{a}} \to \partial K$ as follows: Given a face $\partial K^{\mathbf{a}} \cap V_i^{\mathbf{a}}$ of $\partial K^{\mathbf{a}}$, we split it into a union of simplices obtained by connecting all vertices of $\partial K^{\mathbf{a}} \cap V_i^{\mathbf{a}}$ to the barycenter $b_i^{\mathbf{a}}$ of that face, and then we map each simplex onto the corresponding one in ∂K using an affine map that sends vertices to vertices and the barycenter to the barycenter. This produces a globally Lipschitz, piecewise-affine map ϕ from $\partial K^{\mathbf{a}}$ onto ∂K.

Now, let us consider our piecewise-affine map ϕ on one of the simplices $\Sigma_k^{\mathbf{a}} \subset \partial K^{\mathbf{a}}$. If the vertices corresponding to $\Sigma_k^{\mathbf{a}}$ are in one-to-one correspondence with the ones of the corresponding simplex of ∂K, it means that $D\phi|_{\Sigma_k^{\mathbf{a}}}$ is of the form $\mathrm{Id}_{n-1} + o(1)A_{\mathbf{a}}$, where Id_{n-1} is the identity matrix in $\mathbb{R}^{(n-1)\times(n-1)}$, $o(1)$ is a quantity going to zero as $|\mathbf{a}| \to 0$, and $A_{\mathbf{a}} \in \mathbb{R}^{(n-1)\times(n-1)}$ is a matrix with uniformly bounded entries. Hence, on such a simplex, the tangential divergence of ϕ is given by

$$\big(\mathrm{div}_\tau \phi\big)|_{\Sigma_k^{\mathbf{a}}} = \mathrm{tr}\big(\mathrm{Id}_{n-1} + o(1)A_{\mathbf{a}}\big) = n-1+o(1).$$

On the other hand, if some vertices of $\Sigma_k^{\mathbf{a}}$ are mapped to the same point, then $\phi|_{\Sigma_k^{\mathbf{a}}}$ will be degenerate in some direction. In particular, in some suitable system of coordinates, $D\phi|_{\Sigma_k^{\mathbf{a}}}$

will be of the form $\mathrm{Id}_m + o(1)A_{\mathbf{a}}$, where $\mathrm{Id}_m \in \mathbb{R}^{(n-1)\times(n-1)}$ coincides with the identity matrix in $\mathbb{R}^{m\times m}$ for some $m \le n-2$ and vanishes in the remaining entries, $o(1)$ is a quantity going to zero as $|\mathbf{a}| \to 0$, and $A_{\mathbf{a}} \in \mathbb{R}^{(n-1)\times(n-1)}$ is a matrix with uniformly bounded entries. Hence, in this case,

$$\big(\mathrm{div}_\tau \phi\big)|_{\Sigma_k^{\mathbf{a}}} = \mathrm{tr}\big(\mathrm{Id}_m + o(1)A_{\mathbf{a}}\big) = m + o(1) \le n-1.$$

Thus, we have constructed a map $\phi\colon \partial K^{\mathbf{a}} \to \partial K$ so that, for each $i = 1, \ldots, N$,

$$\phi(\partial K^{\mathbf{a}} \cap V_i^{\mathbf{a}}) = \partial K \cap V_i, \qquad \mathrm{dist}(x, \partial V_i^{\mathbf{a}}) \le C(n,K)\, \mathrm{dist}(\phi(x), \partial V_i) \quad \forall\, x \in \partial K^{\mathbf{a}} \cap V_i^{\mathbf{a}}, \tag{3.1}$$

and the following control on its tangential divergence holds:

$$\mathrm{div}_\tau \phi \le n - 1 + o(1) \quad \text{on } \partial K^{\mathbf{a}}, \tag{3.2}$$

and

$$\mathrm{div}_\tau \phi = n - 1 + o(1) \quad \text{on } \partial K^{\mathbf{a}} \setminus T, \qquad \mathcal{H}^{n-1}(T) = o(1), \tag{3.3}$$

where $T \subset \partial K^{\mathbf{a}}$ is given by the union of the simplices on which the map ϕ is degenerate.

We now consider the vector field $X : \mathbb{R}^n \setminus \{O\} \to \partial K$ defined as

$$X(x) := \phi\bigg(\frac{x}{f_*^{\mathbf{a}}(x)}\bigg).$$

Since $f_*^{\mathbf{a}} \equiv 1$ on $\partial K^{\mathbf{a}}$, it follows by (3.2) and (3.3) that

$$\mathrm{div} X \le \frac{n-1+o(1)}{f_*^{\mathbf{a}}(x)} \qquad \text{on } \mathbb{R}^n, \tag{3.4}$$

and

$$\mathrm{div} X = \frac{n-1+o(1)}{f_*^{\mathbf{a}}(x)} \qquad \text{on } \mathbb{R}^n \setminus \mathcal{C}_T, \tag{3.5}$$

where $\mathcal{C}_T$ denotes the cone generated by T, namely $\mathcal{C}_T = \{tx\colon t > 0,\, x \in T\}$.

Also, since $X(x) \in K$, we have

$$f(\nu) = \sup_{z \in K} \nu \cdot z \ge \nu \cdot X(x) \qquad \forall\, x \in \mathbb{R}^n \setminus \{O\}.$$

Furthermore, we note that $X(x) \in \partial K \cap V_i$ for each $x \in V_i^{\mathbf{a}} \setminus \{O\}$. In particular, $f(\nu) = X \cdot \nu$ on $\partial K^{\mathbf{a}}$. Thanks to this, we get

$$\begin{aligned}
\mathscr{F}(E) - \mathscr{F}(K^{\mathbf{a}}) &= \int_{\partial^* E} f(\nu)\, d\mathcal{H}^{n-1} - \int_{\partial K^{\mathbf{a}}} f(\nu)\, d\mathcal{H}^{n-1} \\
&= \int_{\partial^* E} f(\nu)\, d\mathcal{H}^{n-1} - \int_{\partial K^{\mathbf{a}}} X \cdot \nu\, d\mathcal{H}^{n-1} \\
&= \int_{\partial^* E} [f(\nu) - X \cdot \nu]\, d\mathcal{H}^{n-1} + \int_{\partial^* E} X \cdot \nu\, d\mathcal{H}^{n-1} - \int_{\partial K^{\mathbf{a}}} X \cdot \nu\, d\mathcal{H}^{n-1}
\end{aligned} \tag{3.6}$$

Set

$$I := \int_{\partial^* E} [f(\nu) - X \cdot \nu]\, d\mathcal{H}^{n-1}$$

and

$$II := \int_{\partial^* E} X \cdot \nu\, d\mathcal{H}^{n-1} - \int_{\partial K^{\mathbf{a}}} X \cdot \nu\, d\mathcal{H}^{n-1}.$$

We estimate I and II in the following two subsections, respectively.

