TANGENT CUT LOCI ON SURFACES

A. FIGALLI, L. RIFFORD, AND C. VILLANI

Abstract. Given a smooth compact Riemannian surface, we prove that if a suitable convexity assumption on the tangent focal cut loci is satisfied, then all injectivity domains are semiconvex.

Introduction

Let (M, g) be a smooth compact Riemannian manifold of dimension $n = 2$, $UM \subset TM$ its unit tangent bundle, and exp the associated Riemannian exponential map. We define the distance function to the cut locus, $t_C : UM \to (0, \infty)$ as

$$\begin{aligned} t_C(x,v) &= \textit{cut time of } (x,v) \\ &:= \max\Big\{ t \geq 0;\ (\exp_x(sv))_{0\leq s\leq t} \text{ is a minimizing geodesic} \Big\}. \end{aligned}$$

Then for any $x \in M$, we let

$$\begin{aligned} \mathrm{TCL}(x) &= \textit{tangent cut locus of } x \\ &:= \big\{ t_C(x,v)v;\ v \in U_xM \big\}; \end{aligned} \tag{0.1}$$

$$\begin{aligned} \mathrm{I}(x) &= \textit{injectivity domain of the exponential map at } x \\ &:= \Big\{ tv;\ 0 \leq t < t_C(x,v),\ v \in U_xM \Big\}. \end{aligned} \tag{0.2}$$

Note that $\mathrm{TCL}(x)$ is compact and coincides with the boundary of the open set $I(x)$. Finally, the cut locus of x may be defined as

$$\mathrm{cut}(x) := \exp_x\big(\mathrm{TCL}(x)\big).$$

Many works [2, 8, 9, 10, 11, 14] have been devoted to the cut locus, or the distance function to the cut locus $(x, v) \mapsto t_C(x, v)$. Itoh and Tanaka [10] (see also [2, 11]) proved that the function t_C is Lipschitz continuous on UM. The aim of this paper is to show that under an additional convexity assumption on the tangent focal loci, each function $t_C(x, \cdot)$ is **semiconcave**. Recall that a real-valued function u defined on an open set $U \subset \mathbb{R}^n$ is said to be locally semiconcave on U if for every $\bar{x} \in U$ there exist $\delta, \sigma > 0$ such that

$$tu(x) + (1-t)u(y) - u(tx + (1-t)y) \leq t(1-t)\sigma|x-y|^2, \tag{0.3}$$

for all x, y in the ball $B_\delta(\bar{x})$ and every $t \in [0,1]$. This is equivalent to saying that the function u can be written locally as

$$u(x) = v(x) + \sigma|x|^2 \qquad \forall\, x \in B_\delta(\bar{x}),$$

with v concave on $B_\delta(\bar{x})$. If σ is uniform, we just say that u is semiconcave on U. Properties of locally semiconcave functions are reviewed in [1] and [15, Chapter 10].

As an immediate geometric consequence of the semiconcavity property of $t_C(x,\cdot)$, all sets $I(x)$ are **semiconvex**, which means that there exists $r > 0$ such that around each $w \in \mathrm{TCL}(x)$ the set $I(x) \cap B_r(w)$ is diffeomorphic to a convex set (see [12, Appendix B]); even if r a priori depends on x, it may be chosen uniform by compactness.

This also implies that $\mathrm{TCL}(x)$ is an Alexandrov space with curvature bounded below; in particular our main result is the first partial answer to a question raised by Itoh and Tanaka [10] who asked whether $\mathrm{TCL}(x)$ is in general an Alexandrov space.

To state our main assumption we need some more notation. First, let us define the distance function to the focal locus, $t_F : UM \to (0, \infty]$, by

$$\begin{aligned} t_F(x,v) &= \textit{focalization time} \text{ of } (x,v) \\ &:= \inf\Big\{ t \geq 0;\ \det(d_{tv}\exp_x) = 0 \Big\}. \end{aligned}$$

For a general n-dimensional manifold, $t_F(x,\cdot)$ is semiconcave on its domain [2]. However, since here M is two-dimensional, it is easy to show by the Implicit Function Theorem that $t_F(x,\cdot)$ is smooth on its domain. (See for instance [6, Paragraph 3.1].)

