The optimal partial transport problem

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Abstract

Given two densities f and g, we consider the problem of transporting a fraction $m \in [0, \min\{\|f\|_{L^1}, \|g\|_{L^1}\}]$ of the mass of f onto g minimizing a transportation cost. If the cost per unit of mass is given by $|x-y|^2$, we will see that uniqueness of solutions holds for $m \in [\|f \wedge g\|_{L^1}, \min\{\|f\|_{L^1}, \|g\|_{L^1}\}]$. This extends the result of Caffarelli and McCann in [8], where the authors consider two densities with disjoint supports. The free boundaries of the active regions are shown to be $(n-1)$-rectifiable (provided the supports of f and g have Lipschitz boundaries), and under some weak regularity assumptions on the geometry of the supports they are also locally semiconvex. Moreover, assuming f and g supported on two bounded strictly convex sets $\Omega, \Lambda \subset \mathbb{R}^n$, and bounded away from zero and infinity on their respective supports, $C^{0,\alpha}_{\rm loc}$ regularity of the optimal transport map and local C^1 regularity of the free boundaries away from $\Omega \cap \Lambda$ are shown. Finally, the optimal transport map extends to a global homeomorphism between the active regions.

1 Introduction

In a recent paper [8], Caffarelli and McCann studied the following variant of the Monge-Kantorovich problem: let $f, g \in L^1(\mathbb{R}^n)$ be two nonnegative functions, and denote by $\Gamma_\leq(f,g)$ the set of nonnegative finite Borel measures on $\mathbb{R}^n \times \mathbb{R}^n$ whose first and second marginals are dominated by f and g respectively, i.e.

$$\gamma(A \times \mathbb{R}^n) \leq \int_A f(x)\,dx, \qquad \gamma(\mathbb{R}^n \times A) \leq \int_A g(y)\,dy$$

for all $A \subset \mathbb{R}^n$ Borel. Denoting by $\mathscr{M}(\gamma)$ the mass of γ (i.e. $\mathscr{M}(\gamma) := \int_{\mathbb{R}^n\times\mathbb{R}^n} d\gamma$), fix a certain amount $m \in [0, \min\{\|f\|_{L^1}, \|g\|_{L^1}\}]$ which represents the mass one wants to transport, and consider the following partial transport problem:

$$\text{minimize } C(\gamma) := \int_{\mathbb{R}^n\times\mathbb{R}^n} |x-y|^2\,d\gamma(x,y)$$

among all $\gamma \in \Gamma_\leq(f,g)$ with $\mathscr{M}(\gamma) = m$.

Using weak topologies, one can easily prove existence of minimizers for any fixed amount of mass $m \in [0, \min\{\|f\|_{L^1}, \|g\|_{L^1}\}]$ (see Section 2). We denote by $\Gamma^o(m)$ the set of such minimizers.

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In general one cannot expect uniqueness of minimizers. Indeed, if $m \leq \int_{\mathbb{R}^n} f \wedge g$ (where $(f \wedge g)(x) := \min\{f(x), g(x)\}$), any γ supported on the diagonal $\{x = y\}$ with marginals dominated by $f \wedge g$ is a minimizer with zero cost. To ensure uniqueness, in [8] the authors assume f and g to have disjoint supports. Under this assumption they are able to prove (as in the classical Monge-Kantorovich problem) that there exists a (unique) convex function ψ such that the unique minimizer is concentrated on the graph of $\nabla\psi$ (see [8, Section 2]). This ψ is also shown to solve in a weak sense a Monge-Ampère double obstacle problem (see [8, Section 4]).

Moreover, strengthening the disjointness assumption into the hypothesis on the existence of a hyperplane separating the supports of the two measures, the authors prove a semiconvexity result on the free boundaries (see [8, Sections 5][1]). Furthermore, under some classical regularity assumptions on the measures and on their supports, local $C^{1,\alpha}$ regularity of ψ (which is equivalent to local $C^{0,\alpha}$ regularity of the transport map) and on the free boundaries of the active regions is shown (see [8, Sections 6-7]).

The aim of this paper is to understand what happens if we remove the disjointness assumption. In Section 2 we will see that, although minimizers are non-unique for $m < \int_{\mathbb{R}^n} f \wedge g$ (but in this case the set of minimizers can be trivially described), uniqueness holds for any $m \geq \int_{\mathbb{R}^n} f \wedge g$. Moreover, exactly as in [8], the unique minimizer is concentrated on the graph of the gradient of a convex function. In Remark 2.11 we will also see that our argument for the uniqueness of minimizers extends to more general cost functions on $\mathbb{R}^n$, and also to the case where f and g are two densities on a Riemannian manifold with $c(x, y) = d(x, y)^2$, $d(x, y)$ being the Riemannian distance.

Then, in Section 3 we will prove that the marginals of the minimizers always dominate the common mass $f \wedge g$ (that is all the common mass is both source and target). This property, which has an interest in its own, will also play a crucial role in the regularity results of Section 4. Indeed, thanks to this domination property, in Paragraph 4.1 we can prove a local semiconvexity result on the free boundaries, which reduces to the Caffarelli-McCann result in the disjoint case (see Propositions 4.4 and 4.5 for a precise statement).

Paragraph 4.2 is devoted to the regularity properties of the transport map and the free boundary. First, as in [8], we will prove local $C^{0,\alpha}$ regularity of the transport map (see Theorem 4.8). On the other hand we will see that in our case something completely different happens: usually, assuming C^∞ regularity on the density of f and g (together with some convexity assumption on their supports), one can show that the transport map is C^∞ too. In our case we will show that $C^{0,\alpha}_{\mathrm{loc}}$ regularity is in some sense optimal: we can find two C^∞ densities on $\mathbb{R}$, supported on two bounded intervals and bounded away from zero on their supports, such that the transport map is not C^1 (see Remark 4.9).

Regarding the regularity of the free boundaries, we will prove a local C^1 regularity away from $\mathrm{supp}(f) \cap \mathrm{supp}(g)$ (see Theorem 4.11 for a precise statement). Furthermore, as in [8, Section 6], we will see that the transport map extends to a global homeomorphism between the active regions (see Theorem 4.10).

Finally we will show how one can adapt the proofs in [8, Section 6] to deduce properties like

[1] In [8] the authors speak about the semiconcavity of the free boundary, but this is equivalent to semiconvexity up to a change of coordinates.

the path-connectedness of the active regions, or the fact that free boundary never maps to free boundary. In Remark 4.15 we also discuss a possible improvement of the $C^1_{\rm loc}$ regularity of the free boundaries away from $\mathrm{supp}(f) \cap \mathrm{supp}(g)$ into a $C^{1,\alpha}_{\rm loc}$ regularity.

1.1 Preliminaries on measure theory and convex functions

We first recall some definitions which will play an important role in the paper:

Definition 1.1 (Push-forward) Let X, Y be complete separable metric spaces, μ a finite Borel measure on X, and $F : X \to Y$ a Borel map. The *push-forward* $F_\#\mu$ is the measure on Y defined by $F_\#\mu(B) = \mu(F^{-1}(B))$ for any Borel set $B \subset Y$.

Definition 1.2 (Marginals) Let X, Y be complete separable metric spaces, and let γ be a finite Borel measure on $X \times Y$. We say that μ and ν are, respectively, the first and the second *marginals* of γ if

$$\int_{X\times Y} h_1(x)\, d\gamma(x, y) = \int_X h_1(x)\, d\mu(x), \qquad \int_{X\times Y} h_2(y)\, d\gamma(x, y) = \int_Y h_2(y)\, d\nu(y),$$

for all bounded continuous functions $h_1 : X \to \mathbb{R}$, $h_2 : Y \to \mathbb{R}$.

Definition 1.3 (Minumum and maximum of measures) Let X be a complete separable metric spaces, μ, ν be two finite Borel measure on X. We define $\mu \wedge \nu$ and $\mu \vee \nu$ by

$$\mu \wedge \nu(B) := \inf\big\{\mu(B_1) + \nu(B_2) \,:\, B_1 \cap B_2 = \emptyset,\ B_1 \cup B_2 = B,\ B_1, B_2 \text{ Borel}\big\} \qquad \forall\, B \subset X \text{ Borel},$$

$$\mu \vee \nu(B) := \sup\big\{\mu(B_1) + \nu(B_2) \,:\, B_1 \cap B_2 = \emptyset,\ B_1 \cup B_2 = B, B_1, B_2 \text{ Borel}\big\} \qquad \forall\, B \subset X \text{ Borel}.$$

Moreover, we say that $\mu \le \nu$ if $\mu(B) \le \nu(B)$ for all $B \subset X$ Borel.

It is not difficult to check that $\mu \wedge \nu$ and $\mu \vee \nu$ are still finite Borel measures. Moreover the equality $\mu \wedge \nu + \mu \vee \nu = \mu + \nu$ holds. Indeed, given a Borel set B, assume for simplicity that we have $\mu \wedge \nu(B) = \mu(B_1) + \nu(B_2)$ for some partition B_1, B_2 of B. Then, given any other partition B_1', B_2' of B, we have

$$\mu(B_2)+\nu(B_1) = \mu(B)-\mu(B_1)+\nu(B)-\nu(B_2) \ge \mu(B)-\mu(B_2')+\nu(B)-\nu(B_1') = \mu(B_1')+\nu(B_2'),$$

that is $\mu \vee \nu(B) = \mu(B_2) + \nu(B_1)$, which gives $\mu \wedge \nu(B) + \mu \vee \nu(B) = \mu(B) + \nu(B)$ as wanted (in the general case when the infimum is not attained, it suffices to consider B_1 and B_2 such that $\mu \wedge \nu(B) \ge \mu(B_1) + \nu(B_2) - \varepsilon$ for some $\varepsilon > 0$ arbitrarily small).

Let us also recall the so-called Disintegration Theorem (see for instance [4, Theorem 5.3.1], [10, III-70]). We state it in a particular case, which is however sufficient for our purpose:

Theorem 1.4 (Disintegration of measures) *Let X, Y be complete separable metric spaces, and let γ be a finite Borel measure on $X \times Y$. Denote by μ and ν the marginals of γ on the first* *and second factor respectively. Then there exists two measurable families of probability measures* $(\gamma_x)_{x\in X}$ *and* $(\gamma_y)_{y\in Y}$ *such that*

$$\gamma(dx, dy) = \gamma_x(dy) \otimes d\mu(x) = \gamma_y(dx) \otimes d\nu(y),$$

i.e.

$$\int_{X\times Y} \varphi(x,y)\, d\gamma(x,y) = \int_X \left(\int_Y \varphi(x,y)\, d\gamma_x(y) \right) d\mu(x) = \int_Y \left(\int_X \varphi(x,y)\, d\gamma_y(x) \right) d\nu(y)$$

for all bounded continuous functions $\varphi : X \times Y \to \mathbb{R}$.

We now recall some classical definitions on convex functions:

Definition 1.5 (Subdifferential and Legendre transform) Let $\phi : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be a convex function. If $\phi(x)$ is finite, the *subdifferential* of ϕ at x is defined as

$$\partial\phi(x) := \{p \in \mathbb{R}^n \,:\, \phi(z) - \phi(x) \geq \langle p, z - x\rangle \quad \forall\, z \in \mathbb{R}^n\}.$$

Moreover, for $A \subset \mathbb{R}^n$, we define $\partial\phi(A) := \cup_{x\in A}\partial\phi(x)$.

