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The virtual cohomological dimension of the Mapping Class Groups

Relatore:
Prof.
Gabriele Mondello

Candidato:
Alessio Cela
matricola 1714250

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Dipartimento di Matematica 'Guido Castelnuovo'

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Introduction

The present thesis is centered around the computation of the so called virtual cohomological dimension of Mapping Class Groups of orientable surfaces.
Given any topological group Γ, Milnor has proved that there always exists a classifying Γ-space Y (see [30]). Using Milnor's result it is easy to prove that we can choose such a space to be a CW-complex. Moreover, if we require that Y admits a CW-complex structure, then such a classifying Γ-space is unique up to homotopy equivalence. However, the space produced by Milnor's construction is always enormous and thus a natural question is if, at least in some specific cases, his construction can be improved. Suppose for example that the topology on Γ is the discrete topology. In this case a classifying Γ-space is also called a $K(\Gamma, 1)$-space. Let Y be a $K(\Gamma, 1)$-space that is also a CW complex of dimension n (possibly $n = \infty$). We can take this dimension to be the minimum possible and call it the geometric cohomological dimension of Γ, also denoted by $\mathrm{geom.cd}(\Gamma)$. For example if Γ has non-trivial torsion, then $\mathrm{geom.cd}(\Gamma)$ is always ∞.
There is also an algebraic notion of group cohomological dimension: given a group Γ the cohomological dimension of Γ, also denoted by $\mathrm{cd}(\Gamma)$, is the projective dimension of $\mathbb{Z}$ as a $\mathbb{Z}\Gamma$-module, where Γ acts trivially on $\mathbb{Z}$; i.e. $\mathrm{cd}(\Gamma)$ is the infimum of the set of integers n such that $\mathbb{Z}$ (considered as a $\mathbb{Z}\Gamma$-module) admits a projective resolution of length n. One can show that whenever $\mathrm{cd}(\Gamma) \neq 2$ (the case $\mathrm{cd}(\Gamma) = \infty$ is not excluded) the equality $\mathrm{cd}(\Gamma) = \mathrm{geom.cd}(\Gamma)$ holds (here $\mathrm{geom.cd}(\Gamma)$ refers to Γ when endowed with the discrete topology). For $\mathrm{cd}(\Gamma) = 2$, we still have $\mathrm{cd}(\Gamma) \leq \mathrm{geom.cd}(\Gamma)$ but it is not known if this inequality can be strict. The Eilenberg–Ganea conjecture claims that the equality holds also when $\mathrm{cd}(\Gamma) = 2$.
To compute the cohomological dimension of groups we will stress the followng result.

Theorem 0.0.1. *Let Y be a $K(\Gamma, 1)$-space with a finite dimensional structure of CW-complex. Suppose in addition that Y is a compact topological manifold of dimension d with boundary (possibly $\partial Y = \emptyset$). Then:*
(a) if $\partial Y = \emptyset$, then $\mathrm{cd}\ \Gamma = d$;
(b) if $\partial Y \neq \emptyset$, then cd $\Gamma = d - m - 1$ where m is the minimum integer such that $\widetilde{H}_m(\partial\widetilde{Y}) \neq 0$ (there always exists an m such that $\widetilde{H}_m(\partial\widetilde{Y}) \neq 0$) and $\widetilde{Y}$ is the universal cover of Y.

When a group Γ has non-trivial torsion, in order to get an interesting invariant, one can consider its virtual cohomological dimension. The virtual cohomological dimension of Γ is defined only when Γ admits a finite-index and torsion-free subgroup Γ' and, in this case, it is equal to $\text{vcd}(\Gamma) = \text{cd}(\Gamma')$. A theorem of Serre states that this number does not depend on the choice of the particular subgroup Γ'.

In this thesis we will deal with the problem of computing the virtual cohomological dimension of Mapping Class Groups of orientable surfaces.
On this point, let S_g^b be a compact and orientable surface of genus g and with b boundary components. The Mapping Class Group of S_g^b is defined to be

$$\text{Mod}(S_g^b) = \text{Diff}^+(S_g^b)/\text{Diff}_0(S_g^b).$$

where $\text{Diff}^+(S_g^b)$ is the group of orientation-preserving diffeomorphisms of S_g^b and $\text{Diff}_0(S_g^b)$ is the (normal) subgroup of those diffeomorphisms isotopic to $1_{S_g^b}$. Clearly $\text{Mod}(S_g^b)$ is an algebraic invariant of S_g^b. In order to compute $\text{vcd}(\text{Mod}(S_g^b))$ in the general case we will first consider the cases $(g, b) \in \mathbb{N}_0 \times \{0\} \cup \{(0, 1), (0, 2), (0, 3), (1, 1)\}$ and then use induction on b for fixed g, starting with $(g, b) \in \mathbb{N}_{\geq 2} \times \{0\} \cup \{(0, 3), (1, 1)\}$. The cases $(g, b) \in \{(0, 0), (0, 1), (0, 2), (0, 3), (1, 0), (1, 1)\}$ are treated using directly the definitions, while the cases $g \geq 2$ and $b = 0$ require much more work. Suppose $g \geq 2$ and $b = 0$. The first problem to solve is to find a finite-index and torsion-free subgroup of $\text{Mod}(S_g)$. This is done in Chapter 2 where we find infinitely many of such subgroups, one for each $m \in \mathbb{N}$, $m \geq 3$. These groups are called congruence subgroups of $\text{Mod}(S_g)$ and are denoted by $\text{Mod}(S_g)[m]$. Next we want to find a $K(\text{Mod}(S_g)[m], 1)$-complex that is also a compact topological manifold (possibly with boundary). To that end, consider the Teichmüller space T_{S_g} of S_g i.e.

$$T_{S_g} = \text{HypMet}(S_g)/\text{Diff}_0(S_g).$$

where $\text{HypMet}(S_g)$ is the space of all hyperbolic metrics on S_g (since $\chi(S_g) = 2 - 2g < 0$ this space is non-empty) and $\text{Diff}_0(S_g)$ acts on it by pullback. We can endow T_{S_g} with a differential structure that makes it diffeomorphic to $\mathbb{R}^{6g-6}$. Clearly the Mapping Class Group of S_g acts on T_{S_g} by $f \cdot \mathcal{X} = [\phi^*(h)] \in T_{S_g}$ if $f = [\phi] \in \text{Mod}(S_g)$ and $\mathcal{X} = [h] \in T_{S_g}$. This action is proper discontinuous and with finite stabilizers. In particular, since $\text{Mod}(S_g)[m]$ is torsion-free, the action of $\text{Mod}(S_g)[m]$ on T_{S_g} is proper discontinuous and free, thus the quotient $T_{S_g}/\text{Mod}(S_g)[m]$ is naturally a smooth manifold. The problem is that it fails to be compact. We will thus replace it with a suitable submanifold with corners X_g of T_{S_g} having the following properties.

Theorem 0.0.2. *The space X_g is a smooth manifold with corners of dimension $6g - 6$ such that:*
(a) both X_g and ∂X_g are invariant under the action of $\text{Mod}(S_g)$*;*

(b) $X_g/\mathrm{Mod}(S_g)$ *is compact;*
(c) X_g *is contractible;*
(d) ∂X_g *is homotopically equivalent to* $|C(S_g)|$.

Here $C(S_g)$ is the complex of curves of S_g, i.e. the simplicial complex having as k-simplices the isotopy classes of collections of $k+1$ embedded S^1 in S_g such that each loop is homotopically non-trivial and no two loops are isotopic. In Chapter 3 we prove that the realization $|C(S_g)|$ is $2g-3$ connected, thus, by (d) of the previous theorem, also ∂X_g is.
At this point we easily obtain the upper bound $\mathrm{vcd}(\mathrm{Mod}(S)) \leq 4g-5$ for $g \geq 2$. Indeed, for every fixed $m \geq 3$, the space $Y_g = X_g/\mathrm{Mod}(S_g)[m]$ is a compact smooth manifold with corners of dimension $6g-6$ (in particular a finite CW-complex) and a $K(\mathrm{Mod}(S_g)[m], 1)$-space, its universal cover is X_g whose boundary is $2g-3$ connected and thus $\widetilde{H}_k(\partial X_g) = 0$ for all $k \leq 2g-3$. Applying Theorem 0.0.1, we obtain that $\mathrm{vcd}(\mathrm{Mod}(S_g)) = \mathrm{cd}(\mathrm{Mod}(S_g)[m]) \leq 4g-5$.
The proof of the reverse inequality uses the existence of the so called Mess subgroups B_g of $\mathrm{Mod}(S_g)$. They are torsion-free subgroups of cohomological dimension $\mathrm{cd}(B_g) = 4g-5$. The computation of $\mathrm{cd}(B_g)$ is again done by constructing a $K(B_g, 1)$-space satisfying the properties required to apply Theorem 0.0.1. Using the existence of such subgroups, we have that $B_g \cap \mathrm{Mod}(S_g)[m]$ is a finite-index and torsion-free subgroup of B_g and thus it has cohomological dimension $4g-5$. Hence $\mathrm{vcd}(\mathrm{Mod}(S_g)) = \mathrm{cd}(\mathrm{Mod}(S_g)[m]) \geq \mathrm{cd}(B_g \cap \mathrm{Mod}(S_g)[m]) = 4g-5$ where the inequality $\geq$ is due to the fact that, in general, if Γ is any group and $\Gamma' \subseteq \Gamma$ is any subgroup than $\mathrm{cd}(\Gamma') \leq \mathrm{cd}(\Gamma)$. In conclusion, we will have proved the following result:

Theorem 0.0.3. *For* $g \geq 2$ *we have* $\mathrm{vcd}(\mathrm{Mod}(S_g)) = 4g-5$.

This theorem, as well as the computation of $\mathrm{vcd}(\mathrm{Mod}(S))$ in the general case in which S has non-empty boundary, is due to Harer (see [19]). Another proof is contained in [24]. Instead, the proof we present in this thesis follows the approach in [25]. This proof uses very little about the theory of group cohomology and do not use the full description of the homotopy type of the complex of curves $C(S)$ (use only the description of the homotopy type of $C(S)$ when S is a closed surface of genus $g \geq 2$).

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Chapter 1

The virtual cohomological dimension of a group

The first chapter of this thesis is intended to be a short introduction to the concept of cohomology of groups. It will provide the reader with the necessary results needed to understand the notion of virtual cohomological dimension of a group and present some of the tools necessary for doing some computations. For a much more complete exposition of this topic we recommend [6].

1.1 The cohomological dimension of a group

We begin with some basic definitions and preliminaries about the group cohomology.

Let R be a unitary ring and M a left R-module.

Definition 1.1.1. The **projective dimension** of M over R, denoted by **proj dim**$_R$ **M**, is the infimum of the set of intengers n such that M admits a projective resultion of lenght n.

As the following lemma explains, there many other equivalent definitions :

Lemma 1.1.2. *The following conditions are equivalent:*
(a) proj dim$_R$ $M \leq n$*;*
(b) $\mathrm{Ext}^i_R(M, -) = 0$ *for* $i > n$*;*
(c) $\mathrm{Ext}^{n+1}_R(M, -) = 0$*;*
(d) if $0 \to K \to P_{n-1} \to ... \to P_0 \to M \to 0$ *is any exact sequence of R-modules with each* P_i *projective, then* K *is projective.*

Proof. The implications $(d) \Rightarrow (a) \Rightarrow (b) \Rightarrow (c)$ are clear. We now prove $(c) \Rightarrow (d)$. Complete $P_{n-1} \to ... \to P_0 \to M \to 0$ to a projective resolution

$$... \xrightarrow{d_{n+2}} P_{n+1} \xrightarrow{d_{n+1}} ... \to P_0 \xrightarrow{d_0} M \to 0$$

Then K is isomorphic to the image of $P_n \xrightarrow{d_n} P_{n-1}$. Call L the image of d_{n+1}. Since $P_{n+1} \xrightarrow{d_{n+1}} L$ is in the kernel of $\mathrm{Hom}_R(P_{n+1}, L) \xrightarrow{-\circ d_{n+2}} \mathrm{Hom}_R(P_{n+2}, L)$ and, by hypothesis, $\mathrm{Ext}_R^{n+1}(M, L) = 0$, there exists $\psi : P_n \to L$ such that $\psi \circ d_{n+1} = d_{n+1}$. Hence, being $d_{n+1} : P_{n+1} \to L$ surjective, it must be $\psi|_L = 1_L$ and, considering the short exact sequence $0 \to L \hookrightarrow P_n \xrightarrow{d_n} K \to 0$, we obtain that $P_n = K \oplus L$. Thus K is projective. □

We will only be interested in the case $R = \mathbb{Z}\Gamma$ and $M = \mathbb{Z}$, where Γ is a group that acts trivially on $\mathbb{Z}$.

Definition 1.1.3. The **cohomological dimension** of Γ is cd Γ= proj $\dim_{\mathbb{Z}\Gamma}\mathbb{Z}$ (possibly equal to ∞).

For any group Γ there is a unique (up to homotopy equivalence) connected CW-complex, denoted by $K(\Gamma, 1)$, which has a contractible universal cover and fundamental group isomorphic to Γ (see the Appendix B.2).

Definition 1.1.4. The **geometrical cohomological dimension** of Γ, denoted geom.cd(Γ), is the minimum of the dimensions of a $K(\Gamma, 1)$-complex.

Proposition 1.1.5. cd $\Gamma \leq$ geom.cd(Γ).

Proof. It is enough to observe that if $\widetilde{K(\Gamma, 1)} \to K(\Gamma, 1)$ is the universal cover of a n-dimensional $K(\Gamma, 1)$-complex, then $\widetilde{K(\Gamma, 1)}$ has a natural structure of CW-complex preserved by the action of Γ by deck transformations and with respect to which the augmented cellular chain complex

$$0 \to H_n(\widetilde{K(\Gamma, 1)}^n, \widetilde{K(\Gamma, 1)}^{n-1}) \to ... \to H_0(\widetilde{K(\Gamma, 1)}^0, \emptyset) \xrightarrow{\varepsilon} \mathbb{Z} \to 0$$

is a free $\mathbb{Z}\Gamma$-resolution of $\mathbb{Z}$. □

The simplest cases are when Γ is either a free group or a finite group.

Example 1.1.6. Clearly, if $\Gamma = \{1\}$, cd $\Gamma = 0$. Vice versa, if cd $\Gamma = 0$, then $\mathbb{Z}$ is a projective $\mathbb{Z}\Gamma$-module, thus the canonical projection $\mathbb{Z}\Gamma \xrightarrow{\varepsilon} \mathbb{Z} \to 0$ splits. So $\mathbb{Z}$ is a $\mathbb{Z}\Gamma$-submodule of $\mathbb{Z}\Gamma$. In particular $g \cdot 1 = 1$ for all $g \in \Gamma$. Equivalently, $\Gamma = \{1\}$.

Example 1.1.7. Consider $\Gamma = F(S)$ the free group with generator set $S \neq \emptyset$ and put $X = \bigvee_{s \in S} S^1$. Then X is a 1-dimensional CW-complex with fundamental group Γ. Called $\widetilde{X}$ its universal covering space, $\pi_i(\widetilde{X}) = \pi_i(X) = 0$. To see this we can proceed as follows. Clearly $\pi_1(\widetilde{X}) = 0$ thus, using induction on i, we may assume that $\pi_i(\widetilde{X}) = 0$ and hence, by Hurewicz Theorem, we have $\pi_{i+1}(\widetilde{X}) = H_{i+1}(\widetilde{X}) = 0$ where the last equaility follows from the fact that $\widetilde{X}$ a CW-complex of dimension 1. Since $\Gamma \neq \{1\}$, it follows that cd $\Gamma = 1$.

Example 1.1.8. Let Γ be a finite group of order n, with generator t. Call $N = \sum_{i=0}^{n-1} t^i$. From the free $\mathbb{Z}\Gamma$ resolution

$$... \to \mathbb{Z}\Gamma \xrightarrow{t-1} \mathbb{Z}\Gamma \xrightarrow{N} \mathbb{Z}\Gamma \xrightarrow{t-1} \mathbb{Z}\Gamma \xrightarrow{\varepsilon} \mathbb{Z} \to 0$$

we deduce that, for each $\mathbb{Z}\Gamma$-module M and $i \geq 1$, $\operatorname{Ext}^{2i}_{\mathbb{Z}\Gamma}(\mathbb{Z}, M) = \ker(M \xrightarrow{N} M)/\operatorname{Im}(M \xrightarrow{t-1} M)$. In particular cd $\Gamma = \infty$.

Next, we explore some properties of cd Γ.

Proposition 1.1.9. *If* cd $\Gamma < \infty$ *then* cd $\Gamma = \max\{n \in \mathbb{N} : \operatorname{Ext}^i_{\mathbb{Z}\Gamma}(\mathbb{Z}, F) \neq 0$ for some free $\mathbb{Z}\Gamma-$module $F\}$.

Proof. The inequality $\geq$ is obvious. We will prove $\leq$. Call n the cohomological dimension of Γ and let M be a $\mathbb{Z}\Gamma$-module such that $\operatorname{Ext}^n_{\mathbb{Z}\Gamma}(\mathbb{Z}, M) \neq 0$. For any short exact sequence of $\mathbb{Z}\Gamma$-modules

$$0 \to M^{'} \to F \to M \to 0$$

where F is free, and any projective resolution $P \to \mathbb{Z} \to 0$, we get a short exact sequance of chains complexes

$$0 \to \operatorname{Hom}_{\mathbb{Z}\Gamma}(P, M') \to \operatorname{Hom}_{\mathbb{Z}\Gamma}(P, F) \to \operatorname{Hom}_{\mathbb{Z}\Gamma}(P, M) \to 0$$

(the map $\operatorname{Hom}_{\mathbb{Z}\Gamma}(P, F) \to \operatorname{Hom}_{\mathbb{Z}\Gamma}(P, M)$ is surjective because P is a projective resolution) and, consequently, a long exact sequence

$$... \to \operatorname{Ext}^n_{\mathbb{Z}\Gamma}(\mathbb{Z}, F) \to \operatorname{Ext}^n_{\mathbb{Z}\Gamma}(\mathbb{Z}, M) \to \operatorname{Ext}^{n+1}_{\mathbb{Z}\Gamma}(\mathbb{Z}, M') = 0$$

In particular $\operatorname{Ext}^n_{\mathbb{Z}\Gamma}(\mathbb{Z}, F) \neq 0$. □

Next we would like to find a relation between cd Γ' and cd Γ when $\Gamma' \subseteq \Gamma$ is a subgroup.

Recall that the tensor product $M \otimes_R N$ is defined whenever M is a right R-module and N left R-module. However, unless one between M and N is a R-bimodule, it

has not in general a structure of R-module. In case $R = \mathbb{Z}\Gamma$ we can avoid having to distinguish between left and right modules by using the anti-automorphism $g \mapsto g^{-1}$ of Γ. Treat $\mathbb{Z}\Gamma$ as a right $\mathbb{Z}\Gamma$ (or $\mathbb{Z}\Gamma'$)-module by right multiplication. Then, for any left $\mathbb{Z}\Gamma$-module M, the tensor product $\mathbb{Z}\Gamma \otimes_{\mathbb{Z}\Gamma'} M$ has a natural structure of left $\mathbb{Z}\Gamma$-module by $g' \cdot (g \otimes m) = g'g \otimes m$ for any $g', g \in \gamma$ and $m \in M$.
Consider now the functor Hom. For any left $\mathbb{Z}\Gamma'$-module M, we have the abelian group $\mathrm{Hom}_{\mathbb{Z}\Gamma'}(\mathbb{Z}\Gamma, M)$, where $\mathbb{Z}\Gamma$ is regarded as a left $\mathbb{Z}\Gamma'$-module by the left multiplication. Since the natural right action of $\mathbb{Z}\Gamma$ on itself commutes with the left action of $\mathbb{Z}\Gamma'$, we can make $\mathrm{Hom}_{\mathbb{Z}\Gamma'}(\mathbb{Z}\Gamma, M)$ to be a left $\mathbb{Z}\Gamma$ (and so a $\mathbb{Z}\Gamma'$)-module by $(g \cdot f)(g') = f(g'g)$ for $g, g' \in \Gamma$ and $f \in \mathrm{Hom}_{\mathbb{Z}\Gamma'}(\mathbb{Z}\Gamma, M)$. There is a natural $\mathbb{Z}\Gamma'$-morphism $\pi : \mathrm{Hom}_{\mathbb{Z}\Gamma'}(\mathbb{Z}\Gamma, M) \to M$ defined as $\pi(f) = f(1)$. Moreover, the following property holds:
• given a $\mathbb{Z}\Gamma$-module N and a $\mathbb{Z}\Gamma'$ map $f : N \to M$, there is a unique $\mathbb{Z}\Gamma$-morphism $\widetilde{f} : N \to \mathrm{Hom}_{\mathbb{Z}\Gamma'}(\mathbb{Z}\Gamma, M)$ such that $\pi \circ \widetilde{f} = f$:

$$\begin{array}{ccc} & & \mathrm{Hom}_{\mathbb{Z}\Gamma'}(\mathbb{Z}\Gamma, M) \\ & \overset{\widetilde{f}}{\nearrow} & \downarrow \pi \\ N & \xrightarrow{\quad f \quad} & M. \end{array}$$

Namely, $\widetilde{f}(n)(g) = (g \cdot \widetilde{f}(n))(1) = \widetilde{f}(gn)(1) = \pi(\widetilde{f}(gn)) = f(gn)$ for all $n \in N$ and $g \in \mathbb{Z}\Gamma$.
Thus, we have

$$\mathrm{Hom}_{\mathbb{Z}\Gamma'}(N, M) \cong \mathrm{Hom}_{\mathbb{Z}\Gamma}(N, \mathrm{Hom}_{\mathbb{Z}\Gamma'}(\mathbb{Z}\Gamma, M)) \text{ as groups.} \tag{1.1}$$

for all $\mathbb{Z}\Gamma'$-module M and $\mathbb{Z}\Gamma$-module N.

Lemma 1.1.10. *For any $\mathbb{Z}\Gamma'$-module M, we have*

$$\mathrm{Ext}^i_{\mathbb{Z}\Gamma'}(\mathbb{Z}, M) \cong \mathrm{Ext}^i_{\mathbb{Z}\Gamma}(\mathbb{Z}, \mathrm{Hom}_{\mathbb{Z}\Gamma'}(\mathbb{Z}\Gamma, M))$$

as groups.

Proof. It is a consequence of Equation 1.1, after having observed that a $\mathbb{Z}\Gamma$ projective resolution of $\mathbb{Z}$ is also a projective resolution over $\mathbb{Z}\Gamma'$. □

Proposition 1.1.11. *Let $\Gamma' \subseteq \Gamma$ be a subgroup. Then* cd $\Gamma' \leq$ cd Γ *and equality holds if* cd $\Gamma < \infty$ *and* $[\Gamma : \Gamma'] < \infty$.