We further define

$$\begin{aligned} \mathrm{TFL}(x) &= \textit{tangent focal locus} \text{ of } x \\ &:= \big\{ t_F(x,v)v;\ v \in U_xM \big\}; \end{aligned} \tag{0.4}$$

$$\begin{aligned} \mathrm{NF}(x) &= \textit{nonfocal domain} \text{ at } x \\ &:= \big\{ tv;\ 0 \leq t < t_F(x,v),\ v \in U_xM \big\}. \end{aligned} \tag{0.5}$$

Recall that $t_F \geq t_C$ [7, Corollary 3.77]. Next, for every $x \in M$, we define the *tangent focal cut locus* at x by

$$\mathrm{TFCL}(x) := \mathrm{TCL}(x) \cap \mathrm{TFL}(x),$$

and the *fibered tangent focal cut locus* by

$$\mathrm{TFCL}(M) := \bigcup_{x\in M} \Big(\{x\} \times \mathrm{TFCL}(x)\Big) \subset TM.$$

(Although closed, $\mathrm{TFCL}(M)$ is not necessarily connected, since $\mathrm{TFCL}(x)$ might be disconnected or empty.) Finally, we define

$$\underline{\kappa}(M) := \inf \Big\{ \mathrm{II}_{\mathrm{TFL}(x)}(v);\ (x,v) \in \mathrm{TFCL}(M) \Big\}. \tag{0.6}$$

Here $\mathrm{II}_{\mathrm{TFL}(x)}(v)$ is the second fundamental form of $\mathrm{TFL}(x)$ at v, which in this two-dimensional context is just a fancy notation for the signed curvature of $\mathrm{TFL}(x)$ at v. The above infimum is taken among all *focal cut velocities* in M.

We are now ready to state our main result:

Main Result. *Let (M,g) be a two-dimensional Riemannian manifold such that $\underline{\kappa}(M) > 0$. Then all injectivity domains of M are semiconvex.*

Notice that the above assumption survives perturbation: indeed, if (M,g) satisfies $\underline{\kappa}(M,g) > 0$ and g' is a metric on M which is sufficiently close to g in C^4 topology, then $\underline{\kappa}(M,g') > 0$ as well. (This comes from the upper semicontinuity of TFCL with respect to variations of metric, and the fact that the curvature of TFL depends smoothly on the metric.) In particular the injectivity domains for g' are still semiconvex, although the dependence of the cut locus on the metric is highly irregular.

The proof of the main result is not long, because the main work has already been done in previously published papers [5, 6]. For a start, in Section 1 we shall recall a general semiconvexity result, based on the Implicit Function Theorem, which works near a "genuine" cut velocity, that is when the cut locus property is due to the existence of several distinct minimizing geodesics. This reduces the problem to a local study near "purely focal" cut velocities, which in particular belong to the tangent focal cut locus.

Then in Section 2 we shall see how to exploit the positivity of $\underline{\kappa}(M)$ to prove *convexity* of $I(x)$ in the neighborhood of $\mathrm{TFCL}(x)$. Instead of the Implicit Function Theorem, the reasoning is based on a fourth-order curvature-type condition named after Ma, Trudinger and Wang: see [3] or [15, Chapter 12] for a presentation and survey.