The *Legendre transform* $\phi^* : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ of ϕ is the convex function defined as

$$\phi^*(y) := \sup_{x\in\mathbb{R}^n} \langle y, x\rangle - \phi(x).$$

It is well-known that a convex function and its Legendre transform are related by the following properties:

$$y \in \partial\phi(x) \quad \Leftrightarrow \quad x \in \partial\phi^*(y),$$

$$\phi(x) + \phi^*(y) \geq \langle x, y\rangle \qquad \forall\, x, y \in \mathbb{R}^n \quad \text{with equality if and only if } y \in \partial\phi(x).$$

Since convex functions are locally Lipschitz in the interior of their domain, by Rademacher's Theorem they are differentiable a.e. Indeed, a stronger result holds (see for instance [2] or [17, Chapter 14, First Appendix]):

Theorem 1.6 (Alexandrov) *Let* $A \subset \mathbb{R}^n$ *be an open set,* $\phi : A \to \mathbb{R}$ *be a convex function. Then* ϕ *is twice differentiable (Lebesgue) a.e., that is for almost every* $x \in A$ *there exists a linear map* $\nabla^2\phi(x) : \mathbb{R}^n \to \mathbb{R}^n$ *such that*

$$\phi(x + v) = \phi(x) + \langle\nabla\phi(x), v\rangle + \frac{1}{2}\langle\nabla^2\phi(x)\cdot v, v\rangle + o(|v|^2) \qquad \forall\, v \in \mathbb{R}^n.$$

Moreover $\nabla\phi(x)$ *is differentiable a.e. in* A*, and its differential coincides with* $\nabla^2\phi(x)$*.*

1.2 Notation

In this paper, we will repeatedly adopt the following notation and conventions:

- If μ is a (Borel) measure on $\mathbb{R}^n$ with density h with respect to the Lebesgue measure $\mathscr{L}^n$, we often use h in place of $\mu = h\mathscr{L}^n$. For example, we write $F_\# h$ in place of $F_\#(h\mathscr{L}^n)$.
- If $\gamma \in \Gamma_\leq(f,g)$, we denote by f_γ and g_γ the densities of the first and the second marginal of γ respectively (observe that the constraint $\gamma \in \Gamma_\leq(f,g)$ implies that both marginals are absolutely continuous with respect to the Lebesgue measure).
- If $h_1, h_2 : \mathbb{R}^n \to \mathbb{R}$ are Borel functions, we will often consider the Borel set $\{h_1 > h_2\}$. We remark that if h_1, h_2 are defined up to a set of (Lebesgue) measure zero, then $\{h_1 > h_2\}$ is well-defined up to a set of measure zero.
- If B_1 and B_2 are two Borel set (possibly defined up to a set of measure zero), we say that $B_1 \overset{a.e.}{\subset} B_2$ if the inclusion $B_1 \subset B_2$ holds up to a set of measure zero.

2 Properties of minimizers

As we already said in the introduction, we consider $f, g \in L^1(\mathbb{R}^n)$ two nonnegative Borel function, and we denote by $\Gamma_\leq(f,g)$ the set of nonnegative finite Borel measures on $\mathbb{R}^n \times \mathbb{R}^n$ whose first and second marginals are dominated by f and g respectively. We will always assume for simplicity that both f and g are compactly supported, although many results holds under more general assumptions (for example if f and g have finite second moments). Let $m_{\max} := \min\{\|f\|_{L^1}, \|g\|_{L^1}\}$. We fix $m \in [0, m_{\max}]$, and we consider the minimization problem

$$C(m) := \min_{\gamma \in \Gamma_\leq(f,g),\, \mathscr{M}(\gamma) = m} C(\gamma), \tag{2.1}$$

where $\mathscr{M}(\gamma) = \int_{\mathbb{R}^n \times \mathbb{R}^n} d\gamma$ is the mass of γ, and $C(\gamma) := \int_{\mathbb{R}^n \times \mathbb{R}^n} |x-y|^2\, d\gamma$ is the cost of γ. Since the set

$$\Gamma(m) := \{\gamma \in \Gamma_\leq(f,g),\, \mathscr{M}(\gamma) = m\}$$

is non-empty and weakly* compact, it is simple to prove by the direct method of the calculus of variations existence of minimizers (see [17, Chapter 4] or [8, Lemma 2.2]). Let $\Gamma^o(m) \subset \Gamma(m)$ denotes the set of minimizers. We want to understand their structure.

Let us define $m_{\min} := \int_{\mathbb{R}^n} f \wedge g$.

We observe that for $m \leq m_{\min}$, given any density $0 \leq h \leq f \wedge g$ with mass m (i.e. $\int_{\mathbb{R}^n} h = m$), the plan $\gamma := (\mathrm{Id} \times \mathrm{Id})_\# h$ is optimal, since its cost is zero. Moreover, since all minimizers have zero cost, they are clearly of this form. Thus the set of minimizers is not a singleton except for $m = m_{\min}$, in which case the unique minimizer is given by $(\mathrm{Id} \times \mathrm{Id})_\#(f \wedge g)$.

We now want to study the case $m > m_{\min}$.

The main difference between our strategy and the one developed in [8] is the following: in [8, Section 2] the authors introduce a Lagrange multiplier for the mass constraint, add a point at infinity which acts as a tariff-free reservoir, and study the relations given by classical duality theorems. In this way they are able to deduce existence and uniqueness of minimizers when the supports of f and g are disjoint. Our strategy is instead to attack directly the minimization problem by studying the convexity properties of the function $m \mapsto C(m)$, and then looking at the consequences that follow from them. In this way we will prove that there exists a unique minimizer which is concentrated on the graph of the gradient of a convex function. In particular, if f and g have disjoint support, we recover the uniqueness result in [8].

The proof of our result is divided in two steps:

1. The function $m \mapsto C(m)$ is convex on $[0, m_{\max}]$. Moreover, if m_0 is a point of strict convexity for $C(m)$, then $\Gamma^o(m_0)$ is a singleton.

2. The function $m \mapsto C(m)$ is strictly convex on $(m_{\min}, m_{\max}]$.

Combining the above steps, we immediately deduce the uniqueness of minimizers for $m > m_{\min}$.

We remark that the proof of the second step will require an analysis of the structure of minimizers, that will be obtained applying in a careful way Brenier's Theorem (see Theorem 2.3).

2.1 Step 1: properties of $C(m)$

Here we show some elementary properties of the function $m \mapsto C(m)$.

Lemma 2.1 (Convexity of $C(m)$) *The function $m \mapsto C(m)$ is identically zero on $[0, m_{\min}]$, and it is convex on $[0, m_{\max}]$.*

Proof. The fact that $C(m) = 0$ for $m \in [0, m_{\min}]$ follows from the observation that any γ supported on the diagonal with marginals dominated by $f \wedge g$ is a minimizer with zero cost.

The convexity of $C(m)$ is a simple consequence of the linearity of the functional and the convexity of the constraints: if $\gamma_1 \in \Gamma^o(m_1)$ and $\gamma_2 \in \Gamma^o(m_2)$, for any $\lambda \in [0,1]$ we have $\lambda\gamma_1 + (1-\lambda)\gamma_2 \in \Gamma(\lambda m_1 + (1-\lambda)m_2)$. Therefore we obtain

$$\begin{aligned} C\big(\lambda m_1 + (1-\lambda)m_2\big) &\leq C\big(\lambda\gamma_1 + (1-\lambda)\gamma_2\big) \\ &= \lambda C(\gamma_1) + (1-\lambda)C(\gamma_2) = \lambda C(m_1) + (1-\lambda)C(m_2) \qquad \forall \lambda \in [0,1]. \end{aligned}$$

□

Proposition 2.2 (Strict convexity implies uniqueness) *Let $\gamma_1, \gamma_2 \in \Gamma^o(m_0)$ (possibly $\gamma_1 = \gamma_2$), and define $\gamma_- := \gamma_1 \wedge \gamma_2$, $\gamma_+ := \gamma_1 \vee \gamma_2$. If we denote $m_- = \mathscr{M}(\gamma_1 \wedge \gamma_2)$ and $m_+ = \mathscr{M}(\gamma_1 \vee \gamma_2)$, then $C(m)$ is affine on $[m_-, m_+]$.*

In particular, if $C(m)$ its strictly convex at m_0, then $m_- = m_+$ and $\Gamma^o(m_0)$ is a singleton.

Proof. As $\gamma_- + \gamma_+ = \gamma_1 + \gamma_2$ (recall the discussion after Definition 1.3), and $\gamma_-, \gamma_+ \in \Gamma_{\leq}(f,g)$ thanks to Theorem 2.6 below, we have

$$C(m_-) + C(m_+) \leq C(\gamma_-) + C(\gamma_+) = C(\gamma_1) + C(\gamma_2) = 2C(m_0),$$

that is

$$\frac{C(m_+) + C(m_-)}{2} \leq C(m_0).$$

Since $\frac{m_+ + m_-}{2} = m_0$ and $C(m)$ is convex, we deduce that $C(m)$ is affine on $[m_-, m_+]$. Therefore, if $C(m)$ its strictly convex at m_0, then $m_- = m_+$, which implies $\gamma_- = \gamma_+$. Thus $\gamma_1 = \gamma_2$, and by the arbitrariness of $\gamma_1, \gamma_2 \in \Gamma^o(m_0)$ we deduce that $\Gamma^o(m_0)$ is a singleton. □

2.2 Graph property of minimizers

We will need the following result for the classical Monge-Kantorovich problem (see [5, 6, 14, 15]):

Theorem 2.3 *Let $f', g' \in L^1(\mathbb{R}^n)$ be nonnegative compactly supported functions such that $\int_{\mathbb{R}^n} f' = \int_{\mathbb{R}^n} g'$, and consider the Monge-Kantorovich problem:*

$$\text{minimize} \int_{\mathbb{R}^n \times \mathbb{R}^n} |x-y|^2 \, d\gamma(x,y)$$

among all γ which have f' and g' as first and second marginals, respectively. Then there exists a unique optimal γ^o. Moreover there exists a globally Lipschitz convex function $\psi : \mathbb{R}^n \to \mathbb{R}$ such that $\nabla\psi(x) \in \operatorname{supp}(g')$ for a.e. $x \in \mathbb{R}^n$, and

$$\gamma^o = (\mathrm{Id} \times \nabla\psi)_\# f' = (\nabla\psi^* \times \mathrm{Id})_\# g',$$

where ψ^* denotes the Legendre transform of ψ. This implies in particular

$$\nabla\psi^*(\nabla\psi(x)) = x \quad f'\text{-a.e.}, \qquad \nabla\psi(\nabla\psi^*(y)) = y \quad g'\text{-a.e.} \tag{2.2}$$

Finally ψ solves the Monge-Ampère equation

$$\det(\nabla^2\psi)(x) = \frac{f'(x)}{g'(\nabla\psi(x))} \qquad f'\text{-a.e.} \tag{2.3}$$

Conversely, if $\varphi : \mathbb{R}^n \to \mathbb{R}$ is a convex function such that $\nabla\varphi_\# f' = g'$, then $(\mathrm{Id} \times \nabla\varphi)_\# f'$ solves the Monge-Kantorovich problem.

Let us first prove the following key result, which shows that solutions of the partial optimal transport problem are solutions of an optimal transport problem:

Proposition 2.4 (Minimizers solve an optimal transport problem) *Let $m \in [0, m_{\max}]$, $\bar\gamma \in \Gamma^o(m)$, and consider the Monge-Kantorovich problem:*

$$\text{minimize } C(\gamma) = \int_{\mathbb{R}^n \times \mathbb{R}^n} |x-y|^2 \, d\gamma(x,y)$$

among all γ which have $f + (g - g_{\bar\gamma})$ and $g + (f - f_{\bar\gamma})$ as first and second marginals, respectively. Then the unique minimizer is given by

$$\bar\gamma + (\mathrm{Id} \times \mathrm{Id})_\# \big((f - f_{\bar\gamma}) + (g - g_{\bar\gamma})\big). \tag{2.4}$$

Remark 2.5 Although using Theorem 2.3 the above proposition could be proved in a simpler way, we prefer to give a proof independent of it to show that the minimizing property of the plan defined by (2.4) holds true whenever $c(x,y)$ is a nonnegative cost function.

Proof. Let γ have marginals $\bar f = f + (g - g_{\bar\gamma})$ and $\bar g = g + (f - f_{\bar\gamma})$. The idea is to prove that, since $\int_{\mathbb{R}^n} f = m + \int_{\mathbb{R}^n}(f - f_{\bar\gamma})$, γ has to send at least an amount m of the mass of f onto g. In particular there exists $\gamma_- \le \gamma$ such that $\mathscr{M}(\gamma_-) \ge m$ and $\gamma_- \in \Gamma_\le(f,g)$. From this fact the result will follow easily. Let us therefore prove the existence of γ_-.

We consider the disintegration of γ with respect to its first and second marginals respectively, that is

$$\gamma(dx,dy) = \gamma_x(dy) \otimes \bar f(x)\,dx = \gamma_y(dx) \otimes \bar g(y)\,dy$$

(see Theorem 1.4). Then we define

$$\gamma'(dx,dy) := \gamma_x(dy) \otimes f(x)\,dx,$$

and we denote by f' and g' its marginals. It is clear that $f' = f$, $g' \le g + (f - f_{\bar\gamma})$. Since $\mathscr{M}(\gamma') = \int_{\mathbb{R}^n} f = m + \int_{\mathbb{R}^n}(f - f_{\bar\gamma})$, it is not difficult to see that

$$\int_{\mathbb{R}^n} g' \wedge g \ge m;$$

indeed

$$\int_{\mathbb{R}^n} g' \wedge g \ge \int_{\mathbb{R}^n} g' \wedge \big(g + (f - f_{\bar\gamma})\big) - \int_{\mathbb{R}^n} g' \wedge (f - f_{\bar\gamma}) \ge \int_{\mathbb{R}^n} g' - \int_{\mathbb{R}^n} (f - f_{\bar\gamma}) = m.$$

Thus we immediately get that

$$\gamma_-(dx,dy) := \gamma_y(dx) \otimes (g' \wedge g)(y)\,dy$$

is the desired subplan.

