Proof of the equality $\overline{DRC} = H_{g\mu}$ (lecture 1)

Recap and Goal

Given $\mu \in \mathbb{Z}^{\times n}$, $\mu \neq 0$, $\mu = (m_1, \dots, m_n)$: $\sum m_i = 2g-2$

we defined $H_g(\mu)$ by the fiber square

$$
\begin{tikzcd}
H_g(\mu) \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\square"] & \mathcal{M}_{gn} \arrow[d, "\sigma"] & (C, p_1, \dots, p_n) \arrow[d, mapsto] \\
\mathcal{M}_{gn} \arrow[r, hook, "e"'] & \mathcal{J} & \omega_C(-\sum m_i p_i)
\end{tikzcd}
$$

$$(C, p_1, \dots, p_n) \longmapsto \mathcal{O}$$

Then we introduced a compactification $\tilde{H}_g(\mu)$ of $H_g(\mu)$

$$\overline{H}_g(\mu) \subset \tilde{H}_g(\mu) \subset \overline{\mathcal{M}}_{gn}$$

After that we extended σ in some way and defined $DRL_\diamond$ by

$$\mathcal{M} = \mathcal{M}$$

$$
\begin{tikzcd}
DRL_\diamond \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] \arrow[rr, bend right, "p_\diamond \text{ proper}"] & \hat{\mathcal{M}}^\diamond \arrow[r] \arrow[d, "\sigma_\diamond'"] & \overline{\mathcal{M}} \arrow[dl, bend left, "\sigma"] \\
\overline{\mathcal{M}} \arrow[r, "\overline{e}"'] & \mathcal{J} &
\end{tikzcd}
$$

$(\overline{\mathcal{M}} = \overline{\mathcal{M}}_{gn})$

Then we observed that

$$\begin{tikzcd}
DRL_\diamond \arrow[r] \arrow[d] \arrow[rrr, bend left, "p_\diamond\ \leftarrow \text{proper}"] \arrow[dr, phantom, "\square"] & \mathcal{M}^\diamond \arrow[d, "\sigma_\diamond"] \arrow[dd, bend right, "\sigma'_\diamond"'] & & \overline{\mathcal{M}} \\
\mathcal{M}^\diamond \arrow[r, "e_\diamond"] \arrow[d] \arrow[dr, phantom, "\square"] & J^\diamond := J \times_{\overline{\mathcal{M}}} \mathcal{M}^\diamond \arrow[rr] \arrow[d] \arrow[drr, phantom, "\square"] & & \mathcal{M}^\diamond \arrow[d] \\
\overline{\mathcal{M}} \arrow[r, "\overline{e}"'] & J \arrow[rr] & & \overline{\mathcal{M}}
\end{tikzcd}$$

So we could define $DRL_\diamond$ using this fiber square

Def $\overline{DRC} := p_{\diamond *}\left(\sigma_\diamond^!\left(e_{\diamond *}[\mathcal{M}^\diamond]\right)\right) \in Z_*(\overline{\mathcal{M}})$

this is a section of $J^\diamond \to \mathcal{M}^\diamond$ which is smooth
$\Rightarrow \sigma_\diamond$ is a regular emb and
This is the Gysin homo

a cycle (not just a cycle class)

$p_\diamond$

$DRC^\diamond := [DRL_\diamond] \in CH_*(DRL_\diamond)$

Thm $\boxed{H_{g\mu} = \overline{DRC}}$

Goal: prove this theorem.

We (in this seminars) will also prove that

$$H_{g\mu} = 2^{-g} P^g_g(\tilde{\mu})$$

So:

<u>**Corollary**</u>

$$\boxed{\begin{array}{l} 2^{-g} P^g_g(\tilde{\mu}) = H_{g\mu} = \overline{DRC} \\ \text{in } A^g(\overline{\mathcal{M}}). \\ \text{Here } \tilde{\mu} = (m_1+1, \ldots, m_n+n). \end{array}}$$

■

Plan for today

Step 1 Rewrite $H_{g\mu} \in Z_*(\overline{\mathcal{M}})$ as

$$H_{g\mu} = [\overline{H}_g(\mu)] + \sum_{\substack{Z \subset \widetilde{H}_g(\mu) \\ \text{irreducible comp} \\ Z \subset \partial\overline{\mathcal{M}}}} \partial_Z \, [Z] \qquad ⊛$$

where $\partial_Z = \,?$

Step 2 Define the protagonists: introduce $\overline{\mathcal{M}}^m$ and DRL^m

Rmk As for Federico's talk we define a twist on a leg weighted graph Γ as

$I : H(\Gamma) \to \mathbb{Z}$ s.t.

(i) $I(p_i) = m_i \quad \forall\, i = 1, \dots, n$

(ii) $I(h) = -I(h') \quad \forall\, e = (h, h') \in E(\Gamma)$

(iii) $\displaystyle\sum_{\substack{h \in H(\Gamma) \\ \mathrm{end}(h) = v}} I(h) - k(\mathrm{can}(v)) = 0$

$\mathrm{can}(v) = 2g(v) - 2 + \#\{h \in H(\Gamma) \setminus L(\Gamma) \text{ s.t. } \mathrm{end}(h) = v\}$

⚠ This def slightly different from the one given by Andrea (following [F-P]). They differ by a sign.

Step 1 $\quad H_{g\mu} = [\overline{H}_g(\mu)] + \sum_{\Gamma \in S^{\star}_{g,n}} \sum_{I \in Tw^+(\Gamma)} \dfrac{\prod_{e \in E(\Gamma)} I(e)}{|Aut(\Gamma)|} \cdot$

set of simple non-trivial star graphs

$$\xi_{\Gamma \star}\left(\overline{H}_{g(v_0)}\left(\mu[v_0], + I[v_0] - 1\right) \cdot \prod_{v \in V^{out}(\Gamma)} \overline{\mathcal{H}}_{g(v)}\left(\mu[v], I[v] - 1\right)\right)$$

in [H-J] here we have $-$, but I think it should be a $+$ with the new notatation

It is clear that this is supported on $\bigcup Z$

$Z \subset \widetilde{\mathcal{H}}_g(\mu)$ irr. comp s.t.
$Z \subset \partial \mathcal{M}$

So we clearly have $\star$. for some $a_Z \in \mathbb{Q}$.