3.2. **Estimate on I.** In this section we prove that, if $|\mathbf{a}|$ is sufficiently small, then

$$I \geq c(n,K)\,|E\Delta K^{\mathbf{a}}|. \tag{3.7}$$

By additivity of the integral, it suffices to prove that

$$\int_{\partial^* E\cap V_i^{\mathbf{a}}}[f(\nu)-X\cdot\nu]\,d\mathcal{H}^{n-1} \geq c(n,K)\,|(E\Delta K^{\mathbf{a}})\cap V_i^{\mathbf{a}}| \qquad \forall\, i=1,\ldots,N.$$

To simplify the statement, we remove the subscript i. Thus, we assume that $V^{\mathbf{a}}$ is one of the cones for $K^{\mathbf{a}}$, and V the corresponding cone for K. Also, up to a change of coordinate we can assume that for $x\in V^{\mathbf{a}}$ we have $f_*^{\mathbf{a}}(x)=\alpha x_n$ for some $\alpha>0$ (equivalently, $\partial K^{\mathbf{a}}\cap V^{\mathbf{a}}$ is contained in the hyperplane $\{x_n=\alpha^{-1}\}$). Then our aim is to show

$$\int_{\partial^* E\cap V^{\mathbf{a}}}[f(\nu)-X\cdot\nu]\,d\mathcal{H}^{n-1} \geq c(n,K)\,|(E\Delta K^{\mathbf{a}})\cap V^{\mathbf{a}}|. \tag{3.8}$$

Note that, since $\partial^* E$ is close to ∂K (see Lemma 2.1), $\partial^* E$ is uniformly away from the origin and X is well-defined and uniformly Lipschitz on $\partial^* E$.

By the definition of f and (3.1), for any outer normal ν at $x\in\partial^* E\cap V$ we have

$$f(\nu)-\nu\cdot X \geq \sup_{y\in\partial K\cap\partial V} \nu\cdot y-\nu\cdot X.$$

Since $X(x)\in\partial K\cap V$ for $x\in V^{\mathbf{a}}\setminus\{O\}$, it follows that

$$y-X(x) \text{ is parallel to } \partial K\cap V \qquad \text{for all } x\in\partial^* E\cap V,\ y\in\partial K\cap\partial V.$$

In particular, denoting by ν' the projection of ν onto the first $(n-1)$-variables in $\mathbb{R}^n$ (namely, $\nu=(\nu',\nu_n)\in\mathbb{R}^{n-1}\times\mathbb{R}$), choosing $y\in\partial K\cap\partial V$ so that $y-X$ is parallel to ν' and recalling (3.1), we deduce that

$$\begin{aligned} f(\nu)-\nu\cdot X &\geq \nu\cdot(y-X)\geq c(n,K)\,|\nu'|\,\mathrm{dist}(X(x),\,\partial K\cap\partial V)\\ &\geq c(n,K)\,|\nu'|\,\mathrm{dist}\Big(\frac{x}{f_*^{\mathbf{a}}(x)},\,\partial V^{\mathbf{a}}\Big)\geq c(n,K)\,|\nu'|\,\mathrm{dist}(x,\,\partial V^{\mathbf{a}}), \end{aligned}$$

where the last inequality follows from the fact that $f_*^{\mathbf{a}}\leq 2$ on $\partial^* E$ (since $\partial^* E$ is close to ∂K).

Then, by the coarea formula on rectifiable sets (see e.g. [14, Theorem 18.8]),

$$\begin{aligned} \int_{\partial^* E\cap V^{\mathbf{a}}}[f(\nu)-X\cdot\nu]\,d\mathcal{H}^{n-1} &\geq c(n,K)\int_{\partial^* E\cap V^{\mathbf{a}}}|\nu'|\,\mathrm{dist}(x,\,\partial V^{\mathbf{a}})\,d\mathcal{H}^{n-1}\\ &= c(n,K)\int_{\mathbb{R}}\int_{(\partial^* E\cap V^{\mathbf{a}})_t}\mathrm{dist}(x,\,\partial V^{\mathbf{a}})\,d\mathcal{H}^{n-2}\,dt, \end{aligned} \tag{3.9}$$

where, given a set F, we denote by $(F)_t$ the slice at height t, that is

$$(F)_t := \{x\in F\colon x_n=t\}.$$

Recalling that $\partial K^{\mathbf{a}}\cap V^{\mathbf{a}}\subset\{y\in\mathbb{R}^n\colon y_n=\alpha^{-1}\}$, we now consider two cases, depending on whether

$$\mathcal{H}^{n-1}((E)_{\alpha^{-1}}\cap V^{\mathbf{a}})\leq\frac{1}{2}\mathcal{H}^{n-1}((V^{\mathbf{a}})_{\alpha^{-1}})$$

or not.