Now, since $\gamma_- \in \Gamma_\le(f,g)$ and $\mathscr{M}(\gamma_-) \ge m$, we get $C(\gamma_-) \ge C(m)$. As $C(\gamma) \ge C(\gamma_-)$ and γ was arbitrary, we have proved that the infimum in the Monge-Kantorovich problem is greater or equal than $C(m)$. On the other hand, if we consider the plan $\tilde\gamma$ defined by Equation (2.4), then $C(\tilde\gamma) = C(\bar\gamma) = C(m)$. Thus $\tilde\gamma$ is a minimizer. □

Thanks to the above proposition, we can prove the following:

Theorem 2.6 (Graph structure of minimizers) *Let $m \in [0, m_{\max}]$. There exists a globally Lipschitz convex function ψ such that $\nabla\psi(x) \in \{g > 0\}$ for a.e. $x \in \mathbb{R}^n$, and all $\gamma \in \Gamma^o(m)$ are concentrated on the graph of $\nabla\psi$. Moreover $\nabla\psi$ is injective f-a.e., and for any $\gamma \in \Gamma^o(m)$*

$$\nabla\psi(x) = x \qquad \text{for a.e. } x \in \{f_\gamma < f\} \cup \{g_\gamma < g\}, \tag{2.5}$$

and

$$f_\gamma(x) = g_\gamma(x) \qquad \text{for a.e. } x \in \{f_\gamma < f\} \cup \{g_\gamma < g\}. \tag{2.6}$$

Remark 2.7 The key point of the above statement is that the function ψ is the same for all minimizers. Since we will prove later that for $m \in [m_{\min}, m_{\max}]$ there exists a unique minimizer, while for $m \in [0, m_{\min}]$ the minimizers are all concentrated on the graph of the identity map, the fact that ψ is independent of the minimizer is not interesting in itself. However we preferred to state the theorem in this form, because we believe that the strategy of the proof is interesting and could be used in other situations in which uniqueness of minimizers fails.

Proof. It is simple to see that $\Gamma^o(m)$ is compact with respect to the weak* topology of measures. In particular $\Gamma^o(m)$ is separable, and we can find a dense countable subset $(\gamma_n)_{n=1}^\infty \subset \Gamma^o(m)$. Denote by $\bar\gamma := \sum_{n=1}^\infty \frac{1}{2^n}\gamma_n$. Since the minimization problem (2.1) is linear and the constraints are convex, $\bar\gamma \in \Gamma^o(m)$. The idea now is that, if we prove that $\bar\gamma$ is concentrated on a graph, then all $\gamma \in \Gamma^o(m)$ have to be concentrated on such a graph.

To prove the graph property of $\bar\gamma$ we apply Proposition 2.4: we know that the plan

$$\tilde\gamma := \bar\gamma + (\mathrm{Id}\times\mathrm{Id})_\#\big((f - f_{\bar\gamma}) + (g - g_{\bar\gamma})\big), \tag{2.7}$$

solves the Monge-Kantorovich problem:

$$\text{minimize } C(\gamma) = \int_{\mathbb{R}^n\times\mathbb{R}^n} |x-y|^2\, d\gamma(x,y)$$

among all γ which have $f + (g - g_{\bar\gamma})$ and $g + (f - f_{\bar\gamma})$ as first and second marginals, respectively. By Theorem 2.3 we deduce that $\tilde\gamma$ is concentrated on the graph of the gradient of a convex function ψ, that is

$$\tilde\gamma = (\mathrm{Id}\times\nabla\psi)_\#\big(f + (g - g_{\bar\gamma})\big), \tag{2.8}$$

and by (2.2) we deduce that $\nabla\psi$ is injective f-a.e. Combining (2.8) with (2.7) we deduce that $\tilde\gamma$ is both concentrated on the graph of Id and on the graph of $\nabla\psi$ for a.e. $x \in \{f_{\bar\gamma} < f\}\cup\{g_{\bar\gamma} < g\}$, which implies

$$\nabla\psi(x) = x \qquad \text{for a.e. } x \in \{f_{\bar\gamma} < f\}\cup\{g_{\bar\gamma} < g\}. \tag{2.9}$$

Fix now any $\gamma \in \Gamma^o(m)$, and let us prove that $\nabla\psi(x) = x$ for a.e. $x \in \{f_\gamma < f\}$ (the case $x \in \{g_\gamma < g\}$ being analogous). By (2.9) we know that $\nabla\psi(x) = x$ for a.e. $x \in \bigcup_n\big(\{f_{\gamma_n} < f\}\cup\{g_{\gamma_n} < g\}\big)$. Thus it suffices to prove that, for any $k \in \mathbb{N}$,

$$\Big\{f - f_\gamma > \frac{1}{k}\Big\} \overset{a.e.}{\subset} \bigcup_{n=1}^\infty \Big\{f - f_{\gamma_n} > \frac{1}{2k}\Big\}.$$

First of all we observe that, thanks to the density of $(\gamma_n)_{n=1}^\infty$ in $\Gamma^o(m)$, the set $(f_{\gamma_n})_{n=1}^\infty$ is dense in $\{f_\gamma : \gamma \in \Gamma^o(m)\}$ with respect to the the weak* topology of measures. On the other hand, since $f_\gamma \le f$ for all $\gamma \in \Gamma^o(m)$, the density of $(f_{\gamma_n})_{n=1}^\infty$ holds also with respect to the the weak topology of L^1. Therefore, if by contradiction there exists a Borel set A, with $|A| > 0$, such that $A \subset \{f - f_\gamma > \frac{1}{k}\}$ and $A \cap \{f - f_{\gamma_n} > \frac{1}{2k}\} = \emptyset$ for all $n \ge 1$, we would obtain

$$\int_A f - f_\gamma \ge \frac{1}{k}|A|, \qquad \int_A f - f_{\gamma_n} \le \frac{1}{2k}|A| \quad \forall\, n \ge 1,$$

which contradicts the density of $(f_{\gamma_n})_{n=1}^{\infty}$ in $\{f_\gamma : \gamma \in \Gamma^o(m)\}$ in the weak topology of L^1. This proves (2.5). Observing that $\nabla\psi_\# f_\gamma = g_\gamma$, applying (2.3) with $f' = f_\gamma$ and $g' = g_\gamma$, we get

$$\det(\nabla^2\psi)(x) = \frac{f_\gamma(x)}{g_\gamma(\nabla\psi(x))} \qquad f_\gamma\text{-a.e.}$$

It is a standard measure theory result that, if $\bar{x}$ is a density point for the set $\{\nabla\psi(x) = x\}$, and $\nabla\psi(x)$ is differentiable at $\bar{x}$, then $\nabla^2\psi(\bar{x}) = I_n$, where I_n denotes the identity matrix on $\mathbb{R}^n$ (see [3, Proposition 3.73(c)]). By this fact and (2.5), (2.6) follows easily. □

We now use the above theorem to show a domination property of minimizers which has an interest in itself, and which will play a crucial role in the regularity of the free boundary of the active regions.

Proposition 2.8 (Common mass is both source and target) *Let $m_0 \in [m_{\min}, m_{\max}]$, $\gamma \in \Gamma^o(m)$. Then $f_\gamma \geq f \wedge g$, $g_\gamma \geq f \wedge g$ (that is, all the common mass is both source and target for every minimizer).*

Proof. For $m_0 = m_{\min}$ the result is clear since, as we already said, the unique minimizer is given by $(\mathrm{Id} \times \mathrm{Id})_\#(f \wedge g)$. So we can assume $m_0 > m_{\min}$.

Applying Theorem 2.6 we know that we can write $\gamma = (\mathrm{Id} \times \nabla\psi)_\# f_\gamma$, with $\nabla\psi$ invertible f_γ-a.e. We observe that, since $\{f_\gamma < f \wedge g\} \subset \{f_\gamma < f\}$ and $\{g_\gamma < f \wedge g\} \subset \{g_\gamma < g\}$, by (2.5) we get

$$\nabla\psi(x) = x \qquad \text{for a.e. } x \in \{f_\gamma < f \wedge g\} \cup \{g_\gamma < f \wedge g\}.$$

Moreover, by (2.6), $f_\gamma(x) = g_\gamma(x)$ for a.e. $x \in \{f_\gamma < f \wedge g\} \cup \{g_\gamma < f \wedge g\}$. Thus either both f_γ and g_γ are greater or equal of $f \wedge g$, or $\{f_\gamma < f \wedge g\} = \{g_\gamma < f \wedge g\} \neq \emptyset$ and $f_\gamma = g_\gamma$ on that set.

Suppose by contradiction that $h := [f \wedge g - f_\gamma]_+ = [f \wedge g - g_\gamma]_+$ is not identically zero. Let $m_h > 0$ denote the mass of h, and consider the plan $\gamma_h = (\mathrm{Id} \times \mathrm{Id})_\# h$. Since $f_{\gamma+\gamma_h} = f_\gamma + f_{\gamma_h} = f_\gamma + [f \wedge g - f_\gamma]_+ \leq f$ and $g_{\gamma+\gamma_h} = g_\gamma + g_{\gamma_h} = g_\gamma + [f \wedge g - g_\gamma]_+ \leq g$, we have $\gamma + \gamma_h \in \Gamma_{\leq}(f, g)$. Observing that $C(\gamma_h) = 0$, we get

$$C(m_0 + m_h) \leq C(\gamma + \gamma_h) = C(\gamma) = C(m_0).$$

As $C(m)$ is convex and increasing, it has to be constant on the interval $[0, m_0 + m_h]$, and since $C(0) = 0$, $C(m) \equiv 0$ on $[0, m_0 + m_h]$. This is impossible if $m_0 \geq m_{\min}$, since it would imply that a mass $m_0 + m_h > m_{\min}$ should stay at rest. This contradiction gives the desired result.

□

2.3 Step 2: strict convexity of $C(m)$

In order to prove the strict convexity property, we first need to show that a linear part in the graph of $C(m)$ would imply a monotonicity result on minimizers.

Lemma 2.9 *Let $m_{\min} \leq m_1 < m_2 \leq m_{\max}$, and assume that $C(m)$ is linear on $[m_1, m_2]$. Fix $\gamma_1 \in \Gamma^o(m_1)$. Then there exists $\gamma_2 \in \Gamma^o(m_2)$ such that $\gamma_1 \leq \gamma_2$.*

Proof. Let $\gamma_2 \in \Gamma^o(m_2)$, and assume $\gamma_1 \not\leq \gamma_2$. We will modify γ_2 into $\tilde{\gamma}_2$ so that $\tilde{\gamma}_2 \in \Gamma^o(m_2)$ and $\gamma_1 \leq \tilde{\gamma}_2$.