Proof. The first inequality follows from the fact that a projective resolution of $\mathbb{Z}$ over $\mathbb{Z}\Gamma$ is also a projective resolution of of $\mathbb{Z}$ over $\mathbb{Z}\Gamma'$.
Suppose that cd $\Gamma = n < \infty$ and $[\Gamma : \Gamma'] < \infty$. By Proposition 1.1.9, we can find a free $\mathbb{Z}\Gamma$-module F such that $\mathrm{Ext}^n_{\mathbb{Z}\Gamma}(\mathbb{Z}, F) \neq 0$. Clearly, if $\mathcal{B}$ is a basis of F over $\mathbb{Z}\Gamma$ and

F' is a free $\mathbb{Z}\Gamma'$-module with basis $\mathcal{B}$, then $F \cong \mathbb{Z}\Gamma \otimes_{\mathbb{Z}\Gamma'} F'$. Moreover, by Lemma 1.1.10, $\mathrm{Ext}^n_{\mathbb{Z}\Gamma'}(\mathbb{Z}, F') \cong \mathrm{Ext}^n_{\mathbb{Z}\Gamma}(\mathbb{Z}, \mathrm{Hom}_{\mathbb{Z}\Gamma'}(\mathbb{Z}\Gamma, F'))$. Finally, since $[\Gamma : \Gamma'] < \infty$, we have that $\mathbb{Z}\Gamma$ is a free left $\mathbb{Z}\Gamma'$-module of finite rank, with basis any set of representatives of $\Gamma'\backslash\Gamma$. It follows that (as $\mathbb{Z}\Gamma$-modules)

$$\begin{aligned}\underline{\mathrm{Hom}}_{\mathbb{Z}\Gamma'}(\mathbb{Z}\Gamma, \mathbb{Z}\Gamma') \otimes_{\mathbb{Z}\Gamma'} F' &\cong \mathrm{Hom}_{\mathbb{Z}\Gamma'}(\mathbb{Z}\Gamma, F') \\ f \otimes u &\mapsto \mathbb{Z}\Gamma' - \text{morphism defined by } g \mapsto f(g)u\end{aligned}$$

where $\underline{\mathrm{Hom}}_{\mathbb{Z}\Gamma'}(\mathbb{Z}\Gamma, \mathbb{Z}\Gamma')$ is the group of the $\mathbb{Z}\Gamma'$-morphisms $\mathbb{Z}\Gamma \to \mathbb{Z}\Gamma'$ endowed with the structures of left $\mathbb{Z}\Gamma$-module specified above and right $\mathbb{Z}\Gamma'$-module defined by $(f \cdot h)(g') = f(hg') = h(f(g'))$ for all $g' \in \Gamma$, $h \in \Gamma'$ and $f \in \mathrm{Hom}_{\mathbb{Z}\Gamma'}(\mathbb{Z}\Gamma, \mathbb{Z}\Gamma')$.
Observe that $\underline{\mathrm{Hom}}_{\mathbb{Z}\Gamma'}(\mathbb{Z}\Gamma, \mathbb{Z}\Gamma')$ is isomorphic (with respect to both structures) to $\mathbb{Z}\Gamma$. An isomorphism is

$$\begin{aligned}\underline{\mathrm{Hom}}_{\mathbb{Z}\Gamma'}(\mathbb{Z}\Gamma, \mathbb{Z}\Gamma') &\to \mathbb{Z}\Gamma \otimes_{\mathbb{Z}\Gamma'} \mathbb{Z}\Gamma' \cong \mathbb{Z}\Gamma \\ f &\mapsto \sum_{g\Gamma' \in \Gamma/\Gamma'} g \otimes f(g^{-1}).\end{aligned}$$

To find the inverse map, first consider

$$\begin{aligned}\mathbb{Z}\Gamma' &\to \underline{\mathrm{Hom}}_{\mathbb{Z}\Gamma'}(\mathbb{Z}\Gamma, \mathbb{Z}\Gamma') \\ h &\mapsto \mathbb{Z}\Gamma' - \text{morphism } \varphi_h \text{ defined over } \Gamma \text{ by } g \mapsto \begin{cases} gh & \text{if } g \in \Gamma', \\ 0 & \text{if } g \in \Gamma \smallsetminus \Gamma'; \end{cases}\end{aligned}$$

then our inverse is

$$\begin{aligned}\mathbb{Z}\Gamma \otimes_{\mathbb{Z}\Gamma'} \mathbb{Z}\Gamma' &\to \underline{\mathrm{Hom}}_{\mathbb{Z}\Gamma'}(\mathbb{Z}\Gamma, \mathbb{Z}\Gamma') \\ g_1 \otimes h &\mapsto \mathbb{Z}\Gamma' - \text{morphism defined by } g_2 \mapsto \varphi_h(g_2 g_1).\end{aligned}$$

Therefore,

$$\mathrm{Hom}_{\mathbb{Z}\Gamma'}(\mathbb{Z}\Gamma, F') \cong \underline{\mathrm{Hom}}_{\mathbb{Z}\Gamma'}(\mathbb{Z}\Gamma, \mathbb{Z}\Gamma') \otimes_{\mathbb{Z}\Gamma'} F' \cong \mathbb{Z}\Gamma \otimes_{\mathbb{Z}\Gamma'} F' \cong F$$

as $\mathbb{Z}\Gamma$-modules and thus $\mathrm{Ext}^n_{\mathbb{Z}\Gamma'}(\mathbb{Z}, F') \neq 0$.

□

Corollary 1.1.12. *If* $cd\,\Gamma < \infty$ *then* Γ *is torsion-free.*

Proof. It follows from the previous proposition and Example 1.1.8. □

1.2 The topological point of view

We have seen that the existence of a finite dimensional $K(\Gamma, 1)$-complex gives information about cd Γ. The first theorem of this section shows that, in most of cases, cd $\Gamma =$ geom.cd (Γ).

Theorem 1.2.1. *If* $n = \max\{3, \text{cd}\,\Gamma\}$*, then there exists a n-dimensional* $K(\Gamma, 1)$*-complex. In particular,* cd $\Gamma =$ geom.cd(Γ) *whenever* cd $\Gamma \geq 3$.

Before starting the proof note the following result

Lemma 1.2.2. *If P is a projective module over an arbitrary ring R, then there is a free module F such that $P \oplus F \cong F$.*

Proof. Let $F' \to P \to 0$ be an exact sequence of R-modules, where F' is free. Since this sequence splits, there exists an R-module Q such that $F' \cong P \oplus Q$.
Take

$$F = (P \oplus Q) \oplus (P \oplus Q) \oplus .... = P \oplus (Q \oplus P) \oplus ...$$

. □

Proof of Theorem 1.2.1. Let Y^2 be a 2-complex such that $\pi_1(Y) \cong \Gamma$ (for the existence of such a Γ see [20], Corollary 1.28). Call $\widetilde{Y^2}$ its universal cover and observe that $H_1(\widetilde{Y^2}) = 0$.
We will contruct the k-skeleton of n-dimensional $K(\Gamma, 1)$-complex Y by induction on k. For, suppose we have Y^k and that $H_i(\widetilde{Y^k}) = 0$ for $0 < i < k$, where $\widetilde{Y^k}$ denote the universal cover of Y^k, and choose a set of generators $\{h_\alpha\}$ of the $\mathbb{Z}\Gamma$-module $H_k(\widetilde{Y^k})$. By the Hurewicz Theorem, there exist continuous maps $f_\alpha : S^k \to \widetilde{Y^k}$ such that $H_k(f_\alpha)$ sends a generator of $H_k(S^k)$ to h_α. Define

$$Y^{k+1} = Y^k \cup_\alpha e_\alpha^{k+1}$$

where the cell e_{k+1}^α is attached to Y^k by the map $S^k \xrightarrow{f_\alpha} \widetilde{Y^k} \to Y^k$. Call $\{v_\alpha\}$ the following basis of the $\mathbb{Z}\Gamma$-module $H_k(\widetilde{Y^{k+1}}, \widetilde{Y^k})$: v_α is the image under $H_{k+1}(\chi_\alpha)$ of a generator of $H_{k+1}(D^{k+1}, S^k)$, where $\chi_\alpha : (D^{k+1}, S^k) \to (\widetilde{Y^{k+1}}, \widetilde{Y^k})$ is a characteristic map for the cell attached via f_α.
We now check that $H_i(\widetilde{Y^{k+1}}) = 0$ for $0 < i < k+1$. Observe that the k-skeleton of $\widetilde{Y^{k+1}}$ is $\widetilde{Y^k}$. The long exact sequence of the pair $(\widetilde{Y^{k+1}}, \widetilde{Y^k})$:

$$0 \to H_{k+1}(\widetilde{Y^{k+1}}) \to H_{k+1}(\widetilde{Y^{k+1}}, \widetilde{Y^k}) \xrightarrow{\partial} H_k(\widetilde{Y^k}) \to H_k(\widetilde{Y^{k+1}}) \to 0 = H_k(\widetilde{Y^{k+1}}, \widetilde{Y^k}) \to ...$$
$$... \to 0 = H_{i+1}(\widetilde{Y^{k+1}}, \widetilde{Y^k}) \to H_i(\widetilde{Y^k}) \to H_i(\widetilde{Y^{k+1}}) \to 0 = H_{i+1}(\widetilde{Y^{k+1}}, \widetilde{Y^k}) \to ...$$

gives $H_i(\widetilde{Y^{k+1}}) = 0$ for $0 < i < k$ and that, in order to prove $H_k(\widetilde{Y^{k+1}}) = 0$, it is enough to check that ∂ is surjective. But, the diagram

$$\begin{array}{ccc} H_{k+1}(D^{k+1}, S^k) & \xrightarrow{\partial} & H_k(S^k) \\ \downarrow{\scriptstyle H_{k+1}(\chi_\alpha)} & & \downarrow{\scriptstyle H_k(f_\alpha)} \\ H_{k+1}(\widetilde{Y^{k+1}}, \widetilde{Y^k}) & \xrightarrow[\partial]{} & H_k(\widetilde{Y^k}) \end{array}$$

commutes and thus $\partial(v_\alpha) = \pm h_\alpha$. Hence, ∂ is surjective.
Note also that if $H_k(\widetilde{Y^k})$ is free with basis $\{h_\alpha\}$, then ∂ is an isomorphism and so $H_{k+1}(\widetilde{Y^{k+1}}) = 0$.
Finally if $n = \infty$, set $Y = \bigcup_k Y^k$ so that $\pi_1(Y) \cong \pi_1(Y^2) \cong \Gamma$ and $H_k(\widetilde{Y}) = H_k(\widetilde{Y^{k+1}}) = 0$ for all $k > 0$. Then, by Hurewicz Theorem, $\pi_i(\widetilde{Y}) = 0$ for all $i > 0$ and so $\widetilde{Y}$ is contractible, being a CW-complex (by Whitehead's Theorem).
If instead $n < \infty$, consider Y^{n-1}. The augmented cellular chain complex of $\widetilde{Y^{n-1}}$ gives a partial projective resolution of $\mathbb{Z}$ over $\mathbb{Z}\Gamma$ of length $n-1$:

$$H_{n-1}(\widetilde{Y^{n-1}}, \widetilde{Y^{n-2}}) \xrightarrow{d_{n-1}} \ldots \xrightarrow{d_1} H_0(\widetilde{Y^0}, \emptyset) \xrightarrow{\varepsilon} 0$$

and since $\ker(d_{n-1}) = H_{n-1}(\widetilde{Y^{n-1}})$, Lemma 1.1.2 gives that $H_{n-1}(\widetilde{Y^{n-1}})$ is a projective $\mathbb{Z}\Gamma$-module. Let F be any free module such that $H_{n-1}(\widetilde{Y^{n-1}}) \oplus F \cong F$ and replace Y^{n-1} with $Y^{n-1} \vee S^{n-1} \vee S^{n-1} \vee ..$, one copy of S^{n-1} for each basis element of F. Note that, even in the case $n = 3$, by Van Kampen's Theorem, $\pi_1(Y^{n-1})$ is unchanged after this operation. The effect of doing so on the augmented chain complex of $\widetilde{Y^{n-1}}$ is just changing $H_{n-1}(\widetilde{Y^{n-1}}, \widetilde{Y^{n-2}})$ with $H_{n-1}(\widetilde{Y^{n-1}}, \widetilde{Y^{n-2}}) \oplus F$ while $d_{n-1}|F = 0$. In particular $H_{n-1}(\widetilde{Y^{n-1}})$ is now free. It follows that $Y = Y^n = Y^{n-1} \cup \bigcup e^n_\alpha$ constructed as above using as $\{h_\alpha\}$ a basis of $H_{n-1}(\widetilde{Y^{n-1}})$ works. □

Among all the $\mathbb{Z}\Gamma$-modules there is $\mathbb{Z}\Gamma$ itself. Actually, as the rest of the section explains, we can use it to compute the cohomological dimension of Γ as soon as we know that $\mathrm{cd}\ \Gamma < \infty$.

Assume, for the rest of the section, that $\mathrm{cd}\ \Gamma < \infty$ and let Y be a finite $K(\Gamma, 1)$-complex and $\widetilde{Y}$ its universal cover.

Lemma 1.2.3. $\mathrm{Ext}^i_{\mathbb{Z}\Gamma}(\mathbb{Z}, F) = 0$ *for every free* $\mathbb{Z}\Gamma$*-module* F *if and only if* $\mathrm{Ext}^i_{\mathbb{Z}\Gamma}(\mathbb{Z}, \mathbb{Z}\Gamma) = 0$.

Proof. We need to check the implication $(\Leftarrow)$. First, if $F = \mathbb{Z}\Gamma^m$ then $\mathrm{Ext}^i_{\mathbb{Z}\Gamma}(\mathbb{Z}, F) = \mathrm{Ext}^i_{\mathbb{Z}\Gamma}(\mathbb{Z}, \mathbb{Z}\Gamma)^m = 0$. Now the genearal statement follows in this way:

Step 1 any free $\mathbb{Z}\Gamma$-module F is equal to the direct limit of its finitely generated free submodules: $F = \varinjlim F_j$;

Step 2 if $P_k = H_k(\widetilde{Y^k}, \widetilde{Y^{k-1}})$, then P_i is a projective and finitely generated $\mathbb{Z}\Gamma$-module, thus $\mathrm{Hom}_{\mathbb{Z}\Gamma}(P_k, F) \cong P_k^* \otimes_{\mathbb{Z}\Gamma} F \cong \varinjlim P_k^* \otimes_{\mathbb{Z}\Gamma} F_j \cong \varinjlim \mathrm{Hom}_{\mathbb{Z}\Gamma}(P_k, F_j)$;

Step 3 since taking the homology commutes with the direct limit and $P \to \mathbb{Z} \to 0$ is a free $\mathbb{Z}\Gamma$-resolution of $\mathbb{Z}$, we get $\mathrm{Ext}^i_{\mathbb{Z}\Gamma}(\mathbb{Z}, F) = \varinjlim \mathrm{Ext}^i_{\mathbb{Z}\Gamma}(\mathbb{Z}, F_j) = 0$.

□

Corollary 1.2.4. $\mathrm{cd}\, \Gamma = max\, \{n \in \mathbb{N} : \mathrm{Ext}^n_{\mathbb{Z}\Gamma}(\mathbb{Z}, \mathbb{Z}\Gamma) \neq 0\}$.

Lemma 1.2.5. *Let M be a left $\mathbb{Z}\Gamma$-module. Call $\mathrm{Hom}_c(M, \mathbb{Z}) \subseteq \mathrm{Hom}_{\mathbb{Z}}(M, Z)$ the set of all Abelian group homomorphisms $f : M \to \mathbb{Z}$ such that, for every $m \in M$, $f(gm) = 0$ for all but finitely many $g \in \Gamma$. There is a natural isomorphism of groups*

$$\mathrm{Hom}_{\mathbb{Z}\Gamma}(M, \mathbb{Z}\Gamma) \cong \mathrm{Hom}_c(M, \mathbb{Z}).$$

Proof. Any $\mathbb{Z}\Gamma$-morphism $\varphi : M \to \mathbb{Z}\Gamma$ can be written as $\varphi(m) = \sum_{g\in\Gamma} f_g(m)g$ for some homomorphisms $f_g : M \to \mathbb{Z}$ in $\mathrm{Hom}_c(M, \mathbb{Z})$. Since for $m \in M$ and $h \in \Gamma$ $\varphi(hm) = h\varphi(m)$, it follows that $f_{hg}(hm) = f_g(m)$ for all $h, g \in \Gamma$ and $m \in M$. In particular, $f_g(m) = f_1(g^{-1}m)$ for all $g \in \Gamma$ and $m \in M$. The claimed isomorphism is then obtained by associating φ to f_1. □

Lemma 1.2.6. $\mathrm{Ext}^i_{\mathbb{Z}\Gamma}(\mathbb{Z}, \mathbb{Z}\Gamma)$ *is isomorphic as a group to the cohomology group with compact support $H^i_c(\widetilde{Y})$.*

Proof. Applying the previous lemma with $M = H_k(\widetilde{Y}^k, \widetilde{Y}^{k-1})$ we obtain that the two chain complexes

$$... \leftarrow \mathrm{Hom}_{\mathbb{Z}\Gamma}(H_k(\widetilde{Y}^k, \widetilde{Y}^{k-1}), \mathbb{Z}\Gamma) \leftarrow ... \leftarrow \mathrm{Hom}_{\mathbb{Z}\Gamma}(H_0(\widetilde{Y}^0, \emptyset), \mathbb{Z}\Gamma) \leftarrow 0$$

and

$$... \leftarrow \mathrm{Hom}_c(H_k(\widetilde{Y}^k, \widetilde{Y}^{k-1}), \mathbb{Z}) \leftarrow ... \leftarrow \mathrm{Hom}_c(H_0(\widetilde{Y}^0, \emptyset), \mathbb{Z}) \leftarrow 0$$

are isomorphic. The lemma follows then from the fact that the augmented homology chain complex of $\widetilde{Y}$ is a projective resolution of $\mathbb{Z}$ over $\mathbb{Z}\Gamma$ and that the homology groups of the second complex are the cohomology groups with compact support of $\widetilde{Y}$. □

Note that, in particular, $H^0_c(\widetilde{Y}) = 0$ unless $\Gamma = \{1\}$.

Corollary 1.2.7. *There always exist* $n \in \mathbb{N}$ *such that* $H^n_c(\widetilde{Y}) \neq 0$*. Moreover,* cd $\Gamma =$ *max* $\{n \in \mathbb{N} : H^n_c(\widetilde{Y}) \neq 0\}$.

Finally we can prove the result that will allow us to compute the cohomological dimesion of those groups we will be interested in later.

Theorem 1.2.8. *Suppose, in addition, that* Y *is a compact topological manifold of dimension* d *with boundary (possibly* $\partial Y = \emptyset$*). Then:*
(a) if $\partial Y = \emptyset$*, then* cd $\Gamma = d$*;*
(b) if $\partial Y \neq \emptyset$*, then cd* $\Gamma = d - m - 1$ *where* m *is the minimum integer such that* $\widetilde{H}_m(\partial\widetilde{Y}) \neq 0$ *(there always exists an* m *such that* $\widetilde{H}_m(\partial\widetilde{Y}) \neq 0$*).*

Proof. First assume that Y is orientable. Note that $\partial Y = \emptyset$ if and only if $\partial\widetilde{Y} = \emptyset$. If $\partial Y = \emptyset$, by Poincaré Duality, $H^n_c(\widetilde{Y}) \cong H_{d-n}(\widetilde{Y})$ is different from 0 if and only if $n = d$ ($\widetilde{Y}$ is contractible). Thus the previous corollary yields cd $\Gamma = d$. Assume $\partial Y \neq \emptyset$. The Poincaré-Lefschetz duality and the long exact sequence of the pair $(\widetilde{Y}, \partial\widetilde{Y})$ give $H^n_c(\widetilde{Y}) \cong H_{d-n}(\widetilde{Y}, \partial\widetilde{Y}) \cong \widetilde{H}_{d-n-1}(\partial\widetilde{Y})$. Now apply the previous corollary. This completes the proof for Y orientable.
Now suppose that Y is not orientable. The orientation covering Y' of Y is a compact and orientable manifold (with boundary) of the same dimension of Y, it is a $K(\pi_1(Y'), 1)$-complex, $\pi_1(Y')$ is a subgroup of index 2 in Γ and its universal covering space is $\widetilde{Y}$ the same of Y. Since by hypothesis cd$(\Gamma) < \infty$, by Proposition 1.1.11, we have cd$(\Gamma) =$ cd$(\pi_1(Y')$ and we can apply the previous case. □

1.3 Serre's Theorem

Many groups are not torsion-free and in this case, according to Corollary 1.1.12, their cohomological dimension is not interesting at all. Nevertheless, we can associate to them another quantity, their virtual cohomological dimesion, thanks to the following theorem:

Theorem 1.3.1 (Serre's Theorem)**.** *If* Γ *is torsion-free and* $\Gamma' \subseteq \Gamma$ *is a finite index subgroup, then* cd $\Gamma =$ cd Γ'.

In particular, cd $\Gamma' =$ cd Γ'' for any two torsion free and finite index subgroups of any group Γ.

Definition 1.3.2. Whenever Γ admits a torsion-free and finite index subgroup Γ', we define the **virtual cohomological dimension** of Γ to be vcd $\Gamma =$ cd Γ'.