Here $\quad Tw^+(\Gamma) = \{$ positive twists on $\Gamma\}$ simple star graph.

i.e.: • $I(h) < 0$ for some $H(\Gamma)$

$\Rightarrow$ end$(h) = v_0$ (the central vertex of Γ)

• $I(h) \neq 0 \quad \forall h \in H(\Gamma) \setminus L(\Gamma)$

Fact: (maybe we will prove it next time).

Let $Z = \overline{\{p_0\}}$ be an irreducible component of $\widetilde{\mathcal{H}}_g(\mu)$ contained in $\partial\overline{\mathcal{H}}$.
Then $\exists!$ twist I_0 on the simple star graph Γ_{p_0} s.t. the condition

$$(\dagger) \quad \omega_{C_I} \cong \mathcal{O}_{C_I}\Big(\sum m_i p_i + \sum_{\substack{e=(h,h')\in E(\Gamma):\\ I(e)\neq 0}} (I(h)-1)q_h + (I(h')-1)q_{h'}\Big)$$

holds for $C = C_{p_0}$. Moreover $I_{p_0} \in Tw^+(\Gamma_{p_0})$.

Now we know that $\mathcal{H}_{g'}(\mu')$ is smooth $\Rightarrow$ it is reduced $\Rightarrow \overline{\mathcal{H}}_{g'}(\mu')$ is reduced $\Rightarrow \mathrm{Im}\Big(\prod \overline{\mathcal{H}}_{g'}(\mu') \xrightarrow{\xi_\Gamma} \overline{\mathcal{M}}\Big) \subseteq \overline{\mathcal{M}}$ is also reduced.

Moreover,

Fact $\Rightarrow$ $\forall Z$ we have a unique term in the sum defining $H_{g\mu}$

associated to (Γ_Z, I_Z):

$$\frac{\prod_{e \in E(\Gamma_Z)} \underset{\substack{\wr\\ \wedge \\ 0}}{I_Z(e)}}{\mathrm{Aut}(\Gamma)} \, \xi_{\Gamma_Z *} \left(\overline{\mathcal{H}}_{g(v_0)}(\mu[v_0], I[v_0]-1) \cdot \prod_{v \in V^{out}(\Gamma)} \mathcal{H}_{g(v)}(\mu[v], I[v]-1) \right)$$

which is supported on Z

Dimensional count $\Rightarrow$ the support is exactly Z

$$= \frac{\prod_{e \in E(\Gamma_Z)} I_Z(e)}{\cancel{\mathrm{Aut}(\Gamma_Z)}} \, \cancel{\deg(\xi_{\Gamma_Z})} \, [Z]$$

Prop $\boxed{H_{g_\mu} = [\overline{H}_g(\mu)] + \displaystyle\sum_{\substack{Z \subset \widetilde{\mathcal{H}}_g(\mu) \text{ irr.} \\ \text{comp contained} \\ \text{in } \partial \overline{\mathcal{M}}}} \left(\prod_{e \in E(\Gamma_Z)} I_Z(e) \right) [Z]}$

■

Step 2 : Definition of $\overline{\mathcal{M}}^m$, DRL^m, DRC^m

Let

$$\begin{array}{ccccc} \overline{\mathcal{M}} & \leftarrow & U & \longrightarrow & \mathbb{A}^E \\ p & \leftarrow\!\shortmid & u & \longmapsto & 0 \end{array}$$

be a combinatorial chart.

In particular we also have the data of a graph $\Gamma = \Gamma_{C_p}$.

Now suppose we also have a twist I on Γ.

Define

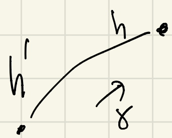
$$\mathbb{A}^E_I := \operatorname{Spec} \mathbb{C}\left[\partial_e, \partial_\gamma : \begin{array}{l} e \in E(\Gamma) \\ \gamma \subset \Gamma \text{ oriented cycle} \end{array}\right]$$

$\mathcal{M}^\diamond \neq \overline{\mathcal{M}}^m$

where for $\gamma \subset \Gamma$ oriented cycle

$$\partial_\gamma := \prod_{e \in E(\Gamma)} \partial_e^{I_\gamma(e)}$$

$$I_\gamma(e) = \begin{cases} 0 & e \notin \gamma \\ \operatorname{sign}_\gamma(e)\, |I_\gamma(e)| & e \in \gamma \end{cases}$$

(if  $\Rightarrow I_\gamma(e) = I_\gamma(h)$)

Form a Cartesian diagram

$$\begin{tikzcd}
 & \mathcal{M}^m_{IU} \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \mathbb{A}^E_I \arrow[d] \\
\overline{\mathcal{M}} & U \arrow[l] \arrow[r] & \mathbb{A}^E
\end{tikzcd}$$

Then we glue the $\mathbb{A}^E_I$ for I running all the twists on Γ the $\mathbb{A}^E_I$ along the torus $\mathbb{C}[\partial_e^{\pm 1} : e \in E(\Gamma)]$ forming $\tilde{\mathbb{A}}^E \to \mathbb{A}^E$

Define

$$\begin{tikzcd}
 & \mathcal{M}^m_U \arrow[r] \arrow[d] \arrow[dl] \arrow[dr, phantom, "\square"] & \tilde{\mathbb{A}}^E \arrow[d] \\
\overline{\mathcal{M}} & U \arrow[l] \arrow[r] & \mathbb{A}^E
\end{tikzcd}$$

and $\overline{\mathcal{M}}^m \to \overline{\mathcal{M}}$ obtained by descending the $\mathcal{M}^m_U \to \mathcal{M}$.

<u>**Facts**</u>

- The map $\overline{\mathcal{M}}^m \to \overline{\mathcal{M}}$ is separated, of finite presentation, an iso on $\mathcal{M}$
- $\sigma : \mathcal{M} \to \mathcal{J}$, $(C, p_1 \ldots p_n) \mapsto \omega_C(-\sum w_i p_i)$ extends uniquely ? to $\overline{\mathcal{M}}^m \xrightarrow[\sigma_m']{} \mathcal{J}$
- $\mathcal{M}^\diamond \to \overline{\mathcal{M}}^m$ is proper

Q) In the paper it is said that this extension is unique. Why?