Let us consider $\frac{\gamma_1+\gamma_2}{2}$. Since $C(m)$ is linear on $[m_1, m_2]$, we immediately get $\frac{\gamma_1+\gamma_2}{2} \in \Gamma^o(\frac{m_1+m_2}{2})$. In particular we can apply Theorem 2.6 to deduce the existemce of a convex function ψ such that both γ_1 and γ_2 are concentrated on the graph of $\nabla\psi$. We now define $\gamma_- = \gamma_1 \wedge \gamma_2 = (\mathrm{Id} \times \nabla\psi)_\#(f_{\gamma_1} \wedge f_{\gamma_2})$, and we write

$$\gamma_1 = \gamma_- + \bar{\gamma}_1, \qquad \gamma_2 = \gamma_- + \bar{\gamma}_2,$$

with $\bar{\gamma}_1 = (\mathrm{Id} \times \nabla\psi)_\#\big([f_{\gamma_1} - f_{\gamma_2}]_+\big)$, $\bar{\gamma}_2 = (\mathrm{Id} \times \nabla\psi)_\#\big([f_{\gamma_2} - f_{\gamma_1}]_+\big)$. Let $\lambda \in (0,1)$ be such that $\mathscr{M}(\lambda\bar{\gamma}_2) = \mathscr{M}(\bar{\gamma}_1)$. Since the function $\nabla\psi$ is injective f-a.e., it is simple to see that

$$\gamma_- + \bar{\gamma}_1 + \bar{\gamma}_2 \in \Gamma_\leq(f, g)$$

(indeed, its marginals are dominated by $f_{\gamma_1} \vee f_{\gamma_2}$ and $g_{\gamma_1} \vee g_{\gamma_2}$ respectively). Thanks to the optimality of γ_1 and γ_2 we have

$$C(\gamma_1) = C(\gamma_- + \bar{\gamma}_1) \leq C(\gamma_- + \lambda\bar{\gamma}_2), \qquad C(\gamma_2) = C(\gamma_- + \bar{\gamma}_2) \leq C(\gamma_- + \bar{\gamma}_1 + (1-\lambda)\bar{\gamma}_2),$$

which implies $C(\bar{\gamma}_1) = C(\lambda\bar{\gamma}_2)$, and therefore $C(\gamma_- + \bar{\gamma}_2) = C(\gamma_- + \bar{\gamma}_1 + (1-\lambda)\bar{\gamma}_2)$. Since

$$\gamma_1 + (1-\lambda)\bar{\gamma}_2 = \gamma_- + \bar{\gamma}_1 + (1-\lambda)\bar{\gamma}_2 \leq \gamma_- + \bar{\gamma}_1 + \bar{\gamma}_2 \in \Gamma_\leq(f, g),$$

we see that $\tilde{\gamma}_2 := \gamma_1 + (1-\lambda)\bar{\gamma}_2 \in \Gamma^o(m_2)$ is the desired minimizer. □

Theorem 2.10 (Strict convexity of $C(m)$) *The function $m \mapsto C(m)$ is strictly convex on $[m_{\min}, m_{\max}]$.*

Proof. Assume by contradiction that there exist $m_{\min} \leq m_1 < m_2 \leq m_{\max}$ such that that $C(m)$ is linear on $[m_1, m_2]$. Thanks to Lemma 2.9 we can find $\gamma_1 \in \Gamma^o(m_1)$ and $\gamma_2 \in \Gamma^o(m_2)$ such that $\gamma_1 \leq \gamma_2$. Let us define $\tilde{f} := f_{\gamma_2} - f_{\gamma_1}$, $\tilde{g} = g_{\gamma_2} - g_{\gamma_1}$. We are now interested in the minimization problem

$$\tilde{C}(m) := \min_{\gamma \in \Gamma_\leq(\tilde{f}, \tilde{g}),\, \mathscr{M}(\gamma)=m} C(\gamma). \tag{2.10}$$

Let $\tilde{m} := m_2 - m_1 = \int_{\mathbb{R}^n} \tilde{f} = \int_{\mathbb{R}^n} \tilde{g}$, and define $\tilde{\gamma} := \gamma_2 - \gamma_1$. Since $\gamma_1 + \lambda\tilde{\gamma} \in \Gamma^o(m_1 + \lambda\tilde{m})$ for $\lambda \in [0,1]$, this easily implies that

$$C(\lambda\tilde{\gamma}) = \tilde{C}(\lambda\tilde{m}),$$

i.e. $\lambda\tilde{\gamma}$ is optimal in the minimization problem (2.10) for all $\lambda \in [0,1]$. We now want to prove that this is impossible. Fix any $\lambda \in (0,1)$, and as in the proof of Theorem 2.6 consider the Monge-Kantorovich problem:

$$\text{minimize } C(\gamma) = \int_{\mathbb{R}^n \times \mathbb{R}^n} |x-y|^2 \, d\gamma(x,y)$$

among all γ which have $\lambda\tilde{f}+(1-\lambda)\tilde{g}$ and $\lambda\tilde{g}+(1-\lambda)\tilde{f}$ as first and second marginals respectively (observe that $(1-\lambda)\tilde{g} = \tilde{g} - g_{\lambda\tilde{\gamma}}$, $(1-\lambda)\tilde{f} = \tilde{f} - f_{\lambda\tilde{\gamma}}$). By Proposition 2.4

$$\gamma_\lambda := \lambda\tilde{\gamma} + (\mathrm{Id}\times\mathrm{Id})_\#\big((1-\lambda)[\tilde{f}+\tilde{g}]\big),$$

solves the Monge-Kantorovich problem. Thus, applying Theorem 2.3, we deduce that γ_λ is concentrated on the graph of the gradient of a globally Lipschitz convex function ψ. In particular, since $\lambda \in (0,1)$, $\tilde{\gamma}$ is concentrated on the graph of $\nabla\psi$ and

$$\nabla\psi(x) = x \qquad \text{for a.e. } x \in \{\tilde{f}+\tilde{g} > 0\}.$$

This clearly implies $\tilde{f} = \tilde{g}$ and $\tilde{\gamma} = (\mathrm{Id}\times\mathrm{Id})_\#\tilde{f}$. Therefore $0 = C(\tilde{\gamma}) = C(m_2) - C(m_1)$, and so $C(m)$ is constant on $[m_1, m_2]$. Since $m_1 \geq m_{\min}$, as in the proof of Proposition 2.8 this is impossible. □

Remark 2.11 (Extension to more general cost functions) We observe that all the above arguments do not really use that the cost is quadratic: all we need is

(1) $c(x,y) \geq 0$ and $c(x,y) = 0$ only for $x = y$;

(2) whenever both the source and the target measure are compactly supported and absolutely continuous with respect to the Lebesgue measure, the Monge-Kantorovich problem with cost $c(x,y)$ has a unique solution which is concentrated on the graph of a function T;

(2') T is injective a.e. on the support of the source measure;

(2") T is differentiable a.e. on the support of the source measure.

Let us remark that condition (2') was used in the proof of Lemma 2.9, while (2") is needed for deducing (2.6) from (2.5).

Some simple assumptions on the cost function which ensure the validity of (2)-(2')-(2") are:

(a) $c \in C^2(\mathbb{R}^n \times \mathbb{R}^n)$;

(b) the map $y \mapsto \nabla_x c(x,y)$ is injective for all $x \in \mathbb{R}^n$;

(b') the map $x \mapsto \nabla_y c(x,y)$ is injective for all $y \in \mathbb{R}^n$;

(c) $\det(\nabla_{x,y} c) \neq 0$ for all $x, y \in \mathbb{R}^n$.

To understand why (a)-(b)-(c) imply (2)-(2"), we recall that (a)-(b) ensure that the existence of an optimal transport map T. Moreover this map is implicitly defined by the identity

$$\nabla_x c(x, T(x)) = \nabla\phi(x),$$

where $\phi(x)$ is given by

$$\phi(x) = \inf_{y\in\mathrm{supp}(\nu)} c(x,y) - C_y,$$

with $y \mapsto C_y$ locally bounded (see [17, Chapter 10]). By (a) we know that ϕ is locally semiconcave, which implies that its gradient (which exists a.e.) is differentiable a.e. Since by (a)-(b)-(c) the map $(x,p) \mapsto (x, [\nabla_x c(x,\cdot)]^{-1} p)$ is a smooth diffeomorphism, we obtain that also T is differentiable a.e. Finally, (a)-(b') give the existence of an optimal transport map also for the Monge-Kantorovich problem with cost $\hat{c}(x,y) := c(y,x)$. From this fact (2') follows easily.

If $c(x,y) = d(x,y)^2$ with $d(x,y)$ a Riemannian distance on a manifold, although $c(x,y)$ does not satisfies the above assumptions, existence and uniqueness of the optimal transport map is still true [16, 12], and the Jacobian identity $\det(\nabla T) = \frac{f}{g \circ T}$ holds almost everywhere [9, 13] (although in the non-compact case, one has to define the gradient of T in an appropriate weak sense). So our existence and uniqueness result for the partial transport problem applies also to this case.

3 Properties of the active regions

Let us denote by Ω and Λ the Borel sets $\{f > 0\}$ and $\{g > 0\}$ respectively. We have $\{f \wedge g > 0\} = \Omega \cap \Lambda$.

For $m \in [m_{\min}, m_{\max}]$ we denote by $\gamma_m = (\mathrm{Id} \times \nabla\psi_m)_\# f_m = (\nabla\psi_m^* \times \mathrm{Id})_\# g_m$ the unique minimizer of the minimization problem (2.1), where f_m and g_m denote the two marginals of γ_m. We define the active source and the active target as

$$F_m := \text{ set of density points of } \{f_m > 0\}, \qquad G_m := \text{ set of density points of } \{g_m > 0\}.$$

We want to study the regularity properties of these sets. First of all, by Proposition 2.8, we have

$$F_m \overset{a.e.}{\supset} \Omega \cap \Lambda, \qquad G_m \overset{a.e.}{\supset} \Omega \cap \Lambda,$$

which will be a key fact to prove the regularity of the boundary of the free regions. We now prove an interior ball condition, which is the analogous in our formalism of [8, Corollary 2.4].

Proposition 3.1 (Structure of the active regions) *There exists a set $\Gamma_m \subset \mathbb{R}^n \times \mathbb{R}^n$ on which γ_m is concentrated such that*

$$F_m \overset{a.e.}{=} (\Omega \cap \Lambda) \cup \bigcup_{(\bar{x},\bar{y}) \in \Gamma_m} \left\{ x \in \Omega \,:\, |x - \bar{y}|^2 < |\bar{x} - \bar{y}|^2 \right\},$$

$$G_m \overset{a.e.}{=} (\Omega \cap \Lambda) \cup \bigcup_{(\bar{x},\bar{y}) \in \Gamma_m} \left\{ y \in \Lambda \,:\, |\bar{x} - y|^2 < |\bar{x} - \bar{y}|^2 \right\}.$$

Proof. Let us recall that $\gamma_m = (\mathrm{Id} \times \nabla\psi_m)_\# f_m = (\nabla\psi_m^* \times \mathrm{Id})_\# g_m$. We denote by D_{ψ_m} and $D_{\psi_m^*}$ the sets where ψ_m and ψ_m^* are respectively differentiable. We recall that the gradient of a convex function is continuous on its domain of definition. We define

$$\begin{aligned} \Gamma_m &:= (\mathrm{Id} \times \nabla\psi_m)(F_m \cap D_{\psi_m}) \cap (\nabla\psi_m^* \times \mathrm{Id})(G_m \cap D_{\psi_m^*}) \\ &= \left\{ (x,y) \,:\, y = \nabla\psi_m(x) \text{ with } x \in F_m \cap D_{\psi_m} \text{ and } x = \nabla\psi_m^*(y) \text{ with } y \in G_m \cap D_{\psi_m^*} \right\}. \end{aligned} \tag{3.1}$$

Since D_{ψ_m} and $D_{\psi_m^*}$ have full measure for f_m and g_m respectively, it is clear that γ_m is concentrated on Γ_m.

We will prove the thesis only for F_m (the case of G_m being analogous). Since D_{ψ_m} has full measure for f_m, the result will follow from the inclusions

$$F_m \cap D_{\psi_m} \overset{a.e.}{\subset} (\Omega \cap \Lambda) \cup \bigcup_{(\bar{x},\bar{y})\in\Gamma_m} \left\{x \in \Omega \,:\, |x-\bar{y}|^2 < |\bar{x}-\bar{y}|^2\right\} \overset{a.e}{\subset} F_m. \tag{3.2}$$

Let us first prove the left inclusion. Let $x \in F_m \cap D_{\psi_m} \setminus \Omega \cap \Lambda$. Then obviously $\nabla\psi_m(x) \neq x$. Define $v_x := \nabla\psi_m(x) - x \neq 0$. Since x is a density point for the set $\{f_m > 0\}$, there exists a sequence of points (x_k) such that

$$(x_k, \nabla\psi_m(x_k)) \in \Gamma_m, \qquad x_k \to x \qquad \text{and} \qquad \langle x - x_k, v_x\rangle \geq \frac{1}{2}|x-x_k||v_x|$$

(the idea is that we want $x - x_k$ "almost parallel" to v_x). Thanks to the choice of x_k it is simple to check that, since $\nabla\psi_m(x_k) \to \nabla\psi_m(x)$, we have

$$|x - \nabla\psi_m(x_k)|^2 < |x_k - \nabla\psi_m(x_k)|^2 \qquad \text{for } k \text{ large enough.}$$

Since $F_m \overset{a.e.}{\subset} \Omega$, this implies the desired inclusion.