Proof of Serre's Theorem. Thanks to Proposition 1.1.11, we need only to show that if cd $\Gamma' < \infty$ then cd $\Gamma < \infty$. Let Y be a finite dimensional $K(\Gamma', 1)$-complex and $\widetilde{Y}$

its universal cover. Denote by $\mathrm{Hom}_{\Gamma'}(-,-)$ the maps in the category of left Γ'-sets. Observe that Γ' acts on $\widetilde{Y}$ by deck transformations and on Γ by left multiplication. Call $X = \mathrm{Hom}_{\Gamma'}(\Gamma, \widetilde{Y})$. Since the right action of Γ on itself commutes with the left action of Γ', there is an induced left action of Γ on X, namely $(h \cdot f)(g) = f(gh)$ for $g, h \in \Gamma$ and $f \in X$. Let $g_1, ..., g_n$ be a set of representatives for $\Gamma' \backslash \Gamma$ and consider the bijection

$$\varphi : X \to \prod_{i=1}^{n} \widetilde{Y}$$

defined by $\varphi(f) = (f(g_1), ..., f(g_n))$.
The space $\prod_{i=1}^{n} \widetilde{Y}$ has a natural structure of CW-complex whose cells are products of the cells of the factors $\widetilde{Y}$ (note that the correspondent topology does not coincide with the product topology in general) and we will endow X with the CW-complex structure induced by φ. This structure is independent of the choice of $g_1, ..., g_n$ because if instead we consider $g'_1 g_1, ..., g'_n g_n$ where $g'_i \in \Gamma'$, the new φ is the composition of the old with the CW-isomorphism $\prod_{i=1}^{n} g'_i : \prod_{i=1}^{n} \widetilde{Y} \to \prod_{i=1}^{n} \widetilde{Y}$. Clearly, neither reordering $g_1, ..., g_n$ affects the induced CW-complex structure.
Note that X has finite dimension.
For any $g \in \Gamma$ we have the commutative triangle

$$\begin{array}{ccc} X & \xrightarrow{\;g\cdot\;} & X \\ & {\scriptstyle\tau}\searrow \quad \swarrow{\scriptstyle\varphi} & \\ & \prod_{i=1}^{n} \widetilde{Y} & \end{array}$$

where $\tau(f) = (f(g_1 g), ..., f(g_n g))$. Since $g_1 g, ..., g_n g$ is still a set of representatives of $\Gamma' \backslash \Gamma$, it follows that the left action of Γ on X preserves this CW-complex structure. Moreover, the action on the cells of X is free. To see this, first observe that the Γ'-morphism $X \to \widetilde{Y}$ given by $f \mapsto f(1)$ takes cells to cells and thus, since Γ' acts freely on the cells of $\widetilde{Y}$, Γ' acts freely on X. In other words, for any cell σ of X, $\{1\} = \mathrm{stab}_{\Gamma'}(\sigma) = \mathrm{stab}_{\Gamma}(\sigma) \cap \Gamma'$. Therefore, since Γ' has finite index in Γ, for every cell σ of X, $\mathrm{stab}_{\Gamma}(\sigma)$ must be finite and, since Γ is torsion free, must be trivial.
Finally X is clearly contractible. It follows that the quotient X/Γ is a finite dimensional $K(\Gamma, 1)$-complex. □

The virtual cohomological dimension of $\mathrm{SL}(2, \mathbb{Z})$

We conclude this section with an example of computation that is interesting for our later purposes.
Before, we recall the so called ping-pong Lemma:

Lemma 1.3.3 (Ping-pong Lemma). *Let Γ be a group acting on a set X and $a, b \in \Gamma$. Suppose there exist non-empty subsets $A, B \subseteq X$ such that $A \cap B = \emptyset$ and $a^n(B) \subseteq A$ and $b^n(A) \subseteq B$ for every $n \in \mathbb{N}_+$. Then the subgroup Γ' generated by a and b is free.*

Proof. We need to prove that any non trivial freely reduced word in a and b represents a non-trivial element of Γ.
If w is one of such words that starts and ends with a non-trivial power of a, then $w(B) \subseteq A$ thus $w \neq 1$. The result follows then by observing that any non-trivial freely reduced word in a and b is conjugated in Γ' to one that starts and ends with a non-trivial power of a. □

Example 1.3.4. Consider now $\Gamma = \mathrm{SL}(2, \mathbb{Z})$.
The ping-pong Lemma applied with

$$a = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}, \ b = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}, \ X = \mathbb{Z}^2,$$

$$A = \{(n, m) \ : \ |n| > |m|\} \text{ and } B = \{(n, m) \ : \ |n| < |m|\}$$

yields that $\Gamma' = \langle a, b \rangle$ is free. If we prove that $[\Gamma : \Gamma'] < \infty$, then $\mathrm{vcd}\ \mathrm{SL}_2(\mathbb{Z}) = 1$.
To see that $[\Gamma : \Gamma'] < \infty$ consider the natural reduction map

$$\mathrm{SL}(2, \mathbb{Z}) \to \mathrm{SL}(2, \mathbb{Z}/2\mathbb{Z}).$$

Let $\Gamma(2)$ be its kernel. We will prove that $\Gamma(2)$ is generated by $-I, a$ and b; so that $[\Gamma : \Gamma'] = [\Gamma : \Gamma(2)][\Gamma(2) : \Gamma'] < \infty$.
The inclusion $\langle -I, a, b \rangle \subseteq \Gamma(2)$ is obvious. To prove the reverse inclusion, first observe that

if $m, n \in \mathbb{Z}$ and $m \neq 0$, we can write $n = qm + r$ where $r, q \in \mathbb{Z}$ and $|r| \leq |m|/2$.
Now pick

$$x = \begin{bmatrix} x_{1,1} & x_{1,2} \\ x_{2,1} & x_{2,2} \end{bmatrix} \in \Gamma(2),$$

so $x_{1,2}$ and $x_{2,1}$ are even and $x_{1,1}$ and $x_{2,2}$ are odd. If $x_{2,1} = 0$, then

$$x = \pm \begin{bmatrix} 1 & \pm 2k \\ 0 & 1 \end{bmatrix} = \pm a^{\pm k}$$

for some $k \in \mathbb{N}$. If $x_{2,1} \neq 0$, we distinguish two cases: either $|x_{1,1}| > |x_{2,1}|$ or $|x_{1,1}| < |x_{2,1}|$ ($|x_{1,1}| \neq |x_{2,1}|$ because $x_{1,1}$ is odd and $x_{2,1}$ is even). In the first case, write $x_{1,1} = (2x_{2,1})q + r$ with $q, r \in \mathbb{Z}$ and $|r| \leq |x_{2,1}|$, and consider

$$a^{-q}x = \begin{bmatrix} 1 & -2q \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_{1,1} & x_{1,2} \\ x_{2,1} & x_{2,2} \end{bmatrix} = \begin{bmatrix} r & x_{1,2} - 2qx_{2,2} \\ x_{2,1} & x_{2,2} \end{bmatrix}.$$

In the second case, write $x_{2,1} = (2x_{1,1})q + r$ with $q, r \in \mathbb{Z}$ and $|r| \leq |x_{1,1}|$, and consider

$$b^{-q}x = \begin{bmatrix} 1 & 0 \\ -2q & 1 \end{bmatrix} \begin{bmatrix} x_{1,1} & x_{1,2} \\ x_{2,1} & x_{2,2} \end{bmatrix} = \begin{bmatrix} x_{1,1} & x_{1,2} \\ r & x_{2,2} - 2qx_{1,2} \end{bmatrix}.$$

It follows by induction on $\max\{|x_{1,1}|, |x_{2,1}|\}$ that $x \in \langle -I, a, b \rangle$.

Chapter 2

Preliminary results about surfaces

This second chapter is dedicated to a brief summary of the main results about surfaces that we will use throughtout the rest of the thesis.

2.1 The classification theorem

A surface is a 2-dimensional topological manifold, possibly with boundary. We assume manifolds to be Hausdorff and to have a countable basis.

Our tratment will constantly rely on the following theorem:

Theorem 2.1.1. *(a) Any closed, connected, orientable surface is homeomorphic to the connected sum of a sphere with* $g \geq 0$ *tori;*
(b) any compact, connected, orientable surface is homeomorphic to the complement of $b \geq 0$ *open disks with disjoint closure inside the connected sum of a sphere with* $g \geq 0$ *tori.*

The g in the theorem is called the **genus** of the surface and the b is the **number of component boundaries**. The compact surface of genus g and b boundary components will be denoted by S_g^b and in the case $b = 0$ simply by S_g. Note that the Euler characteristic of S_g^b is $\chi(S_g^b) = 2 - 2g - b$.

The following two theorems will allow us to use differential techniques to study surfaces.

Theorem 2.1.2. *Every surface has a smooth structure.*

Proof. See [22]. □

Theorem 2.1.3. *Homeomorphic surfaces are diffeomorphic.*

Proof. See [22]. □

Indeed, a much stronger result is true. See the section 2.4 of this chapter.

In the rest of the thesis, for a **surface** will always mean a compact, connected and orientable 2-dimensional smooth manifold S_g^b. Sometimes we will also consider **surfaces with corners** that are compact, connected and orientable 2-dimensional smooth manifold with corners.

2.2 Hyperbolic metrics

Definition 2.2.1. A hyperbolic metric on a surface S is a complete Riemannian metric curvature -1 for which the boundary of S is totally geodesic (this means that the geodesics of ∂S are geodesics of S).

Remark 2.2.2. One can prove that S admits a hyperbolic metric if and only if $\chi(S) < 0$. Suppose S is endowed with a hyperbolic metric and call $\widetilde{S}$ the universal cover of S. If $\partial S = \emptyset$, then $\widetilde{S}$ is a simply connected riemannian 2-manifold with constant curvature -1, thus it is isometric to the hyperbolic plane $\mathbb{H}^2$. If instead $\partial S \neq \emptyset$, then $\widetilde{S}$ is isometric to a totally geodesic convex subspace of $\mathbb{H}^2$.

The orientation-preserving isometries of $\mathbb{H}^2$

The orientation-preserving isometries of $\mathbb{H}^2$ are classified: making the identification $\mathbb{H}^2 = \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$, we have $\mathrm{Isom}^+(\mathbb{H}^2) \cong \mathrm{PSL}(2, \mathbb{R})$ where

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot z = \frac{az + b}{cz + d}$$

It is useful to compactify $\mathbb{H}^2$. For this, consider the inclusions $\mathbb{H}^2 \subseteq \mathbb{C} \subseteq P^1(\mathbb{C})$ where $\mathbb{C}$ is identified with $\{[z, 1] \in P^1(\mathbb{C}) : z \in \mathbb{C}\}$, and take the closure $\overline{\mathbb{H}^2}$ of $\mathbb{H}^2$ in $P^1(\mathbb{C})$. Note that $\overline{\mathbb{H}^2}$ is homeomorphic to a disk and that the action of $\mathrm{PSL}(2, \mathbb{R})$ preserves $\partial\overline{\mathbb{H}^2} = \overline{\mathbb{H}^2} \smallsetminus \mathbb{H}^2$. It follows that every $A \in \mathrm{PSL}(2, \mathbb{R})$ has a fixed point in $\overline{\mathbb{H}^2}$. Let $A \in \mathrm{PSL}(2, \mathbb{R})$. Then exactly one of the following holds:

1. A has three fixed point in $\overline{\mathbb{H}^2}$. In this case A must be the identity isometry;

2. A has exactly two fixed points in $\overline{\mathbb{H}^2}$ and they lie in $\partial\overline{\mathbb{H}^2}$. In this case, A is said to be a **hyperbolic isometry**. Moreover, A is conjugated in $\mathrm{Isom}^+(\mathbb{H}^2)$ to an isometry of the form $z \mapsto kz$ for some positive real number $k \neq 1$ and thus $|\mathrm{tr}(A)| > 2$;

3. A has exactly one fixed point in $\overline{\mathbb{H}^2}$ and it lies in $\partial\overline{\mathbb{H}^2}$. In this case A is said to be a **parabolic isometry**. Moreover, A is conjugated in $\mathrm{Isom}^+(\mathbb{H}^2)$ to an isometry of the form $z \mapsto z + b$ for some $b \in \mathbb{R} \smallsetminus \{0\}$ and thus $|\mathrm{tr}(A)| = 2$;

4. A has exactly one fixed point in $\overline{\mathbb{H}^2}$ and it lies in $\mathbb{H}^2$. In this case, A is said to be an **elliptic isometry**. Moreover A is conjugated in $\mathrm{Isom}^+(\mathbb{H}^2)$ to an isometry of the form $z \mapsto \frac{\cos(\theta)z-\sin(\theta)}{\sin(\theta)z+\cos(\theta)}$ with $\theta \in (0, 2\pi) \smallsetminus \{\pi\}$ and thus $|\mathrm{tr}(A)| < 2$.

Note that A cannot have a fixed point in $\mathbb{H}^2$ and one in $\partial\mathbb{H}^2$ unless $A = 1_{\mathbb{H}^2}$. Note also that being the identity, hyperbolic, parabolic or elliptic is invariant under conjugation.

2.2.1 Two useful coordinate systems on $\mathbb{H}^2$

Fermi coordinates

Let γ be an oriented geodesic line in $\mathbb{H}^2$ (identified with the Poincarè half-space). In particular, using the standard orientation on $\mathbb{H}^2$, it makes sense to talk about the left and the right side of γ. Suppose $\mathbb{R} \ni t \mapsto \gamma(t)$ is a parametrization by arc length of γ and for all $x = \gamma(t)$ $(t \in \mathbb{R})$, let $X_x \in T_x\mathbb{H}^2$ be the unique unitary vector such that $(\gamma'(t), X_x)$ is a positive basis of $T_x\mathbb{H}^2$.

Definition 2.2.3. We will call

$$\begin{aligned}\varphi : \mathbb{R} \times \mathbb{R} &\to \mathbb{H}^2 \\ (\rho, t) &\mapsto \exp_{\gamma(t)}(\rho X_{\gamma(t)})\end{aligned}$$

the **Fermi coordinates** on $\mathbb{H}^2$.

Lemma 2.2.4. *The metric tensor in the Fermi coordinates (ρ, t) is $ds^2 = d\rho^2 + \cosh(\rho)^2 dt^2$.*

Proof. Clearly, up to an orientation-preserving isometry of $\mathbb{H}^2$, me may assume that $\gamma(t) = ie^t$. Also we will use the notation (u_1, u_2) instead of (ρ, t). Since the curves $u_1 = \rho$ and $u_2 = \text{const}$ are unit-speed geodesics, we have $g_{11} = 1$ and

$$0 = \frac{\partial^2 u_k}{\partial \rho^2} + \sum_{i,j} \Gamma^k_{i,j} \frac{\partial u_i}{\partial \rho}\frac{\partial u_j}{\partial \rho} \qquad u_1 = \rho,\ u_2 = \text{const}$$

for $k = 1, 2$. Therefore $\Gamma^1_{11} = \Gamma^2_{11} = 0$. But

$$0 = \Gamma^1_{11} = \frac{1}{2}\sum_l g^{1l}\{\frac{\partial g_{1l}}{\partial u_1} + \frac{\partial g_{l1}}{\partial u_1} - \frac{\partial g_{11}}{\partial u_l}\} = g^{12}\frac{\partial g_{12}}{\partial u_1}$$

for $k = 1, 2$ and $g^{12} = -\frac{g_{12}}{\det(g_{ij})}$, thus $g_{12}\frac{\partial g_{12}}{\partial u_1} = 0$. Since $g_{12}(0, u_2) = 0$, we obtain $g_{12} = 0$. This proves that $ds^2 = du_1^2 + g_{22}du_2^2$, where (necessarily) $g_{22} > 0$. Finally we have

$$-1 = K = \frac{R_{1212}}{g_{22}} = \frac{\sum_l R^l_{121}g_{l2}}{g_{22}} = \frac{g_{22}R^2_{121}}{g_{22}} = R^2_{121} =$$
$$= \sum_l \Gamma^l_{11}\Gamma^2_{2l} - \sum_l \Gamma^l_{21}\Gamma^2_{1l} + \frac{\partial \Gamma^2_{11}}{\partial u_2} - \frac{\partial \Gamma^2_{21}}{\partial u_1} = -\Gamma^2_{21}\Gamma^2_{12} - \frac{\partial \Gamma^2_{21}}{\partial u_1}$$

and since $\Gamma^2_{12} = \Gamma^2_{21} = \frac{1}{2}g^{22}\frac{\partial g_{22}}{\partial u_1}$ we have obtained

$$-1 = -\frac{1}{4g_{22}^2}(\frac{\partial g_{22}}{\partial u_1})^2 - \frac{1}{2}\frac{\partial}{\partial u_1}(\frac{\frac{\partial g_{22}}{\partial u_1}}{g_{22}}) = -\frac{1}{2}\frac{(\frac{\partial^2 g_{22}}{\partial u_1^2})^2}{g_{22}} + \frac{1}{4}\frac{(\frac{\partial g_{22}}{\partial u_1})^2}{g_{22}^2} = -\frac{\frac{\partial^2 \sqrt{g_{22}}}{\partial u_1^2}}{\sqrt{g_{22}}}$$

and imposing $g_{22}(0, 1) = 1$ we have $g_{22}(u_1, u_2) = \cosh(u_1)^2$. □

Polar coordinates

Another coordinate system are the polar coordinates. Fix a unitary vector $v \in T_x\mathbb{H}^2$.

Definition 2.2.5. The polar coordinates centered at a point x on $\mathbb{H}^2$ are

$$(0, \infty) \times (0, 2\pi) \to \mathbb{H}^2$$
$$(\rho, \theta) \mapsto \exp_x(\rho R_\theta(v))$$

where $R_\theta : T_x\mathbb{H}^2 \to T_x\mathbb{H}^2$ is the rotation of angle θ.

Reasoning as above, we obtain the following lemma.

Lemma 2.2.6. *The metric tensor in the polar coordinates (ρ, θ) becomes $ds^2 = d\rho^2 + \sinh(\rho)^2 d\theta^2$.*

2.3 Circles and arcs on surfaces

Let S be a surface.

2.3.1 Some terminology

Definition 2.3.1. For a **circle** C in S we will mean a smooth 1-dimensional submanifold of $S \setminus \partial S$ diffeomorphic to S^1.

Definition 2.3.2. Two circles C_0 and C_1 (or a circle and a boundary component of S or two boundary components of S) are said to be **homotopic** if there is a smooth homotopy $F : S^1 \times [0,1] \to S$ such that $F|_{S^1\times\{t\}}$ is a diffeomorphism onto C_t for $t = 0, 1$; while are said **isotopic** if they are homotopic through a smooth homotopy F such that $F(-, t)$ is an embedding in S for each fixed $t \in [0, 1]$.

Remark 2.3.3. Clearly, being isotopic is an equivalence relation.

Remark 2.3.4. Let C_0 and C_1 be two circles, or a circle and a boundary component or two boundary components of S and suppose that there is a continuous homotopy $F : S^1 \times [0,1] \to S$ such that $F|_{S^1\times\{t\}}$ is a diffeomorphism onto C_t for $t = 0, 1$. Then C_0 and C_1 are homotopic, i.e. there is a smooth homotopy $\widetilde{F} : S^1 \times [0,1] \to S$ such that $\widetilde{F}|_{S^1\times\{t\}}$ is a diffeomorphism onto C_t for $t = 0, 1$. To see this, first replace F with a new continuous homotopy $G : S^1 \times [0,1] \to S$ defined by $G(x,t) = F(x, \rho(t))$, where $\rho : [0,1] \to [0,1]$ is smooth bump function that is 0 on a neighborhood of 0 and 1 on a neighborhood of 1. The advantage is that G is smooth on a neighborhood of $S^1 \times \partial[0,1] = S^1 \times \{0,1\}$ and thus we can find a smooth function $\widetilde{F} : S^1 \times [0,1] \to S$ that agrees with G on a neighborhood of $S^1 \times \{0,1\}$ (see the proof of Theorem 6.29 in Chapter 6 of [29]. Note that in the proof of this theorem the author assumes M without boundary only because in this way M can be embedded in some $\mathbb{R}^N$ and it is possible to find an open set U of $\mathbb{R}^N$ containing M and a smooth retraction $r : U \to M$. In our case $M = S$ can have non-empty boundary, but the existence of such an open set U in $\mathbb{R}^3$ and of a smooth retraction r is obvious). Clearly $\widetilde{F}$ has the required properties.

Definition 2.3.5. Let C be a circle in S. The circle C is said **trivial** if it is homotopic to a point, is said **peripheral** if it is homotopic to a boundary component of S and finally is said **essential** if it is neither trivial nor peripheral.

Definition 2.3.6. A circle C in S is said **non-separating** if $S \setminus C$ is connected, otherwise C is said **separating**.

Definition 2.3.7. Let C be a circle in S. The **result of cutting** S **along** C is the compact, orientable 2-dimensional smooth manifold S_C having as many connected components as $S \setminus C$ and for which exists an orientation-preserving diffeomorphisms $f : \partial_1 \to \partial_2$ between two of its boundary components such that the quotient $S_C/x \sim f(x)$ is diffeomorphic to S via an orientation-preserving diffeomorphism that sends the image of ∂_1 and ∂_2 under the quotient map to the circle C. Finally, if Δ is a collection of disjoint circles on S, we will denote by S_Δ the result of cutting S along each of the circles in S.

Definition 2.3.8. Two circles will be said to be of the **same type** if there exists an orientation-preserving diffeomorphism of S taking one to the other.

Definition 2.3.9. By an **arc** of S we will mean a submanifold diffeomorphic to $[0,1]$. An arc γ will be said **proper** if it is a neat submanifold of S, that is a submanifold with the following further properties: $\partial\gamma = \gamma \cap \partial S$ and $\gamma \pitchfork \partial S$.

Definition 2.3.10. Two proper arcs γ_0 and γ_1 of S are said to be **isotopic** if there exists a smooth homotopy $F : [0,1]\times[0,1] \to S$ such that $F(-,t)$ is a C^∞-embedding for each $t \in [0,1]$, $F(0,t)$, $F(1,t) \in \partial S$ for all $t \in [0,1]$ and $F([0,1] \times \{0\}) = \gamma_0$, $F([0,1] \times \{1\}) = \gamma_1$.

Remark 2.3.11. Clearly being isotopic is an equivalence relation.

Definition 2.3.12. Let γ be a proper arc in S. The **result of cutting** S **along** γ is the surface S_γ for which exists an orientation-preserving diffeomorphisms $f : \gamma_1 \to \gamma_2$ between two arcs of S_γ contained in ∂S_γ such that the quotient $S_\gamma/x \sim f(x)$ is a surface diffeomorphic to S via an orientation-preserving diffeomorphism that sends the image of γ_1 and γ_2 under the quotient map to the arc γ. Finally, if Δ is a collection of disjoint proper arcs of S, we will denote by S_Δ the result of cutting S along each of the arcs in S.

Definition 2.3.13. Let $\Delta = \{C_1, ..., C_k, \gamma_1, ..., \gamma_m\}$ be a collection of circles and proper arcs of S such that every arc of Δ is disjoint from every other element of Δ and the circles of Δ are pairwise transverse. We define the **result of cutting** S **along the circles and arcs of** Δ inductively on k. If $k = 0$ we have already defined S_Δ. Suppose $k > 0$. Let $S' = S_{C_1}$ and identify S with a quotient of S' as explained in Definition 2.3.7. Let $\Delta' = \{C'_1, ..., C'_{k'}, \gamma'_1, ..., \gamma'_{m'}\}$ be a collection of circles and proper arcs of the connected components of S' (that are surfaces) satisfying the same conditions as Δ and such that $\pi^{-1}(\cup_{i=1}^{k} C_i \cup \cup_{j=1}^{m} \gamma_j) = \cup_{i=1}^{k'} C'_i \cup \cup_{j=1}^{m'} \gamma'_j$ where $\pi : S' \to S$ is the quotient map. Note that $k' = k - 1$. Define $S_\Delta = (S')_{\Delta'}$.

2.3.2 Intersection of circles

Given two circles C_0 and C_1 in S there are at least two ways of counting the number of intersection points between them.

Definition 2.3.14. If C_0 and C_1 are oriented, choose an orientation preserving diffeomorphism $f : S^1 \to C_1$ and a smooth map $\widetilde{f}$ homotopic to f and transverse to C_0, then the **algebraic intersection number** of C_0 and C_1 is

$$\hat{\imath}(C_0, C_1) = \sum_{\theta \in \widetilde{f}^{-1}(C_0)} \mathrm{ind}_\theta \widetilde{f},$$

where $\mathrm{ind}_\theta \widetilde{f} \in \{\pm 1\}$ is $+1$ if, for positive vectors $w_{\widetilde{F}(\theta)} \in T_{\widetilde{F}(\theta)} C_0$ and $v_\theta \in T_\theta S^1$, the pair $(w_{\widetilde{F}(\theta)}, d_\theta \widetilde{F}(v_\theta))$ is a positive basis of $T_{\widetilde{F}(\theta)} S$, and it is -1 otherwise.