While $\mathcal{M}^\circ$ was reduced, $\overline{\mathcal{M}}^m$ is not reduced in general

so I cannot apply the fact that

if an extension exists $\Rightarrow$ it is unique.

$$\begin{tikzcd} X \arrow[dr, dashed] & \\ X^0 \arrow[u, phantom, "\cup"] \arrow[r] & Y \end{tikzcd}$$

open and dense

reduced

separated

<u>One example</u>

$$X = \mathbb{C}[x,y]/(xy,y^2) \longleftarrow \mathbb{C}[t]/(t^2)$$

$$y \xleftarrow{f_1} t$$

$$0 \xleftarrow{f_2} t$$

Then on $X_x = \left(\mathbb{C}[x,y]/(xy,y^2)\right)_x$ we have

$f_1|_{X_x} = f_2|_{X_x}$ but $f_1 \neq f_2$.

The pb here is that we can only conclude that f_1 and f_2 agree on a closed subscheme of X containing X_x (i.e. on $\mathbb{C}[x,y]/(y) = \mathbb{C}[x]$) but not necessarily on X

This cannot happen for

$$\begin{tikzcd} \overline{\mathcal{M}}^m \arrow[r] & Y \\ \mathcal{M} \arrow[u, phantom, "\cup" rotate=90] \arrow[ur] & \end{tikzcd}$$

because if $\mathcal{M} \subset V \subset \overline{\mathcal{M}}^m \Rightarrow V = \overline{\mathcal{M}}^m$ (as schemes)

closed subscheme

Why? Is it true?

Def DRL^m

$$\begin{tikzcd}
\mathrm{DRL}^m \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] \arrow[rr, bend left, "p_m"] & \overline{\mathcal M}^m \arrow[r] \arrow[d, "\sigma_m"] \arrow[dd, bend left=60, "\sigma_m'"] & \overline{\mathcal M} \\
\overline{\mathcal M}^m \arrow[r, "e^m"] \arrow[d] \arrow[dr, phantom, "\square"] & \mathcal J^m = \mathcal J \times_{\overline{\mathcal M}} \overline{\mathcal M}^m \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \overline{\mathcal M}^m \arrow[d] \\
\overline{\mathcal M} \arrow[r, "\bar e"'] & \mathcal J \arrow[r] & \overline{\mathcal M}
\end{tikzcd}$$

Obs Since $\mathcal M^\diamond \to \overline{\mathcal M}^\diamond$ is proper and surjective, we have

$$\begin{tikzcd}
\mathrm{DRL}^\diamond \arrow[r] \arrow[d, hook] \arrow[dr, phantom, "\substack{\text{proper}\\ +\text{surj}}"] \arrow[rr, bend left, "\text{proper}"] & \mathrm{DRL}^m \arrow[r, "\substack{\text{separated}\\ +\text{fin. pres}}"'] \arrow[d, hook] & \overline{\mathcal M} \\
\mathcal M^\diamond \arrow[r] & \overline{\mathcal M}^m \arrow[ur, "\text{separated + finite pres}"'] &
\end{tikzcd}$$

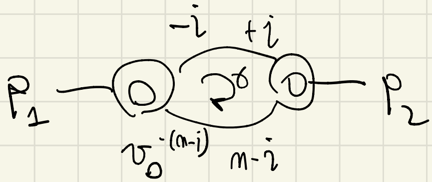
$\Rightarrow$ $\mathrm{DRL}^m \to \overline{\mathcal M}$ is also proper

Def $\mathrm{DRC}^m := p_{m*}\Big(\underbrace{\sigma_m}^{!}\big(e_{m*}\overbrace{[\overline{\mathcal M}^m]}^{[\mathrm{DRL}^m] \,=\, \leftarrow \text{later}}\big)\Big) \in Z_*(\overline{\mathcal M})$

Again this makes sense because σ_m is a reg. emb being $\mathcal J^m \to \overline{\mathcal M}^m$ smooth

Why $\mathcal{M}^{\diamond} \neq \overline{\mathcal{M}}^{m}$

$(\Gamma, I_i) =$



$\mu = (n, -n)$

$(k=0)$

$i \in \mathbb{Z}$

$\mathbb{A}^{E}_{I_i}$:

$0 < i < n$: $\quad \mathbb{A}^{E}_{I_i} = \mathrm{Spec}\, \mathbb{C}\left[\partial_{e_1}, \partial_{e_2}, \underset{\substack{\uparrow \\ \partial_{\gamma}}}{\partial_{e_1}^{i}\,\partial_{e_2}^{-(n-i)}}, \underset{\substack{\uparrow \\ \partial_{\gamma^{-1}}}}{\partial_{e_1}^{-i}\,\partial_{e_2}^{(n-i)}}\right]$

while

$$\mathcal{M}^{I}_{\Gamma} = \mathrm{Spec}\, \mathbb{C}\left[\partial_{e_1}, \partial_{e_2}, \partial_{e_1}^{\frac{i}{k_i}}\,\partial_{e_2}^{-\frac{n-i}{k_i}}, \partial_{e_1}^{-\frac{i}{k_i}}\,\partial_{e_2}^{\frac{n-i}{k_i}}\right]$$

$k_i := MCD(i, n-i)$

Proof of $H_{g\mu} = \overline{DRC}$ (second lecture)

Comparing the various double ramification cycles and loci

At this point we defined

$$\begin{tikzcd}
DRL^{\diamond} \arrow[r, "\text{proper}", "\text{and surj}"'] \arrow[d, phantom, "\text{closed } \cap"] \arrow[rr, bend left, "p_{\diamond}\ \text{proper}"] & DRL^{m} \arrow[d, phantom, "\cap \text{ closed}"] \arrow[r, "\text{proper}", "p_m"'] & \overline{\mathcal{M}} \\
\mathcal{M}^{\diamond} \arrow[r, "\text{proper}"', "\text{and surj}"'] & \overline{\mathcal{M}}^{m} \arrow[r] & \overline{\mathcal{M}}
\end{tikzcd}$$

Lemma 1

The map $DRL^{m} \to \overline{\mathcal{M}}$ factors set theoretically via $\tilde{\mathcal{H}}_g(\mu) \subset \overline{\mathcal{M}}$ and is surjective.