We now have to prove the right inclusion in (3.2). The heuristic idea is simple: if a point $x \in \Omega$ is such that $|x-\bar{y}|^2 < |\bar{x}-\bar{y}|^2$ but $x \notin F_m$, then we can replace $(\bar{x},\bar{y})$ with $(x,\bar{y})$ to obtain a measure on $\mathbb{R}^n \times \mathbb{R}^n$ with the same mass than γ_m but which pays less, and this would contradict the optimality of γ_m. Here is the rigorous argument: let us consider the set

$$E := \bigcup_{(\bar{x},\bar{y})\in\Gamma_m} \left\{x \in \Omega \,:\, |x-\bar{y}|^2 < |\bar{x}-\bar{y}|^2\right\} \setminus \Omega \cap \Lambda.$$

First of all, since the set Γ_m is separable, we can find a dense countable subset $\big((x_k,y_k)\big)_{k\in\mathbb{N}} \in \Gamma_m$, so that

$$E = \bigcup_{k\in\mathbb{N}} \left\{x \in \Omega \,:\, |x-y_k|^2 < |x_k-y_k|^2\right\} \setminus \Omega \cap \Lambda.$$

Assume by contradiction that $E \overset{a.e.}{\not\subset} F_m$. This implies that there exist $k \in \mathbb{N}$ and a Borel set A with $|A| > 0$ such that

$$A \subset \left\{x \in \Omega \,:\, |x-y_k|^2 < |x_k-y_k|^2\right\} \setminus \Omega \cap \Lambda, \qquad A \cap F_m = \emptyset.$$

Fix $\varepsilon > 0$. First of all, since

$$A_\varepsilon := A \cap \left\{x \in \Omega \,:\, |x-y_k|^2 < |x_k-y_k|^2 - \varepsilon\right\} \to A$$

as $\varepsilon \to 0$ (i.e. $\chi_{A_\varepsilon} \to \chi_A$ in L^1 as $\varepsilon \to 0$), by monotone convergence there exists $\varepsilon > 0$ small such that $|A_\varepsilon| > 0$. Fix $\eta > 0$ small (the smallness will depend on ε, as we will see below). Since

$y_k = \nabla\psi_m(x_k)$, $\nabla\psi_m$ is continuous at x_k, and x_k is a density point for F_m, we can find a small ball $B_\delta(x_k)$ such that

$$|x-\nabla\psi_m(x)|^2 \geq |x_k-y_k|^2-\frac{\varepsilon}{8}, \quad |\nabla\psi_m(x)-\nabla\psi_m(x_k)|^2 \leq \eta \qquad \forall\, x \in B_\delta(x_k)\cap F_m \cap D_{\psi_m}. \tag{3.3}$$

Therefore, for $\eta = \eta(\varepsilon)$ small enough,

$$|x-y|^2 \leq \left(\sqrt{|x_k-y_k|^2-\varepsilon}+\eta\right)^2 \leq |x_k-y_k|^2-\frac{\varepsilon}{4} \qquad \forall\, x \in A_\varepsilon,\, y \in \nabla\psi_m(B_\delta(x_k)\cap F_m \cap D_{\psi_m}). \tag{3.4}$$

Up to choosing δ smaller, we can assume $\int_{A_\varepsilon} f \geq \int_{B_\delta(x_k)} f_m > 0$. Thus, if we consider the density $g_\delta := \nabla\psi_{m\#}(f_m\chi_{B_\delta(x_k)})$, we can replace the plan

$$\gamma_\delta := (\mathrm{Id}\times\nabla\psi_m)_\#(f_m\chi_{B_\delta(x_k)})$$

with any plan $\tilde\gamma_d$ such that $f_{\tilde\gamma_d} \leq f\chi_{A_\varepsilon}$ and $g_{\tilde\gamma_d} = g_\delta$. In this way, since $\mathscr{M}(\tilde\gamma_\delta) = \mathscr{M}(\gamma_\delta)$, thanks to (3.3) and (3.4) we get

$$C(\tilde\gamma_\delta) \leq \left(|x_k-y_k|^2-\frac{\varepsilon}{4}\right)\mathscr{M}(\tilde\gamma_\delta) < \left(|x_k-y_k|^2-\frac{\varepsilon}{8}\right)\mathscr{M}(\gamma_\delta) \leq C(\gamma_\delta)$$

Since $\gamma_m - \gamma_\delta + \tilde\gamma_\delta \in \Gamma(m)$, the above inequality would contradict the optimality of γ_m. Thus $E \subset F_m$ and also the second inclusion is proved. □

Remark 3.2 (The case of general cost) The above result can be generalized to the cost functions considered in Remark 2.11, obtaining that

$$F_m \overset{a.e.}{=} (\Omega\cap\Lambda) \cup \bigcup_{(\bar x,\bar y)\in\Gamma_m} \{x \in \Omega \,:\, c(x,\bar y) \leq c(\bar x,\bar y)\},$$

$$G_m \overset{a.e.}{=} (\Omega\cap\Lambda) \cup \bigcup_{(\bar x,\bar y)\in\Gamma_m} \{y \in \Lambda \,:\, c(x,\bar y) \leq c(\bar x,\bar y)\},$$

with Γ_m defined as $(\mathrm{Id}\times T)(F_m\cap D_T)\cap(T^{-1}\times\mathrm{Id})(G_m\cap D_{T^{-1}})$, where T is the optimal transport map, and D_T and $D_{T^{-1}}$ denote the set of continuity points for T and T^{-1} respectively.

Remark 3.3 (Equality everywhere) Let us define

$$U_m := (\Omega\cap\Lambda) \cup \bigcup_{(\bar x,\bar y)\in\Gamma_m} B_{|\bar x-\bar y|}(\bar y),$$

$$V_m := (\Omega\cap\Lambda) \cup \bigcup_{(\bar x,\bar y)\in\Gamma_m} B_{|\bar x-\bar y|}(\bar x).$$

By the proof of above proposition it is not difficult to deduce that, if Ω and Λ are open sets, then

$$F_m \cap D_{\psi_m} \subset U_m \cap \Omega \subset F_m$$

(and analogously for G_m). Since $F_m \cap D_{\psi_m}$ is of full Lebesgue measure inside F_m, and U_m is open, recalling the definition of F_m one easily obtains $F_m = U_m \cap \Omega$, that is the equality is true everywhere and not only up to set of measure zero (and analogously $G_m = V_m \cap \Lambda$). In particular, the inclusions $F_m \supset \Omega \cap \Lambda$ and $G_m \supset \Omega \cap \Lambda$ hold.

Remark 3.4 (Monotone expansion of the active regions) Thanks to the uniqueness of minimizers for $m \in [m_{\min}, m_{\max}]$, we can apply [8, Theorem 3.4] to show the monotone expansion of the active regions. More precisely one obtains

$$f_{m_1} \leq f_{m_2} \quad \text{and} \quad g_{m_1} \leq g_{m_2} \qquad \text{for } m_{\min} \leq m_1 \leq m_2 \leq m_{\max},$$

so that in particular

$$F_{m_1} \subset F_{m_2} \quad \text{and} \quad G_{m_1} \subset G_{m_2} \qquad \text{for } m_{\min} \leq m_1 \leq m_2 \leq m_{\max}.$$

4 Regularity results

By the duality theory developed in [8], as for $m \geq m_{\min}$ there exists a unique optimal transport map $\nabla\psi_m$, one could prove that ψ_m is the unique *Brenier solution* to a Monge-Ampère obstacle problem[2] (see [8, Section 4]). However we prefer here to not make the link with the Monge-Ampère obstacle problem (which would require to use the duality theory in [8, Section 2] in order to construct the obstacles), and instead we concentrate directly on the regularity of the maps ψ_m and of the free boundaries of the active regions F_m and G_m.

Since for $m \leq m_{\min}$ the map ψ_m is trivial (just take $\psi_m(x) = |x|^2/2$ so that $\nabla\psi_m(x) = x$ everywhere), from now on we will consider only the case $m > m_{\min}$.

4.1 Partial semiconvexity of the free boundary

We will assume $\Omega = \{f > 0\}$ and $\Lambda = \{g > 0\}$ to be open and bounded sets. We define the free boundaries of F_m and G_m as $\partial F_m \cap \Omega$ and $\partial G_m \cap \Lambda$ respectively. Thanks to Remark 3.3, they respectively coincide with

$$\partial U_m \cap \Omega \qquad \text{and} \qquad \partial V_m \cap \Lambda.$$

Therefore, to prove regularity results on the free boundaries, we need to study the regularity properties of the sets ∂U_m and ∂V_m inside Ω and Λ, respectively.

Definition 4.1 Let E and F be open sets. We say that $\partial E \cap F$ is *locally semiconvex* if, for each $x \in \partial E \cap F$, there exists a ball $B_r(x) \subset F$ such that $\partial E \cap B_r(x)$ can be written in some system of coordinates as the graph of a semiconvex function, and $E \cap B_r(x)$ is contained in the epigraph of such a function.

In [8, Section 5] the authors prove the semiconvexity of the free boundaries of the active regions assuming the existence of a hyperplane which separates the supports of f and g. Their

[2]In [8] the authors adopt the terminology of *weak** *solution*, instead of *Brenier solution*.

proof is based on an analogue of Proposition 3.1. Indeed the existence of a separating hyperplane, together with the fact that in their case $|x - \nabla\psi(x)| \geq \delta > 0$, allows to write $U_m \cap \Omega$ as the epigraph of a function $u : \mathbb{R}^{n-1} \to \mathbb{R}$ such that, for each point $x = (x', u(x')) \in \partial U_m \cap \Omega$, there exists a ball of radius $\geq \delta$ which touches the graph of u at x from below. Thanks to this property, they can show the semiconvexity of the free boundary. In particular they deduce that the Lebesgue measure of ∂U_m and ∂U_m is zero, and so f_m and g_m give no mass to ∂U_m and ∂V_m respectively (this property is crucial to apply Caffarelli's regularity theory in this context, see [8, Theorem 6.3] and the proof of Theorem 4.8 in the next paragraph).

Remark 4.2 The assumption of the existence of a separating hyperplane plays an important role in the semiconvexity property. Indeed assume that g is supported on the ball $B_1(0) = \{|x| \leq 1\}$, and f has a connected support which contains the points $x_+ := (2, 0, \ldots, 0)$ and $x_- := (-2, 0, \ldots, 0)$. It could be possible that the maps $\nabla\psi$ send the points x_+ and x_- into two (distinct) points y_+ and y_- such that $|x_+ - y_+| = |x_- - y_-| = 2$. In this case the active target region V_m contains $A := B_1(0) \cap \big(B_2(x_+) \cup B_2(x_-)\big)$, and so ∂V_m could be not a graph near the origin (and so it would not be locally semiconvex). However it is still possible to prove that ∂V_m is $(n-1)$-rectifiable, so that in particular it has zero Lebesgue measure (see Proposition 4.4 below).

In our case everything becomes a priori much difficult for two reasons: first of all, since we do not want to assume the supports of f and g to be disjoint, of course we cannot have a separating hyperplane. Furthermore, as we do not have any lower bound on the quantity $|x - \nabla\psi(x)|$, the condition of having a ball touching from inside at each point of the boundary becomes a priori useless.

We will solve these problems in two ways, depending on which of the two following results we want to prove:

(a) The free boundaries have zero Lebesgue measure (resp. are $(n-1)$-rectifiable, i.e. countable union of Lipschitz hypersurfaces).

(b) Each free boundary can be written as the union of a locally semiconvex hypersurface together with a closed subset of a Lipschitz hypersurface.

The idea behind both results is the following: by what we already proved, the active regions contain $\Omega \cap \Lambda$ (Proposition 3.1). Therefore, since $|x - \nabla\psi(x)|$ can be 0 only for $x \in \Omega \cap \Lambda$, at the points where radius of the ball touching from inside goes to 0 we will use the information that the set $\Omega \cap \Lambda$ is touching ∂U_m from inside.

This observation is more or less sufficient to prove (a). On the other hand, by Remark 4.2, it is clear that to prove (b) we need some geometric assumption on the supports f and g. To deal with this problem, we will assume that there exists an open convex set C such that $\Lambda \subset C$ and $\Omega \setminus \Lambda \subset \mathbb{R}^n \setminus C$. This hypothesis, which generalizes the assumption of a separating hyperplane, is satisfied for instance if Λ is convex, since it suffices to take $C = \Lambda$.

We will need the following:

Lemma 4.3 *Let A be a bounded Borel subset of $\mathbb{R}^n$, and define*

$$E := \bigcup_{x \in A} B_{r(x)}(x),$$

where $x \mapsto r(x) \in (0, +\infty)$ is a Borel function. Assume that there exist $\delta, R > 0$ such that $\delta \le r(x) \le R$ for all $x \in A$. Then ∂E is a $(n-1)$-rectifiable set, and in particular has zero Lebesgue measure.