In the first case, since $\partial E \cap V^{\mathbf{a}}$ is almost a graph with respect to the x_n variable (see Lemma A.2), for all $t \geq \alpha^{-1}$ and for ϵ small enough we have

$$\mathcal{H}^{n-1}((E)_t \cap V^{\mathbf{a}}) \leq \mathcal{H}^{n-1}((E)_{\alpha^{-1}} \cap V^{\mathbf{a}}) + C\epsilon \leq \frac{3}{4}\mathcal{H}^{n-1}((V^{\mathbf{a}})_{\alpha^{-1}}).$$

Hence, since

$$\mathcal{H}^{n-2}((\partial^* E \cap V^{\mathbf{a}})_t) = \mathcal{H}^{n-2}(\partial^*(E)_t \cap V^{\mathbf{a}}_t)$$

for a.e. t (see e.g. [14, Theorem 18.11]), we can apply Lemma A.1 with $d = n-1$, $F = (E)_t \cap V^{\mathbf{a}}$, and $\Omega = (V^{\mathbf{a}})_t$, to get

$$\begin{aligned}\int_{\mathbb{R}} \int_{(\partial^* E \cap V^{\mathbf{a}})_t} \mathrm{dist}(x,\, \partial V^{\mathbf{a}})\, d\mathcal{H}^{n-2}\, dt &\geq \int_{t \geq \alpha^{-1}} \int_{(\partial^* E \cap V^{\mathbf{a}})_t} \mathrm{dist}(x,\, (\partial V^{\mathbf{a}})_t)\, d\mathcal{H}^{n-2}\, dt \\ &\geq c(n,K) \int_{t \geq \alpha^{-1}} \mathcal{H}^{n-1}((E)_t \cap V^{\mathbf{a}})\, dt \\ &= c(n,K)\, |(E \setminus K^{\mathbf{a}}) \cap V^{\mathbf{a}}|.\end{aligned}$$

Since $|E \cap V^{\mathbf{a}}| = |K^{\mathbf{a}} \cap V^{\mathbf{a}}|$ (see Lemma 2.2), it follows that

$$|(E \setminus K^{\mathbf{a}}) \cap V^{\mathbf{a}}| = \frac{1}{2}|(E \Delta K^{\mathbf{a}}) \cap V^{\mathbf{a}}|,$$

and we conclude that

$$\int_{\partial^* E \cap V^{\mathbf{a}}} [f(\nu) - \nu \cdot X]\, d\mathcal{H}^{n-1} \geq c(n,K)\, |(E \cap V^{\mathbf{a}}) \Delta K^{\mathbf{a}}|,$$

as desired.

In the second case, namely when

$$\mathcal{H}^{n-1}((E)_{\alpha^{-1}} \cap V^{\mathbf{a}}) > \frac{1}{2}\mathcal{H}^{n-1}((V^{\mathbf{a}})_{\alpha^{-1}}),$$

we simply apply the argument above to $V^{\mathbf{a}} \setminus E$, and conclude as before.

3.3. **Estimate on II.** In this section we prove that

$$II \geq -\big(C\epsilon^{\frac{1}{n}} + o(1)\big)\, |E \setminus K^{\mathbf{a}}|, \tag{3.10}$$

where $C = C(n,K)$, and $o(1)$ is a quantity that goes to 0 as $|\mathbf{a}| \to 0$.

Note that, by the volume constraint $|E| = |K^{\mathbf{a}}|$, it follows that

$$|E \setminus K^{\mathbf{a}}| = |K^{\mathbf{a}} \setminus E|.$$

Also, since ∂E and ∂K are $(C\epsilon^{\frac{1}{n}})$-close (see Lemma 2.1), it follows by (3.4) and (3.5) that

$$|\mathrm{div} X - (n-1)| \leq C\epsilon^{\frac{1}{n}} + o(1) \qquad \text{on } (E \Delta K^{\mathbf{a}}) \setminus \mathcal{C}_T,$$

and

$$-C \leq \mathrm{div} X - (n-1) \leq C\epsilon^{\frac{1}{n}} + o(1) \qquad \text{on } (E \Delta K^{\mathbf{a}}) \cap \mathcal{C}_T,$$

where $C = C(n,K)$. Hence, by the divergence theorem we get

$$
\begin{aligned}
II = \int_{\partial^* E} X\cdot\nu\, d\mathcal{H}^{n-1} - \int_{\partial K^{\mathbf{a}}} X\cdot\nu\, d\mathcal{H}^{n-1}
&= \int_{E\setminus K^{\mathbf{a}}} \mathrm{div} X\, d\mathcal{H}^{n-1} - \int_{K^{\mathbf{a}}\setminus E} \mathrm{div} X\, d\mathcal{H}^{n-1} \\
&= \int_{E\setminus K^{\mathbf{a}}} [\mathrm{div} X - (n-1)]\, d\mathcal{H}^{n-1} \\
&\quad - \int_{K^{\mathbf{a}}\setminus E} [\mathrm{div} X - (n-1)]\, d\mathcal{H}^{n-1} \\
&\geq \int_{(E\setminus K^{\mathbf{a}})\cap \mathcal{C}_T} [\mathrm{div} X - (n-1)]\, d\mathcal{H}^{n-1} - \big(C\epsilon^{\frac{1}{n}} + o(1)\big)|K^{\mathbf{a}}\Delta E| \\
&\geq -C|(E\setminus K^{\mathbf{a}})\cap \mathcal{C}_T| - \big(C\epsilon^{\frac{1}{n}} + o(1)\big)|K^{\mathbf{a}}\Delta E|,
\end{aligned}
\tag{3.11}
$$

Thus, to conclude the proof, we need to show that $|(E\setminus K^{\mathbf{a}})\cap \mathcal{C}_T|$ is small compared to $|K^{\mathbf{a}}\Delta E|$.