proof This follows from the definition of $\overline{p}^{m}$. ∎

Lemma 2

The map $DRL^{m} \to \overline{\mathcal{M}}$ is quasi finite

proof For any chart

$$\begin{tikzcd}
 & DRL^{m} \cap \overline{\mathcal{M}}^{m}_{IU} \arrow[dl, "p"'] & \subset \ \mathcal{M}^{m}_{IU} \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \mathbb{A}^{E}_{I} \arrow[d] \\
\overline{\mathcal{M}} & & U \arrow[ll] \arrow[r] & \mathbb{A}^{E}
\end{tikzcd}$$

$p \hookleftarrow u \longmapsto 0$

there is at most one point of the fiber of p in $DRL^{m} \cap \overline{\mathcal{M}}^{m}_{IU}$

Reason: the coordinate a_γ on $\mathbb{A}^E_I$ parametrize the way we glue the degree 0 line bundles

$$\omega_{C_I}\Big(-\sum m_i p_i - \sum_{e=(h,h')\in E(\Gamma_C)} (I(h)-1)q_h + (I(h')-1)q_{h'}\Big)$$

to a line bundle on C and there is at most one way to obtain $\mathcal{O}_C$.

∎

Lemma 3 $\boxed{DRL^m \hookrightarrow \overline{\mathcal{M}}^m \text{ has pure cod } g}$

proof

$$\begin{array}{ccc} DRL^m & \rightarrow & \overline{\mathcal{U}}^m \\ \downarrow & \square & \downarrow \\ \overline{\mathcal{M}} & \rightarrow & \mathcal{Y} \end{array}$$

$\Rightarrow$ DRL^m is locally cutted out by g equations in $\overline{\mathcal{M}}^m$

$\Rightarrow$ cod $\leq g$ at every point.

Moreover $DRL^m \rightarrow \underbrace{\widetilde{\mathcal{H}}_g(\mu)}_{\substack{\text{pure}\\ \text{cod } g}} \subset \overline{\mathcal{M}}$ is quasi finite

$\Rightarrow$ Every irr comp of DRL^m has cod $\geq g$ in $\overline{\mathcal{M}}^m$.

∎

Corollary 1

$$\boxed{\text{Every generic point of } DRL^{m} \text{ lies over a generic point of } \widetilde{\mathcal{H}}_g(\mu).}$$

■

Lemma 4

$$\boxed{\overline{DRC} = DRC^{m}}$$

proof

Consider

$$\begin{tikzcd}
\mathcal{J}^{\diamond} \arrow[r, "f_J"] \arrow[d] \arrow[dr, phantom, "\square"] & \mathcal{J}^{m} \arrow[d] \\
\mathcal{M}^{\diamond} \arrow[r, "f"'] \arrow[u, bend left=60, "{\sigma_\diamond, e_\diamond}"] & \overline{\mathcal{M}}^{m} \arrow[u, bend right=60, "{e^m, \sigma^m}"']
\end{tikzcd}$$

proper and birational

Then

$$\sigma^{m!}\Big(e^{m}_{*}\underbrace{[\overline{\mathcal{M}}^{m}]}_{f_*[\mathcal{M}^\diamond]}\Big) = \sigma^{m!}\big(f_{J*}\, e_{\diamond *}[\mathcal{M}^{\diamond}]\big) =$$

$$= f_{*}\big(\sigma^{m!}\, e_{\diamond *}[\mathcal{M}^{\diamond}]\big) \overset{\text{excess intersection formula}}{=}$$

$E = f^{*}N_{\sigma^m}/N_{\sigma^\diamond} = 0$

$$\overset{!}{=} f_{*}\Big(\underbrace{e(\overset{0}{\overset{\|}{E}})}_{=1} \cap \sigma_{\diamond}^{!}\big(e_{\diamond *}[\mathcal{M}^{\diamond}]\big)\Big)$$

■

We will show that $DRC^m = H_{g\mu}$.

Computation of DRC^m

Proposition 1

$p_0 \in \tilde{\mathcal{H}}_g(\mu)$ generic point contained in $\partial\overline{\mathcal{M}}$. Then

$$\mathrm{mult}_{p_0}(DRC^m) = \sum_{\substack{p \in DRL^m \\ \text{lying over } p_0}} \ell(\mathcal{O}_{DRL^m, p})$$

necessarily a generic point

Proof

Fact || $\overline{\mathcal{M}}^m$ is C-M at the generic points of DRL^m mapped to $\partial\overline{\mathcal{M}}$

So Fulton's intersection theory tells us

$$\begin{tikzcd}
DRL^m \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\square"] & \overline{\mathcal{M}}^m \arrow[d, "\sigma^m"] \\
\overline{\mathcal{M}}^m \arrow[r, "e^m"'] & \mathcal{J}^m
\end{tikzcd}$$

for $Z = \overline{\{p\}}$ irr comp of DRL^m

coefficient of $\{\overline{p}\}$ in the class of $[e^m].[\sigma^m]$

$$\mathrm{mult}_p([e^m].[\sigma^m]) = \ell\left(\mathcal{O}_{\overline{\mathcal{M}}^m, p} / \text{ideal of } DRL^m\right) =$$
$$= \ell(\mathcal{O}_{DRL^m, p})$$

Therefore

$$\alpha_m^!\left(e_{m\star}[\overline{\mathcal{M}}^m]\right) = \sum_{\substack{p\in DRL^m \\ \text{generic point}}} \ell(\mathcal{O}_{DRL^m,p})\,\overline{\{p\}} = [DRL^m]$$

$$\Big\downarrow p_{m\star}$$

$$DRC^m = \sum_{\substack{p\in DRL^m \\ \text{generic point}}} \ell(\mathcal{O}_{DRL^m,p})\,\underbrace{\deg\left(p_m\Big|_{\overline{p}}^{\overline{p_m(p)}}\right)}_{\overset{?}{=}\,1}\,\overline{p_m(p)}$$

$$k(p) \overset{?}{=} k(p_m(p))$$

∎

At this point:

$$DRC^m = [\overline{\mathcal{H}}_g(\mu)] + \sum_{\substack{P_0 \text{ generic} \\ \text{point of } \partial\mathcal{H}_g(\mu) \\ \overline{\{P_0\}} = Z}} \underbrace{\mathrm{mult}_{P_0}(DRC^m)}_{= \sum_{p\in DRL^m \text{ over } P_0} \ell(\mathcal{O}_{DRL^m,p})} [Z]$$

only one term appears here by Lemma 5

Lemma 5

$P_0 \in \partial\mathcal{H}_g(\mu)$ generic point contained in $\partial\overline{\mathcal{M}}$.