Proof. First of all, let $\Gamma \subset \mathbb{R}^n \times \mathbb{R}$ denotes the graph of $x \mapsto r(x)$. Then

$$E = \bigcup_{(x,r) \in \Gamma} B_r(x).$$

Denoting by $\overline{\Gamma}$ the closure of Γ, it is simple to check that

$$E = \bigcup_{(x,r) \in \overline{\Gamma}} B_r(x). \tag{4.1}$$

Indeed, if $y \in \bigcup_{(x,r) \in \overline{\Gamma}} B_r(x)$, then there exist $(x, r) \in \overline{\Gamma}$ such that $y \in B_r(x)$. Since the ball $B_r(x)$ is open, it is clear that if $(x_k, r(x_k))$ is a sequence of points in Γ that converges to (x, r), then $y \in B_{r(x_k)}(x_k)$ for k large enough. This proves (4.1).

Let now $y \in \partial E$. This implies that for all $\varepsilon > 0$ the following holds:

$$|y - x| \ge r - \varepsilon \qquad \forall\, (x, r) \in \overline{\Gamma}$$

and there exists $(x_\varepsilon, r_\varepsilon) \in \overline{\Gamma}$ such that $|y - x_\varepsilon| \le r_\varepsilon + \varepsilon$. Since $\overline{\Gamma}$ is compact (recall that A is bounded and $r \in [\delta, R]$ for all $(x, r) \in \overline{\Gamma}$), it is simple to deduce that there exists $(x_0, r_0) \in \overline{\Gamma}$ such that $|y - x_0| = r_0$, that is $y \in \partial B_{r_0}(x_0)$. As $r_0 \ge \delta$, we have proved that at each point $y \in \partial E$ we can find a ball of radius δ touching ∂E at y from the interior. This condition, called Interior Ball Property, implies that ∂E is $(n-1)$-rectifiable[3].

On the other hand we remark that, to prove just that ∂E has zero Lebesgue measure, it suffices to show that ∂E has no density points. Indeed we recall that, given a Borel set A, it is a well-known fact that a.e. point $x \in A$ is a density point (see for instance [11, Paragraph 1.7.1, Corollary 3]). To see that ∂E has no density points it suffices to observe that if $y \in \partial E$ then $y \in \partial B_{|y-x|}(x)$ for some ball $B_{|y-x|}(x) \subset E$, and so

$$\lim_{r \to 0} \frac{\mathscr{L}^n\big(\partial E \cap B_r(y)\big)}{\mathscr{L}^n\big(B_r(y)\big)} \le \lim_{r \to 0} \frac{\mathscr{L}^n\big(B_r(y) \setminus B_{|y-x|}(x)\big)}{\mathscr{L}^n\big(B_r(y)\big)} = \frac{1}{2}.$$

□

[3]Using [1, Lemma 2.1] one can prove for instance a stronger result: let $\{v_1, \ldots, v_{k_n}\}$ be a subset of the unit sphere of $\mathbb{R}^n$ such that, for each vector $p \in \mathbb{R}^n$ with unit norm, there exists $i \in \{1, \ldots, k_n\}$ with $\langle v_i, p \rangle \ge 1/2$. Then, if to any $x \in \partial E$ we associate a unit vector $p_x \in \mathbb{R}^n$ such that $B_\delta(x - \delta p_x) \subset E$, we can decompose ∂E as the union of A_i, with $A_i := \{x \in \partial E : \langle v_i, p_x \rangle \ge 1/2\}$, and thanks to [1, Lemma 2.1] one can show that each A_i is locally semiconvex.

We can now prove (a).

Proposition 4.4 (No mass on the free boundary) *Assume $\Omega \cap \Lambda$ open, with $\mathscr{L}^n\big(\partial(\Omega \cap \Lambda)\big) = 0$. Let $(\Omega \cap \Lambda)_\varepsilon := \{x \in \mathbb{R}^n \,:\, \mathrm{dist}(x, \Omega \cap \Lambda) < \varepsilon\}$, and assume that for any $\varepsilon > 0$*

$$\mathrm{dist}\big(\Omega \setminus (\Omega \cap \Lambda)_\varepsilon, \Lambda \setminus \Omega\big) =: \delta(\varepsilon) > 0. \tag{4.2}$$

Then $\mathscr{L}^n\big(\partial U_m\big) = 0$, and in particular both f_m and g_m give no mass to ∂U_m. Moreover, if $\partial(\Omega \cap \Lambda)$ is Lipschitz, then ∂U_m is $(n-1)$-rectifiable.

Proof. Decompose the boundary of U_m as

$$\partial U_m = \partial U_m^e \cup \partial U_m^b,$$

where

$$\partial U_m^e := \partial U_m \setminus \partial(\Omega \cap \Lambda), \qquad \partial U_m^b := \partial U_m \cap \partial(\Omega \cap \Lambda).$$

Obviously ∂U_m^b has measure zero. Since by definition $U_m \supset \Omega \cap \Lambda$, and $\Omega \cap \Lambda$ is open, we easily get $\partial U_m \cap (\Omega \cap \Lambda) = \emptyset$, that is ∂U_m cannot enter inside $\Omega \cap \Lambda$. Therefore we can write ∂U_m^e as the increasing union of the sets

$$B_\varepsilon^e := \partial U_m^e \setminus (\Omega \cap \Lambda)_\varepsilon.$$

Consider the open set

$$E_\varepsilon := \bigcup_{(\bar{x},\bar{y}) \in \Gamma_m,\, \bar{x} \notin (\Omega \cap \Lambda)_\varepsilon} B_{|\bar{x}-\bar{y}|}(\bar{y}),$$

where Γ_m is the set defined in (3.1). Thanks to (4.2), we can apply Lemma 4.3 to deduce that $\mathscr{L}^n\big(\partial E_\varepsilon\big) = 0$. Observing that

$$\partial U_m^e \setminus (\Omega \cap \Lambda)_\varepsilon \subset \partial E_\varepsilon,$$

we obtain that $\mathscr{L}^n\big(B_\varepsilon^e\big) = 0$ for all $\varepsilon > 0$, and so $\mathscr{L}^n\big(\partial U_m^e\big) = 0$.

Suppose now that $\partial(\Omega \cap \Lambda)$ is Lipschitz. Then ∂U_m^b is clearly $(n-1)$-rectifiable. Moreover, by Lemma 4.3, also ∂E_ε is $(n-1)$-rectifiable for all $\varepsilon > 0$. From this we conclude easily. $\square$

Let us now prove (b).

Proposition 4.5 (Local semiconvexity away from the intersection of the supports) *Assume Ω open, and suppose there exists an open convex set C such that $\Lambda \subset C$ and $\Omega \setminus \Lambda \subset \mathbb{R}^n \setminus C$. We decompose the boundary of U_m inside Ω as*

$$\partial U_m \cap \Omega = \big(\partial U_m^e \cap \Omega\big) \cup \big(\partial U_m^b \cap \Omega\big),$$

where

$$\partial U_m^e := \partial U_m \setminus \partial C, \qquad \partial U_m^b := \partial U_m \cap \partial C.$$

Then ∂U_m^e is locally semiconvex inside Ω.

Proof. By the definition of C, one can easily check that $\Omega \cap \Lambda = \Omega \cap C$. In particular $\Omega \cap \Lambda$ is open. Therefore, since by Remark 3.3 we have $U_m \cap \Omega = F_m \supset \Omega \cap \Lambda = \Omega \cap C$, we easily obtain $(\partial U_m \cap \Omega) \cap C = \emptyset$, that is the free boundary cannot enter inside C. Thus we can write ∂U_m^e as the increasing union of the sets

$$B_\ell^e := \partial U_m^e \cap \{x \in \mathbb{R}^n \,:\, \mathrm{dist}(x, C) \geq 1/\ell\}.$$

Let us write C as the intersection of a countable set of halfspaces:

$$C = \bigcap_k H_k, \qquad H_k = \{x \in \mathbb{R}^n \,:\, \langle x, v_k \rangle < 0\}.$$

We see that each set B_ℓ^e can be written as

$$B_\ell^e = \bigcup_k B_{\ell,k}^e, \qquad B_{\ell,k}^e := \partial U_m^e \cap \{x \in \mathbb{R}^n \,:\, \mathrm{dist}(x, H_k) \geq 1/\ell\}.$$

We now remark that, since for any $k \in \mathbb{N}$ the set $\Omega \cap \{\mathrm{dist}(x, H_k) \geq 1/\ell\}$ is separated from Λ by a hyperplane, by the same proof in [8, Section 5] we deduce that $B_{\ell,k}^e \cap \Omega$ is contained in a semiconvex graph. This easily implies that the set $\partial U_m^e \cap \Omega$ is locally semiconvex. □

Remark 4.6 If we exchange the role of Ω and Λ in the above propositions, the above regularity properties hold for ∂V_m in place of ∂U_m.

4.2 Regularity of the transport map and of the free boundary

By the results of Section 2, we know that the optimal plan γ_m is induced by a convex function ψ_m via $\gamma_m = (\mathrm{Id} \times \nabla\psi_m)_\# f_m$. Since $\nabla\psi_{m\#} f_m = g_m$, we will say that ψ_m is a *Brenier solution* to the Monge-Ampère equation

$$\det(\nabla^2\psi_m)(x) = \frac{f_m(x)}{g_m(\nabla\psi_m(x))} \qquad \text{on } F_m, \qquad \nabla\psi_m(F_m) \subset G_m. \tag{4.3}$$

We recall that the function ψ_m was constructed applying Theorem 2.3 to the Monge-Kantorovich problem:

$$\text{minimize } C(\gamma) = \int_{\mathbb{R}^n \times \mathbb{R}^n} |x - y|^2 \, d\gamma(x, y)$$

among all γ which have $f + (g - g_m)$ and $g + (f - f_m)$ as first and second marginals, respectively (see Proposition 2.4). In particular $\nabla\psi_m$ is unique f-a.e. Therefore, if f is strictly positive on an open connected set $\Omega \subset \mathbb{R}^n$, then ψ_m is unique on Ω up to additive constants.

We now want to deduce regularity properties on ψ_m and ∂F_m. Exchanging the role of f and g, all results on ψ_m and F_m will be true also for ψ_m^* and G_m. We recall that, thanks to Remark 3.3, $F_m = U_m \cap \Omega$, $G_m = V_m \cap \Lambda$.

Assumption 1: *we assume that f and g are supported on two bounded open sets Ω and Λ respectively. Moreover we assume that f and g are bounded away from zero and infinity on Ω and Λ respectively.*

In order to prove regularity results on transport maps arising from the Monge-Kantorovich problem it is well known that one needs to assume at least convexity of the target domain (see [7]), but even assuming Λ to be convex one cannot expect $G_m \cap \Lambda$ to be convex. However we can adapt the strategy used in [8, Section 6] in order to prove local $C^{1,\alpha}$ regularity of ψ_m. On the other hand in our case, even assuming f and g to be C^∞, we cannot expect ψ_m to be C^∞ in the interior of Ω, while this is the case if Ω and Λ are disjoint (see Remark 4.9 below).