Remark 2.3.15. In the case $S = S_g$ with $g \geq 1$, there is also another, sometimes more convenient, description of the algebraic intersection number of the oriented circles C_0 and C_1. Consider the **standard symplectic basis** $\alpha_1, \beta_1, ..., \alpha_g, \beta_g$ of $H_1(S_g, \mathbb{Z})$ presented below:

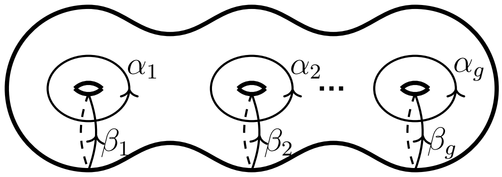


Figure 2.1 The standard symplectic basis of $H_1(S_g, \mathbb{Z})$.

and call $\omega = \sum_{i=1}^{g} [\alpha_i]^* \wedge [\beta_i]^* \in \bigwedge^2 (H^1(S_g, \mathbb{Z}))$. Here $[\alpha_1]^*, [\beta_1]^*, ..., [\alpha_g]^*, [\beta_g]^*$ is the dual basis of $\alpha_1, \beta_1, ..., \alpha_g, \beta_g$ in $H^1(S_g, \mathbb{Z}) \cong \mathrm{Hom}_{\mathbb{Z}}(H_1(S_g, \mathbb{Z}), \mathbb{Z})$. Assume that S_g is oriented in such a way that α_1 and β_1 intersect positively. Then

$$\hat{i}(C_0, C_1) = \omega([C_0], [C_1])$$

where $[C_i] \in H_1(S_g, \mathbb{Z})$ is the homology class of C_i for $i = 0, 1$ (remember that the C_i are oriented).

Another way of counting the number of intersection points between two circles consists of considering their geometric intersection number.

Definition 2.3.16. Let C_0 and C_1 be two circles in S. The **geometric intersection number** between C_0 and C_1 is

$$i(C_0, C_1) = \min\{|\widetilde{C_0} \cap \widetilde{C_1}| : \widetilde{C}_i \text{ is a circle of } S \text{ belonging to the homotopy class of } C_i \text{ for } i = 0, 1\}.$$

We will say that C_0 and C_1 are in **minimal position** if they are transverse and $i(C_0, C_1) = |C_0 \cap C_1|$.

There is a pratical way to check that two transverse circles C_0 and C_1 are in minimal position, it is the bigon criterion.
First a definition.

Definition 2.3.17. Two circles C_0 and C_1 form a **bigon** if there is an embedded disk in S (the bigon) whose boundary is the union of two connected pieces of C_0 and C_1 intersecting in exactly two points:

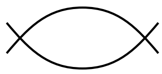


Figure 2.2: A bigon.

Proposition 2.3.18. *Two transverse circles C_0 and C_1 are in minimal position if and only if they do not form a bigon.*

Proof. See Proposition 1.7 in chapter 1 of [12]. □

2.3.3 Circles and hyperbolic geometry

Now suppose that S admits a hyperbolic metric and thus endow S with a hyperbolic metric.

Geodesic circles

Definition 2.3.19. For a **geodesic circle** of S we will mean either a circle of S that is also the image of some geodesic of S or a boundary component of S, that, by hypothesis, is the image of a geodesic of S.

The next results show the utility of this notion.

Proposition 2.3.20. *Any non-trivial circle C is homotopic to a unique geodesic circle of S. Moreover this geodesic circle is characterized by the property of being the shortest circle in the homotopy class of C.*

Proof. See Proposition 1.3 in chapter 1 of [12]. □

Corollary 2.3.21. *A geodesic circle in S is never homotopic to a point of S.*

Proposition 2.3.22. *Distinct geodesic circles of S are in minimal position.*

Proof. Note that two distinct geodesic circles are always transversal. Suppose they form a bigon. Then, since the bigon is simply connected, we can lift it to a bigon in the hyperbolic plane bounded by two geodesics. But, this contradicts the fact that the geodesic between any two points of $\mathbb{H}^2$ unique. □

Corollary 2.3.23. *Let C_0 and V_0 be two homotopically non-trivial, disjoint circles in a hyperbolic surface S which are not homotopic to each other. Then the unique geodesic circles C_1 and V_1 in the isotopy class of C_0 and V_0 respectively are disjoint.*

Remark 2.3.24. Let $\alpha : S^1 \to C \subseteq S_g$ $(g \geq 2)$ be a parametrization of a non-trivial circle C in S_g and fix a universal covering map $p : \mathbb{H}^2 \to S_g$ in such a way that p is a local isometry. The deck transformation of $\mathbb{H}^2$ associated to α (that is defined modulo conjugation in $\mathrm{Isom}^+(\mathbb{H}^2)$) is a hyperbolic isometry. Indeed, it cannot be the identity otherwise C would be trivial, it cannot be elliptic because the group of deck tranformations acts in a free way on $\mathbb{H}^2$ and it cannot be parabolic otherwise in S_g there would be closed geodesics (that fail to be smooth in at most one point) of arbitrary small length and if this length is small enough we could lift these geodesics to closed geodesics in $\mathbb{H}^2$ having a contradiction (see also Lemma 2.3.27).
Now extend α by periodicity to a map $\mathbb{R} \to S_g$, that we still call α, and let $\widetilde{\alpha} : \mathbb{R} \to \mathbb{H}^2$ be a lift of α. Then the limits $\lim_{t\to\pm\infty} \widetilde{\alpha}(t)$ exist and belong to $\partial\overline{\mathbb{H}^2}$. To see this use the fact that any hyperbolic isometry of $\mathbb{H}^2$ is conjugated to an isometry of the form $z \mapsto kz$ where $k > 0$. We will call these two limits the **end points** of $\widetilde{\alpha}$.

Injectivity radius

We now restrict our attention to the case $S = S_g$ with $g \geq 2$. The surface S is still endowed with a fixed hyperbolic metric.

Definition 2.3.25. For each x in S_g define the **injectivity radius of S_g at** x, denoted by $r_x(S_g)$, to be the supremum of all r such that $\{y \in S_g : d(x, y) < r\}$ is isometric to a hyperbolic disk. Define also the **injectivity radius of** S_g, denoted by $r(S_g)$, to be the infimum of all $r_x(S_g)$ where $x \in S_g$.

Remark 2.3.26. The function $r : S_g \to [0, \infty)$ defined by $x \mapsto r_x(S_g)$ is continuous. Indeed if $d(x, y) < r_p(S_g)$, then $r_p(S_g) - d(p, q) \leq r_q(S_g)$.

Lemma 2.3.27. *There exists a shortest geodesic circle C in S_g. Moreover the injectivity radius of S_g is equal to $r(S_g) = \frac{l(C)}{2}$.*

Proof. First fix $x \in S_g$ and let $r = r_x(S_g)$. Then the lifts of x in $\mathbb{H}^2$ have pairwise distance of at least $2r$ and there are two such lifts, say x_1 and x_2, having distance exactly $2r$. The geodesic arc between x_1 and x_2 is sent under the covering map $\mathbb{H}^2 \to S_g$ to a geodesic circle C_x (that may fail to be smooth at x) of length $2r$. Note that C_x does not autoinsersect. Clearly C_x is the shortest simple loop in S_g based at x. Now consider the function r of the previous remark. Since S_g is compact, r has a minimum at some point $x_0 \in S_g$. It follows that C_{x_0} is the shortest loop in S_g, thus the unique geodesic circle C in its homotopy class has length equal $l(C) = l(C_{x_0})$. This proves the lemma. □

2.3.4 Isotopies and homotopies of circles

In this subsection we investigate the relation between homotopies and isotopies of circles and prove the so called change of variable principle, that is the analogous for understanding change of basis in a vector space.

Proposition 2.3.28. *Let C_0 and C_1 be two homotopic circles (or a circle and a boundary component or two boundary components) in a surface S and suppose that they are homotopically non-trivial in S and that their intersection is empty. Then they bound an annulus in S.*

Proof. First assume $S = S_g$ with $g \geq 2$. Then S can be endowed with an hyperblic metric and its universal cover $\widetilde{S}$ is isometric to $\mathbb{H}^2$. Let F be an homotopy between C_0 and C_1. Extend F by periodicity to a map $\mathbb{R} \times [0,1] \to S_g$, that we still call F, and let $\widetilde{F} : \mathbb{R} \times [0,1] \to \mathbb{H}^2$ be a lift of F. Then $\mathrm{Im}(\widetilde{F_0})$ and $\mathrm{Im}(\widetilde{F_1})$ do not intersect and $\widetilde{F_0}$ and $\widetilde{F_1}$ have the same endpoints. The deck transformation ϕ of $\mathbb{H}^2$ determined by $\widetilde{F_0}|_{[0,1]}$ is a hyperbolic isometry of $\mathbb{H}^2$ that acts as a translation on both $\mathrm{Im}(\widetilde{F_i})$ for $i = 0, 1$. Let $\widetilde{R}$ be the connected region delimited by $\mathrm{Im}(\widetilde{F_0})$ and $\mathrm{Im}(\widetilde{F_1})$ in $\mathbb{H}^2$. Then, ϕ preserves $\widetilde{R}$. Indeed, ϕ fix (as set) the boundary of $\widetilde{R}$ and the complementary of $\widetilde{R}$ in $\mathbb{H}^2$ has two connected component each with boundary one boundary component of $\widetilde{R}$. Call $R = \widetilde{R}/\langle\phi\rangle$ and $p : R \to p(R) \subseteq S_g$ the map induced by the universal cover map. Note that R is an annulus and that p is a covering map that can have only one sheet, thus p must be an homeomorphism. In particular C_0 and C_1 bound an annulus in S_g. Finally we deal with the general case. The case in which $\partial S \neq \emptyset$ is easily reduced to the case $S = S_g$ with $g \geq 2$ by glueing a surface of genus 2 and one hole at each boundary component of S. Then, by the case $S = S_g$ ($g \geq 2$), cutting the resulting surface S_g along C_0 and C_1 we obtain at least one component that is an annulus and this annulus is necessarily contained in S. What remains are the cases $S = S^2$ and $S = T^2$. If $S = S^2$, any circle in S is trivial. If $S = T^2$ is the torus, then cutting S along C_0 we obtain an annulus A (use the classification theorem of surfaces) and cutting agian along C_1 we obtain two annuli (use the classification theorem of surfaces). This completes the proof. □

Lemma 2.3.29. *If a circle or a boundary component is homotopic to a point then it bounds a disk in S.*

Proof. This is Theorem 1.7 of [11]. □

Corollary 2.3.30. *Two circles (or a circle and a boundary component or two boundary components) C_0 and C_1 are homotopic if and only if they are isotopic.*

Remark 2.3.31. Note that for us circles are not maps, but submanifolds. This is crucial here. For example, the clockwise parametrization of a circle C around 0 in the closed disk D^2 is not isotopic (as a map) to the counterclockwise parametrization of the same circle, while they are both homotopic to a constant map and thus are homotopic. To see that they are not isotopic observe that, by the Theorem 2.3.34, if they were isotopic, we would have a diffeomorphism of the annulus bounded by ∂D^2 and C whose differential is the identity in the points of ∂D^2 and has negative deteminant in those of C, contradicting the description of the connected components of $\mathrm{GL}(2,\mathbb{R})$.

Proof of the Corollary. If C_0 and C_1 are homotopic to a point the Corollary follows from Lemma 2.3.29 and the Disk Theorem (see Theorem C.1.3 in the Appendix C). Suppose that C_0 and C_1 are not homotopic to a point. By an isotopy of C_0 we can make C_0 to be transverse to C_1. Then, C_0 and C_1 form a bigon or are disjoint. If they form a bigon, we can take this bigon to be innermost with respect to C_0 and C_1. This means that it is an embedded disk D in S bounded by one arc of C_0 and one arc of C_1 and $\mathring{D} \cap (C_0 \cup C_1) = \emptyset$. Such a bigon prescribes an isotopy of C_0 that reduces the intersection. Thus we can remove the bigons one by one by isotopies of C_0 untill $C_0 \cap C_1 = \emptyset$. But then C_0 and C_1 bound an annulus of S and so are isotopic. □

Now we explain a technique, called **change of coordinates principle**, that is used quite frequently and that will allow us to reduce the proof of statements about general situations to the proof of the same statement for a few easier specific situations.

Definition 2.3.32. Let M be smooth manifold and $N \subseteq M$ a smooth submanifold. A smooth map $F : N \times [0,1] \to M$ is said a **smooth isotopy** if F_t is a smooth embedding for all $t \in [0,1]$. The **support** of the isotopy F is the closure of the set of $x \in N$ such that $F_t(x) = x$ for all $t \in [0,1]$. Similarly, for a **smooth diffeotopy** we mean is a smooth map $M \times [0,1] \to M$ such that F_t is a diffeomorphism of M for all $t \in [0,1]$.

Remark 2.3.33. Note that if M is compact and connected (for example if $M = S$ is a surface) any smooth embedding $M \to M$ is a diffeomorphism. Indeed, being an open and closed map it is necessarily surjective. Thus any smooth isotopy $M \times [0,1] \to M$ is actually a smooth diffeotopy.

Theorem 2.3.34. *Let M be a manifold and $N \subseteq M$ a submanifold. If $\partial N \neq \emptyset$ we require N to be a neat submanifold. Let $F : N \times [0,1] \to M$ be a smooth isotopy of N such that $F(N \times [0,1]) \subseteq M \setminus \partial M$ or $F(N \times [0,1]) \subseteq \partial M$. Then F extends to a smooth diffeotopy of M having compact support. Moreover, in the case $F(N \times [0,1]) \subseteq M \setminus \partial M$, the diffeotopy can be chosen with support in $M \setminus \partial M$.*

Proof. See Theorem 1.3 in chapter 8 of [23]. □

Corollary 2.3.35. *If C_0 and C_1 are two isotopic circles of S, then there exists a smooth diffeotopy of S taking C_0 to C_1 and fixing pointwise ∂S.*

Lemma 2.3.36 (Change of coordinates principle)**.** *Two circles C_1 and C_2 are of the same type if and only if S_{C_0} and S_{C_1} are diffeomorphic.*

Proof. The implication ($\Rightarrow$) is clear. We prove ($\Leftarrow$). Since there exist an orientation-reversing diffeomorphisms of S_{C_0}, composing with such a diffeomorphism, we see that there is an orientation-preserving diffeomorphism between S_{C_0} and S_{C_1}. Our argument is based upon the following two observations:

Obs 1 Every two orientation-preserving diffeomorphisms of S^1 are isotopic. Indeed, given $f \in \mathrm{Diff}^+(S^1)$ we may first replace f with another diffeomorphism, that we still call f, that is isotopic to f and that has a fixed point $x \in S^1$. Then f acts as the identity on $\pi_1(S^1, x)$ and thus the lift $\widetilde{f} : \mathbb{R} \to \mathbb{R}$ of $f \circ p$ (where $p : \mathbb{R} \to S^1$ is the usual covering map $p(\theta) = e^{2\pi i\theta}$) that sends 0 in 0 has the property that $\widetilde{f}(n + \theta) = \widetilde{f}(\theta) + n$ for all $\theta \in \mathbb{R}$ and $n \in \mathbb{Z}$. It follows that the straight-line homotopy F between $\widetilde{f}$ and $1_{\mathbb{R}}$ induces an homotopy between f and 1_{S^1}. Finally the derivative of F_t with respect to θ is always positive, so our homotopy happens to be an isotopy.

Obs 2 If S is any surface with at least two boundary components ∂_1 and ∂_2, then there exists an orientation-preserving diffeomorphism of S taking ∂_1 to ∂_2 and fixing pointwise the other boundary components of S. Indeed, let γ be a proper arc in S connecting ∂_1 and ∂_2. We can choose a closed neighborhood N of $\partial_1 \cup \partial_2 \cup \gamma$ diffeomorphic to the closed disk with two holes corresponding to ∂_1 and ∂_2. In particular, there exists a diffeomorphism of N taking ∂_1 to ∂_2 and fixing the other boundary component. Extend this diffeomorphism to a diffeomorphism of S to be the identity out of N.

Using these two observations it is easy to conclude. Identify S_{C_i} with $S \setminus \mathring{N}_i$ where N_i is a closed neighborhood of C_i in S such that there is an orientation-preserving diffeomorphism $(N_i, C_i) \cong (S^1 \times [0,1], S^1 \times \{1/2\})$ $(i = 0, 1)$. By hypothesis, there is an orientation-preserving diffeomorphism $S \setminus \mathring{N}_0 \to S \setminus \mathring{N}_1$ and, by the second observation, we may assume that it takes ∂N_0 to ∂N_1. Using the first observation, we finally extend this diffeomorphism to an orientation-preserving diffeomorphism $S \to S$ taking $C_0 \cong S^1 \times \{1/2\} \subseteq S^1 \times [0,1]$ to $C_1 \cong S^1 \times \{1/2\} \subseteq S^1 \times [0,1]$. □

Corollary 2.3.37. *Two nonseparating circles in S are of the same type.*

Proof. If C is a non-separating circle of $S = S_g^b$, then $S_C = S_{g'}^{b+2}$ for some g', that can be determined by looking at $\chi(S_C)$. Write $S = (S \setminus C) \cup N$ where $N \cong S^1 \times [0,1]$ is a closed neighborhood of C in S. Then $\chi(S) = \chi(S \setminus C) + \chi(N) - \chi(N \setminus C) = \chi(S_C)$. In particular, $g' = g - 1$ and the corollary follows from the Lemma 2.3.36. □

2.3.5 Alexander Method

In order to study a diffeomorphism of a surface S one can try to find some circles in S that remain fixed, possibly modulo isotopy, under the action of it, then cut S along such circles obtaining a collection of simpler surfaces and study the behaviour of the induced diffeomorphism on this collection.

One important result in this direction is the following proposition.

Proposition 2.3.38 (Alexander method). *Let S be a surface and let $\mathcal{F}_i$ $i = 1, 2$ be two families of k non-trivial circles of S such that for each $i = 1, 2$:*

1. *every two distinct circles in $\mathcal{F}_i$ are in minimal position;*
2. *every two distinct circles in $\mathcal{F}_i$ are not isotopic;*
3. *for distinct C_{i_1}, C_{i_2} and C_{i_3} circles in $\mathcal{F}_i$ at least one of $C_{i_1} \cap C_{i_2}$, $C_{i_1} \cap C_{i_3}$ and $C_{i_2} \cap C_{i_3}$ is empty.*

Suppose, in addiction, that every circle C_i of $\mathcal{F}_1$ is isotopic to a (unique) circle C'_i of $\mathcal{F}_2$. Then there exists a smooth diffeotopy of S relative to ∂S taking simultaneously each C_i to C'_i .

Proof. We proceed by induction on k. If $k = 1$ then this is Corollary 2.3.35. Suppose $k > 1$. By the inductive hypothesis, there is a smooth diffeotopy of S relative to ∂S taking C_i to C'_i for all $i < k$. Thus, we may assume from the beginning that $C_i = C'_i$ for all $i < k$ and prove that there is a smooth diffeotopy of S relative to ∂S taking C_k to C'_k and fixing setwise each C_i for $i < k$. Call $\Delta = \cup_{i<k} C_i$. We can think to Δ as a graph with vertices at the intersection points of two circles C_i and C_j with $i < j < k$. First of all, we perform a smooth diffeotopy rel$(\partial S \cup \Delta)$ that makes C_k to be transverse to C'_k. This can be done as follows. Observe that from the hypothesis 3, C_k is disjoint from the vertices of Δ and that, by the hypothesis 1, both C_k and C'_k intersect the edges of Δ in a finite number of points, thus we can make C_k transverse to C'_k along these edges through a relative smooth diffeotopy of S that fixes $\partial S \cup \Delta$. Finally, using Theorem 2.3.34, we perform a smooth diffeotopy of S relative to $\partial S \cup \Delta$ that perturbs C_k to intersect C'_k transversely in $S \setminus \Delta$.
Next we perform a relative smooth diffeotopy of S that fixes setwise Δ and pointwise ∂S and takes C_k to be disjoint from C'_k as follows. If at this point $C_k \cap C'_k \neq \emptyset$, then they form a bigon. We can take this bigon to be innermost with respect to C_k and C'_k. By the hypothesis 3 and non-triviality of the circles in question, the intersection of Δ with this bigon is a collection of disjoint arcs that, by the hypotesis 1, connect one boundary arc of the bigon with the other. Thus there is a smooth diffeotopy of S relative to ∂S fixing setwise each C_i for $i < k$ that pushes C_k across

the bigon. Repeating this process we makes C_k to have empty intersection with C'_k. From Proposition 2.3.4, C_k and C'_k bound an annulus in S. As before and using 2, the intersection of Δ with this annulus is a collection of disjoint arcs connecting C_k to C'_k and thus there is a smooth diffeotopy of S relative to ∂S that takes C_k to C'_k and fixed setwise each C_i for $i < k$. □

Two immediate corollaries that are worth to be pointed out are the following.

Corollary 2.3.39. *Let $\mathcal{F}_i$ $i = 1, 2$ be two families of k non-trivial circles in a surface S, each such that every two distinct circles of $\mathcal{F}_i$ have empty intersection and are not isotopic ($i = 1, 2$). Suppose that every circle C_i of $\mathcal{F}_1$ is isotopic to a (unique) circle C'_i of $\mathcal{F}_2$. Then there exists a smooth diffeotopy of S relative to ∂S taking simultaneously each C_i to C'_i .*

Corollary 2.3.40. *Let $\mathcal{F} = \{C_1, C_2\}$ and $\mathcal{F}' = \{C'_1, C'_2\}$ be two families each consisting of two non-trivial and non-isotopic circles of S that are in minimal position and such that C_i and C'_i are isotopic for $i = 1, 2$. Then, there exists a smooth diffeotopy of S relative to ∂S taking each C_i to C'_i for $i = 1, 2$.*

2.4 Isotopies and homotopies of diffeomorphisms and homeomorphisms

In this subsection we collect some useful facts about diffeomorphisms and homeomorphisms of surfaces that will allow us to replace one kind of map with a better one.

Definition 2.4.1. An **isotopy** of diffeomorphisms (respectively of homeomorphisms) between two diffeomorphisms (respectively homeomorphisms) $f, g : S \to S$ is a continuous homotopy $F : S \times [0, 1] \to S$ between f and g such that for all $t \in [0, 1]$ $F(-, t)$ is a diffeomorphism (respectively a homeomorphism) of S.

Remark 2.4.2. Clearly, being isotopic is an equivalence relation.

Any homeomorphism of a surface S is isotopic to a diffeomorphism of S

Theorem 2.4.3. *Any homeomorphism of a surface S is isotopic to a diffeomorphism of S. Moreover if a homeomorphism restricts to the identity on ∂S, then the isotopy can be chosen to be relative to ∂S.*

Proof. See [22]. For the second part of the statement, note that if a homeomorphism of a surface S restricts to the identity in ∂S, then it is isotopic to a homeomorphism that is the identity on a neighborhood of ∂S (in particular it is smooth in a neighborhood of ∂S). Thus we can apply what is proved in [22]. To see this, glue to each boundary component of ∂S an annulus, obtaining a new surface S' homeomorphic to S. Consider the continuous map $F : S \times [0,1] \to S'$ obtained by stretching S to become S'. See the figure below.