$\Rightarrow \exists!\, p \in DRL^m$ (necessarily a generic point) lying over P_0.

proof

Existence: (and a first result of uniqueness) Let $\mathcal{M} \leftarrow U \rightarrow \mathbb{A}^E$ be a combinatorial chart around p_0. Since $p_0 \in \widetilde{\mathcal{H}}_g(\mu) \Rightarrow \exists$ twist on C_{p_0} s.t. (†) holds.

Consider

$$\begin{tikzcd} & \overline{\mathcal{M}}^m_{IU} \arrow[r] \arrow[dl] \arrow[d] \arrow[dr, phantom, "\square"] & \mathbb{A}^E_I \arrow[d] \\ \overline{\mathcal{M}} & U \arrow[l] \arrow[r] & \mathbb{A}^E \end{tikzcd}$$

the fiber of p_0 in $\overline{\mathcal{M}}^m_{IU}$ corresponds to the ways of glueing the 0 degree line bundles

$$\omega_{(C_{p_0})_I} \otimes \mathcal{O}_{(C_{p_0})_I}\left(-\sum m_i p_i - \sum_{\substack{e=(h,h')\in E(\Gamma):\\ I(e)\neq 0}} (I(h)-1)q_h + (I(h')-1)q_{h'}\right)$$

to a line bundle on C_{p_0}.

Since $p_0 \in \widetilde{\mathcal{H}}_g(\mu) \Rightarrow \exists!$ way of glueing them in such a way to obtain $\mathcal{O}_{C_{p_0}}$.

i.e. $\exists!\ p \in \overline{\mathcal{M}}^m_{IU} \subset \overline{\mathcal{M}}^m$.

$$p \mapsto p_0$$

Uniqueness **Obs** The previous argument gives that for fixed I $\exists$ at most one $p \in DRL^m \cap \overline{\mathcal{M}}^m_{I\sigma}$ over p_0.

So we will prove that:

Claim there exist at most one I on $\Gamma_{C_{p_0}}$ s.t. $DRL^m \cap \overline{\mathcal{M}}^m_{I\sigma}$ contains a point over p_0.

Equivalently, $\Gamma_{C_{p_0}}$ admits a unique twist I_{p_0} s.t. $(+)$ holds (This is a fact we used in the previous lecture)

proof of the claim

This uses the following

Lemma Fix $g \geq 1$, $n \geq 0$ and $\mu_1, \mu_2 \vdash 2g-2$, $\mu_1, \mu_2 > 0$ of length n. Then

$$\begin{bmatrix} \mathcal{H}_g(\mu_1) \text{ and } \mathcal{H}_g(\mu_2) \\ \text{share an irr. component} \end{bmatrix} \Longleftrightarrow \left[\mu_1 = \mu_2 \right]$$

So for $v \in V^{out}(\Gamma)$ let C_v be the corresponding component considered with marked points all the marked points coming from those of C_{p_0} and all the points $C_v \cap \overline{C_{p_0} \setminus C_{v_0}}$

<u>Obs</u> || $g(v) \geq 1$

<u>reason</u>: otherwise the condition

$$\sum_{end(h)=v} I(h) - \underbrace{2g(v)}_{=0} + 2 - val(v) = 0$$

cannot be satisfied. ∎

Let I be a twist on C_{p_0} s.t. (†) holds. Then I gives $C_v \in \mathcal{H}_{g(v)}(\mu')$ for some $\mu' \geq 0$.

Also, C_v must be a generic point of $\mathcal{H}_{g(v)}(\mu')$ otherwise C_{p_0} would not be a generic point of $\widetilde{\mathcal{H}}_g(\mu)$.

$\underset{\text{Lemma}}{\Longrightarrow}$ μ' is completely determined by the fact that $C_v \in \mathcal{H}_{g(v)}(\mu')$.

$\Rightarrow$ claim.

∎

Proposition 2

For $p \in DRL^m$
$\downarrow J$
$p_0 \in \widetilde{\mathcal{H}_g}(\mu)$ generic point $p_0 \in \partial \overline{\mathcal{M}}$

we have

$$\tau\left(\vartheta_{DRL^m, p}\right) = \prod_{e \in E(\Gamma_{p_0})} I(e)$$

unique twist on Γ_{p_0} s.t. (†) holds

The proof is tricky:

Choose a combinatorial chart

$$\mathcal{M} \leftarrow U \longrightarrow \mathbb{A}^E$$

and I twist on Γ inducing $\overline{\mathcal{M}}^m_{IU} \to \overline{\mathcal{M}}$ containing p_0

Obs The twisted induced on Γ_{p_0} must be $I_{\Gamma_{p_0}}$.

Since Γ_{p_0} is obtained from Γ contracting some edges $\Rightarrow$ we can assume $(\Gamma, I) = (\Gamma_{p_0}, I_{p_0})$

Obs From

$$\begin{array}{ccccccc} p \in \overline{\mathcal{M}}^m_{IU} & \to & \mathbb{A}^E_I & \subseteq & \mathbb{A}^K \\ \downarrow & \square & \downarrow & & \\ U & \longrightarrow & \mathbb{A}^E & & \\ \cap\!\mid & & & & \\ \mathbb{A}^M & & & & \end{array}$$

we can assume it's affine

$\Rightarrow \overline{\mathcal{M}}^m_{IU} \subseteq U \times \mathbb{A}^K$

$\subseteq \mathbb{A}^N$ is affine.

Let p' be a general point of $\overline{\{p\}}$ and $H \subseteq \mathbb{A}^N$ a general hyperplane of cod $2g-3+m$ in $\mathbb{A}^N$ passing through p'.