1. Semiconvexity around genuine cut velocities

We write $\exp^{-1}$ for the inverse of the exponential map: by convention, for every $x \in M$, $\exp_x^{-1}(y)$ is the set of *minimizing* velocities v such that $\exp_x v = y$. In particular $\mathrm{TCL}(x) = \exp_x^{-1}(\mathrm{cut}(x))$ and $\mathrm{I}(x) = \exp_x^{-1}(M \setminus \mathrm{cut}(x))$. For every $x \in M$ we define the multivalued mapping $\mathcal{W}_x : U_xM \to U_xM$ by

$$\forall\, v \in U_xM, \quad \mathcal{W}_x(v) := \frac{\exp_x^{-1}\left(\exp_x\big(t_C(x,v)\,v\big)\right)}{t_C(x,v)} \subset U_xM. \tag{1.1}$$

In words, $\mathcal{W}_x$ is the set of initial velocities of minimizing geodesics which will "cut" the geodesic $t \mapsto \exp_x(tv)$: if $w \in \mathcal{W}_x(v)$ then $t_C(x,w) = t_C(x,v)$ and $\exp_x\big(t_C(x,w)\,w\big) = \exp_x\big(t_C(x,v)\,v\big)$.

If A is a given set in T_xM, we denote by $\mathrm{diam}(A)$ its diameter with respect to the metric g. The quantity $\Delta(x,v) := \mathrm{diam}(\mathcal{W}_x(v))$ vanishes if and only if the geodesic γ starting from x with velocity v is purely focal, in the sense that γ is the only minimizing geodesic joining $\overline{x}$ to its cut point along γ. Conversely, the positivity of Δ quantifies the fact that v is a true cut velocity, in the sense that two distinct minimizing geodesics join x to $\exp_x(t_C(x,v)v)$.

The meaning of Proposition 1.1 below is that the semiconcavity of the injectivity domain is essentially controlled by a lower bound on Δ.

Proposition 1.1. *Let $\overline{x} \in M$ be fixed and $\overline{v} \in U_{\overline{x}}M$ such that $\Delta(\overline{x},\overline{v}) > 0$. Then there are $\delta > 0$ and a smooth function $\tau : U_{\overline{x}}M \cap B(\overline{v},\delta) \to \mathbb{R}_+$ such that $t_C(\overline{x},\overline{v}) = \tau(\overline{v})$ and $t_C(\overline{x},v) \leq \tau(v)$ for every $v \in U_{\overline{x}}M \cap B(\overline{v},\delta)$. Moreover the C^2 norm of τ is bounded by $C/\Delta(\overline{x},\overline{v})^6$, where C depends only on an upper bound on $\|g\|_{C^3}$ and* $\mathrm{diam}(M)$.

In the above statement the C^3 norm of g is computed in a choice of local coordinates, fixed in advance; regularity bounds on the charts defining M are implicitly involved.

Proposition 1.1 is a slightly more explicit variant of [12, Proposition C.6(b)], and the argument also appeared in [2]. Anyway the proof is short, so we shall provide it in its entirety.

Proof of Proposition 1.1. First, for every $(x,v) \in T_xM$ we set $\phi_x(v) := -d_v \exp_x(v) \in T_{\exp_x(v)}M$ (so that if $\gamma : [0,1] \to M$ is a constant-speed minimizing geodesic going from x to y, with initial velocity v_0 and final velocity v_1, the map ϕ_x is defined by $v_0 \mapsto -v_1$). As in [12, Lemma 4.2], there is a constant $L > 0$, depending only on the C^2 norm of the exponential map on $I(M) = \cup_{x\in M}(\{x\} \times I(x))$, such that if

$v, w \in \mathrm{TCL}(x)$ satisfy $\exp_x v = \exp_x w$ then

$$L^{-1}|v-w|_x \leq \big|\phi_x(v) - \phi_x(w)\big|_{\exp_x v} \leq L|v-w|_x.$$

Set $\bar{t} := t_C(\bar{x}, \bar{v})$, $\bar{y} := \exp_{\bar{x}}\big(\bar{t}\bar{v}\big)$ and $\Delta := \mathrm{diam}\big(\mathcal{W}_{\bar{x}}(\bar{v})\big)$. Let further $d_{\bar{x}}(y) := d(\bar{x}, y)$ be the distance function to $\bar{x}$. By assumption, there is $w \in \mathcal{W}_{\bar{x}}(\bar{v})$ such that $|w - \bar{v}|_{\bar{x}} = \Delta > 0$. Since $d_{\bar{x}}$ is locally semiconcave on $M \setminus \{\bar{x}\}$ and $-\phi_{\bar{x}}\big(\bar{t}w\big)/\bar{t}$ is a supergradient for $d_{\bar{x}}$ at $\bar{y}$ [15, Definition 10.5 and Proposition 10.15], there is a smooth function $h : M \to \mathbb{R}$ such that