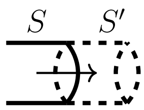


Figure 2.3: Definition of F.

In particular, F_0 is the inclusion $S \hookrightarrow S'$, F_t is a topological embedding for all $t \in [0,1]$, F_1 is a homeomorphism $S \to S'$. Then, given a homeomorphism $\phi : S \to S$ that is the identity on ∂S, we can extend ϕ to a homeomorphism $\bar{\phi} : S' \to S'$ that is the identity on $S' \setminus S$ and observe that $F_1^{-1} \circ \bar{\phi} \circ F_1$ is a homeomorphism of S isotopic to ϕ (in S) and is the identity on a neighborhood of ∂S. Here we are using the fact that the map $S \times [0,1] \to S$ defined by $(x,t) \mapsto F_t^{-1}(\hat{\phi}(F_t(x))$ is continuous. This follows from the fact that the group the homeomorphisms of S endowed with the compact-open topology is a topological group. $\square$

Continuous and smooth homotopies and isotopies

There is also a further peculiarity in dimension 2. That is, continuous and smooth homotopies and isotopies are not truly different.

Here is the main result.

Theorem 2.4.4. *Let $f : S \to S$ be a diffeomorphism. Then:*
(a) if f is continuously homotopic to 1_S through an homotopy that, if $\partial S \neq \emptyset$, preserves setwise each boundary component of S, then f is smoothly isotopic to 1_S;
(b) if $\partial S \neq \emptyset$, $f|_{\partial S} = 1_{\partial S}$ and f is continuously homotopic to 1_S via an homotopy relative to ∂S, then f is smoothly isotopic to 1_S via an isotopy relative to ∂S.

Remark 2.4.5. If $\partial S \neq \emptyset$, and $f \in \mathrm{Diff}(S)$ preserves setwise each component of ∂S, using Observation 1 in Lemma 2.3.36 and Theorem 2.3.34, we see that f is smoothly isotopic to a new diffeomorphism that restricts to the identity on ∂S. Thus in the case $\partial S \neq \emptyset$, it is enough to prove the statement (b) of the theorem.

We now give the proof of this theorem, starting with some special cases.

The main ingredient is the following result due to the mathematician Stephen Smale.

Theorem 2.4.6 (Smale's Theorem)**.** *The space* $\mathrm{Diff}(D^2, \partial D^2)$ *of the diffeomorphisms of the disk fixing pointwise* ∂D^2*, endowed with the* C^∞*-topology, is contractible.*

Corollary 2.4.7. *Theorem 2.4.4 holds for* $S = D^2$*.*

Another useful result is the following.

Lemma 2.4.8. *Let* $\mathrm{Homeo}([0,1], \{0,1\})$ *and* $\mathrm{Diff}([0,1], \{0,1\})$ *be respectively the space of homeomorphisms and diffeomorphisms of* $[0,1]$ *fixing the points* 0 *and* 1*. Endow* $\mathrm{Homeo}([0,1], \{0,1\})$ *with the compact-open topology and* $\mathrm{Diff}([0,1], \{0,1\})$ *with one between the compact-open topology or the* C^∞*-topology. Then all these spaces are contractible.*

Proof. All of them deformation retract to the identity map with deformation map $F(f,t)(x) = tf(x) + (1-t)x$. □

Lemma 2.4.9. *Theorem 2.4.4 holds for* $S = A = S^1 \times [0,1]$*.*

Proof. Let $f \in \mathrm{Diff}(A, \partial A)$. Let γ be the straight-line arc from $(1,0)$ to $(1,1)$ in A. Then $f(\gamma)$ is a proper arc of A with $\partial f(\gamma) = \{(1,0),(1,1)\} = \partial\gamma$. We claim that there is a smooth isotopy of A relative to ∂A taking $f(\gamma)$ to γ. To see this, let $\bar{A}$ be the surface diffeomorphic to the torus obtained by glueing the two boundary components of A in such a way that $(1,0)$ and $(1,1)$ correspond. Then γ and ∂A become non-trivial circles in $\bar{A}$. Call these circles C_1 and C_2 respectively. Note that f defines a diffeomorphism $\bar{f}$ of $\bar{A}$ that restricts to the identity on C_2. The circles $\bar{f}(C_1)$ is homotopic to C_1 and $\bar{f}|_{C_2} = 1_{C_2}$. An argument similar to the one in Proposition 2.3.38, show that $\bar{f}$ is smoothly isotopic to a diffeomorphism that fix setwise each C_i for $i = 1,2$ via an isotopy that does not move the points in C_2. Since $\mathrm{Diff}([0,1], \{0,1\})$ is contractible and $\bar{f}$ fixes a point of C_1, we can also assume that $\bar{f}|_{C_1} = 1_{C_1}$. Equivalently, we may assume that $f|_\gamma = 1_\gamma$. Cutting A along γ we obtain a square Q. The proof of Theorem 2.4.3, shows that the map induced by f on this square is smoothly isotopic to one that is the identity in a neigborhood of ∂Q via an isotopy rel ∂Q. Equivalently we may have assumed from the beginning that f were the identity on a neighborhood of $\gamma \cup \partial A$. To conclude apply Smale's Theorem to the map induced by f on Q [1]. □

Lemma 2.4.10. *Theorem 2.4.4 holds for* $S = S^2$*.*

We will give two proofs of this lemma.

[1] A priori, Q has corners, thus it is not diffeomorphic to a disk D^2. However, since f is the identity on a neighborhood U of ∂Q, we can apply Smale's Theorem to a smaller disk $D' \subseteq Q$ containing $Q \smallsetminus \bar{U}$.

Proof 1. We will prove that any orientation-preserving diffeomorphism of S^2 is smoothly isotopic to 1_{S^2}. First we note that every orientation-preserving diffeomorphisms of S^2 is smoothly isotopic to another diffeomorphism that is the identity on a disk of S^2. Indeed, let $f \in \mathrm{Diff}^+(S^2)$ be a diffeomorphism and $D \subseteq S^2$ a disk. Then, by Theorem C.1.3, the inclusion map $i : D \hookrightarrow S^2$ and $f|_D$ are smoothly isotopic. Using Theorem 2.3.34, it follows that we can find a smooth isotopy F between $F_0 = 1_{S^2}$ and F_1 such that $F_1 \circ f$ is the identity on a disk $D' \subseteq D$.
Thus we may assume from the beginning that f is the identity on a disk D of S^2. To conclude the proof, we can apply Smale's Theorem to the map obtained by restricting f to $S^2 \setminus \mathring{D}$. □

Proof 2. Consider an orientation-preserving diffeomorphism $f : S^2 \to S^2$. Let C be the equator of S^2. Orient C and call U and V respectively the two open sets of $S^2 \setminus C$ on the left and on the right of C. Since $f(C)$ and C are homotopic we may assume that $\phi(C) = C$ and, possibly composing f with a rotation, that $f(U) = U$ and $f(V) = V$. In particular, now $f|_C$ is an orientation preserving diffeomorphism of C. By the observation 1 in Lemma 2.3.36 and the Theorem 2.3.34, we can actually assume that $f|_C = 1_C$. Finally, applying Smale's Theorem to $U \cup C$ and $V \cup C$, we obtain that f is smoothly isotopic to 1_{S^2}. □

Lemma 2.4.11. *Theorem 2.4.4 holds for $S = S^3_0$.*

Proof. Let f be any diffeomorphism of S^3_0 that restricts to the identity on each component of ∂S^3_0. We will prove that f is smoothly isotopic to $1_{S^3_0}$ through an isotopy rel ∂S^3_0. Let ∂_1, ∂_2 and ∂_3 be the boundary components of S^3_0. For $1 \leq i < j \leq 3$, let γ_{ij} be a proper arc of S^3_0 connecting ∂_i and ∂_j. We can choose these arcs to be disjoint. As in the proof of Lemma 2.4.9, considering the double of S^3_0 and using an argument similar to the one in Proposition 2.3.38, we see that, up to a smooth isotopy, we can assume that f fixes setwise each of the γ_{ij}. Hence, since $\mathrm{Diff}([0,1], \{0,1\})$ is contractible, we can actually assume that $f|_{\gamma_{ij}} = 1_{\gamma_{ij}}$. Cutting S^3_0 along every γ_{ij} we obtain two disks, D_1 and D_2, each of which is preserved by f, being f the identity on ∂S^3_0. As in the proof of Theorem 2.4.3, we see that, for $i = 1, 2$, the restriction of f to D_i is smoothly isotopic to a diffeomorphism that is the identity on a neighborhood of ∂D_i through an isotopy relative to ∂D_i. Thus we can apply Smale's Theorem to conclude the proof. □

To finally prove the theorem for all the remaining cases we will use a particular subdivision of the surface in disks and annuli, as explained in the next lemma.

Lemma 2.4.12. *Let S be a surface, $S \neq S^2, D^2, A, S^3_0$. Then there exists a collection $\mathcal{F}$ of circles of S satisfying the following conditions:*

1. *$\mathcal{F}$ satisfy the hypotheses of Proposition 2.3.38;*

2. *the result of cutting S along all the circles in $\mathcal{F}$ is a collection $\mathcal{C}$ of disks and annuli. In addiction, $\mathcal{C}$ contains exactly n distinct annuli $A_1, ..., A_n$, one for each boundary component ∂_i of S and $\partial_i \subseteq \partial A_i$ for all $i = 1, ..., n$;*

3. *if $C_i \neq C_j$ are two circles in $\mathcal{F}$, then $|C_i \cup C_j| \leq 1$;*

4. *for every $C \in \mathcal{F}$, there exists $V \neq C$ in $\mathcal{F}$ such that $C \cap V \neq \emptyset$.*

Proof. Using the following circles

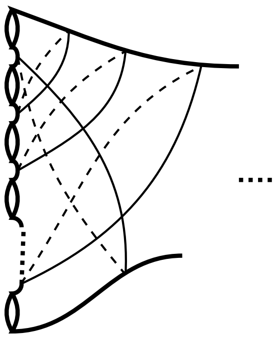


Figure 2.4: Reduction of the problem to the case $b \leq 1$ and $g \geq 1$.

we reduce ourself to prove the case $b \leq 1$ and $g \geq 1$. If $b = 0$ we can choose $\mathcal{F}$ as below

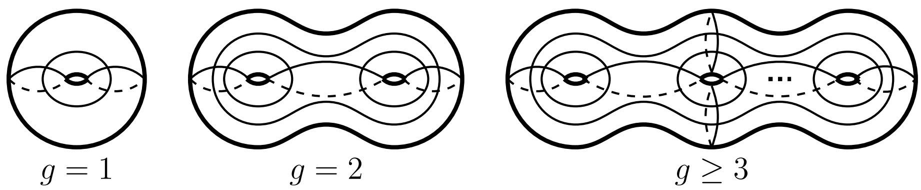


Figure 2.5: A possible choice of $\mathcal{F}$ for $b = 0$.

Finally when $b = 1$ we can choose $\mathcal{F}$ to be as below

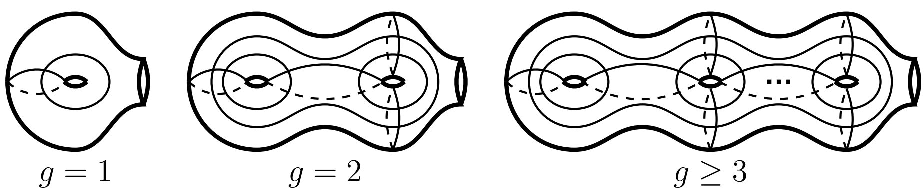


Figure 2.6: A possible choice of $\mathcal{F}$ for $b = 1$.

The proof is complete. $\square$

Proof of Theorem 2.4.4 in the remaining cases. Let $S \neq D^2, S^2, A, S^3_0$ be a surface. Let $f \in \mathrm{Diff}(S)$ be as in (a) if $\partial S = \emptyset$ or as in (b) if $\partial S \neq \emptyset$. Suppose $\partial S = \partial_1 \sqcup ... \sqcup \partial_n$ where $n \geq 0$. Let $\mathcal{F}$ be a family of circles of S satifying the conditions of the previous lemma. By Proposition 2.3.38, we may assume that $f(C) = C$ for all $C \in \mathcal{F}$. In particular, f fixes every point of S of the form $C_i \cap C_j$ where $C_i \neq C_j \in \mathcal{F}$. Since for all $C \in \mathcal{F}$, the set $C \smallsetminus \cup_{V \in \mathcal{F}: V \neq C} V$ is a collection of intervals whose boundary points are fixed by f, it follows that f preserves each of these intervals and, since $\mathrm{Diff}([0,1],\{0,1\})$ is contractible, we can actually assume that f is the identity on each $C \in \mathcal{F}$. Now cut S along all the circles in $\mathcal{F}$ and observe that f must preserves each component of the cutting surface. As in the proof of Theorem 2.4.3, the restriction of f on each component Q is smoothly isotopic to a diffeomorphism that is the identity on a neighborhood of Q via an isotopy relative to ∂Q, equivalently we may have assumed from the beginning that f is the identity on a neighborhood of each $C \in \mathcal{F}$. Applying Smale's Theorem to the disks in $\mathcal{C}$, we see that f is smoothly isotopic to a map that is supported in a collection of annuli corresponding to the boundary components of S. In particular if $\partial S = \emptyset$ we are done. The proof of statemet (b) will be finished after we have introduced Dehn twists. See Corollary 3.2.9. $\square$

Reformulation of the previous results

We will now reformulate the content of Theorem 2.4.4 and Theorem 2.4.3 in terms of spaces of functions.

Define

$\mathrm{Homeo}(S) = \{f : S \to S : f \text{ is a homeomorphism}\}$;
$\mathrm{Diff}(S) = \{f : S \to S : f \text{ is a diffeomorphism}\}$;
$\mathrm{Homeo}(S, \partial S) = \{f : S \to S : f \text{ is a homeomorphism and } f|_{\partial S} = 1_{\partial S}\}$;
$\mathrm{Diff}(S, \partial S) = \{f : S \to S : f \text{ is a diffeomorphisms and } f|_{\partial S} = 1_{\partial S}\}$.

Endow the spaces of homeomorphisms with the compact-open topology and the spaces of diffeomorphisms indifferently with the compact-open topology or the C^∞-topology.

Proposition 2.4.13. *The inclusions* $\mathrm{Diff}(S) \hookrightarrow \mathrm{Homeo}(S)$ *and* $\mathrm{Diff}(S, \partial S) \hookrightarrow \mathrm{Homeo}(S, \partial S)$ *induce bijections*

$$\pi_0(\mathrm{Diff}(S)) \to \pi_0(\mathrm{Homeo}(S))$$

and

$$\pi_0(\mathrm{Diff}(S, \partial S)) \to \pi_0(\mathrm{Homeo}(S, \partial S)).$$

If we fix an orientation on S, we can also consider the subspaces (with the induced topology)

$$\begin{aligned}
&\text{Homeo}^+(S) = \{f : S \to S : f \text{ is an orientation preserving homeomorphism}\};\\
&\text{Diff}^+(S) = \{f : S \to S : f \text{ is an orientation preserving diffeomorphism}\};\\
&\text{Homeo}^+(S, \partial S) = \text{Homeo}^+(S) \cap \text{Homeo}(S, \partial S);\\
&\text{Diff}^+(S, \partial S) = \text{Diff}^+(S) \cap \text{Diff}(S, \partial S);
\end{aligned}$$

By an orientation-preserving homeomorphism f we mean that the orientation class $[S, \partial S] \in H_2(S, \partial S)$ is preserved under $H_2(f)$. Since isotopic homeomorphisms are in particular homotopic through an homotopy that preserves setwise ∂S, they are necessarily both orientation-preserving or both orientation-reversing. Thus we have also the following proposition.

Proposition 2.4.14. *The inclusions* $\text{Diff}^+(S) \hookrightarrow \text{Homeo}^+(S)$ *and* $\text{Diff}^+(S, \partial S) \hookrightarrow \text{Homeo}^+(S, \partial S)$ *induce bijections*

$$\pi_0(\text{Diff}^+(S)) \to \pi_0(\text{Homeo}^+(S))$$

and

$$\pi_0(\text{Diff}^+(S, \partial S)) \to \pi_0(\text{Homeo}^+(S, \partial S)).$$

2.5 Pants decomposition of a surface

As we have seen it can be useful to cut a surface along some circles and arcs in such a way to obtain a collection of disks on which to work. This section introduces another useful decomposition of surfaces.

Definition 2.5.1. A **pair of pants** is a surface diffeomorphic to sphere with three holes.

There are at least two equivalent definitions of what it is a pants decomposition of S.
Let $\chi(S) < 0$.

Definition 2.5.2. A **pants decomposition** of S is a collection Δ of disjoint circles of S with the property that S_Δ is disjoint union of pair of pants. Equivalently, a pants decomposition of S is a maximal collection Δ of disjoint essential circles that are pairwise not isotopic.

The equivalence between the two definitions is easy. First, suppose Δ is as in the first definition. Then the circles in Δ must be essential. They must be non-trivial otherwise there would be a circle that cuts S in two components one of which is a disk not containing any other circle of Δ, but then this means that Δ does not satisfy the first definition. They cannot be isotopic to a boundary component of S otherwise S_Δ would have a component that is an annulus or a disk. It is also clear that the circles in Δ must be pairwise not isotopic, otherwise S_Δ would have a component that is an annulus or a disk. Moreover, since every circle in a pair of pants is either homotopic to a boundary component or to a point, we cannot add to Δ an essential circle of S disjoint from and not isotopic to the circles in Δ. Vice versa, suppose Δ is as in the second definition and, by contradiction, that there is a component of S_Δ that is not a pair of pants. Note that S_Δ cannot have components that are diffeomorphic to a sphere with $b = 0, 1$ or 2 boundary components (here we use that $\chi(S) < 0$), thus S_Δ has a component that either has positive genus or is a sphere with more than three boundary components. But then such component contains a essential circle and this would contradict the maximality of Δ.

Definition 2.5.3. We will say that two pants decompositions Δ_1 and Δ_2 are of the **same type** if there exists an orientation-preserving diffeomorphism ϕ of S such that $\phi(\Delta_1) = \Delta_2$.

Remark 2.5.4. An application of Lemma 2.3.36 gives that there is only a finite number of types of pants decomposition of S.

Lemma 2.5.5. *Let Δ be a pants decomposition of $S = S_g^b$. Then $|\Delta| = 3g + b - 3$.*

Proof. Consider the decomposition in components $S_\Delta = P_1 \sqcup ... \sqcup P_k$. Then

$$2 - 2g - b = \chi(S) = \chi(S_\Delta) = \chi(P_1) + ... + \chi(P_k) = -k$$

(note that $\chi(P_i) = 1$ for all $i = 1, ..., k$) thus

$$|\Delta| = \frac{3k - b}{2} = \frac{6g - 6 + 2b}{2} = 3g - 3 + b.$$

□

Suppose S is endowed with a fixed hyperbolic metric.

Proposition 2.5.6. *Let Δ be a pants decomposition of S. Substituting each circle C in Δ with the unique geodesic circle in the isotopy class of C we obtain another pants decomposition of S.*

Proof. By Corollary 2.3.23, the new geodesic circles are pairwise disjoint. Now check the conditions in the second definition. Note that maximality follows from Lemma 2.5.5. □

Chapter 3

The Mapping Class Group

This third chapter introduces the main protagonist of our story, namely the Mapping Class Group of a surface, $\mathrm{Mod}(S)$. We will study some first examples and properties of this group that, as it will immediately be clear, encode many information about the surface. The main reference for this chapter is [12].

3.1 Definition and first examples

Definition 3.1.1. The **Mapping Class Group** of a surface S, denoted by $\mathrm{Mod}(S)$, can be defined in many equivalent ways. Here are some equivalent definitions:

1. $\mathrm{Mod}(S)$ is the group of isotopy (or homotopy, where homotopies are required to fix setwise the boundary) classes of orientation-preserving diffeomorphisms of S;

2. $\mathrm{Mod}(S) = \mathrm{Diff}^+(S)/\mathrm{Diff}_0^+(S)$, where $\mathrm{Diff}_0^+(S)$ denotes the normal subgroup of $\mathrm{Diff}^+(S)$ consisting of those diffeomorphism isotopic (or homotopic through an homotopy fixing setwise the boundary) to the identity;

3. $\mathrm{Mod}(S) = \pi_0(\mathrm{Diff}^+(S))$, where $\mathrm{Diff}^+(S)$ is endowed with the compact-open topology;

4. $\mathrm{Mod}(S) = \pi_0(\mathrm{Diff}^+(S))$, where $\mathrm{Diff}^+(S)$ is endowed with the C^∞-topology;

5. $\mathrm{Mod}(S)$ is the group of isotopy (or homotopy, where homotopies are required to fix setwise the boundary) classes of orientation-preserving homeomorphisms of S;

6. $\mathrm{Mod}(S) = \mathrm{Homeo}^+(S)/\mathrm{Homeo}_0^+(S)$, where $\mathrm{Homeo}_0^+(S)$ denotes the normal subgroup of $\mathrm{Homeo}^+(S)$ consisting of those homeomorphisms isotopic (or homotopic through an homotopy fixing setwise the boundary) to the identity;

7. $\mathrm{Mod}(S) = \pi_0(\mathrm{Homeo}^+(S))$, where $\mathrm{Homeo}^+(S)$ is endowed with the compact-open topology.

Remark 3.1.2. We will use the equivalence of these definitions from the very beginning even if, a priori, we are not allowed to untill we will have finished the proof of Theorem 2.4.4. The reader who wants to read the end of the proof before proceeding any further can read now Example 3.1.9, Definition 3.2.3, Proposition 3.2.6 and Corollary 3.2.9 in this order.

There are many other variants of $\mathrm{Mod}(S)$. Here are the other variants we will be interested in.

Definition 3.1.3. Define

$$\mathrm{Mod}(S, \partial S) = \mathrm{Diff}^+(S, \partial S)/\mathrm{Diff}_0^+(S, \partial S)$$

where $\mathrm{Diff}_0^+(S, \partial S)$ is the normal subgroup of $\mathrm{Diff}^+(S, \partial S)$ consisting of those diffeomorphisms homotopic relatively to ∂S to 1_S.

Clearly, similarly to $\mathrm{Mod}(S)$, one obtain many equivalent definitions of $\mathrm{Mod}(S, \partial S)$ using the results in section 2.4.

Remark 3.1.4. Note that, if $\partial S \neq \emptyset$, a diffeomorphism of S fixing pointwise ∂S is necessarily orientation preserving. Moreover, if $\partial S = \emptyset$, the two definitions coincide.

Another interesting variant arises when considering distinguished points in S. Let $\{x_1, ..., x_n\}$ be distinct points in $S \setminus \partial S$.

Definition 3.1.5. Define

$$\mathrm{Mod}(S; \{x_1, ..., x_n\})$$

to be the group of the relative homotopy classes of orientation-preserving homeomorphisms of S fixing setwise $\{x_1, ..., x_n\}$, where homotopies are required to be relative to $\{x_1, ..., x_n\}$ and preserve setwise ∂S.