To this aim, we write

$$K^{\mathbf{a}}_{1+r} = (1+r)K^{\mathbf{a}}$$

and note that

$$\partial K^{\mathbf{a}}_{1+r} = \{x\in\mathbb{R}^n\colon f^{\mathbf{a}}_*(x) = 1+r\}.$$

Then, since $|\nabla f^{\mathbf{a}}_*| \leq C(n,K)$ and $f(\nu)\leq C(n,K)$, by the classical coarea formula (see for instance [7, Section 3.4.4, Proposition 3]) we get

$$
\begin{aligned}
|E\setminus K^{\mathbf{a}}| &= \int_0^\infty \int_{\partial K^{\mathbf{a}}_{1+r}} \frac{1}{|\nabla f^{\mathbf{a}}_*|}\chi_{E\setminus K^{\mathbf{a}}}\, d\mathcal{H}^{n-1}\, dr \\
&\geq c(n,K)\int_0^\infty \mathcal{H}^{n-1}(E\cap\partial K^{\mathbf{a}}_{1+r})\, dr \geq c(n,K)\int_0^\infty\int_{E\cap\partial K^{\mathbf{a}}_{1+r}} f(\nu)\, d\mathcal{H}^{n-1}\, dr.
\end{aligned}
$$

Thus, setting for simplicity $\mathbf{v} := |E\setminus K^{\mathbf{a}}|$, we get

$$\mathbf{v} \geq c(n,K)\int_0^\infty\int_{E\cap\partial K^{\mathbf{a}}_{1+r}} f(\nu)\, d\mathcal{H}^{n-1}\, dr \geq c(n,K)\int_0^{M\mathbf{v}}\int_{E\cap\partial K^{\mathbf{a}}_{1+r}} f(\nu)\, d\mathcal{H}^{n-1}\, dr,$$

for some large constant M to be determined. Then by the mean value theorem, there exists $r_0\in[0,M\mathbf{v}]$ so that

$$\int_{E\cap\partial K^{\mathbf{a}}_{1+r_0}} f(\nu)\, d\mathcal{H}^{n-1} \leq \frac{1}{M}.$$

Since ∂E is almost a Lipschitz graph by Lemma A.2, we conclude that

$$\int_{E\cap\partial K^{\mathbf{a}}_{1+r}} f(\nu)\, d\mathcal{H}^{n-1} \leq \frac{1}{M} + C\epsilon \qquad \forall\, r\geq r_0. \tag{3.12}$$

Moreover, recalling that $\mathcal{C}_T$ is the cone over a set $T \subset \partial K^{\mathbf{a}}$ satisfying $\mathcal{H}^{n-1}(T) = o(1)$, we further have

$$|(E \setminus K^{\mathbf{a}}) \cap \mathcal{C}_T \cap K^{\mathbf{a}}_{1+r_0}| \le C(K) r_0 \mathcal{H}^{n-1}(T \cap K^{\mathbf{a}}) \le o(1) M \mathbf{v}. \tag{3.13}$$

We now claim that, if $M = M(n, K)$ is sufficiently large, then

$$E \subset K^{\mathbf{a}}_{1+r_0} \tag{3.14}$$

(recall that r_0 depends on M). Towards this, we first show the following lemma.

Lemma 3.1. *Let V be a cone for ∂K, and denote by x_0 the barycenter of the face $\partial K \cap V$. Then, for any $\nu \in \mathbb{R}^n$ we have*

$$f(\nu) \ge \nu \cdot x_0 + c(n, K) \left|\nu'\right|,$$

where ν' denotes the projection of ν onto the hyperplane parallel to $\partial K \cap V$.

Proof. By the definition of f we have

$$f(\nu) \ge \sup_{x \in \partial K \cap \partial V} \nu \cdot x \ge \nu \cdot x_0 + \sup_{x \in \partial K \cap \partial V} \nu \cdot (x - x_0).$$

Notice that $x - x_0$ is parallel to the face $\partial K \cap V$. Thus, by choosing $x \in \partial K \cap \partial V$ so that $x - x_0$ is parallel to ν', and noticing that $|x - x_0| \ge c(n, K) > 0$ (since x_0 is the barycenter of $\partial K \cap V$), we obtain

$$f(\nu) \ge \nu \cdot x_0 + \nu \cdot (x - x_0) \ge \nu \cdot x_0 + c(n, K)|\nu'|,$$

as desired. $\square$

Now, we fix one of the cones $V^{\mathbf{a}} = V_i^{\mathbf{a}}$, and apply Lemma 3.1 to $V = V_i$. Up to a change of variables we can assume that the normal of $\partial K \cap V$ is given by e_n. Hence, denoting by ν' the projection of ν onto the first $(n-1)$-variables (i.e., $\nu = (\nu', \nu_n) \in \mathbb{R}^{n-1} \times \mathbb{R}$), we have

$$\begin{aligned} &\int_{(\partial^* E \cap V^{\mathbf{a}}) \setminus K^{\mathbf{a}}_{1+r_0}} f(\nu)\, d\mathcal{H}^{n-1} \\ &\qquad \ge \int_{(\partial^* E \cap V^{\mathbf{a}}) \setminus K^{\mathbf{a}}_{1+r_0}} \nu \cdot x_0 \, d\mathcal{H}^{n-1} + c(n, K) \int_{(\partial^* E \cap V^{\mathbf{a}}) \setminus K^{\mathbf{a}}_{1+r_0}} |\nu'|\, d\mathcal{H}^{n-1}. \end{aligned}$$

For the first term, by the divergence theorem applied to the constant vector field x_0 inside the set

$$(E \cap V^{\mathbf{a}}) \setminus K^{\mathbf{a}}_{1+r_0},$$

we get

$$\int_{(\partial^* E \cap V^{\mathbf{a}}) \setminus K^{\mathbf{a}}_{1+r_0}} \nu \cdot x_0 \, d\mathcal{H}^{n-1} = \int_{\partial K^{\mathbf{a}}_{1+r_0} \cap V^{\mathbf{a}} \cap E} e_n \cdot x_0 \, d\mathcal{H}^{n-1} - \int_{(\partial V^{\mathbf{a}} \cap E) \setminus K^{\mathbf{a}}_{1+r_0}} \nu \cdot x_0 \, d\mathcal{H}^{n-1}.$$