$$\begin{cases} d\big(\bar{x}, \bar{y}\big) = h\big(\bar{y}\big) = \bar{t}, \\ \nabla h(\bar{y}) = -\phi_{\bar{x}}\big(\bar{t}w\big)/\bar{t} \\ d\big(\bar{x}, y\big) < h(y), \ \forall\, y \in M \setminus \{\bar{y}\}. \end{cases}$$

Define $\Psi : [0, \infty) \times U_{\bar{x}}M \to \mathbb{R}$ by

$$\Psi(t, v) := h\big(\exp_{\bar{x}}(tv)\big) - t.$$

The function Ψ is smooth and satisfies $\Psi(\bar{t}, \bar{v}) = 0$. Moreover, one has

$$\begin{aligned} -\frac{\partial \Psi}{\partial t}\big(\bar{t}, \bar{v}\big) &= \Big\langle \nabla h\big(\bar{y}\big), \frac{\phi_{\bar{x}}\big(\bar{t}\bar{v}\big)}{\bar{t}} \Big\rangle_{\bar{y}} - 1 \\ &= -\frac{1}{\bar{t}^2} \big\langle \phi_{\bar{x}}\big(\bar{t}w\big), \phi_{\bar{x}}\big(\bar{t}\bar{v}\big) \big\rangle_{\bar{y}} - 1 \\ &= \frac{1}{2\bar{t}^2} \Big| \phi_{\bar{x}}\big(\bar{t}w\big) - \phi_{\bar{x}}\big(\bar{t}\bar{v}\big) \Big|_{\bar{y}}^2 \geq \frac{\Delta^2}{2L^2} > 0. \end{aligned} \tag{1.2}$$

Therefore, by the Implicit Function Theorem, there are an open neighborhood $\mathcal{V}$ of $\bar{v}$ in $U_{\bar{x}}M$ and a smooth function $\tau : \mathcal{V} \to \mathbb{R}$ such that

$$\Psi\big(\tau(v), v\big) = 0 \qquad \forall\, v \in \mathcal{V}. \tag{1.3}$$

If $v \in \mathcal{V}$ and $y := \exp_{\bar{x}}\big(\tau(v)v\big)$, then we have $d(\bar{x}, y) < h(y) = \tau(v)$, so necessarily $t_C(\bar{x}, v) < \tau(v)$.

On the other hand, by construction $\tau(\bar{v}) = g\big(\bar{y}\big) = \bar{t}$. Moreover, differentiating (1.3) twice yields

$$\tau' = -\left(\frac{\partial \Psi}{\partial t}\right)^{-1} \left(\frac{\partial \Psi}{\partial v}\right),$$

$$\tau'' = -\left(\frac{\partial \Psi}{\partial t}\right)^{-3} \left[\left(\frac{\partial \Psi}{\partial t}\right)^2 \left(\frac{\partial^2 \Psi}{\partial v^2}\right) - 2\left(\frac{\partial \Psi}{\partial t}\right)\left(\frac{\partial \Psi}{\partial v}\right)\left(\frac{\partial^2 \Psi}{\partial t\, \partial v}\right) + \left(\frac{\partial \Psi}{\partial v}\right)^2 \left(\frac{\partial^2 \Psi}{\partial t^2}\right)\right].$$

By (1.2), this implies that the C^2 norm of τ is controlled by C/Δ^6, where C depends only on L and the C^2 norms of h and exp. Recalling that L depends on the C^2 norm of exp, which in turn depends on the C^3 norm of the metric g, we obtain the claimed result. □