Remark 3.1.6. At least when $n = 1$, the group $\mathrm{Mod}(S, \{x\})$ can also be described to be the relative isotopy classes of orientation-preserving homeomorphisms of S, where isotopies are required to be relative to $\{x\}$. This follows from Theorem 6.3 of [11] or one can improve the results of the previous chapter.

Now we present some examples of computations that can be worked out directly from the definitions.

Example 3.1.7 (Alexander's Lemma)**.** By Smale's Theorem $\mathrm{Mod}(D^2, \partial D^2) = \{1\}$. Moreover if $\phi \in \mathrm{Diff}^+(D^2)$, then ϕ is smoothly isotopic to a diffeomorphism that restricts to the identity on ∂D^2, thus $\mathrm{Mod}(D^2) = \{1\}$. There are many other possible strategies to prove this same result once we have the equivalence of all the definitions of Mapping Class Group. For example if $\phi \in \mathrm{Diff}(D^2, \partial D^2)$, then the straight-line homotopy between ϕ and 1_{D^2} is a smooth homotopy rel ∂D^2. A further possibility is to consider

$$F : I \times D^2 \to D^2 \quad F(t, x) = \begin{cases} (1-t)\phi(\frac{x}{1-t}) & 0 \leq |x| < 1-t \\ x & 1-t \leq |x| \leq 1 \end{cases}$$

that is an isotopy (of homeomorphisms) rel ∂D^2 between ϕ and 1_{D^2}.
Finally, we note that using this last isotopy we also obtain $\mathrm{Mod}(D^2; \{0\}) = \{1\}$.

Example 3.1.8. The proofs of Lemma 2.4.10 show that that $\mathrm{Mod}(S^2) = \{1\}$.

Example 3.1.9. We show that $\mathrm{Mod}(A, \partial A) \cong \mathbb{Z}$, where A is the annulus $A = S^1 \times [0, 1]$.
Let $p : \widetilde{A} = \mathbb{R} \times [0, 1] \to A = S^1 \times [0, 1]$ be the universal cover of A, $p(\theta, t) = (e^{2\pi i\theta}, t)$. For every diffeomorphism ϕ of A fixing pointwise ∂A, call $\widetilde{\phi} : \widetilde{A} \to \widetilde{A}$ the lift of $\phi \circ p$ such that $\widetilde{\phi}(0) = 0$ and $\widetilde{\phi}_i : \mathbb{R} \times \{i\} = \mathbb{R} \to \mathbb{R} \times \{i\} = \mathbb{R}$ the restriction of $\widetilde{\phi}$ to $\mathbb{R} \times \{i\}$ for $i \in \{0, 1\}$. Note that $\widetilde{\phi}_i$ is a lift of $\mathbb{R} \to S^1$ defined by $\theta \mapsto e^{2\pi i\theta}$, in particular it is a translation of $\mathbb{R}$ by an integer and, since $\widetilde{\phi}(0) = 0$, $\widetilde{\phi}_0$ must be the identity.
Consider the homomorphism of groups

$$\tau : \mathrm{Diff}(A, \partial A) \to \mathbb{Z}$$

defined by the formula $\tau(\phi) = \widetilde{\phi}_1(1)$.
We claim that it is surjective and has kernel $\mathrm{Diff}_0(A, \partial A)$. Surjectivity is easily established. Indeed, every matrix

$$M = \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix}$$

defines a diffeomorphism of $\widetilde{A}$ sending 0 to 0. This diffeomorphism passes to the quotient and gives a diffeomorphism ϕ_M of A such that $\tau(\phi_M) = n$.
In order to find the kernel of τ, we give another equivalent definition of τ. Call δ the straight oriented arc of A from $(1, 0)$ to $(1, 1)$. For every $\phi \in \mathrm{Diff}(A, \partial A)$, the loop $(\phi \circ \delta) \cdot \delta^{-1}$ is based at $\delta(0) = (1, 0)$. Identifying $\pi_1(A, (1, 0)) = \mathbb{Z}[\alpha] \cong \mathbb{Z}$, where $\alpha(t) = (e^{2\pi it}, 0)$ $0 \leq t \leq 1$. We have $\tau(\phi) = [(\phi \circ \delta) \cdot \delta^{-1}] \in \pi_1(A, (1, 0))$ does not depend on the choice of a representative in $[\phi] \in \mathrm{Mod}(A, \partial A)$ and thus

$\mathrm{Diff}_0(A, \partial A) \subseteq \ker(\tau)$. Finally, suppose that $\tau(\phi) = 0$. Then $\widetilde{\phi}_1 = 1_{\mathbb{R}}$. Note that the straight-line homotopy between $\widetilde{\phi}$ and $1_{\mathbb{R}\times[0,1]}$ is always equivariant under the action of deck transformations. This means that for every $n \in \mathbb{Z}$ and $((\theta, t), s) \in \widetilde{A} \times [0, 1]$

$$p(s\widetilde{\phi}(\theta + n, t) + (1 - s)(\theta + n, t)) = p(s\widetilde{\phi}(\theta, t) + (1 - s)(\theta, t)).$$

Indeed, $(\theta, t) \mapsto \widetilde{\phi}(\theta + n, t)$ and $(\theta, t) \mapsto \widetilde{\phi}(\theta, t) + (n, 0)$ always coincide since, for fixed n, both are lifts of $\phi \circ p$ sending 0 to $(n, 0)$. But, under our hypothesis, it is also true that the straight-line fixes $\partial\widetilde{A}$ and thus induced an homotopy rel ∂A between ϕ and 1_A. Therefore $\mathrm{Diff}_0(A, \partial A) = \ker(\tau)$ and we have completed the proof.

Before starting next example, we derive a useful lemma from the Examples 3.1.7 and 3.1.9.

Lemma 3.1.10. *Let S be a surface with at least one boundary component and call S' the surface obtained from S by collapsing each boundary component ∂_i of S to a point x_i (one point for each boundary component). Then, the natural homomorphism of groups*

$$\mathrm{Mod}(S) \to \mathrm{Mod}(S'; \{x_1, ..., x_n\})$$

is an isomorphism.

Proof. Note that the quotient map restricts to a homomorphism $S \smallsetminus \partial S \to S' \smallsetminus \{x_1, ..., x_n\}$. Using this identification, if $\phi \in \mathrm{Homeo}^+(S)$, then the class of ϕ is sent to the class of $\hat{\phi} \in \mathrm{Homeo}^+(S', \{x_1, ...x_n\})$, where $\hat{\phi}$ is equal to ϕ in $S' \smallsetminus \{x_1, ..., x_n\} = S \smallsetminus \partial S$ and sends x_i to x_j if ϕ sends the boundary component ∂_i to the boundary component ∂_j. We first prove that our homomorphism is injective. Suppose that $\phi \in \mathrm{Diff}^+(S)$ is such that $[\hat{\phi}] = [1_{S'}] \in \mathrm{Mod}(S'; \{x_1, ..., x_n\})$. In particular ϕ must fix setwise each boundary component of S. From Observation 1 in Lemma 2.3.36 and Theorem 2.3.34, we may assume that $\phi|_{\partial S} = 1_{\partial S}$ and thus, as in the proof of Theorem 2.4.3, that ϕ is the identity on a neighborhood of ∂S. This is the same that saying that $\hat{\phi}$ is the identity on a neighborhood U of $\{x_1, ..., x_n\}$. Let $\widetilde{G}$ be an homotopy of S' relative to $\{x_1, ..., x_n\}$ such that $\widetilde{G}_0 = \hat{\phi}$ and $\widetilde{G}_1 = 1_{S'}$ and let $d : S' \to [0, 1]$ be a smooth map that is 0 on a neighborhood $V \subseteq U$ of $\{x_1, ..., x_n\}$ and 1 outside U. The map $G : S' \times [0, 1] \to S'$ defined by $G(x, t) = \widetilde{G}(x, d(x)t)$ is an homotopy between $\hat{\phi}$ and a map that is the identity out of $U \smallsetminus V$. We may assume that $\bar{U} \smallsetminus V$ is a collection of annuli around the x_i and thus we have proved that ϕ is homoopic to a map that is the identity of S out of a collection of annuli with boundary components that are isotopic to a boundary component of S. Since the homotopies used to define $\mathrm{Mod}(S)$ are not required to fix pointwise ∂S, Example 3.1.9 shows that $[\phi] = [1_S]$ in $\mathrm{Mod}(S)$. Finally, we prove that our homomorphism is surjective. Let $\psi \in \mathrm{Homeo}^+(S, \{x_1, ..., x_n\})$. For $i = 1, ..., n$, let D_i be closed disks in S' around

x_i. Fix i and consider $\psi|_{D_i} : D_i \to \psi(D_i)$. Observe that $\psi(D_i)$ is a closed disk in S' around $\psi(x_i) = x_j$ for some j. According to Example 3.1.7, we can modify ψ inside D_i as we want not changing the class of ψ in $\mathrm{Mod}(S', \{x_1, ..., x_n\})$, as long as D_i is sent homeomorphically onto $\psi(D_i)$ and x_i is sent to x_j. For a suitable changes, the restriction of ψ to $S \setminus \partial S = S \setminus \{x_1, ..., x_n\}$ can be extended to an homeomorphism $\phi : S \to S$ and clearly $[\phi] \in \mathrm{Mod}(S)$ is sent to $[\psi] \in \mathrm{Mod}(S, \{x_1, ..., x_n\})$ under our homomorphism. The proof is complete.

□

Example 3.1.11. Now we compute $\mathrm{Mod}(S^2, \{x_1, x_2\})$, where x_1, x_2 are two distinguished points in S^2. By Lemma 3.1.10, $\mathrm{Mod}(S^2, \{x_1, x_2\}) = \mathrm{Mod}(A)$. Call Σ_2 the group of permutations of $\{x_1, x_2\}$. We claim that the homomorphism

$$\mathrm{Mod}(S^2, \{x_1, x_2\}) \to \Sigma_2$$

given by the action of $\mathrm{Mod}(S^2, \{x_1, x_2\})$ on $\{x_1, x_2\}$ is an isomorphism of groups. Indeed, it is clearly surjective and injectivity follows from the fact that, by Example 3.1.9, if $\phi \in \mathrm{Diff}^+(A)$ preserves setwise each component of ∂A, then ϕ is isotopic to 1_S.

Example 3.1.12. Next we compute $\mathrm{Mod}(S^2, \{x_1, x_2, x_3\})$, where x_1, x_2 and x_3 are three distinguished points in S^2. Note that, thanks to Lemma 3.1.10, we have $\mathrm{Mod}(P) \cong \mathrm{Mod}(S^2, \{x_1, x_2, x_3\})$ where P is a pair of pants, that is a surface homeomorphic to S^3_0.
Call Σ_3 the group of permutations of $\{x_1, x_2, x_3\}$. We claim that the natural homomorphism of groups

$$\mathrm{Mod}(S^2, \{x_1, x_2, x_3\}) \to \Sigma_3$$

given by the action of $\mathrm{Mod}(S^2, \{x_1, x_2, x_3\})$ on $\{x_1, x_2, x_3\}$ is an isomorphism of groups. It is clearly a surjection. We only need to prove it is injective. But if $\phi \in \mathrm{Diff}^+(S^3_0)$ preserves each boundary component of S^3_0, then it is diffeotopic to a diffeomorphism that restricts to the identity on ∂S^3_0 and injectivity follows from the proof of Lemma 2.4.11.

Example 3.1.13. We now compute the Mapping Class Group of the torus $T^2 = S^1 \times S^1$. There is an homomorphism of groups

$$\tau : \mathrm{Mod}(T^2) \to \mathrm{GL}(2, \mathbb{Z})$$

given by the action of $\mathrm{Mod}(T^2)$ on $H_1(T^2, \mathbb{Z}) \cong \mathbb{Z}^2$.
We claim that this is injective with image $\mathrm{SL}(2, \mathbb{Z})$; in particular $\mathrm{Mod}(T^2) \cong \mathrm{SL}(2, \mathbb{Z})$.

First, we see that τ takes values in $\mathrm{SL}(2,\mathbb{Z})$. Let $[\alpha],[\beta] \in \pi_1(T^2,\bar{0})$ be the standard symplectic basis of $H_1(T^2)$ and

$$\omega : H_1(T^2) \times H_1(T^2) \to \mathbb{Z}$$

be the intersection form $\omega = [\alpha]^* \wedge [\beta]^*$. Then for every orientation-preserving diffeomorphism ϕ of T^2,

$$1 = \hat{i}(\alpha,\beta) = \hat{i}(\phi(\alpha),\phi(\beta)) = \omega(H_1(\phi)([\alpha]), H_1(\phi)([\beta])) = \det(H_1(\phi)).$$

Therefore $\mathrm{Im}\tau \subseteq \mathrm{SL}(2,\mathbb{Z})$.
Moreover, every $M \in \mathrm{SL}(2,\mathbb{Z})$ defines an orientation-preserving diffeomorphisms of $\mathbb{R}^2$ and this diffeormiphism induces, by passing to the quotient, a diffeomorphism ϕ_M of $T^2 = \mathbb{R}^2/\mathbb{Z}^2$ such that $\tau([\phi_M]) = A$. So $\mathrm{Im}\tau = \mathrm{SL}(2,\mathbb{Z})$.
It remains to prove that τ is injective. Suppose that $\phi \in \mathrm{Diff}^+(T^2)$ is such that $H_1(\phi) = 1_{H_1(T^2)}$. We may assume that $\phi(x) = x$ where $x = \alpha \cap \beta$, otherwise we replace ϕ with $h \circ \phi$ where $h \in \mathrm{Diff}_0(T^2)$ is such that $h(\phi(x)) = x$. Then, since $\pi_1(T^2, x) \cong H_1(T^2)$ is abelian, it follows that ϕ acts trivially on it. Equivalently, if $p : \mathbb{R}^2 \to \mathbb{R}^2/\mathbb{Z}^2$ is the quotient map and $\widetilde{\phi} : \mathbb{R}^2 \to \mathbb{R}^2$ is the lift of $\phi \circ p$ such that $\widetilde{\phi}(0) = 0$, then $\widetilde{\phi}$ is such that $\widetilde{\phi}(x+n) = \widetilde{\phi}(x) + n$ for all $x \in \mathbb{R}^2$ and $n \in \mathbb{Z}^2$ and thus the straight-line homotopy between $\widetilde{\phi}$ and $1_{\mathbb{R}^2}$ induces an homotopy between ϕ and 1_{T^2}, by passing to the quotient. The proof is complete.

Example 3.1.14. Finally we compute the Mapping Class Group of the torus with one point distinguished, or equivalently the Mapping Class Group of S_1^1. We have an homomorpshism of groups τ obtained by composition

$$\tau : \mathrm{Mod}(T^2, \{x\}) \to \mathrm{Mod}(T^2) \xrightarrow{\cong} \mathrm{SL}(2,\mathbb{Z}).$$

Observe that τ is exactly the homomorphism given by the action of $\mathrm{Mod}(T^2,\{x\})$ on $H_1(T^2 \smallsetminus x) \cong H_1(T^2) \cong \mathbb{Z}^2$. We claim that this is an isomorphism. It is surjective since any element M of $\mathrm{SL}(2,\mathbb{Z})$ fixes the origin and discends to a diffeomorphism of T^2 fixing the point $\bar{0} = p(0)$ in T^2, where p the covering map $p : \mathbb{R}^2 \to \mathbb{R}^2/\mathbb{Z} = T^2$, and whose class in $\mathrm{Mod}(T^2,\{x\})$ is sent to M under τ. Now we prove that τ is injective. Maybe in this case it is easier to work with $\mathrm{Mod}(S_1^1)$ instead of $\mathrm{Mod}(T^2,\{x\})$. Thus, suppose that ϕ is a diffeomorphism of S_1^1 such that $H_1(\phi) = 1_{H_1(S_1^1)}$. We want to prove that ϕ is isotopic to $1_{S_1^1}$. Let α,β be the standard symplectic basis of $H_1(T^2)$. Clearly, identifying α and β with their images, we may assume that $\alpha \cup \beta \subseteq S_1^1$, where $S_1^1 \subseteq T^2$ is considered as a submanifold. Then, since $\pi_1(S_1^1) = H_1(S_1^1)$ and $H_1(\phi) = 1_{H_1(S_1^1)}$, we have that $\phi(\alpha)$ and $\phi(\beta)$ are isotopic (in S_1^1) to α and β respectively and, by Corollary 2.3.40, we can modify ϕ by an isotopy so that it fixes setwise α and β. In particular ϕ fixes $\alpha \cap \beta = \{x\}$

and, since ϕ is orientation-preserving, the restrictions maps $\phi|_\alpha : \alpha \to \alpha$ and $\phi|_\beta : \beta \to \beta$ are necessarily both orientation-preserving or both orientation-reversing. But, $\pi_1(\phi) = 1_{\pi_1(S^1_1,x)} : \pi_1(S^1_1, x) \to \pi_1(S^1_1, x)$ and thus $\phi|_\alpha$ and $\phi|_\beta$ are necessarily both orientation-preserving. Since $\mathrm{Diff}([0,1]\{0,1\})$ is contractible, we may actually assume that $\phi|_\alpha = 1_\alpha$ and $\phi|_\beta = 1_\beta$. Now cut S^1_1 along $\alpha \cup \beta$, obtaining an annulus A. Since the restriction of the map induced by ϕ on A is the identity when restricted to the boundary component ∂ of A corresponding to $\alpha \cup \beta$, from Example 3.1.9, it follows that this map is isotopic to 1_A through an isotopy relative to ∂, equivalently ϕ is isotopic to $1_{S^1_1}$.

3.2 Dehn twists

In this section we introduce some types of mapping classes, called **Dehn twists** and study some of their properties.

First we consider the annulus.

Definition 3.2.1. Let $A = S^1 \times [0,1]$ be the annulus, oriented using the counter-closkwise orientation on S^1. Let $\rho : [0,1] \to [0,1]$ be a smooth bump function such that $\rho(0) = 0$ and $\rho(1) = 1$. Define

$$T : A \to A \quad T(e^{2\pi i\theta}, t) = (e^{2\pi i(\theta+\rho(t))}, t).$$

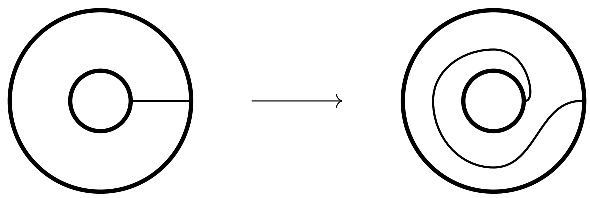


Figure 3.1: Spiegazione del Dehn Twist.

We will call the mapping class of T the **Dehn twist** about the circle $C = S^1 \times \{1/2\}$.

Remark 3.2.2. Observe that, from Example 3.1.9, the mapping class determined by T does not depend on the choice of ρ.

Now consider an arbitrary surface S and a circle C in S. Let N be a closed tubular neighborhood of C in $S \setminus \partial S$. Thus, we have an orientation-preserving diffeomorphism $\phi : (N, C) \cong (A, S^1 \times \{1/2\})$.

Definition 3.2.3. The **Dehn twist about C** is the class in $\mathrm{Mod}(S)$ of

$$T_C : S \to S \quad T_C(x) = \begin{cases} \phi^{-1} \circ T \circ \phi(x) & x \in N; \\ x & x \notin N. \end{cases}$$

Remark 3.2.4. This definition does not dependent on the choice of the closed tubular neighborhood. To see this, first note that, for fixed N, it does not depend on the choice of ϕ. This follows from Example 3.1.9. Then, use Theorem 6.5 in chapter 4 and Theorem 1.7 in chapter 8 of [23] to see that the definition is independent of the choice of N. Finally, an application of Theorem 2.3.34 gives that this definition does not even depend on the choice of the representative in the isotopy class of C.

Example 3.2.5. Let $A = S^1 \times [0,1]$ Then Example 3.1.9 can be reformulated by saying that $\mathrm{Mod}(A, \partial A) \cong \mathbb{Z}[T]$ where T is the Dehn twist about the circle $S^1 \times \{1/2\}$.

Dehn twists can be studied by looking at their action on the circles of S.

Proposition 3.2.6. *Let C_0 and C_1 be essential circles in S and $k \in \mathbb{Z}$. Then*

$$i(T^k_{C_0}(C_1), C_1) = |k|\, i(C_0, C_1)^2$$

Proof. Choose representatives $\widetilde{C_0}$ and $\widetilde{C_1}$ in the homotopy class of C_0 and C_1 respectively that are in minimal position. Observe that C_i and $\widetilde{C_i}$ are isotopic, so $T_{\widetilde{C_i}} = T_{C_i}$ for $i = 0, 1$. We will indicate $\widetilde{C_i}$ with C_i for $i = 0, 1$.
We now describe $T^k_{C_0}(C_1)$. Take $k\, i(C_0, C_1)$ curves $C_0^{(i)}$ $i = 1, ..., k\, i(C_0, C_1)$ parallel to $C_0 = C_0^{(1)}$, each in minimal position with C_1. If at each intersection point between C_1 and $C_0^{(i)}$ we do the surgery in figure

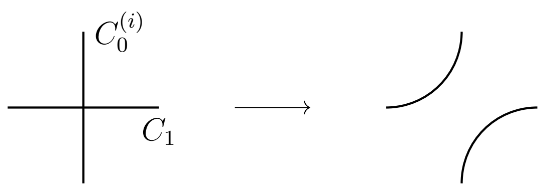


Figure 3.2: The operation of surgery.

we obtain a circle C_1' of S in the isotopy class of $T^k_{C_0}(C_1)$ with $|C_1' \cap C_0| = |k|\, i(C_0, C_1)^2$. The following figure should explain the situation:

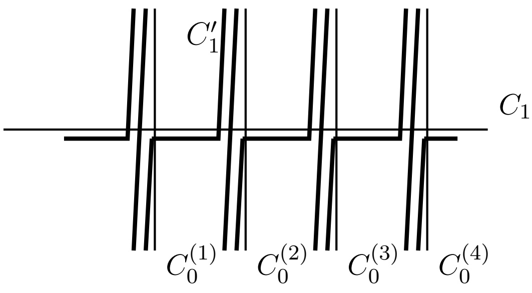


Figure 3.3: The circles C_0, C_1 and C_1' for $k = 2$ and $i(C_0, C_1) = 2$.

To conclude the proof it is enough to check that C_1' and C_1, as shown in the figure, do not form a bigon. There a two types of candidate bigons. They are presented in the figure below

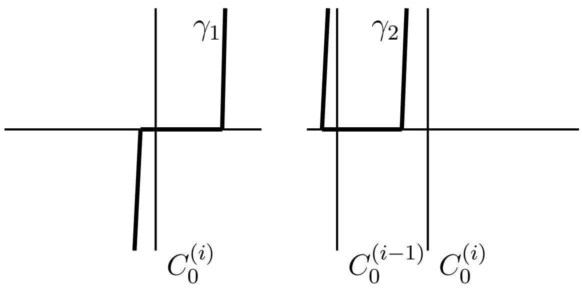


Figure 3.4: Candidate bigons

γ_1 corresponds to the case in which the two intersection point of the two arcs of C_1' and C_1 have the same orientation, γ_2 to the case this orientation is opposite. But, γ_1 cannot border a bigon because the orientations of the intersection points in γ_1 are the same; while if γ_2 bordered a bigon then C_0 and C_1 would form a bigon and this would be a contradiction.