Set

$$\begin{tikzcd} p' \arrow[dd, mapsto, "\text{we can assume}"'] & \in DRL' := \operatorname{Spec}\left(\mathcal{O}_{DRL^m \cap H, p'}\right) \arrow[d] \\ & DRL^m \arrow[d] \\ 0 & \mathbb{A}^E \end{tikzcd}$$

local artinian ring of length:

$$e\left(\mathcal{O}_{DRL^m \cap H, p'}\right) = e\left(\mathcal{O}_{DRL^m, p}\right)$$

Prop 3

$DRL^m \hookrightarrow \mathbb{A}^E$ is a closed embedding with associated ideal $\left(a_e^{I(e)} : e \in E(\Gamma)\right)$

Using this proposition:

Corollary

$$e\left(\mathcal{O}_{DRL^m, p}\right) = \prod_{e \in E(\Gamma)} I(e)$$

■

Sketch of proof of prop 3

The main ingredient of the proof is the following

Lemma Call $R := \mathbb{C}[\partial_e : e \in E] = \mathcal{O}_{\mathbb{A}^E}(\mathbb{A}^E)$, and let $b \subset R$ be an ideal containing some power of $m := (\partial_e : e \in E)$. Call $B := R/b$. Then

$$\left[\begin{tikzcd} & DRL' \arrow[d] \\ \operatorname{Spec} B \arrow[r, hook] \arrow[ur, dashed, "\exists \text{ lift}"] & \mathbb{A}^E \end{tikzcd}\right] \iff \left[b \supset \partial_e^{I(e)} \quad \forall e \in E \right]$$

Assuming this lemma we prove the theorem.

$b := (\partial_e^{I(e)} : e \in E)$. Then

$$\begin{tikzcd} & DRL' \arrow[d] \\ \operatorname{Spec} B \arrow[r] \arrow[ur, "\exists \text{ lift}", "\varphi"'] & \mathbb{A}^E \end{tikzcd}$$

So the map $DRL' \xrightarrow{f} \operatorname{Spec}(B)$ What is this map? (See next pages)

has a section $\Rightarrow$ it is surjective on tangent spaces

Fact (that we won't prove)

> For $p \in DRL^m$ s.t. Γ_p is a simple star graph
>
> $$\dim T_p DRL^m = 2g-3+m + \#\{e \in E(\Gamma_p) \mid I(e) > 1\}$$
>
> $p \in \overline{\mathcal{M}}^m_{I,U}$

In particular

$$\dim T_{p'} DRL^1 = \#\{e \in E(\Gamma) \mid I(e) > 1\}$$

and

$$\dim T_0 \operatorname{Spec} \mathbb{C}[\partial_e : e \in E] / (\partial_e^{I(e)}, e \in E) =$$

$$= \dim \frac{(\partial_e : e \in E) / (\partial_e^{I(e)})}{\left((\partial_e : e \in E) / (\partial_e^{I(e)})\right)^2} = \#\{e \in E(\Gamma) \mid I(e) > 1\}$$

$\Rightarrow \varphi$ induces an isomorphism of tangent spaces

$\Rightarrow \varphi$ is unramified

$\Rightarrow \exists$ (General fact)

$$\begin{tikzcd}
X = \bigsqcup \operatorname{Spec} A_i \sqcup \operatorname{Spec} B \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \operatorname{Spec} A = \{q''\} \arrow[d, "\text{étale}"] \\
DRL^1 = \{p'\} \arrow[r, "\varphi"] & \operatorname{Spec} B = \{q'\}
\end{tikzcd}$$

$\{p'\} \mapsto \{q'\}$, $q'' \mapsto q'$

$\Rightarrow A$ artin local

s.t. $\forall i$ $\quad \mathrm{Spec}\, A_i \xhookrightarrow[\text{closed emb}]{} \mathrm{Spec}\, A$ and

no points of $\mathrm{Spec}\, B$ are sent to q"

then it must be $X = \mathrm{Spec}\, A_1$ i.e.

$$
\begin{tikzcd}
X = \mathrm{Spec}\, A_1 \arrow[r, hook, "\text{closed emb}"] \arrow[d] \arrow[dr, phantom, "\square"] & \mathrm{Spec}\, A \arrow[d, "\text{étale}"] \\
DRL' \arrow[r] & \mathrm{Spec}\, B
\end{tikzcd}
$$

$\Rightarrow$ $DRL' \overset{\varphi}{\hookrightarrow} \mathrm{Spec}\, B$ is a closed emb. $\overset{\text{previous lemma again}}{\Rightarrow}$ φ is an iso

■

Federico suggested the following solution to the existence of the map φ

The only thing we want is to find an isomorphism of tangent spaces $\quad T_0\left(\mathrm{Spec}\; R/_{(\partial_e^{I(e)} : e \in E)}\right) \cong T_{p'}\, DRL'$.

As we said in the seminar, since the $\partial_e \in \mathcal{O}_{DRL \cap H, p'} =: A$ belong to the maximal ideal and A is local Artin

$\Rightarrow \exists\, N > 0 : \quad \partial_e^N = 0 \quad \forall e \in E$.

Let $\mathfrak{b}' := (\partial_e^N : e \in E)$. Then by the lemma

$$\begin{tikzcd} & DRL' \arrow[d] \\ \operatorname{Spec} R/(\partial_e^{I(e)} : e \in E) \arrow[r, hook] \arrow[ur, "\exists \text{ lift}"] & \operatorname{Spec} R/\mathfrak{b}' \end{tikzcd}$$

This implies that

$$\operatorname{Spec} R/(\partial_e^{I(e)} : e \in E) \longrightarrow DRL'$$

is injective on tangent spaces and so by dimensional reasons it is an isomorphism.