2. Convexity near focal cut velocities

Before starting the proof of our main result, let us first introduce some few more notation. Let $x \in M$, $v \in \mathrm{NF}(x)$, and $(\xi, \eta) \in T_xM \times T_xM$ be fixed. Since $y := \exp_x v$ is not conjugate to x, by the Inverse Function Theorem there are an open neighborhood $\mathcal{V}$ of (x, v) in TM, and an open neighborhood $\mathcal{W}$ of (x, y) in $M \times M$, such that

$$\begin{array}{rcl} \Psi_{(x,v)} : \mathcal{V} \subset TM & \longrightarrow & \mathcal{W} \subset M \times M \\ (x', v') & \longmapsto & \big(x', \exp_{x'}(v')\big) \end{array}$$

is a smooth diffeomorphism from $\mathcal{V}$ to $\mathcal{W}$. Then we may define $\widehat{c}_{(x,v)} : \mathcal{W} \to \mathbb{R}$ by

$$\widehat{c}_{(x,v)}(x', y') := \frac{1}{2}\big|\Psi^{-1}_{(x,v)}(x', y')\big|^2_{x'} \qquad \forall\, (x', y') \in \mathcal{W}. \tag{2.1}$$

If $v \in \mathrm{I}(x)$ then for y' close to $\exp_x v$ and x' close to x we have

$$\widehat{c}_{(x,v)}(x', y') = c(x', y') := d(x', y')^2/2.$$

Then for any $x \in M$, $v \in \mathrm{NF}(x)$, and $(\xi, \eta) \in T_xM \times T_xM$, we define

$$\begin{aligned} \overline{\mathfrak{S}}_{(x,v)}(\xi, \eta) &:= -\frac{3}{2}\ \frac{d^2}{ds^2}\bigg|_{s=0}\ \frac{d^2}{dt^2}\bigg|_{t=0}\ \widehat{c}_{(x,v)}\Big(\exp_x(t\xi),\ \exp_x(v + s\eta)\Big) \\ &= -\frac{3}{2}\ \frac{d^2}{ds^2}\bigg|_{s=0} \Big\langle \nabla^2_x \widehat{c}_{(x,v)}\Big(\cdot\,, \exp_x(v + s\eta)\Big) \cdot \xi, \xi \Big\rangle_x. \end{aligned}$$

This tensor was introduced in [4]; it is a generalization of the original Ma–Trudinger–Wang (MTW) tensor defined in [13]. (The conventions here are the same as in [15, Chapter 12].)

We note that $\nabla^2_x \widehat{c}_{(x,v)}(x, \exp_x v)$ blows up as v approaches $\mathrm{TFL}(x)$, in the sense that one of its eigenvalues approaches $-\infty$. In contrast, all the x-derivatives of $c(x, \exp_x v)$ remain bounded (but not continuous) if v approaches a nonfocal cut velocity. So the behaviour of the MTW tensor is nontrivial only near focalization.

It was shown in [6] by a Jacobi fields analysis that the strict convexity of the nonfocal domain implies the "positivity" of the MTW tensor (in the sense of (2.5) below). In turn this positivity implies the convexity of the injectivity domain [5]: this property (independent of the dimension) is established by a maximum principle showing that if v_0 and v_1 belong to the injectivity domain at x, $v_t = (1-t)v_0 + v_1$ is the segment joining both, and $y_t = \exp_x(v_t)$, then $[0,1] \ni t \longmapsto d(x,y_t)^2 - |v_t|^2$ cannot achieve its maximum within $(0,1)$. Much more information is in [5].

So our strategy consists in putting together two cases:

• Near pure focalization ($\Delta \simeq 0$), we use the convexity assumption $\underline{\kappa}(M) > 0$ and the results of [6] to deduce the convexity of the injectivity domain;

• In the regime of true cut velocities ($\Delta > 0$), we apply Proposition 1.1.