□

Corollary 3.2.7. *(a) Dehn twists about essential circles have infinite order in* $\mathrm{Mod}(S)$*;*
(b) Dehn twists about peripheral but non-trivial circles have infinite order in $\mathrm{Mod}(S, \partial S)$*.*

Proof. We start with the first statement. It is enough to prove that for each essential circle C_0 there is another essential circle C_1 such that $i(C_0, C_1) > 0$. Using Lemma 2.3.36 we reduce to check few possible cases for C_0.
If C_0 is non-separating, then the surface must have at least genus 1 and we may assume the situation is that presented below and choose C_1 as in the figure:

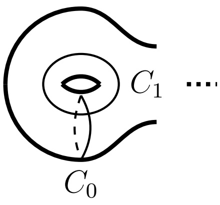


Figure 3.5: Case in which C_0 is non-separating

If C_0 separates $S = S_g^b$, write $S_C = S_1 \sqcup S_2$ where $S_i = S_{g_i}^{b_i}$ $i = 1, 2$ and $b_1 + b_2 = b + 2$ and $g = g_1 + g_2$. Again the relation $g = g_1 + g_2$ can be obtained from the identity $\chi(S_1) + \chi(S_2) = \chi(S_C) = \chi(S)$.
If $g_1, g_2 > 0$, we may assume we are in the situation presented below and choose C_1 as shown:

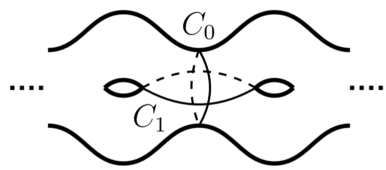


Figure 3.6: Case in which C_0 is separating.

where on the left side of C_0 we have S_1 and on the right side S_2.
The case in which $g_1 \cdot g_2 = 0$ can be treated similarly. Note that in this case at least one between b_1 and b_2 is ≥ 3. This concludes the proof of the first statement.
Now suppose that C_0 is peripheral and non-trivial. Consider the double $\bar{S}$ of S. In $\bar{S}$ the circle C_0 is essential. If T_{C_0} were trivial in $\mathrm{Mod}(S, \partial S)$ then it would be trivial in $\mathrm{Mod}(S)$ contradicting the previous cases. □

The previous result can also be generalized as follows.

Proposition 3.2.8. *Let $C_1, ..., C_k$ be a collection of disjoint non-trivial and non-isotopic circles of S. Then the subgroup of $\mathrm{Mod}(S, \partial S)$ generated by $T_{C_1}, ..., T_{C_k}$ is $\oplus_{i=1}^k \mathbb{Z} T_{C_i}$.*

Proof. If $\chi(S) \geq 0$ the result is trivial. Assume $\chi(S) < 0$. Clearly the T_{C_i} commute in $\mathrm{Mod}(S, \partial S)$ thus we only need to prove that if $T = T_{C_1}^{n_1} \cdot ... T_{C_k}^{n_k} = 1$ in $\mathrm{Mod}(S, \partial S)$ then $n_1 = ... = n_k = 0$. If $\partial S \neq \emptyset$, let $\bar{S}$ be the double of S, otherwise set $\bar{S} = S$. Clearly $T = 1$ in $\mathrm{Mod}(\bar{S})$. Moreover, the circles $C_1, ..., C_k$ are essential and pairwise non-isotopic in $\bar{S}$. We will find for each $i = 1, ..., k$ a circle V_i of $\bar{S}$ such that $V_i \cap C_j = \emptyset$ for $i \neq j$ and $i(V_i, C_i) > 0$. After that, for all $i = 1, ..., k$, we will have $0 = i(V_i, V_i) = i(V_i, T(V_i)) = i(V_i, T_{C_i}^{n_i}(V_i)) = |n_i| i(V_i, C_i)^2$ from which $n_i = 0$. Therefore we only need to find the circles V_i. To do this, let Δ be a pants decomposition of $\bar{S}$ with $\{C_1, ..., C_k\} \subseteq \Delta$. Call $\Delta_i = \Delta \setminus \{C_i\}$. The cutting surface $\bar{S}_{\Delta_i}$ has a component R that is either homeomorphic to S_0^4 or to S_1^1. In the first case there is a circle V_i in $\bar{S}$ disjoint from every circle in Δ_i and such that $i(C_i, V_i) = 1$ and in the second case there is a circle V_i of $\bar{S}$ disjoint from all the circles in Δ_i and such that $i(C_i, V_i) = 2$. This completes the proof. $\square$

We can finally prove the statement (b) in Theorem 2.4.4. It will follow from the following result.

Corollary 3.2.9. *Let S be a surface with $\chi(S) < 0$ and $\partial S = \partial_1 \sqcup ... \sqcup \partial_n$ where $n \geq 1$. Let ϕ be a diffeomorphism of S such that:*

1. *ϕ is the identity out of some annular neighborhoods A_i of ∂_i for $i = 1, ..., n$;*
2. *$\phi|_{\partial S} = 1_{\partial S}$;*
3. *ϕ is continuously homotopic to 1_S via an homotopy relative to ∂S.*

Then ϕ is smoothly isotopic to 1_S through an isotopy relative to ∂S.

Proof. For $i = 1, ..., n$, call ϕ_i the restriction of ϕ to A_i. We will prove that each ϕ_i is smoothly isotopic to the identity through an isotopy relative to ∂A_i. Thanks to Lemma 2.4.9, we know that Theorem 2.4.4 holds for the cylinder. Thanks to Example 3.2.5, we know that the class of ϕ_i in $\mathrm{Mod}(A_i, \partial A_i)$ is a power of the Dehn twist about a circle isotopic to a boundary component of A_i. Thus, it is enough to prove that this power is 0. This follows from the previous proposition. $\square$

3.2.1 Pairs of filling circles

We now apply the theory of Dehn twists to prove the existence of pairs of filling circles in S_g for $g \geq 2$.

Definition 3.2.10. We say that a pair of circles $\{C_1, C_2\}$ **fills** a surface S if they are in minimal position and $S \setminus (C_1 \cup C_2)$ is union of disks or, equivalently, for any non-trivial circle C in S we have $|C \cap (C_1 \cup C_2)| > 0$.

Lemma 3.2.11. *Let C_1 and C_2 be two circles in a suface S that are in minimal position. Given a third circle C, there exists in the homotopy class of C a representative that is simultaneously in minimal position with C_1 and C_2.*

Proof. Clearly, by pertubating C, we may assume that C is transverse to both C_1 and C_2. If C is not in minimal position with C_1 then they form a bigon. By transversality and compactness, $C \cap C_1$ is finite and since C and C_1 do not autointersect, we can take this bigon D to be innermost with respect to C and C_1. By assumption, C_1 and C_2 are in minimal position, thus every intersection of C_2 with this bigon is a colletion of arcs of C_2 each either connecting the C_1-side of the bigon with the C-side or two points of the C-side of the bigon. If there is an arc of the second type, then we have a bigon formed by C_2 and C inside our original bigon. We can take this bigon to be innermost with respect to C and C_2. Moreover, since it is contained in our original bigon D, this new bigon does not intersect C_1 and thus we can push C by homotopy across this new bigon reducing the number of bigon formed by C_2 and C inside D and preserving the number of intersection points of C with C_1. Repeating this operation we reduce ourself to the case in which every arc of C_2 in D connects the C side of D with the C_1 side. In this case we can push C by homotopy across the bigon D. This procedure gives a circle homotopic to C that is in minimal position with C_1. Now repeat the argument with C_2 replacing C_1 and observe that, since C and C_1 do not form a any bigon, now the intersection of C_1 with any innermost bigon formed by C_2 and C is a collection of arcs connecting the C_2-side of the bigon with the C-side. □

Proposition 3.2.12. *Let $g \geq 2$. There exists a pair of filling circles in S_g.*

Proof. Let $\Delta = \{C_1, ..., C_{3g-3}\}$ and C as shown below in figure. Then Δ is a pants decomposition of S_g and C is a non-trivial circle in S_g.

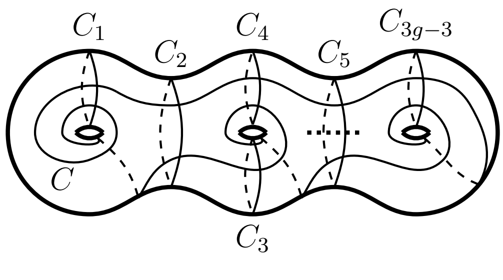


Figure 3.7 Definition of Δ and C.

By the bigon criterion, C is in minimal position with each C_i. Let $M = T_{C_1} \circ ... \circ T_{C_{3g-3}}$. We claim that for any non-trivial circle V of S_g

$$|i(M(C), V) - \sum_{i=1}^{3g-3} i(C_i, V)| \leq i(C, V).$$

From which it easily follows that C and the representative C' constructed as in the proof of Proposition 3.2.6 in the homotopy class of $M(C)$ are a pair of filling circles for S_g. Indeed, by the bigon criterion, C and C' are in minimal position. Let V be any non-trivial circle in S_g. If $i(V,C) = i(V,M(C)) = 0$, then it must be $i(C_i,V) = 0$ for all $i = 1,...,k$ and, by maximality of Δ, the circle V must be isotopic to some C_i. But, then, $i(V,C) > 0$ and this would be a contradiction. Thus either $|V \cap C| \geq i(V,C) > 0$ or $|V \cap C'| \geq i(V,C') > 0$.
We need only to prove the claim. Fix a circle V' in S_g in the homotopy class of V that is simultaneously in minimal position with both C and C'. This is possible thanks to the previous lemma. By perturbing V' we may assume that it does not pass through $C \cap C'$. Hence, we have the following inequalities:

$$\sum_{i=1}^{3g-3} i(C_i,V') \leq \sum_{i=1}^{3g-3} |C_i \cap V'| \leq |(C \cup C') \cap V'| \leq |V' \cap C| + |V' \cap C'| = i(V,C) + i(V,M(C)).$$

It remains to prove that

$$i(M(C),V) \leq \sum_{i=1}^{3g-3} i(C_i,V) + i(C,V).$$

But this is quite obvious: take as a representative of $[M(C)]$ a circle that lies in the union of C and small tubular neighborhoods of the C_i and as a representative of $[V]$ one that intersects C in $i(C,V)$ points and cuts across every C_i-annulus in $i(C_i,V)$ arcs; then they intersect in $\sum_{i=1}^{3g-3} i(C_i,V) + i(C,V)$ points.

□

3.3 Congruence subgroups

In this section we define the **congruence subgroups** $\mathrm{Mod}(S_g)[m]$ for $m \geq 2$. They will be finite-index subgroups and, for $m \geq 3$, torsion-free. Our interest for this kind of subgroups should be clear from the content of chapter 1.

First of all, we note that, for $g \geq 1$, $\mathrm{Mod}(S_g)$ has always non-trivial torsion.

Example 3.3.1. The rotation ϕ by π about the indicated axes gives a non-trivial mapping class of finite order.

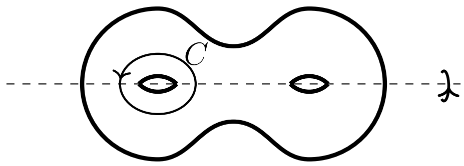


Figure 3.8: Example of a non-trivial mapping class with finite order.

To see that ϕ determines a non-trivial element of $\mathrm{Mod}(S_g)$ observe that if $[C] \in H_1(S)$ is the homology class of the oriented circle C, then $[C] \neq 0$ and $H_1(\phi)([C]) = -[C]$.

3.3.1 The symplectic representation of $\mathrm{Mod}(S_g)$

Recall that the **linear symplectic group** $\mathrm{Sp}(2g, \mathbb{Z})$ is the subgroup of $\mathrm{GL}(2g, \mathbb{Z})$ consisting of those matrix A satisfying $A^T JA = J$, where J is the block diagonal matrix having diagonal blocks

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

Let $g \geq 1$. Observe that the matrix of the symplectic intersection form ω with respect to the standard symplectic basis is exactly the matrix J. We identify $H_1(S_g; \mathbb{Z}) \cong \mathbb{Z}^{2g}$, using the standard symplectic basis of $H_1(S_g; \mathbb{Z})$. The action of $\mathrm{Mod}(S_g)$ on $H_1(S_g; \mathbb{Z})$ defines an homomorphism of groups

$$\Psi : \mathrm{Mod}(S_g) \to \mathrm{Aut}(H_1(S_g; \mathbb{Z})) \cong \mathrm{Aut}(\mathbb{Z}^{2g}) \cong \mathrm{GL}(2g, \mathbb{Z})$$

Since every orientation-preserving diffeomorphism of S_g preserves $\hat{i}$, it also preserves ω and thus Ψ takes values in $\mathrm{Sp}(2g, \mathbb{Z})$. For this reason, Ψ is called the **symplectic representation** of $\mathrm{Mod}(S_g)$. Using Ψ, we will deduce some properties of $\mathrm{Mod}(S_g)$ from those of $\mathrm{Sp}(2g, \mathbb{Z})$.

Congruence subgroups of $\mathrm{Sp}(2g, \mathbb{Z})$

Let $m \geq 2$ and $g \geq 1$. The **level m congruence subgroup** $\mathrm{Sp}(2g, \mathbb{Z})[m]$ is defined to be the kernel of the reduction homomorphism of groups

$$\mathrm{Sp}(2g, \mathbb{Z}) \to \mathrm{Sp}(2g, \mathbb{Z}/m\mathbb{Z})$$

Proposition 3.3.2. $\mathrm{Sp}(2g, \mathbb{Z})[\mathrm{m}]$ *is torsion-free for* $m \geq 3$.

Observe that, when $m = 2$, $-I \in \mathrm{Sp}(2g, \mathbb{Z})[2]$, so it is not torsion-free.

Proof. We first note that it is enough to prove the case $m = p^a$ where either $p = 2$ and $a > 1$ or p is an odd prime and a=1. Indeed, if n is a divisor of m, $\mathrm{Sp}(2g, \mathbb{Z})[\mathrm{m}] \subseteq \mathrm{Sp}(2g, \mathbb{Z})[\mathrm{n}]$.
Consider $I \neq h \in \mathrm{Sp}(2g, \mathbb{Z})[\mathrm{m}]$ and $k \geq 2$ an integer. We want to prove that $h^k \neq I$. Clearly it enough to check the case in which k is a prime number.
Write $h = I + p^d T$ where $d \geq a$ and T is a matrix with at least one entry that is not divisible by p. There are two cases to be considered:
Case 1: if $p = k$, then

$$h^k = (I+p^dT)^k = I+kp^dT+\binom{k}{2}p^{2d}T^2+... \equiv I+kp^dT = I+p^{d+1}T \not\equiv I \pmod{p^{d+2}}$$

where in the first congruence we used that $m \neq 2$.
Case 2: if $p \neq k$, then

$$h^k = (I + p^dT)^k = I + kp^dT + \binom{k}{2}p^{2d}T^2 + ... \equiv I + kp^dT \not\equiv I \pmod{p^{d+1}}$$

This completes the proof. □

Congruence subgroups of Mod(S_g)

Let $m \geq 2$ and $g \geq 1$.

Definition 3.3.3. The **level m congruence subgroup** of $\mathrm{Mod}(S_g)$ is the kernel of the composition

$$\mathrm{Mod}(S_g) \xrightarrow{\Psi} \mathrm{Sp}(2g, \mathbb{Z}) \to \mathrm{Sp}(2g, \mathbb{Z}/m\mathbb{Z})$$

Equivalently, it is the subgroup of those elements acting trivially on $H_1(S_g, \mathbb{Z}/m\mathbb{Z})$.

Theorem 3.3.4. *Let $g \geq 1$. If $\phi \in \mathrm{Diff}(S_g)$ is an orientation-preserving diffeomorphism of finite order that defines a non-trivial mapping class in $\mathrm{Mod}(S_g)$, then it does not act trivially on $H_1(S_g, \mathbb{Z})$.*

Proof. If $g = 1$ the result follows from follows from Example 3.1.13. Assume $g \geq 2$ and suppose $\phi^k = 1_{S_g}$.
First, we prove that ϕ has isolated fixed points. Let $x \in \mathrm{Fix}(\phi)$ any fixed point of ϕ. Choose any riemannian metric h on S_g and average it by taking $h + \phi^*(h) + ... + (\phi^{k-1})^*(h)$, so that ϕ becomes an isometry of S_g with this new metric. Since ϕ is an isometry between two compact connected riemannian manifold, ϕ is completely determined by $\phi(x) = x$ and $d_x\phi$. The differential $d_x\phi$ is an orthogonal transformation with positive determinant (this is because ϕ is orientation-preserving), thus it is a rotation. Observe that it cannot be the trivial rotation, otherwise ϕ would be the

identity. Thus 1 is not an eigenvalue of $d_x\phi$ and x must be an isolated fixed point of ϕ.
Since ϕ has isolated fixed points we can apply Lefschetz fixed point Theorem, obtaining

$$0 \leq \sum_{x\in \mathrm{Fix}(\phi)} \mathrm{ind}_x(\phi) = \sum_{i=0}^{2}(-1)^i \mathrm{tr}(H_i(\phi)) = 2 - \mathrm{tr}(H_1(\phi))$$

where in the first inequality we have used the fact that if R is a non-trivial rotation of the plane then $R-I$ is orientation-preserving and thus $\mathrm{ind}_x(\phi) = 1$ for all $x \in \mathrm{Fix}(\phi)$. Since $g \geq 2$, it follows that $H_1(\phi) \neq 1_{H_1(S_g,\mathbb{Z})}$. □

This theorem has strong implications for the study of $\mathrm{Mod}(S_g)$. In fact, any periodic element of $\mathrm{Mod}(S_g)$ has a representative that is a periodic diffeomorphism of S_g, for $g \geq 2$.

Theorem 3.3.5. *Let $g \geq 2$. If $f \in \mathrm{Mod}(S_g)$ is an element of finite order k, then there is a representative $\phi \in \mathrm{Diff}(S_g)$ of f so that $\phi^k = 1_{S_g}$.*

Proof. See Theorem 7.1 in chapter 7 of [12]. □

We can now easily deduce the following

Corollary 3.3.6. *Let $g \geq 2$ and $m \geq 3$. The group $\mathrm{Mod}(S_g)[\mathrm{m}]$ is torsion-free.*

Proof. If there exist $1 \neq f \in \mathrm{Mod}(S_g)[\mathrm{m}]$ having finite order, then $\Psi(f) \neq 1$ belongs to $\mathrm{Sp}(2g,\mathbb{Z})[\mathrm{m}]$ and has finite order. This contradicts the fact that $\mathrm{Sp}(2g,\mathbb{Z})[\mathrm{m}]$ is torsion-free. □

Chapter 4

The complex of curves $C(S)$

To any surface S we can associate a simplicial complex defined as follows.

Definition 4.0.1. The **complex of curves** of a surface S is the simplicial complex $C(S)$ specified by the following data:

vertices: the vertices are the isotopy classes of essential circle in S;

k-simplices: $k+1$ distict isotopy classes of essential circles $C_0, ..., C_k$ in S form a k-simplex of $C(S)$ if and only if $i(C_i, C_j) = 0$ for all $0 \leq i \neq j \leq k$.

We will restrict our attention to the case $S = S_g$ with $g \geq 2$ and thus endow S with an auxiliary hyperbolic metric.
In this case, $C(S)$ can also be described as the simplicial complex having one vertex for each geodesic circle of S and as k-simplices the sets of $k+1$ geodesic circles of S that are pairwise non-intersecting. This is because each geodesic circle is necessarily essential being $\partial S = \emptyset$, each essential circle of S is isotopic to a unique geodesic circle of S and the geodesic circles corresponding to two disjoint, non-isotopic and non-trivial circles of S are disjoint.

4.1 Connectivity of $C(S)$

It is known (see [18]) that the geometric realization of $C(S)$ is $e(S)$-connected, where

$$e(S) = \begin{cases} -\chi(S) - 1 = 2g - 3 & \text{if } \partial S = \emptyset; \\ -\chi(S) - 2 = 2g - 4 + b & \text{if } \partial S \neq \emptyset. \end{cases}$$

Note that if $g \geq 2$ then $e(S) > 0$.

We will prove this result only for $S = S_g$ and $g \geq 2$. This is all we need for our purposes.

For the rest of the chapter assume $S = S_g$ where $g \geq 2$. The objective of this chapter is to give a proof of the following theorem:

Theorem 4.1.1. *Let $g \geq 2$. Then the complex of curves $C(S_g)$ is $-\chi(S_g)-1 = 2g-3$ connected.*

The proof presented here follows [24].

4.1.1 Reformulation of the problem

Let $f : S \to \mathbb{R}$ be a smooth function.

Definition 4.1.2. We will say that a component of the level set $f^{-1}(a)$ is **non-singular** if it does not contain a critical point of f.

In particular, this component must be a circle C in S.
Note that if g is close to f in the C^∞-topology, then $g^{-1}(a)$ has a non-singular component that is a circle in S isotopic to C.

Definition 4.1.3. We will say that f is **non-degenerate** if there is a level set of f containing a non-singular component that is a non-trivial circle of S.

Definition 4.1.4. Let P be a topological space. For a **family of functions** $\{f_t : S \to \mathbb{R}\}_{t \in P}$ we will mean a continuous map $P \to C^\infty(S, \mathbb{R})$ sending t to f_t, where $C^\infty(S, \mathbb{R})$ is endowed with the C^∞-topology.

Note that, since S is compact, the strong and the weak C^∞-topology on $C^\infty(S; \mathbb{R})$ coincide.

From functions to circles

To any family of non-degenerate functions $\{f_t : S \to \mathbb{R}\}_{t \in P}$ it can be associated (in a non-unique way) a certain simplicial complex C_P and a simplicial map $C_P \to C(S)$ as now we explain.
For each $t \in P$ choose $a_t \in \mathbb{R}$ such that $f_t^{-1}(a_t)$ contains a non-singular component that is a non-trivial circle C_t of S. For each $t \in P$, there exists an open neighborhood U_t of t in P such that for all $u \in U_t$ the level set $f_u^{-1}(a_t)$ has a non-singular component that is a non-trivial circle $C_{t,u}$ of S isotopic to C_t. Let $\{U_t\}_{t \in V}$ be any subcover of $\{U_t\}_{t \in P}$ and consider the nerve C_P of this subcover and

$$V \to C(S)$$
$$t \mapsto C_t$$

We claim that this map induces a simplicial map $C_P \to C(S)$, i.e. if $t_0, ..., t_k \in V$ are such that $U_{t_0} \cap ... \cap U_{t_k} \neq \emptyset$, then $\{C_{t_0}, ..., C_{t_k}\}$ is a simplex of $C(S)$. Indeed, if $u \in U_{t_0} \cap ... \cap U_{t_k}$ we have that $C_{t_i,u} = C_{t_i}$ in $C(S)$ for each $i = 0, ..., k$ and, since $C_{t_0,u}, ..., C_{t_k,u}$ are components of some level set of a unique function, f_u, any two of them either do not intersect at all or coincide. This proves our claim.