Note that, since $x_0 \in V^{\mathbf{a}}$ (for $|\mathbf{a}| \ll 1$) and $V^{\mathbf{a}}$ is a convex cone, it follows that $\nu \cdot x_0 \le 0$ on $(\partial V^{\mathbf{a}} \cap E) \setminus K^{\mathbf{a}}_{1+r_0}$. Also, $x_0 \cdot e_n = f(e_n) = f(\nu)$ on $\partial K^{\mathbf{a}}_{1+r_0}$ (see Figure 4). Hence

$$\int_{(\partial^* E \cap V^{\mathbf{a}}) \setminus K^{\mathbf{a}}_{1+r_0}} \nu \cdot x_0 \, d\mathcal{H}^{n-1} \ge \int_{\partial K^{\mathbf{a}}_{1+r_0} \cap V^{\mathbf{a}} \cap E} f(\nu)\, d\mathcal{H}^{n-1}.$$

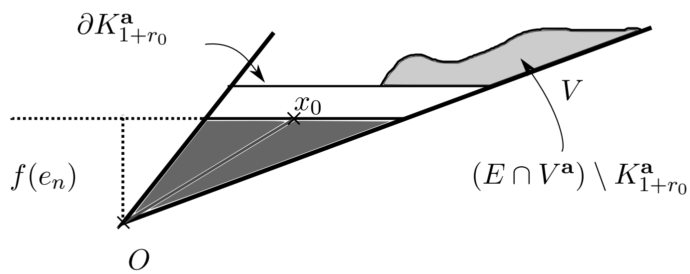

FIGURE 4. The vector x_0 and the length of $f(e_n)$ are shown in the figure. The dark grey part represents $K^{\mathbf{a}} \cap V^{\mathbf{a}}$. In our proof, we apply the divergence theorem to the constant vector field x_0 inside the set $(E \cap V^{\mathbf{a}}) \setminus K^{\mathbf{a}}_{1+r_0}$.

For the second term, we apply first the coarea formula for rectifiable sets (see e.g. [14, Theorem 18.8]) to get

$$\int_{(\partial^* E \cap V_i^{\mathbf{a}}) \setminus K^{\mathbf{a}}_{1+r_0}} |\nu'| \, d\mathcal{H}^{n-1} = \int_{r_0}^{\infty} \int_{\partial^* E \cap V_i^{\mathbf{a}} \cap \partial K^{\mathbf{a}}_{1+r}} d\mathcal{H}^{n-2} \, dr.$$

Then, provided M^{-1} and ϵ are small enough, thanks to (3.12) we can apply the relative isoperimetric inequality to $E \cap V^{\mathbf{a}} \cap \partial K^{\mathbf{a}}_{1+r}$ inside the convex set $\partial K^{\mathbf{a}}_{1+r}$ for $r \geq r_0$ to obtain

$$\begin{aligned}\int_{r_0}^{\infty} \int_{\partial^* E \cap V^{\mathbf{a}} \cap \partial K^{\mathbf{a}}_{1+r}} d\mathcal{H}^{n-2} \, dr &\geq c(n,K) \int_{r_0}^{\infty} \left(\mathcal{H}^{n-1}(E \cap V^{\mathbf{a}} \cap \partial K^{\mathbf{a}}_{1+r})\right)^{\frac{n-2}{n-1}} \, dr \\ &\geq c(n,K) \left(M^{-1} + C\epsilon\right)^{-\frac{1}{n-1}} \int_{r_0}^{\infty} \mathcal{H}^{n-1}(E \cap V^{\mathbf{a}} \cap \partial K^{\mathbf{a}}_{1+r}) \, dr \\ &= c(n,K) \left(M^{-1} + C\epsilon\right)^{-\frac{1}{n-1}} |(E \cap V^{\mathbf{a}}) \setminus K^{\mathbf{a}}_{1+r_0}|,\end{aligned}$$

where the second inequality follows by (3.12).

Combining all the previous estimates, we proved that

$$\begin{aligned}\int_{(\partial^* E \cap V^{\mathbf{a}}) \setminus K^{\mathbf{a}}_{1+r_0}} f(\nu) \, d\mathcal{H}^{n-1} - \int_{\partial K^{\mathbf{a}}_{1+r_0} \cap V^{\mathbf{a}} \cap E} f(\nu) \, d\mathcal{H}^{n-1}& \\ \geq c(n,K) \left(M^{-1} + C\epsilon\right)^{-\frac{1}{n-1}} |(E \cap V^{\mathbf{a}}) \setminus K^{\mathbf{a}}_{1+r_0}|&,\end{aligned}$$

and by summing this inequality over all cones $V_i^{\mathbf{a}}$ we conclude that

$$\int_{\partial^* E \setminus K^{\mathbf{a}}_{1+r_0}} f(\nu) \, d\mathcal{H}^{n-1} - \int_{\partial K^{\mathbf{a}}_{1+r_0} \cap E} f(\nu) \, d\mathcal{H}^{n-1} \geq c(n,K) \left(M^{-1} + C\epsilon\right)^{-\frac{1}{n-1}} |E \setminus K^{\mathbf{a}}_{1+r_0}|. \tag{3.15}$$

On the other hand, if we test the (ϵ, R)-minimality of E against the set

$$G = (1+\lambda)[E \cap K^{\mathbf{a}}_{1+r_0}],$$

where λ is chosen so that

$$\big|(1+\lambda)[E\cap K^{\mathbf{a}}_{1+r_0}]\big| = |E|$$

(note that this set is admissible for ϵ, and hence $|E\Delta K|$, small enough), we obtain

$$\mathscr{F}(E) \le (1+\lambda)^{n-1}\mathscr{F}(E\cap K^{\mathbf{a}}_{1+r_0}) + C\lambda\epsilon \le \mathscr{F}(E\cap K^{\mathbf{a}}_{1+r_0}) + C(n,K)\lambda.$$

Also, by the definition of λ and Lemma 2.1, we easily get the bound

$$\lambda \le C(n,K)|E\setminus K^{\mathbf{a}}_{1+r_0}|,$$

therefore

$$\mathscr{F}(E) \le \mathscr{F}(E\cap K^{\mathbf{a}}_{1+r_0}) + C(n,K)|E\setminus K^{\mathbf{a}}_{1+r_0}|.$$