The following lemma will allow us to separate between these two situations:

Lemma 2.1. *Let M be a compact Riemannian manifold and $\mathcal{V}$ a neighborhood of* $\mathrm{TFCL}(M)$ *in TM; then there is $\nu > 0$ with the following property. If $(\overline{x},\overline{v}) \in TM\setminus\{0\}$ satisfies $\Delta(\overline{x},\overline{v}/|\overline{v}|) < \nu$ then for all $v_0, v_1 \in \overline{\mathrm{I}(\overline{x})}$, $t \in [0,1]$ with $|v_0 - \overline{v}| < \nu$, $|v_1 - \overline{v}| < \nu$, $v_t = (1-t)v_0 + tv_1$, $y_t = \exp_{\overline{x}}((1-t)v_0 + tv_1)$, $\overline{q}_t = -d_{v_t}\exp_{\overline{x}} v_t$, for all $q_t \in (\exp_{y_t})^{-1}(\overline{x})$ and any $s \in [0,1]$, one has*

$$\big(y_t,\, (1-s)\overline{q}_t + sq_t\big) \in \mathcal{V}. \tag{2.2}$$

Proof of Lemma 2.1. First recall that $\phi(x,v) = (\exp_x v, -d_v \exp_x(v))$ defines a bi-Lipschitz involution sending $\mathrm{TCL}(M)$ to itself [12, Lemma 5.1] and $\mathrm{TFL}(M)$ to itself. These properties are immediate from the interpretation given at the beginning at the proof of Proposition 1.1. In particular it follows easily that $\Delta(x,v) \le C\,\Delta(\phi(x,v))$ for some constant $C > 0$.

Now, if the claim is false, then for each $k \in \mathbb{N}$ there are $(x^k, v^k) \in TM\setminus\{0\}$, v_0^k, v_1^k in $\overline{I(x^k)}$ and $t_k \in [0,1]$, such that $\Delta(x^k, v^k/|v^k|) < 1/k$, $|v_0^k - v^k| < 1/k$, $|v_1^k - v^k| < 1/k$ and, with $w^k = (1-t_k)v_0^k + t_k v_1^k$, $y^k = \exp_{x^k} w^k$, $\overline{q}^k = -d_{w_k}\exp_{x^k}(w^k)$, there is $q^k \in (\exp_{y^k})^{-1}(x^k)$ and $s_k \in [0,1]$ such that

$$\Big(y^k, (1-s_k)\overline{q}^k + s_k q^k\Big) \notin \mathcal{V}. \tag{2.3}$$

Take $k \to \infty$, and up to subsequence assume that $x^k \to x$, $v^k \to v$, $v_0^k \to v$, $v_1^k \to v$, $t_k \to t$, $s_k \to s$, $y^k \to y = \exp_x v$, $\overline{q}^k \to \overline{q} = -d_v \exp_x(v)$. Also $\Delta(y^k, \overline{q}^k) = \Delta(\phi(x^k, w^k)) \le C\,\Delta(x^k, w^k) \to 0$, so

$$\left| t_C\left(y^k, \frac{\overline{q}_k}{|\overline{q}^k|}\right) - t_C\left(y^k, \frac{q_k}{|q^k|}\right) \right| \xrightarrow[k\to\infty]{} 0,$$

which implies $|q^k - \overline{q}^k| \to 0$, hence $q^k \to \overline{q}$. Then, by taking the limit in (2.3) we find

$$(y, \overline{q}) \notin \mathcal{V}. \tag{2.4}$$

The fact that $\Delta(x^k, v^k)$ converges to 0 does not allow us to deduce $\Delta(x, v) = 0$; but for sure it implies that $(x, v) \in \mathrm{TFCL}(M)$. Hence $(y, \overline{q}) \in \mathrm{TFCL}(M)$, which contradicts (2.4). Then the proof is complete. □