$\blacksquare$

Concluding the proof of $H_{g\mu} = \overline{DRC}$

Thm $\boxed{\text{We have } \quad H_{g\mu} = \overline{DRC} \quad \text{in} \quad A^g(\overline{\mathcal{M}})}$

proof

$$H_{g\mu} = [\overline{H_g(\mu)}] + \sum_{\substack{Z \subset \widetilde{\mathcal{H}}_g(\mu) \\ \text{irr comp} \\ \text{contained in } \partial\overline{\mathcal{M}}}} \left(\prod_{e \in E(\Gamma_Z)} I_Z(e) \right) [Z]$$

$$\overline{DRC} = \sum_{\substack{p_0 \in \widetilde{\mathcal{H}}_g(\mu) \\ \text{generic points} \\ \text{contained in } \partial\overline{\mathcal{M}}}} \underbrace{\ell(\mathcal{O}_{DRL^m,\ \text{unique point } p \text{ in } DRL^m \text{ over } p_0})}_{\substack{\text{mult}_{p_0}(\overline{DRC}) \\ \shortparallel}} [Z]$$

$$\shortparallel$$

$$\left(\prod_{e \in E(\Gamma_{p_0})} I_{p_0}(e) \right)$$

$$+ [\overline{\mathcal{H}_g(\mu)}]$$

$$\uparrow$$

$$\sigma^m \Big|_{\mathcal{M}} = \alpha : (C, p_1 \ldots p_n) \mapsto \omega_C\left(-\sum m_i p_i\right)$$

$$\Rightarrow \overline{DRC}\Big|_{\mathcal{M}} = [\mathcal{H}_g(\mu)]$$

∎

A few words about the computation of the tangent space of DRL^m at a simple star

Let $p \in DRL^m$ be a closed point with Γ_p a simple star graph.
Say $p \in \overline{\mathcal{M}}^m_{IU}$.

Now since p is not a generic point of DRL^m I do not know if $\exists!$ twist I on C_p s.t. $(\dagger)$ holds. This is somehow linked to the pb of computing the degree of $p_m|_{\overline{\{p\}}}$

Anyway the following argument works for any I s.t. $p \in DRL^m \cap \overline{\mathcal{M}}^m_{IU}$ and so this seems to suggest that the twist on C_p making $(\dagger)$ to hold is unique.

<u>Goal</u>: We want ~~to~~ find kernel and image of

$$T_p \overline{\mathcal{M}}^m \to T_p \overline{\mathcal{M}}$$

so that we have an exact sequence

$$0 \to \operatorname{Ker} \to T_p \overline{\mathcal{M}}^m \to T_p \overline{\mathcal{M}} \to \operatorname{Im} \to 0$$

We will see that this splits

and so

$$T_p \overline{\mathcal{M}}^m = \operatorname{Ker} \oplus \operatorname{Im}.$$

So consider a combinatorial chart $\overline{\mathcal{M}} \leftarrow U \to \mathbb{A}^E$, $p \leftarrow\!\shortmid u \mapsto 0$

and identify $T_p \overline{\mathcal{M}} = T_u U$.

Consider

$$\begin{tikzcd}
\operatorname{Spec} \mathbb{C} \arrow[r, "p"] \arrow[d] & \overline{\mathcal{M}}^m_{IU} \arrow[d] \\
\operatorname{Spec} \mathbb{C}[t]/(t^2) \arrow[r, "v"'] \arrow[ur, dashed, "\tilde{v}"] & U
\end{tikzcd}$$

This gives $\forall e \in E(\Gamma)$ $a_e \in \mathbb{C}$ and $\forall \gamma \subset \Gamma$ oriented loop $a_\gamma \in \mathbb{C}^*$
Since $p \mapsto 0$ in $\mathbb{A}^E$
$\Rightarrow a_e = 0$ $\forall e \in E$

This gives $\forall e \in E(\Gamma)$ $\ell_e \in t \cdot \mathbb{C}[t]/(t^2)$

Data of $\overline{v}$ $\overset{1:1}{\longleftrightarrow}$ data for all γ oriented loop in Γ

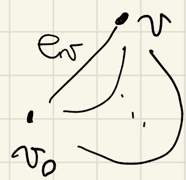
$$\ell_\gamma \in \left(\mathbb{C}[t]/(t^2)\right)^* \text{ s.t.}$$

(i) $\ell_\gamma(t=0) = a_\gamma$ ($\Leftrightarrow$ $\overline{v}_\gamma$ is invertible in $k[t]/t^2$)

So $\ell_\gamma = a_\gamma + t\, b_\gamma$

(ii) $$\ell_\gamma \prod_{e: I_\gamma(e)<0} \ell_e^{-I_\gamma(e)} - \prod_{e: I_\gamma(e)>0} \ell_e^{I_\gamma(e)} = 0$$

<u>Claim</u> Fix $v \in V^{out}(\Gamma)$



$E_v := \{$ directed edges from v_0 to $v\}$

φ

e_v is a distinguished edge.

We have the following three possibilities:

(1): $\forall e \in E_v$ $\ell_e^{I(e)} = 0$

(2) $\forall e \in E_v$ $\ell_e^{I(e)} \neq 0$

(3) $\exists e \neq e' \in E_v$: $\ell_e^{I(e)} = 0$ $\ell_{e'}^{I(e')} \neq 0$.

In case (1): in order to give $\overline{v}$ we have to choose $\ell_{\gamma(e_v, e')}$ $\in \left(\mathbb{C}[t]/(t^2)\right)^*$ satisfying (1) for $e' \in E_v - \{e_v\}$

as we want.

In case ②: $\quad \ell_e^{I(e)} \neq 0 \Rightarrow I(e) = 1 \quad \forall e \in E_v$

Necessary condition for the existence of $\bar{v}$ is that

$\forall e' \in E_v - \{e_v\} \quad \partial_{\gamma(e_v, e')} \ell_{e'} - \ell_{e_v} = 0$

in which case to define $\bar{v}$ we have to

specify $b_{\gamma(e_v, e')} \in \mathbb{C}^*$ for $e' \in E_v - \{e_v\}$

(chosen as we want)

In case ③: $\quad \nexists \, \bar{v}$.