The idea to prove interior regularity for ψ_m is to apply Caffarelli's regularity theory. To this aim we have to prove, as in [8, Section 6], that ψ_m solves a Monge-Ampère equation with convex target domain. Let us recall the interior regularity result of Caffarelli [7]:

Theorem 4.7 *Let f and g be nonnegative densities supported on two bounded open sets Ω and Λ respectively. Assume Λ convex, and that f and g are bounded away from zero and infinity on Ω and Λ respectively. If $\psi : \mathbb{R}^n \to (-\infty, +\infty]$ is a convex function such that $\nabla\psi_\# f = g$, then there exists $\alpha > 0$ such that $\psi \in C^{1,\alpha}_{\text{loc}}(\Omega)$. Moreover ψ is strictly convex on Ω.*

Let us therefore assume Λ convex. Since $\nabla\psi_m$ solves the Monge-Kantorovich problem from $f_m+(f-f_m)+(g-g_m)$ to $g_m+(f-f_m)+(g-g_m)$, and $\nabla\psi_m(x) = x$ for $x \in \{f > f_m\}\cup\{g > g_m\}$, we have

$$\nabla\psi_{m\#}(f_m + (g - g_m)) = g. \tag{4.4}$$

This implies (see Theorem 2.3) that $\nabla\psi_m$ solves a Monge-Kantorovich problem where the target measure g is bounded from away from zero and infinity on the bounded open convex set Λ. Since, under Assumption 1, $f_m + (g - g_m)$ is bounded from above, in order to apply Caffarelli's interior regularity theory we need to prove the existence of an open bounded set on which this density is concentrated, and to show that it is bounded away from zero on this set. We observe that

$$\nabla\psi_m(x) = x \qquad \text{and} \qquad f_m(x) = g_m(x) = (f \wedge g)(x)$$

for a.e. $x \in \{f - f_m > 0\} \cup \{g - g_m > 0\}$ (see Theorem 2.6). Thus

$$f_m = g_m = f \wedge g \qquad \text{if } 0 < f_m < f \text{ or } 0 < g_m < g.$$

This easily implies that

$$f_m + (g - g_m) = f \quad \text{or} \quad f_m + (g - g_m) = g \qquad \text{if } f_m = f \text{ or } f_m = f \wedge g,$$

$$f_m + (g - g_m) = g \qquad \text{if } f_m = g_m = 0.$$

Therefore

$$f_m + (g - g_m) = \begin{cases} f \text{ or } g & \text{in } U_m \cap \Omega, \\ g & \text{in } \Lambda \setminus \overline{V_m}. \end{cases} \tag{4.5}$$

Thus $f_m + (g - g_m)$ is bounded away from zero on the domain $(U_m \cap \Omega) \cup (\Lambda \setminus \overline{V_m})$, and this domain has full mass since ∂V_m has zero Lebesgue measure (see Proposition 4.4 and Remark 4.6). Combining this with Theorem 4.7, we obtain:

Theorem 4.8 (Interior $C^{1,\alpha}$ regularity) *Suppose that Assumption 1 holds, and that Λ is convex. If ψ_m is a Brenier solution to (4.3), then $\psi_m \in C^{1,\alpha}_{\mathrm{loc}}(U_m \cap \Omega)$ and is strictly convex on $U_m \cap \Omega$.*

Remark 4.9 (Smooth densities need not to have a smooth solution) In [8], the authors show that, if f and g are C^∞ with disjoint supports, and if the support of g is convex, then $\psi_m \in C^\infty$. This property strongly relies on the fact that the supports are disjoint. Indeed, even if we assume f (resp. g) to be C^∞ we cannot expect f_m (resp. g_m) to be continuous on its support. In particular we cannot expect for higher regularity results. Consider for instance the following 1-dimensional example: let $h \geq 0$ be a even function on $\mathbb{R}$ of class C^∞ with support contained in $[-1, 1]$. Define

$$f(x) = \chi_{[-4,4]}(x) + h(x + 2), \qquad g(x) = \chi_{[-4,4]}(x) + h(x - 2)$$

(where χ_A denotes the indicator function of a set A). If $m = m_{\min} = 8$, then all the common mass stay at rest. If now $m = 8 + \varepsilon$ with $0 < \varepsilon < \int_{-1}^{1} h$, then the following happens: let $\delta = \delta(\varepsilon) \in (0, 2)$ be such that $\int_{-1}^{-1+\delta} h = \int_{1-\delta}^{1} h = \varepsilon$, and let T_δ denote the optimal transport map (for the classical Monge-Kantorovich problem) which sends $f\chi_{[-1-\delta,1+\delta]}$ into $g\chi_{[-1-\delta,1+\delta]}$. Then it is not difficult to see that

$$\psi'_m(x) = \begin{cases} x & \text{if } x \in [-4, -1-\delta], \\ T_\delta(x) & \text{if } x \in [-1-\delta, 1+\delta], \\ x & \text{if } x \in [1+\delta, 4], \end{cases}$$

$$f_m = \chi_{[-4,4]}(x) + h(x + 2)\chi_{[-1-\delta,-1]}(x), \qquad g_m = \chi_{[-4,4]}(x) + h(x - 2)\chi_{[1,1+\delta]}(x).$$

Therefore, although f and g are C^∞ on their supports (which are smooth and convex), f_m and g_m are not continuous. Since

$$\psi''_m = \frac{f_m}{g_m \circ \psi'_m} \qquad \text{on } [-4, 4],$$

and ψ'_m is continuous on $[-4, 4]$ (so that $\psi'_m(-1-\delta) = -1-\delta$), we get that

$$\lim_{x \to (-1-\delta)^-} \psi''_m(x) = 1,$$

$$\lim_{x \to (-1-\delta)^+} \psi''_m(x) = \lim_{x \to (-1-\delta)^+} \frac{f_m(x)}{g_m(\psi'_m(x))} = \frac{f(-1-\delta)}{g(-1-\delta)} = 1 + h(1-\delta) > 1,$$

and so ψ_m is not C^2.

We now want to prove global C^1 regularity of ψ_m up to the free boundary. Moreover we will prove, following the lines of the proof of [8, Theorem 6.3], that the free boundary is locally a C^1 hypersurface away from $\Omega \cap \Lambda$, and the transport map displaces along it in the perpendicular direction. To this aim, we will need to assume strict convexity of the domains:

Assumption 2: Ω *and* Λ *are strictly convex.*

We remark that Assumption 2 implies in particular that Ω is connected. Thus ψ_m is uniquely defined on Ω up to additive constants, and it makes sense to speak about *the* Brenier solution to (4.3).

Theorem 4.10 (Global C^1 regularity) *Suppose that Assumptions 1 and 2 hold. Let ψ_m be the Brenier solution to* (4.3)*. Then there exists $\tilde{\psi}_m \in C^1(\mathbb{R}^n) \cap C^{1,\alpha}_{\mathrm{loc}}(U_m \cap \Omega)$ such that $\tilde{\psi}_m = \psi_m$ on $U_m \cap \Omega$, $\nabla\tilde{\psi}_m(x) = x$ on $\Lambda \setminus \overline{V_m}$, and $\nabla\tilde{\psi}_m(\mathbb{R}^n) = \overline{\Lambda}$. Moreover $\nabla\tilde{\psi}_m : \overline{U_m \cap \Omega} \to \overline{V_m \cap \Lambda}$ is a homeomorphism.*

Proof. Since the assumptions on f and g are symmetric, by Proposition 2.4 we deduce that $\nabla\psi_m^*$ is the unique optimal transport map for the Monge-Kantorovich problem from $g_m + (f - f_m)$ to f. By Theorem 4.8 applied to ψ_m^* we know that $\psi_m^* \in C^{1,\alpha}_{\mathrm{loc}}(V_m \cap \Lambda)$ and is strictly convex on $V_m \cap \Lambda$. Since Ω and Λ are both convex, we can apply Proposition 4.5 (with $C = \Omega$) to deduce that $(\partial V_m \setminus \partial\Omega) \cap \Lambda$ is locally semiconvex. Moreover, as $\nabla\psi_m^*(y) = y$ for a.e. $y \in \Lambda \setminus \overline{V_m} \subset \{g > g_m\}$, we obtain $\psi_m^*(y) = \frac{|y|^2}{2} + C$ on each connected component of the open set $\Lambda \setminus \overline{V_m}$. Thus it is not difficult to see that ψ_m^* is strictly convex on the full domain Λ, and thanks to the strict convexity of Λ ψ_m^* is strictly convex also on $\overline{\Lambda}$. Let us consider the strictly convex function

$$\phi_m = \begin{cases} \psi_m^* & \text{on } \overline{\Lambda}, \\ +\infty & \text{on } \mathbb{R}^n \setminus \overline{\Lambda}, \end{cases}$$

and define $\tilde{\psi}_m := \phi_m^*$. Exactly as in [8, Theorem 6.3], the strict convexity of ϕ_m implies that $\tilde{\psi}_m \in C^1(\mathbb{R}^n)$. Furthermore, since $y \in \partial\tilde{\psi}_m(x)$ if and only if $x \in \partial\phi_m(y)$, we deduce that $\partial\tilde{\psi}_m(\mathbb{R}^n) \subset \overline{\Lambda}$, which implies that $\tilde{\psi}_m$ is globally Lipschitz. Finally, as $\phi_m \geq \psi_m^*$ with equality on $\overline{\Lambda}$, we deduce that

$$\tilde{\psi}_m(x) \leq \psi_m(x) \qquad \forall\, x \in \mathbb{R}^n \text{ with equality if } \partial\psi_m(x) \cap \overline{\Lambda} \neq \emptyset.$$

Since by (4.4) and (4.5) $\nabla\psi_m(x) \in \overline{\Lambda}$ for a.e. $x \in (U_m \cap \Omega) \cup (\Lambda \setminus \overline{V_m})$, we deduce that $\tilde{\psi}_m$ gives the desired extension of ψ_m. This implies in particular that $\nabla\psi_m : U_m \cap \Omega \to \overline{\Lambda}$ extends to a continuous map from $\overline{U_m \cap \Omega}$ to $\overline{V_m \cap \Lambda}$. Indeed the extension cannot takes values outside $\overline{V_m}$ since f_m does not vanish on $U_m \cap \Omega$ and $\nabla\psi_{m\#} f_m = g_m$ is supported on $\overline{V_m \cap \Lambda}$.

By the symmetry of the assumptions on f and g, the above argument implies that also $\nabla\psi_m^* : V_m \cap \Lambda \to \overline{\Omega}$ extends to a continuous map from $\overline{V_m \cap \Lambda}$ to $\overline{U_m \cap \Omega}$. Since by (2.2) $\nabla\psi_m^*(\nabla\psi_m(x)) = x$ a.e. inside $U_m \cap \Omega$, and $\nabla\psi_m(\nabla\psi_m^*(y)) = y$ a.e. inside $V_m \cap \Lambda$, by continuity both equalities hold everywhere inside $U_m \cap \Omega$ and $V_m \cap \Lambda$ respectively. Thus $\nabla\psi_m : U_m \cap \Omega \to V_m \cap \Lambda$ is a homeomorphism with inverse given by $\nabla\psi_m^*$. Since both maps extend continuously up to the boundary, we deduce that $\nabla\tilde{\psi}_m : \overline{U_m \cap \Omega} \to \overline{V_m \cap \Lambda}$ is a homeomorphism. ☐

We can now prove the C^1 regularity of the free boundary away from $\Omega \cap \Lambda$.

Theorem 4.11 (Free boundary is C^1 away from $\Omega \cap \Lambda$) *Suppose that Assumptions 1 and 2 hold. Let $\partial U_m \cap \Omega = \big(\partial U_m^e \cap \Omega\big) \cup \big(\partial U_m^b \cap \Omega\big)$ be the decomposition provided by Proposition 4.5* *with $C = \Lambda$, and let $\tilde{\psi}_m$ be the C^1 extensions of ψ_m provided by Theorem 4.10. Then $\partial U_m^e \cap \Omega$ is locally a C^1 surface, and for all $x \in \partial U_m^e \cap \Omega$ the vector $\nabla\tilde{\psi}_m(x) - x$ is different from 0, and gives the direction of the inward normal to U_m.*

Proof. Let us recall that

$$U_m := (\Omega \cap \Lambda) \cup \bigcup_{(\bar{x},\bar{y})\in\Gamma_m} B_{|\bar{x}-\bar{y}|}(\bar{y}),$$

where Γ_m was defined in (3.1). Since by Theorem 4.8 $\nabla\psi_m : U_m \cap \Omega \to V_m \cap \Lambda$ and $\nabla\psi_m^* : V_m \cap \Lambda \to U_m \cap \Omega$ are both continuous, we can write

$$U_m = (\Omega \cap \Lambda) \cup \bigcup_{x\in U_m\cap\Omega} B_{|x-\nabla\psi_m(x)|}\big(\nabla\psi_m(x)\big).$$

Moreover, since $\nabla\tilde{\psi}_m : \overline{U_m \cap \Omega} \to \overline{V_m \cap \Lambda}$ continuously extends $\nabla\psi_m$ (see Theorem 4.10), as in the proof of Lemma 4.3 we obtain that

$$U_m = (\Omega \cap \Lambda) \cup \bigcup_{x\in\overline{U_m\cap\Omega}} B_{|x-\nabla\tilde{\psi}_m(x)|}\big(\nabla\tilde{\psi}_m(x)\big).$$

Let us fix $z \in \partial U_m^e \cap \Omega$. First of all, since z is at a positive distance from $\overline{\Lambda}$, it is clear that $\nabla\tilde{\psi}_m(z) - z \neq 0$, and that $B_{|z-\nabla\tilde{\psi}_m(z)|}\big(\nabla\tilde{\psi}_m(z)\big) \subset U_m$ touches $\partial U_m^e \cap \Omega$ at z. Moreover, with the same notation as in the proof of Proposition 4.5, there exists $\ell, k \in \mathbb{N}$ such that $z \in B_{\ell,k}^e \cap \Omega$, and $B_{\ell,k}^e \cap \Omega$ is the semiconvex graph of a function $u_{\ell,k} : \mathbb{R}^{n-1} \to \mathbb{R}$, with U_m is contained in the epigraph of $u_{\ell,k}$. These two facts together imply that, if $z = (z', u_{\ell,k}(z'))$ and z' is a point where $u_{\ell,k}$ is differentiable, then the vectors $(-\nabla' u_{\ell,k}(z'), 1)$ and $\nabla\tilde{\psi}_m(z) - z$ are parallel (here ∇' denotes the gradient with respect to the first $n-1$ coordinates). Writing $\nabla\tilde{\psi}_m(z) - z$ with respect to the system of coordinates induces by $u_{\ell,k}$, that is

$$\nabla\tilde{\psi}_m(z) - z = (\nabla'\tilde{\psi}_m(z) - z', \partial_n\tilde{\psi}_m(z) - z_n),$$

we obtain that, if z' is a point of differentiability for $u_{\ell,k}$, then $\partial_n\tilde{\psi}_m(z) - z_n = \partial_n\tilde{\psi}_m(z', u_{\ell,k}(z')) - u_{\ell,k}(z') \neq 0$ and the gradient of $u_{\ell,k}$ is given by

$$\nabla' u_{\ell,k}(z') = -\frac{\nabla'\tilde{\psi}_m(z', u_{\ell,k}(z')) - z'}{\partial_n\tilde{\psi}_m(z', u_{\ell,k}(z')) - u_{\ell,k}(z')}.$$

From this and the uniform continuity of $\nabla\tilde{\psi}_m$, we deduce that $z' \mapsto \nabla' u_{\ell,k}(z')$ is uniformly continuous on its domain of definition. Thus $\nabla' u_{\ell,k}$ has a unique continuous extension, and the function $u_{\ell,k}$ is C^1. This easily implies the thesis. □

Following the strategy used in the proof of [8, Corollary 6.7], we can prove the path-connectedness of the active regions (observe that with respect to the proof in [8] we cannot use [8, Corollary 2.4], so we need to slightly change the argument, although the strategy of the proof is the same).