Proof of Main Result. Let M be a two-dimensional smooth compact manifold satisfying $\underline{\kappa}(M) > 0$. By [6, Proposition 3.1 and Remark 3.2], there is a neighborhood $\mathcal{V}$ of $\mathrm{TFCL}(M)$ in TM and there are constants $K, C > 0$ such that for all $(x, v) \in \mathcal{V}$ with $v \in \mathrm{NF}(x)$,

$$\forall\, (\xi, \eta) \in T_xM \times T_xM, \qquad \overline{\mathfrak{S}}_{(x,v)}(\xi, \eta) \geq K\, |\xi|_x^2\, |\eta|_x^2\,, - C\, \langle \xi, \eta\rangle_x^2. \tag{2.5}$$

Reducing $\mathcal{V}$ if necessary, we also deduce from $\underline{\kappa}(M) > 0$ that

(2.6)
$$v \in \mathcal{V} \cap \mathrm{NF}(x), \quad w \in \mathcal{V} \cap \overline{\mathrm{NF}(x)} \quad \Longrightarrow\ \forall\, t \in (0,1), \quad (1-t)v + tw \in \mathrm{NF}(x).$$

The neighborhood $\mathcal{V}$ determines a positive number ν with the properties stated in Lemma 2.1.

Now let $(\overline{x}, \overline{v}) \in \mathrm{TCL}(M)$; the goal is to prove the semiconvexity of $\mathrm{I}(\overline{x})$ near $\overline{v}$.

- If $\Delta(\overline{x}, \overline{v}) \geq \nu$ then by Proposition 1.1 $t_C(\overline{x}, \cdot\,)$ is semiconcave around $\overline{v}/|\overline{v}|$.
- If $\Delta(\overline{x}, \overline{v}) < \nu$ then let us show that $I(\overline{x})$ is *convex* around $\overline{v}$. Let $v_0, v_1 \in I(\overline{x}) \cap B(\overline{v}, \nu)$, and let $v_t = (1-t)v_0 + tv_1$. Since $\mathrm{NF}(\overline{x})$ is uniformly convex in $B(\overline{v}, \nu)$, v_t belongs to $\mathrm{NF}(\overline{x})$ for all t; so $\overline{q}_t \in \mathrm{NF}(y_t)$. On the other hand $q_t \in \overline{\mathrm{I}(y_t)} \subset \overline{\mathrm{NF}(y_t)}$, so by (2.6) we have

$$(1-s)\, \overline{q}_t + s\, q_t \in \mathrm{NF}(y_t) \qquad \text{for all } s \in (0,1). \tag{2.7}$$

Furthermore by (2.2) $(y_t, (1-s)\overline{q}_t + sq_t) \in \mathcal{V}$, so (2.5) gives

$$\forall\, (\xi, \eta) \in T_{y_t}M \times T_{y_t}M, \qquad \overline{\mathfrak{S}}_{(y_t,(1-s)\overline{q}_t+sq_t)}(\xi, \eta) \geq K\, |\xi|_{y_t}^2\, |\eta|_{y_t}^2\ -\ C\, \langle \xi, \eta\rangle_{y_t}^2. \tag{2.8}$$

The combination of (2.7) and (2.8) implies, by the reasoning of [5, Theorem 2.7, Proof in Section 6] that $v_t \in I(\overline{x})$ for all $t \in (0,1)$. This proves the convexity of $I(\overline{x})$ near $\overline{v}$, and the proof is complete. □

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Alessio Figalli

Department of Mathematics
The University of Texas at Austin
1 University Station, C1200
Austin TX 78712, USA

email: figalli@math.utexas.edu

Ludovic Rifford

Université de Nice–Sophia Antipolis
Labo. J.-A. Dieudonné, UMR 6621
Parc Valrose, 06108 Nice Cedex 02
FRANCE

EMAIL: rifford@unice.fr

Cédric Villani

ENS Lyon & Institut Universitaire de France
UMPA, UMR CNRS 5669
46 allée d'Italie
69364 Lyon Cedex 07
FRANCE

E-MAIL: cvillani@umpa.ens-lyon.fr