Combining this bound with (3.15), we conclude that

$$\begin{aligned} c(n,K)\left(M^{-1}+C\epsilon\right)^{-\frac{1}{n-1}}|E\setminus K^{\mathbf{a}}_{1+r_0}| &\le \int_{\partial^* E\setminus K^{\mathbf{a}}_{1+r_0}} f(\nu)\,d\mathcal{H}^{n-1} - \int_{\partial K^{\mathbf{a}}_{1+r_0}\cap E} f(\nu)\,d\mathcal{H}^{n-1} \\ &= \mathscr{F}(E) - \mathscr{F}(E\cap K^{\mathbf{a}}_{1+r_0}) \le C(n,K)|E\setminus K^{\mathbf{a}}_{1+r_0}|. \end{aligned}$$

Thus, for $M = M(n,K) > 0$ sufficiently large and $\epsilon \le \epsilon_0(n,K)$ small enough, we conclude that $|E\setminus K^{\mathbf{a}}_{1+r_0}| = 0$, and (3.14) follows.

Finally, (3.10) is an immediate consequence of (3.11), (3.13), and (3.14).

3.4. **Conclusion.** Since E is an (ϵ,R)-minimizer with $R\ge n+1$, for $\epsilon \ll 1$ we have that $|\mathbf{a}|\ll 1$, and therefore

$$\mathscr{F}(E) \le \mathscr{F}(K^{\mathbf{a}}) + \epsilon|E\Delta K^{\mathbf{a}}|.$$

On the other hand, combining (3.6), (3.7), and (3.10), we get

$$\mathscr{F}(E) - \mathscr{F}(K^{\mathbf{a}}) \ge c(n,K)|E\Delta K^{\mathbf{a}}|.$$

Choosing ϵ sufficiently small we conclude that $|E\Delta K^{\mathbf{a}}| = 0$, which proves Theorem 1.1.

4. Proof of Theorem 1.5 and 1.6

Proof of Theorem 1.5. Let $K^{\mathbf{a}} \in \mathscr{C}(K)$ with small $|\mathbf{a}|$, and let ϵ_0 be the constant in Theorem 1.1. Then, for any $\epsilon \le \epsilon_0$, we consider the variational problem

$$\min\big\{\mathscr{F}(E) + \epsilon|E\Delta K^{\mathbf{a}}| \colon |E| = |K|\big\}. \tag{4.1}$$

By the concentration-compactness principle, if $\{E_m\}_{m\ge1}$ is a minimizing sequence then, up to a subsequence, $E_m \to \hat{E}$ in $L^1_{\mathrm{loc}}(\mathbb{R}^n)$, where $\delta = 1 - \frac{|\hat{E}|}{|K|}$ is nonnegative, and[3]

$$\mathscr{F}(\hat{E}) + \epsilon|\hat{E}\Delta K^{\mathbf{a}}| + n|K|\delta^{\frac{n-1}{n}} \le \liminf_{m\to\infty}\big(\mathscr{F}(E_m) + \epsilon|E_m\Delta K^{\mathbf{a}}|\big). \tag{4.2}$$

Note that $|\hat{E}\Delta K^{\mathbf{a}}| \ge |K^{\mathbf{a}}| - |\hat{E}| = \delta|K|$. Hence, since $\{E_m\}_{m\ge1}$ is a minimizing sequence, using both $K^{\mathbf{a}}$ and $(1-\delta)^{-\frac{1}{n}}\hat{E}$ as competitors, applying Wulff inequality (1.1) to $\hat{E}$, and recalling Lemma 2.3, we get

$$n|K|(1-\delta)^{\frac{n-1}{n}} + \frac{\epsilon}{2}\delta|K| + \frac{\epsilon}{2}|\hat{E}\Delta K^{\mathbf{a}}| + n|K|\delta^{\frac{n-1}{n}} \le \mathscr{F}(K^{\mathbf{a}}) \le n|K| + \omega(|\mathbf{a}|)|\mathbf{a}|$$

[3]The bound (4.2) follows from the lower semicontinuity of $\mathscr{F}$ applied to the part of E_m that does not escape to infinity, plus the Wulff inequality (1.1) applied to the part of E_m that may have escaped to infinity.

and

$$\mathscr{F}(\hat{E})+\epsilon|\hat{E}\Delta K^{\mathbf{a}}|+n|K|\delta^{\frac{n-1}{n}}\leq(1-\delta)^{-\frac{n-1}{n}}\mathscr{F}(\hat{E})+\epsilon\Big|\big((1-\delta)^{-\frac{1}{n}}\hat{E}\big)\Delta K^{\mathbf{a}}\Big|.$$

The first inequality implies that $\delta\to 0$ and $|\hat{E}\Delta K^{\mathbf{a}}|\to 0$ as $|\mathbf{a}|\to 0$, and then the second inequality yields

$$\begin{aligned}n|K|\delta^{\frac{n-1}{n}}&\leq\Big[(1-\delta)^{-\frac{n-1}{n}}-1\Big]\mathscr{F}(\hat{E})+\epsilon\Big[\Big|\big((1-\delta)^{-\frac{1}{n}}\hat{E}\big)\Delta K^{\mathbf{a}}\Big|-|\hat{E}\Delta K^{\mathbf{a}}|\Big]\\&=\Big[(1-\delta)^{-\frac{n-1}{n}}-1\Big]\mathscr{F}(\hat{E})+\epsilon\Big[(1-\delta)^{-1}\Big|\hat{E}\Delta\big((1-\delta)^{\frac{1}{n}}K^{\mathbf{a}}\big)\Big|-|\hat{E}\Delta K^{\mathbf{a}}|\Big]\\&\leq C(n,K)\mathscr{F}(\hat{E})\delta+\epsilon\left[(1-\delta)^{-1}-1\right]\Big|\hat{E}\Delta\big((1-\delta)^{\frac{1}{n}}K^{\mathbf{a}}\big)\Big|+\epsilon\Big|K^{\mathbf{a}}\Delta\big((1-\delta)^{\frac{1}{n}}K^{\mathbf{a}}\big)\Big|\\&\leq C(n,K)\Big(\mathscr{F}(\hat{E})+\epsilon|\hat{E}|+\epsilon|K^{\mathbf{a}}|\Big)\delta\leq C(n,K)\delta,\end{aligned}\tag{4.3}$$

where in the last inequality we used (4.2) to get a uniform bound on $\mathscr{F}(\hat{E})$.