Corollary 1

$$\boxed{\mathrm{Ker}\,(T_p \bar{\mathcal{M}}^m \to T_p \bar{\mathcal{M}}) = \mathrm{Hom}\,(H_1(\Gamma, \mathbb{C}), \mathbb{C}) = H^1(\Gamma, \mathbb{C})}$$

proof

if $v = 0 \Rightarrow$ we are always in case ① and so we only have to

specify $\quad b_{\gamma(e_v, e')} \in \mathbb{C} \quad \forall v \in V$ and $e' \in E_v - \{e_v\}$

i.e. on a basis of $H_1(\Gamma, \mathbb{C})$. ∎

Corollary 2

locally trivial deformations

$$\mathrm{Im}\left(T_p\overline{\mathcal{M}}^m \to T_p\overline{\mathcal{M}}\right) \cong H^1(C_p, \Omega(-P)) \oplus \bigoplus_{\substack{e\in E(\Gamma):\\ I(e)>1}} \mathbb{C} \oplus \bigoplus_{v\in V^{out}(\Gamma)} L_v$$

where

$$L_v := \begin{cases} 0 & \exists e\in E_v : I(e)>1 \\ \left\{ (\ell_e)_{e\in E} \in \mathbb{C}^{\#E} \mid \ell_e \partial_{\gamma(e',e)} = \ell_{e'} \ \forall e,e'\in E_v \right\} & \text{otherwise} \end{cases}$$

$$L_v \subset \bigoplus_{e\in E_v} \mathbb{C}$$

Therefore we have

$$0 \to H^1(\Gamma,\mathbb{C}) \to T_p\overline{\mathcal{M}}^m \to H^1(C_p,\Omega(-P)) \oplus \bigoplus_{e: I(e)>1} \mathbb{C} \oplus \bigoplus_{v\in V^{out}} L_v \to 0$$

$$\dim = \begin{cases} 1 \\ 0 \end{cases}$$

$(\gamma \mapsto b_\gamma) \leftarrow v$

b_γ = degree 1 coefficient of ℓ_γ.

$$\Rightarrow T_p\overline{\mathcal{M}}^m \xrightarrow{\sim} H^1(\Gamma,\mathbb{C}) \oplus H^1(C_p,\Omega(-P)) \oplus \bigoplus_{e: I(e)>1} \mathbb{C} \oplus \bigoplus_{v\in V^{out}} L_v$$

Recall that the final goal is to compute

$$\dim T_p DRL^m = \,?$$

Consider

$$\begin{tikzcd} p \in DRL^m \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \overline{\mathcal{M}}^m \arrow[d, "\sigma'_m"] \\ \mathcal{M} \arrow[r, "\bar{e}"'] & \mathcal{J} \end{tikzcd}$$

$\Rightarrow$ $\sigma'_m(p) = \bar{e}(p) =: e \in \mathcal{J}$. So we have

$$\begin{tikzcd}[column sep=small] 0 & & & & & \\ & \operatorname{coker}(b) \arrow[ul] & & & & \\ & 0 \arrow[r] & T_e \mathcal{J}_p \arrow[r] \arrow[ul] & T_e \mathcal{J} \arrow[r] & T_p \overline{\mathcal{M}} \arrow[r] & 0 \\ & & & T_p \overline{\mathcal{M}}^m \arrow[ul, dashed, "b"] \arrow[u, "T_p\sigma'_m - T_p\bar{e}"'] \arrow[ur, bend right, "0"'] & & \\ & & & & \ker(b) = T_p DRL^m \arrow[ul] & \\ & & & & & 0 \arrow[ul] \end{tikzcd}$$

$\Rightarrow$ To know $\dim(T_p DRL^m)$ is the same as to know $\dim(\operatorname{coker}(b))$

Finally dualizing the sequence in green we get

$$\begin{tikzcd}[column sep=small] 0 \arrow[r] & \operatorname{coker}(b^\vee) \arrow[r] \arrow[d, phantom, "\shortparallel"] & (T_p \mathcal{J}_p)^\vee \arrow[r] \arrow[d, phantom, "\shortparallel"] & (T_p \overline{\mathcal{M}}^m)^\vee \arrow[r] & (T_p DRL^m)^\vee \arrow[r] & 0 \\ & \ker(b^\vee) & H^0(C_p, \omega_{C_p}) & & & \end{tikzcd}$$

$$\begin{tikzcd}[column sep=small]
T_p \overline{\mathcal{M}}^m \arrow[d, "b"'] & = & H^1(\Gamma,\mathbb{C}) \arrow[dll, "b_\Gamma"] & \oplus & H^1(C_p,\Omega(-p)) \arrow[dllll, "b_\Omega"] & \oplus & \displaystyle\bigoplus_{e:I(e)>1}\mathbb{C} \arrow[dllllll, "b_{>1}"] & \oplus & \displaystyle\bigoplus_{v\in V^{out}} \underbrace{L_v} \arrow[dllllllll, bend left, "b_v"] \\
T_e \mathcal{J}_p & & & & & & & &
\end{tikzcd}$$

$$\Rightarrow \ker(b^\vee) = \ker(b_\Gamma^\vee) \cap \ker(b_\Omega^\vee) \cap \ker(b_{>1}^\vee) \cap \bigcap_{v\in V^{out}} \ker(b_v^\vee)$$

To identify $\ker(b^\vee)$ we introduce some more notation.

__Notation__ $V^{>1} := \{ v \in V^{out}(\Gamma) \mid I(e)>1 \ \forall e \in E_v \}$

__Notation / Remark__

Since $\omega(-\sum m_i p_i) \otimes \Upsilon = \mathcal{O}_{C_p}$ we can find $\varphi_0 \in H^0(C_p, \omega(-\sum m_i p_i) \otimes \Upsilon)$ generating section. Now Υ is trivial on C_p^{sm}, let $\mathbb{1}$ be a generating section. Then

$$\frac{\varphi}{\mathbb{1}} \in H^0(C_p, \omega(-\sum m_i p_i) \otimes \mathcal{M})$$

and when restricted to C_v for $v \in V^{out}(\Gamma)$

$$\left.\frac{\varphi}{\mathbb{1}}\right|_{C_v} \in H^0(C_v, \omega)$$

__Reason__: $I > 0$ on $h \in H(\Gamma)$ s.t. $\mathrm{end}(h) = v$.

Thm

$$\ker(b^\vee) \cong \mathrm{Im}\Big(\bigoplus_{v \in V^{>1}} \mathbb{C} \hookrightarrow H^0(C_p, \omega_{C_p}) \Big)$$

$\cap\!|$

$H^0(C_p, \omega_{C_p})$

$(c_v)_{v \in V^{>1}} \longmapsto$ section of ω_{C_p} which is equal to 0 on C_{v_0} and $c_v \underbrace{\varphi_0}_{1}$ on C_v

■

Corollary $\boxed{\dim T_p DRL^m = 2g + m - 3 + \#\{e \in E \mid I(e) > 1\}}$

proof dimensional count.

■