Corollary 4.12 (Path-connectedness of the active region) *Suppose that Assumptions 1 and 2 hold, and let $\tilde\psi_m$ be the C^1 extensions of ψ_m provided by Theorem 4.10. Then $\overline{U_m \cap \Omega}$ is path-connected.*

Proof. Fix $x_0, x_1 \in \overline{U_m \cap \Omega}$, and let $x_t := (1-t)x_0 + tx_1$ the segment joining them. Let us assume that the segment x_t is not entirely contained in $\overline{U_m \cap \Omega}$ (otherwise the thesis is trivially true), and let $[t', t''] \subset [0,1]$ be a maximal subinterval such that $x_t \notin \overline{U_m \cap \Omega}$ for all $t \in [t', t'']$. As in the proof of [8, Corollary 6.7], we will construct a path in $\overline{U_m \cap \Omega}$ which connects $x_{t'}$ to $x_{t''}$. Iterating this construction on all intervals $[t', t'']$, the path connectedness follows.

Since all x_t belongs to the strictly convex set $\overline{\Omega}$ for $t \in [0,1]$, both $x_{t'}$ and $x_{t''}$ necessarily lie on the free boundary $\partial U_m \cap \Omega$. We will prove that the segment $y_s := (1-s)\nabla\tilde\psi_m(x_{t'}) + s\nabla\tilde\psi_m(x_{t''})$ lies in $\overline{V_m \cap \Lambda}$. Then the homeomorphism $\nabla\tilde\psi_m$ will give the desired path $[t', t''] \ni t \mapsto \nabla\tilde\psi_m^{-1}\big(y_{\frac{t-t'}{t''-t'}}\big)$.

Let $z_s := (1-s)x_{t'} + sx_{t''}$ reparameterize the segment between $x_{t'}$ and $x_{t''}$. By construction we know that $z_s \notin \overline{U_m \cap \Omega}$ for all $s \in (0,1)$. Assume by contradiction that there exists $\bar s \in (0,1)$ such that $y_{\bar s} \notin \overline{V_m \cap \Lambda}$. We observe that, by the strict convexity of $|\cdot|^2$,

$$|z_{\bar s} - y_{\bar s}|^2 < (1-\bar s)|x_{t'} - \nabla\tilde\psi_m(x_{t'})|^2 + \bar s|x_{t''} - \nabla\tilde\psi_m(x_{t''})|^2.$$

Without loss of generality we can assume $|x_{t'} - \nabla\tilde\psi_m(x_{t'})| \le |x_{t''} - \nabla\tilde\psi_m(x_{t''})|$. Thus we have

$$|z_{\bar s} - y_{\bar s}|^2 < |x_{t'} - \nabla\tilde\psi_m(x_{t'})|^2$$

The idea is to remove some mass near $(x_{t'}, \nabla\tilde\psi_m(x_{t'}))$ and to add it to $(z_{\bar s}, y_{\bar s})$ to contradict the optimality of $\gamma_m = (\mathrm{Id} \times \nabla\tilde\psi_m)_\# f_m$.

We first prove that for any $\delta' > 0$ the set $B_{\delta'}(x_{t'}) \cap \big(U_m \cap \Omega\big)$ has positive mass with respect to f_m. Observe that two possibilities arise: either $x_{t'} \in (\partial U_m \cap \Omega) \setminus \overline{\Lambda}$, or $x_{t'} \in \partial(\Omega \cap \Lambda)$. Since $U_m \supset \Omega \cap \Lambda$ and $(\partial U_m \cap \Omega) \setminus \overline{\Lambda}$ is locally a C^1 hypersurface, it is simple to see that in both cases $x_{t'}$ belongs to the support of f_m.

Thanks to this fact, if we fix $\varepsilon > 0$ small enough, we can find $\delta_x, \delta_y, \delta' > 0$ small such that

$$B_{\delta_x}(z_{\bar s}) \cap \overline{U_m \cap \Omega} = \emptyset, \qquad B_{\delta_x}(z_{\bar s}) \cap B_{\delta'}(x_{t'}) = \emptyset, \qquad B_{\delta_y}(y_{\bar s}) \cap \overline{V_m \cap \Lambda} = \emptyset,$$

$$\int_{B_{\delta_x}(z_{\bar s})} f = \int_{B_{\delta_y}(y_{\bar s})} g = \int_{B_{\delta'}(x_{t'})} f_m = \varepsilon,$$

and

$$|x - y|^2 < |x' - \nabla\tilde\psi_m(x')| \qquad \forall\, x \in B_{\delta_x}(z_{\bar s}),\ y \in B_{\delta_y}(y_{\bar s}),\ x' \in B_{\delta'}(x_{t'}).$$

Thus it is not difficult to check that, if we define

$$\bar\gamma := (\mathrm{Id} \times \nabla\tilde\psi_m)_\#\big((1 - \chi_{B_{\delta'}(x_{t'})})f_m\big) + \frac{1}{\varepsilon}\big(f\chi_{B_{\delta_x}(z_{\bar s})}\big) \otimes \big(g\chi_{B_{\delta_y}(z_{\bar s})}\big)$$

then $\bar\gamma \in \Gamma(m)$, and $C(\bar\gamma) < C(\gamma_m) = C(m)$. This contradiction gives the desired result. □

As in [8, Lemma 6.8], we can also prove that $\nabla\tilde{\psi}_m$ does not map $\partial U_m \cap \Omega$ on $\overline{\partial V_m \cap \Lambda}$:

Proposition 4.13 (Free boundary never maps to free boundary) *Suppose that Assumptions 1 and 2 hold, and let $\tilde{\psi}_m$ be the C^1 extensions of ψ_m provided by Theorem 4.10. Then*

(a) if $x \in \partial U_m \cap \Omega$, then $\nabla\tilde{\psi}_m(x) \notin \overline{\partial V_m \cap \Lambda}$;

(b) if $x \in \partial U_m \cap \partial\Omega$, then $\nabla\tilde{\psi}_m(x) \notin \partial V_m \cap \Lambda$.

Proof. We first prove (a). Let $x \in \partial U_m \cap \Omega$, and assume by contradiction that $\nabla\tilde{\psi}_m(x) \in \overline{\partial V_m \cap \Lambda}$. First of all we observe that, since $\partial V_m \cap (\Omega \cap \Lambda) = \emptyset$, then $\overline{\partial V_m \cap \Lambda} \cap \Omega = \emptyset$. Therefore $\nabla\tilde{\psi}_m(x) - x \neq 0$, and we have

$$B_{|\nabla\tilde{\psi}_m(x)-x|}(x) \subset V_m,$$

with $B_{|\nabla\tilde{\psi}_m(x)-x|}(x)$ touching ∂V_m at $\nabla\tilde{\psi}_m(x)$. We can now use the argument in the proof of [8, Lemma 6.8] to deduce a contradiction. Indeed, let us define $x_r = x + r(\nabla\tilde{\psi}_m(x) - x)$ (so that $x_0 = x$). Since $B_{|x-\nabla\tilde{\psi}_m(x)|}\big(\nabla\tilde{\psi}_m(x)\big) \subset U_m$, we have that $x_r \in U_m \cap \Omega$ for r small enough. Let $y_r := \nabla\tilde{\psi}_m(x_r)$ (so that $y_0 = \nabla\tilde{\psi}_m(x)$). Since $\nabla\tilde{\psi}_m$ is a homeomorphism from $\overline{U_m \cap \Omega}$ onto $\overline{V_m \cap \Lambda}$, we have $y_r \neq y$ for $r > 0$. Moreover, by the monotonicity of the gradient of a convex function,

$$0 \leq \langle y_r - y_0, x_r - x_0\rangle = r\langle y_r - y_0, y_0 - x_0\rangle.$$

This implies that y_r lies in the halfspace $\{y \in \mathbb{R}^n : \langle y - y_0, y_0 - x_0\rangle \geq 0\}$ for all $r > 0$, and y_0 is the (unique) closest point to x_r. Thus $y_0 \in B_{|y_r - x_r|}(x_r)$ for $r > 0$ small. Since any point $\tilde{y} \in V_m \cap \Lambda$ sufficiently close to y_0 must also belongs to $B_{|y_r - x_r|}(x_r) \cap \Lambda$, this fact contradicts $y_0 = y \in \overline{\partial V_m \cap \Lambda}$.

To prove (b) observe that, by the symmetry of Assumptions 1 and 2 and by (a), the inverse $\nabla\tilde{\psi}_m^*$ of $\nabla\tilde{\psi}_m$ satisfies that, if $y \in \partial V_m \cap \Lambda$, then $\nabla\tilde{\psi}_m^*(y) \notin \overline{\partial U_m \cap \Omega}$. This implies that, if $\nabla\tilde{\psi}_m(x) \in \partial V_m \cap \Lambda$, then $x \notin \overline{\partial U_m \cap \Omega}$. Since $\overline{\partial U_m \cap \Omega} = \partial U_m \cap \overline{\Omega} = (\partial U_m \cap \Omega) \cup (\partial U_m \cap \partial\Omega)$, (b) follows from (a). □

Thanks to the above proposition, one can show that inactive region maps to target boundary.

Corollary 4.14 (Inactive region maps to target boundary) *Suppose that Assumptions 1 and 2 hold, and let $\tilde{\psi}_m$ be the C^1 extensions of ψ_m provided by Theorem 4.10. If $x \in \overline{\Omega \setminus U_m}$, then $\nabla\tilde{\psi}_m(x) \in \partial\Lambda$.*

Proof. If $x \in \partial U_m \cap \overline{\Omega}$, by Proposition 4.13 we have $\nabla\tilde{\psi}_m(x) \notin \partial V_m \cap \Lambda$. Since $\nabla\tilde{\psi}_m : \overline{U_m \cap \Omega} \to \overline{V_m \cap \Lambda}$ is a homeomorphism which maps the boundary on the boundary, this implies that $\nabla\tilde{\psi}_m(x) \in \partial(U_m \cap \Lambda) \setminus (\partial V_m \cap \Lambda) \subset \partial\Lambda$. Thus it remains to consider the case $x \in \overline{\Omega} \setminus \overline{U_m \cap \Omega}$, and the proof is exactly the same as in [8, Corollary 6.9]. □

Remark 4.15 In [8, Sections 7] Caffarelli and McCann prove that the free boundary is locally $C^{1,\alpha}$ (and not only locally C^1). A key fact to achieve this result is that the free boundary never maps to the free boundary, so that if $x \in \overline{\partial U_m \cap \Omega}$ one can exploit the convexity of $V_m \cap \Lambda$ near $\nabla\tilde{\psi}_m(x)$ to ensure that the Monge-Ampère measure associated to $\nabla\tilde{\psi}_m^*$ has a doubling property (see the discussion at the beginning of [8, Sections 7]). It seems therefore plausible that one could adapt their proof in our situation, improving the C^1_{loc} regularity of the free boundaries away from $\Omega \cap \Lambda$ into a $C^{1,\alpha}_{\mathrm{loc}}$ regularity.

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