Note that (4.3) is impossible for δ sufficiently small, unless $\delta=0$. This proves that $|\hat{E}|=|K|$ provided $|\mathbf{a}|$ is sufficiently small, therefore $\hat{E}$ is a minimizer of (4.1). It is then immediate to show that $\hat{E}$ is an ϵ-minimizer, and so it follows by Theorem 1.1 that $\hat{E}=K^{\mathbf{a}'}$ for some $|\mathbf{a}'|\ll 1$. In particular

$$\mathscr{F}(K^{\mathbf{a}'})+\epsilon|K^{\mathbf{a}'}\Delta K^{\mathbf{a}}|\leq\mathscr{F}(K^{\mathbf{a}}).$$

Since

$$|K^{\mathbf{a}'}\Delta K^{\mathbf{a}}|\geq c(n,K)|\mathbf{a}'-\mathbf{a}|$$

(this follows easily by the argument used in the proof of Lemma 2.3), Lemma 2.3 yields

$$c(n,K)\epsilon|\mathbf{a}'-\mathbf{a}|\leq\mathscr{F}(K^{\mathbf{a}})-\mathscr{F}(K^{\mathbf{a}'})\leq\omega(|\mathbf{a}|+|\mathbf{a}'|)|\mathbf{a}-\mathbf{a}'|,$$

which proves that $\mathbf{a}'=\mathbf{a}$ for $|\mathbf{a}|+|\mathbf{a}'|$ sufficiently small. Therefore, $K^{\mathbf{a}}$ is the unique minimizer of (4.1), and the theorem follows. □

Proof of Theorem 1.6. Choose σ_0 small so that we can apply Lemma 2.2 to obtain a set $K^{\mathbf{a}}$ for E. We apply the idea of [12], i.e. the selection principle.

Towards this let us assume that the conclusion of the theorem fails. Then there exist a sequence of sets of finite perimeter E_k, and sequence of vectors $\mathbf{a}_k\in\mathbb{R}^N$ with $|\mathbf{a}_k|$ small, such that $|E_k\Delta K|\to 0$, $|E_k\cap V_i^{\mathbf{a}_k}|=|K^{\mathbf{a}_k}\cap V_i^{\mathbf{a}_k}|$ for any $1\leq i\leq N$, $\mathscr{F}(E_k)-\mathscr{F}(K^{\mathbf{a}_k})\to 0$, and

$$0<\mathscr{F}(E_k)-\mathscr{F}(K^{\mathbf{a}_k})<\frac{\lambda}{2}|E_k\Delta K^{\mathbf{a}_k}|\tag{4.4}$$

for some $\lambda>0$ small enough (the smallness will be fixed later). Notice that, since $|E_k\Delta K|\to 0$, we have $|\mathbf{a}_k|\to 0$.

Consider the following variation problem:

$$\min\Big\{\mathscr{F}(E)+\lambda\big||E\Delta K^{\mathbf{a}_k}|-|E_k\Delta K^{\mathbf{a}_k}|\big|+C_0\lambda\sum_{1\leq i\leq N}\big||E\cap V_i^{\mathbf{a}_k}|-|K^{\mathbf{a}_k}\cap V_i^{\mathbf{a}_k}|\big|:|E|=|K|\Big\},\tag{4.5}$$

with $C_0 = C_0(n,K) > 0$ to be determined. By the same argument as in the proof of Theorem 1.5, there exists a minimizer F_k among sets of finite perimeter. Thus, for any set $G \subset \mathbb{R}^n$ with $|G| = |F_k|$ we have

$$\begin{aligned}\mathscr{F}(F_k) &\le \mathscr{F}(G) + \lambda\Big(\big||G\Delta K^{\mathbf{a}_k}| - |E_k\Delta K^{\mathbf{a}_k}|\big| - \big||F_k\Delta K^{\mathbf{a}_k}| - |E_k\Delta K^{\mathbf{a}_k}|\big|\Big)\\ &\qquad + C_0\lambda \sum_{1\le i\le N} \Big(\big||G\cap V_i^{\mathbf{a}_k}| - |K^{\mathbf{a}_k}\cap V_i^{\mathbf{a}_k}|\big| - \big||F_k\cap V_i^{\mathbf{a}_k}| - |K^{\mathbf{a}_k}\cap V_i^{\mathbf{a}_k}|\big|\Big)\\ &\le \mathscr{F}(G) + \lambda\big||G\Delta K^{\mathbf{a}_k}| - |F_k\Delta K^{\mathbf{a}_k}|\big| + C_0\lambda\sum_{1\le i\le N}\big||G\cap V_i^{\mathbf{a}_k}| - |F_k\cap V_i^{\mathbf{a}_k}|\big|\\ &\le \mathscr{F}(G) + \lambda|F_k\Delta G| + C_0\lambda\sum_{1\le i\le N}|(F_k\Delta G)\cap V_i^{\mathbf{a}_k}| = \mathscr{F}(G) + (1+C_0)\lambda|F_k\Delta G|.\end{aligned}$$

This proves that F_k is a $\big((1+C_0)\lambda\big)$-minimizer, and by choosing λ small enough so that $(1+C_0)\lambda \le \epsilon_0$ as in Theorem 1.1, there exists $\mathbf{a}_k' \in \mathbb{R}^N$ such that $F_k = K^{\mathbf{a}_k'}$.

Now, since $K^{\mathbf{a}_k'}$ is the minimizer of (4.5), we have

$$\mathscr{F}(K^{\mathbf{a}_k'}) + \lambda\big||K^{\mathbf{a}_k'}\Delta K^{\mathbf{a}_k}| - |E_k\Delta K^{\mathbf{a}_k}|\big| \le \mathscr{F}(E_k).$$

Thus, noticing that $\mathscr{F}(E_k) \to \mathscr{F}(K)$, $E_k \to K$, and $|\mathbf{a}_k| \to 0$, it follows by the inequality above that $|\mathbf{a}_k'| \to 0$.

Therefore, by Lemma 2.3, and by testing the minimality of $K^{\mathbf{a}_k'}$ in (4.5) against $K^{\mathbf{a}_k}$, we have

$$\begin{aligned}\omega(|\mathbf{a}_k| + |\mathbf{a}_k'|)|\mathbf{a}_k' - \mathbf{a}_k| &\ge \mathscr{F}(K^{\mathbf{a}_k}) - \mathscr{F}(K^{\mathbf{a}_k'})\\ &\ge \lambda\big||K^{\mathbf{a}_k'}\Delta K^{\mathbf{a}_k}| - |E_k\Delta K^{\mathbf{a}_k}|\big| - \lambda|E_k\Delta K^{\mathbf{a}_k}|\\ &\qquad + C_0\lambda\sum_{1\le i\le N}\big||K^{\mathbf{a}_k'}\cap V_i^{\mathbf{a}_k}| - |K^{\mathbf{a}_k}\cap V_i^{\mathbf{a}_k}|\big|\\ &\ge C_0\lambda\sum_{1\le i\le N}\big||K^{\mathbf{a}_k'}\cap V_i^{\mathbf{a}_k}| - |K^{\mathbf{a}_k}\cap V_i^{\mathbf{a}_k}|\big| - \lambda|K^{\mathbf{a}_k'}\Delta K^{\mathbf{a}_k}|.\end{aligned}$$

As in the proof of Theorem 1.5 one can note that

$$\sum_{1\le i\le N}\big||K^{\mathbf{a}_k'}\cap V_i^{\mathbf{a}_k}| - |K^{\mathbf{a}_k}\cap V_i^{\mathbf{a}_k}|\big| \ge c(n,K)|\mathbf{a}_k' - \mathbf{a}_k|, \qquad |K^{\mathbf{a}_k'}\Delta K^{\mathbf{a}_k}| \le C(n,K)|\mathbf{a}_k' - \mathbf{a}_k|.$$

Hence, choosing $C_0 = C_0(n,K)$ sufficiently large, we deduce

$$\omega(|\mathbf{a}_k| + |\mathbf{a}_k'|)|\mathbf{a}_k' - \mathbf{a}_k| \ge c(n,K)\lambda|\mathbf{a}_k' - \mathbf{a}_k|,$$

which proves that $|\mathbf{a}_k| = |\mathbf{a}_k'|$ for $k \gg 1$. Hence, testing the minimality of $K^{\mathbf{a}_k}$ in (4.5) against E_k, and recalling (4.4), we get

$$\mathscr{F}(K^{\mathbf{a}_k}) + \lambda|E_k\Delta K^{\mathbf{a}_k}| \le \mathscr{F}(E_k) \le \mathscr{F}(K^{\mathbf{a}_k}) + \frac{\lambda}{2}|E_k\Delta K^{\mathbf{a}_k}|,$$

which implies $|E_k\Delta K^{\mathbf{a}_k}| = 0$ and $\mathscr{F}(E_k) = \mathscr{F}(K^{\mathbf{a}_k})$. This contradicts (4.4), and the theorem follows.

□

Appendix A. Technical results

In this appendix we prove some technical results used in the paper.

First of all, we prove a weighted relative isoperimetric inequality.

Lemma A.1. *Let $\Omega \subset \mathbb{R}^d$ be a Lipschitz domain, and $F \subset \Omega$ a set of finite perimeter. Then there exists a constant $C = C(d, \Omega) > 0$ such that*

$$|F \cap \Omega| \leq C \int_{\partial^* F \cap \Omega} \operatorname{dist}(x, \partial\Omega)\, d\mathcal{H}^{d-1},$$

whenever $|F \cap \Omega| \leq \frac{3}{4}|\Omega|$.

Proof. Let $u = \chi_{F\cap\Omega}$ and u_k be a sequence of smooth functions approximating u strongly in L^1, $\|Du_k\| \rightharpoonup \|Du\|$ weakly* as measures; see e.g. [7, Theorems 2&3, Chapter 5.2] and [7, Chapter 5.4].

Then, applying the weighted Poincaré-type inequality from [2] to the functions u_k yields

$$\int_\Omega |u_k|\, dx \leq C \int_\Omega |Du_k| \operatorname{dist}(x, \partial\Omega)\, dx,$$

where $C = C(d, \Omega)$. Letting $k \to \infty$, we obtain the result. □

We now state a Lipschitz regularity result for ϵ-minimizers.

Lemma A.2. *For any $i = 1, \ldots, N$, let H_i denotes the hyperplane containing $\partial K \cap V_i$. Let E be an $(\epsilon, n+1)$-minimizer of $\mathscr{F}$ with $|E| = |K|$. There exist $\bar\epsilon = \bar\epsilon(n, K) > 0$ and $L = L(n, K) > 0$ such that if $\epsilon \leq \bar\epsilon$ then the following holds: For any $i = 1, \ldots, N$ there exists a neighborhood $\mathcal{U}_i \subset \mathbb{R}^n$ of $\partial K \cap V_i$ such that, up to a translation of E,*

$$\partial E \subset \bigcup_{1 \leq i \leq N} \mathcal{U}_i$$

and

$$(\partial E \cap \mathcal{U}_i) \setminus \Gamma_i \text{ is a } L\text{-Lipschitz graph with respect to } H_i,$$

where $\Gamma_i \subset \partial E$ satisfies $\mathcal{H}^{n-1}(\Gamma_i) \leq C(n, K)\epsilon$.

Proof. Thanks to Lemma 2.1, up to a translation we have that ∂E is uniformly close to ∂K. This allows us to apply [1, Proposition 4.6] and deduce that, for ϵ sufficiently small, we can cover almost all the boundary of E with uniformly Lipschitz graphs. □

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