Definition 4.1.5. We will call a simplicial map constructed in this way a **realization** of the family $\{f_t\}_{t\in P}$.

This construction has the following **extension property**: suppose we have been given a realization $C_Q \to C(S)$ of the family $\{f_t\}_{t\in Q}$ where $Q \subseteq P$ is a subspace; then, there exists a realization $C_P \to C(S)$ of $\{f_t\}_{t\in P}$ such that C_Q is a subcomplex of C_P and $C_P \to C(S)$ extends $C_Q \to C(S)$.
Indeed, C_Q is the nerve of some covering $\{U_t\}_{t\in V}$ ($V \subseteq Q$) of Q and $C_Q \to C(S)$ is determined by the association $V \ni t \mapsto C_t \in C(S)$, where C_t is a non-singular component of some $f_t^{-1}(a_t)$ that is also a non-trivial circle of S. Now, every U_t is of the form $U_t = U'_t \cap Q$ for some open neighborhood of t in P and U'_t can also be choosen such that for every $u \in U'_t$ the level set $f_u^{-1}(a_t)$ has a non-singular component that is a non-trivial circle of S isotopic to C_t. By using for the construction of $C_P \to C(S)$ any covering containing $\{U'_t\}_{t\in V}$ we have done.

From circles to functions

Actually, every simplicial map $C \to C(S)$, with C a finite simplex, arises as a realization of some family of non-degenerate functions.

To prove this, we start by recalling a very simple lemma.

Lemma 4.1.6. *Let C be a simplicial complex and let $v_0, ..., v_k$ be vertices of C. For $i = 0, ..., k$, let $\mathrm{St}(v_i)$ denote the closed star of v_i in the first barycentric subdivision of C. Then $\{v_0, ..., v_k\}$ is a simplex of C if and only if $\mathrm{St}(v_0) \cap ... \cap \mathrm{St}(v_k) \neq \emptyset$.*

Proposition 4.1.7. *Let C be a finite simplicial complex. Any simplicial map $C \to C(S)$ is a realization of some family of non-degenerate function $\{f_t : S \to \mathbb{R}\}_{t\in|C|}$. In addition, if $|C|$ is a smooth manifold, the family $\{f_t\}_{t\in|C|}$ can be chosen such that the evaluation map*

$$f : |C| \times S \to \mathbb{R} \quad f(t, x) = f_t(x)$$

is smooth.

Proof. Suppose $C \to C(S)$ sends the vertex v to the geodesic circle C_v. For each C_v choose a closed tubular neighborhood N_v of C_v in S such that $N_v \cap N_w = \emptyset$ if $C_v \cap C_w = \emptyset$. This is possible because C has a finite number of vertices. Next, for

each N_v choose a smooth function $g_v : N_v \to [0, \infty)$ such that $C_v = g_v^{-1}(a_v)$ for some regular value $a_v \in (0, \infty)$ of g_v and $g_v^{-1}(0) = \partial N_v$.
Consider the map

$$g : \bigcup_{v \in V} \mathrm{St}(v) \times N_v \to \mathbb{R} \quad g(t, x) = g_v(x) \text{ if } (t, x) \in \mathrm{St}(v) \times N_v,$$

where V is the set of vertices of C and $\mathrm{St}(v)$ is, as in the previous lemma, the closed star of v in the first barycentric subdivision of C. Extend g to a continuous map $f : |C| \times S \to \mathbb{R}$ such that for every $t \in |C|$ f_t is smooth as follows. First consider the functions

$$f_v : \mathrm{St}(v) \times S \to \mathbb{R} \quad f_v(t, x) = \begin{cases} g_v(x) & \text{if } x \in N_v, \\ 0 & \text{otherwise;} \end{cases}$$

then extends each f_v to a smooth function $\widetilde{f_v}$ that is 0 out of $|C| \times N_v$, by means of bump functions, and finally set $f(t, x) = \sum_{v : t \in \mathrm{St}(v)} \widetilde{f_v}(x)$ for $(t, x) \in |C| \times S$.
What is more, $\{\mathrm{St}(v)\}_{v \in V}$ is a covering of $|C|$ whose nerve, by the previous lemma, is exactly C. We can choose open neighborhoods U_v of $\mathrm{St}(v)$ small enough such that the nerve of the open covering $\{U_v\}_{v \in V}$ is still C and for each $u \in U_v$ the level set $f_u^{-1}(a_v)$ contains a non-singular component that is a non-trivial circle of S isotopic to C_v. The realization of $\{f_t\}_{t \in |C|}$ associated to the cover $\{U_v\}_{v \in V}$ and circles $\{C_v\}_{v \in V}$ solves the problem.

□

The strategy for the proof

Finally, we explain how the previous construction can be used to investigate the connectivity of the complex of curves of S.

Notation 4.1.1. We will denote by C^∞_W and by C^∞_S the space of smooth maps from two manifolds endowed with the weak and the strong C^∞-topology respectively.

Notation 4.1.2. For $d \geq 0$, call $X(d)$ the subset of $C^\infty(\mathbb{R}^d \times S, \mathbb{R})$ consisting of those functions f such that for all $t \in \mathbb{R}^d$ f_t is non-degenerate.
We introduce, for $\varepsilon > 0$, a particular open neighborhood $\mathcal{N}(f, \varepsilon)$ of a function f in $C^\infty_S(\mathbb{R}^d \times S, \mathbb{R})$ defined as follows. Let $\{(U_1, \varphi_1), ..., (U_k, \varphi_k)\}$ be an adequate atlas on S. This means that it is an atlas of S with $\varphi_i(U_i) = \mathbb{R}^2$ for every $i = 1, ..., k$ and $\{\varphi_i^{-1}(D^2)\}_{i=1}^k$ is a cover of S. Then $\mathcal{N}(f, \varepsilon)$ consists of those functions g such that

$$|f \circ (1_{\mathbb{R}^d} \times \varphi_i^{-1})(t, x) - g \circ (1_{\mathbb{R}^d} \times \varphi_i^{-1})(t, x)| < \varepsilon$$

and

$$|D(f \circ (1_{\mathbb{R}^d} \times \varphi_i^{-1}))(t, x) - D(g \circ (1_{\mathbb{R}^d} \times \varphi_i^{-1}))(t, x)| < \varepsilon$$

for all $(t, x) \in \mathbb{R}^d \times D^2$ and $i = 1, ..., k$.

To see that $\mathcal{N}(f, \varepsilon)$ is an open set, consider the atlas Φ of $\mathbb{R}^d \times S$ given by the charts $(\mathbb{R}^d \times U_i, 1_\mathbb{R} \times \varphi_i)$ for $i = 1, ..., k$ and the compact subsets $K_{n,i} = D^d(n) \times \varphi_i^{-1}(D^2) \subseteq \mathbb{R}^d \times U_i$ where $n \in \mathbb{N}_0$ and $D^d(n)$ is the closed disk of radius n and center 0 in $\mathbb{R}^d$. Then $\mathcal{N}(f, \varepsilon)$ is exactly $\mathcal{N}^1(f, \Phi, , \Psi, K, \varepsilon)$ as defined in [23], where Ψ is the trivial atlas on $\mathbb{R}$ and $K = \{K_{i,n}\}$.

Theorem 4.1.8. *Suppose that for* $d = 0, ..., m$ *the space* $X(d)$ *is dense in* $C_S^\infty(\mathbb{R}^d \times S, \mathbb{R})$*, then* $C(S)$ *is* $(m-1)$*-connected.*

Proof. Let $n \leq m-1$ and $g : S^n \to |C(S)|$ be a continuous map. Consider a triangulation of $S^n \cong |C|$, where C is a finite simplicial complex. If the triangulation is fine enough, then g is homotopic to the geometric realization of some simplicial map $h : C \to C(S)$. Moreover, h is a realization of some family of non-degenerate functions $\{f_t\}_{t \in |C| = S^n}$ such that the evaluation map $f : |C| \times S \to \mathbb{R}$ defined by $f(t, x) = f_t(x)$ is smooth. Note that, since $\mathbb{R}$ is convex, f con be extended to a smooth function $\tilde{f} : \mathbb{R}^d \times S \to \mathbb{R}$. For example, if $\rho : [0, \infty) \to [0, 1]$ is a smooth function that is equal to 0 in a neighborhood of 0 and equal to 1 in 1, we can take $\tilde{f}(t, x) = \rho(t) f(t/|t|, x)$ (well defined also for $t = 0$ being $\rho(0) = 0$). Now, $\tilde{f}$ need not to be a family of non-degenerate functions. But, by hypothesis, there exists an arbitrary close approximation f' of $\tilde{f}$, with f'_t non-degenerate functions for all $t \in \mathbb{R}^d$. If $\varepsilon > 0$ is sufficiently small and $f' \in \mathcal{N}(f, \varepsilon)$, then h is also a realization of $\{f'_t\}_{t \in S^n}$. Indeed, with the notations of the previous proposition, it is clear that, for small ε, for every vertex v of C and for all $t \in \mathrm{St}(v)$ the level set $f_t'^{-1}(a_v)$ contains a non-singular component that is a non-trivial circle of S isotopic to C_v. Now, maybe replacing the U_v with smaller open neighborhoods of $\mathrm{St}(v)$, repeat the argument of the proposition to conclude.
By applying the extension property with $Q = S^n \subseteq P = D^{n+1}$ we obtain an extension of h, $\bar{h} : C_P \to C(S)$. This can be done in such a way that $|C_P| = D^{n+1}$. For example, we can use as cover of D^{n+1} one containing for each vertex v of C a convex open set U'_v as in the definition of C_P and take all the other open sets to be finitely many open balls of $D^{n+1} \setminus \partial D^{n+1}$. This would be a finite good cover of D^{n+1} and thus the geometric realization of its nerve is D^{n+1}. Finally, $|\bar{h}|$ extends $|h|$ to D^{n+1} and the proof is complete. □

4.1.2 Proof of the connectivity of C(S)

First of all, we recall a result from Differential Topology. See Appendix A for the details.

In this subsection will make use of the notion of maps of finite type (see Definition A.2.16). The properties of maps of finite type we will use are:

1. Let M be a smooth manifold and let $f : M \to \mathbb{R}$ be a smooth map of finite type. Then, the singular points of f are isolated and, around each singular point, there is a chart of M with respect to which it is representated by a polynomial. See Definition A.2.4 and Corollary A.2.5. Note also that, by the Implicit Function Theorem, around a non-singular point f is represented, in some chart, by a polynomial of degree 1.

2. Let M be a smooth manifold. Every smooth map $\mathbb{R}^d \times M \to \mathbb{R}$ can be approximated arbitrarily well (in $C^\infty_S(\mathbb{R}^d \times M, \mathbb{R})$) by another smooth map $f : \mathbb{R}^d \times M \to \mathbb{R}$ such that for every $t \in \mathbb{R}^d$ the map f_t is of finite type. See Theorem A.2.17.

Branch numbers

Suppose we have been given a map of finite type $f : S \to \mathbb{R}$. Then, around each singular point, in some coordinates, it is a polynomial in two variables.

Lemma 4.1.9. *Let* $(0, 0)$ *be a point of a real 1-dimesional (with respect to the Zariski topology) irreducible algebraic variety* $V \subseteq \mathbb{R}^2$*. Then, in a suitable neighborhood of* $(0, 0)$*,* V *is the union of finitely many topological subspaces* $B_1, ..., B_n$*, called* ***branches*** *of* V *at* $(0, 0)$*, with the following properties:*

1. *for* $i \neq j$*, we have* $B_i \cap B_j = \{(0, 0)\}$*;*

2. *each branch is homeomorphic to an open interval of* $\mathbb{R}$ *under a homeomorphism that, unless* $V \subseteq \{x_1 = 0\}$*, can be chosen of the form*

$$r(t) = (x(t), y(t)) = \pm(t^\mu, a_1 t + a_2 t^2 + ...) \quad |t| < \varepsilon$$

where $a_i \in \mathbb{R}$ *and* $\mu \in \mathbb{N}_+$*.*

This lemma is Lemma 3.3 on page 27 of [31]. We copy here the proof for completeness.

Proof. We assume $V \not\subseteq \{x_1 = 0\}$ and we prove the existence of such a parametrization. We will use the fact that the lemma holds in the complex case. Precisely, a complex irreducible curve $V_\mathbb{C}$ in $\mathbb{C}^2$ with $0 \in V$ a non-isolated point is, around 0, union of finitely many topological subspaces $B_1, ..., B_n$, called branches of $V_\mathbb{C}$ at $(0, 0)$, with the following properties:

1. for $i \neq j$, we have $B_i \cap B_j = \{(0, 0)\}$;

2. each branch is homeomorphic to an open disk of $\mathbb{C}$ under a homeomorphism that, unless $V \subseteq \{z_1 = 0\}$, can be chosen of the form

$$r(t) = (x(t), y(t)) = \pm(t^\mu, a_1 t + a_2 t^2 + ...) \quad |t| < \varepsilon$$

where $a_i \in \mathbb{C}$ and $\mu \in \mathbb{N}_+$.

Call $V_\mathbb{C}$ the closure of V in $\mathbb{C}^2$ (with respect to the Zariski topology of $\mathbb{C}^2$). Then $V_\mathbb{C}$ is irreducible, for if $V_\mathbb{C} = V_1 \cup V_2$ is union of two closed sets of $\mathbb{C}^2$, then $V = (V_1 \cap \mathbb{R}^2) \cup (V_2 \cap \mathbb{R}^2)$ and each $V_i \cap \mathbb{R}^2$ $i = 1, 2$ is closed in the Zariski topology of $\mathbb{R}^2$. Thus $V = V_i \cap \mathbb{R}^2$ for some i and it must be $V_\mathbb{C} = V_i$.
Since $V_\mathbb{C} \not\subseteq \{z_1 = 0\}$, we can parametrize its branches as above stated. Moreover, if $(t^\mu, a_1 t + a_2 t^2 + ...) \in \mathbb{R}^2$ then $t = \xi s$ where $\xi^\mu = \pm 1$ and $s \in \mathbb{R}$. Substituting $t = \xi s$ in $a_1 t + a_2 t^2 + ...$ we obtain a new power serie $(a_1\xi)s + (a_2\xi^2)s^2 + ...$ that is real for all small s if all $a_i\xi^i$ are real and otherwise is not real for all small $s \neq 0$. Indeed, suppose there is a sequence $\{s_n\} \subseteq \mathbb{R} \setminus \{0\}$ convergent to 0 on which this power serie takes real values. Then we can use this sequence to compute the derivatives of our power serie at 0 obtaining that they all are real and thus that all the $a_i\xi^i$ are real, too.
In conclusion the real case has been reduced to the complex one and the lemma follows. □

Definition 4.1.10. Let V be a 1-dimensional algebraic variety in $\mathbb{R}^2$ and $(x_1, x_2) \in V$ a non-isolated point of V. Note that V is finite union of 1-dimensional irreducible algebraic varieties. Let $B_1, ..., B_n$ be the branches of V at (x_1, x_2) and for $i = 1, ..., n$ let $r_i : (-\varepsilon, \varepsilon) \to \mathbb{R}^2$ be parametrizations of $B_1, ..., B_n$ respectively as above explained. We will call the images of the half-intervals $[0, \varepsilon)$ and $(-\varepsilon, 0]$ under r_i an **half-branch**.

Remark 4.1.11. Note that if $V = V_1 \cup ... \cup V_k$ is the decomposition of V in irreducible components and for $i = 1, ..., k$ the sets $B^i_1, ..., B^i_{j_i}$ are the branches of V_i in $(x_1, x_2) \in V$, then for all $1 \leq s, t \leq k$ and $1 \leq h \leq j_s$, $1 \leq i \leq j_t$ such that $s \neq t$ or $s = t$ but $h \neq i$ we have $|B^s_h \cap B^t_i| < \infty$. Thus we can actually assume $B^s_h \cap B^t_i = \{(x_1, x_2)\}$. This is beacause if $s \neq t$ then $|V_s \cap V_t| < \infty$ and if $s = t$ but $h \neq i$ from Lemma 4.1.9 we have $B^s_h \cap B^t_i = \{(x_1, x_2)\}$.

Definition 4.1.12. For each critical point x of a finite type map f, we define the **branch number** of f in x to be $b_f(x)$ =number of branches at x of the level set $f^{-1}(f(x))$ (this number is 0 if x is an isolated point of $V = f^{-1}(f(x))$).

Lemma 4.1.13. *Suppose $b_f(x) \geq n \geq 1$. Then there exist coordinates u_1 and u_2 around x such that for all $0 < i + j < n$ we have $\frac{\partial^{i+j} f}{\partial u_1^i \partial u_2^j}(x) = 0$. Consequently, this equality holds for all coordinates around x.*

Proof. Let u_1 and u_2 be coordinates around x with respect to which f is a polynomial and $f(0)=0$. According to the previous lemma, every branch of $f^{-1}(0)$ is parametrized by $a^{(j)}_{k(j)}t^{k(j)}+a^{(j)}_{k(j)+1}t^{k(j)+1}+...$ with $a^{(j)}_i\in\mathbb{R}^2$ and $a^{(j)}_{k(j)}\neq 0$, thus, up to a further linear change of coordinates that brings all the $a^{(j)}_{k(j)}$ in the right half plane, we may assume that each $a^{(j)}_{k(j)}$ actually belongs to the right half plane. In particular, the open right half plane contains at least half of the half branches of $f^{-1}(0)$.
Call $m=\deg(f)$ the degree of f and write $f=\sum_{1\le i+j\le m}a_{ij}u_1^iu_2^j$. Using the substitution $(u_1,u_2)\mapsto(u_1+\gamma u_2,u_2)$ where $\gamma\in\mathbb{R}$, the coefficient of u_2^m becomes $\sum_{j=1}^m a_{m-j,j}\gamma^{m-j}$ where $a_{m-j,j}\neq 0$ for some j, being $\deg(f)=m$. Thus we may assume that the coefficient a_{0m} of u_2^m is different from 0 and, if γ is small enough, the open right half plane still contains each of the $a^{(j)}_{k(j)}$ for $j=1,...,n$ and thus at least half of the half branches of $f^{-1}(0)$.
Now write $f=p_m(u_1)u_2^m+p_{m-1}(u_1)u_2^{m-1}+...+p_0(u_1)$ where $p_j(u_1)=\sum_{i\le m-j}a_{ij}u_1^i$ and in particular $p_m(u_1)=a_{0m}\neq 0$. Thus, fixed u_1, the polynomial $f=f(u_1,u_2)$ has exactly m complex roots (up to multiplicity) and we can find $\varepsilon>0$ and continuous functions $r_j:[0,\varepsilon)\to\mathbb{C}$ $j=1,...,m$ such that, for all $t\in[0,\varepsilon)$, the $\{r_j(t)\}_{j=1,...,m}$ are exactly the roots of $f(t,u_2)$. To see this consider the set $V=\{(u_1,z_2)\in[0,\infty)\times\mathbb{C}:f(u_1,z_2)=0\}$. From the proof of the previous lemma $V\cap[0,\varepsilon)\times\mathbb{C}$ is equal to the union m sets of the form $B=\{\pm(t^\mu,a_0+a_1t+a_2t^2+...):t\in\mathbb{C},\ |t|<\varepsilon,t^\mu\in[0,\infty)\}$ for some $a_0,a_1,a_2,...\in\mathbb{C}$ and $\mu\in\mathbb{N}_+$. Since $t^\mu\in[0,\infty)$, it must be $t=\xi s$ where $\xi^\mu=1$ and $s\in[0,\infty)$. Thus, up to exchanging each a_i with $a_i\xi^i$ for all i, we can assume $t=s\in[0,\varepsilon)$. Equivalently we have $B=\{(u_1,a_0+a_1u_1^{1/\mu}+....):u_1\in[0,\varepsilon^{1/\mu})\}$. So the r_j can be chosen of the form $r(t)=a_0+a_1t^{1/\mu}+a_2t^{2/\mu}+....$ Note that the r_j are smooth in $(0,\varepsilon)$. Between the r_j there are at least n corresponding to half branches of $f^{-1}(0)$ in the open right half plane. Suppose that such r_j are the first n. For them we have that $a^{(j)}_{k(j)}$ belongs to the right half plane, thus $\mu\le k(j)$ and $r_j(u_1)=O(u_1)$ for $u_1\to 0$.
Therefore, in the strip $[0,\varepsilon)\times\mathbb{R}$, we can write $f=a_{0,m}(u_2-r_1(u_1))\cdot...\cdot(u_2-r_m(u_1))$. We claim that from this it follows that $\frac{\partial^{i+j}}{\partial u_1^i\partial u_2^j}f(0,0)=0$ for all $0\le i+j\le n-1$. Indeed, for all $0\le j\le n-1$ we have

$$\frac{\partial^j f}{\partial u_2^j}(u_1,u_2)=\sum_{0\le i_1<...<i_j\le m}\frac{f(u_1,u_2)}{\prod_{t\in\{i_1,...,i_j\}}(u_2-r_t(u_1))}$$

and thus

$$\frac{\partial^j f}{\partial u_2^j}(u_1,0)=(-1)^j\sum_{0\le i_1<...<i_j\le m}\frac{f(u_1,0)}{\prod_{t\in\{i_1,...,i_j\}}r_t(u_1)}=(-1)^{m-j}a_{0,m}\sum_{0\le i_1<...<i_j\le m}\prod_{t\notin\{i_1,...,i_j\}}r_t(u_1)$$

Note that this equality holds also for $j = 0$. Since $j < n$, each product $\prod_{t \notin \{i_1,...,i_j\}} r_t(u_1)$ contains at least one $r_j(u_1)$ with $j \le n$, and, since $r_j(0) = 0$ for $j \le n$, we have $\frac{\partial^j f}{\partial u_2^j}(0,0) = 0$ for all $0 \le j < n$. Now let $i + j \le n-1$. We will prove by induction on i (while j remains fixed) that $\frac{\partial^{i+j}}{\partial u_1^i \partial u_2^j} f(0,0) = 0$. If $i = 0$ we have already proved the result. Let $i > 0$. Then

$$\begin{aligned}
\frac{\partial^i}{\partial u_1^i}\left(\frac{\partial^j}{\partial u_2^j} f(u_1,0)\right) &= \lim_{u_1 \to 0} \frac{\frac{\partial^{i-1}}{\partial u_1^{i-1}}\left(\frac{\partial^j}{\partial u_2^j} f(u_1,0)\right)}{u_1} = \\
&= i! \lim_{u_1 \to 0} \frac{\frac{\partial^j}{\partial u_2^j} f(u_1,0)}{u_1^i} = \\
&= i!(-1)^{m-j} a_{0,m} \lim_{u_1 \to 0} \sum_{0 \le i_1 < \ldots < i_j \le m} \frac{\prod_{t \notin \{i_1,...,i_j\}} r_t(u_1)}{u_1^i} = 0
\end{aligned}$$

where in the first equality we have used the inductive hypothesis and the definition of derivative, in the second equality we have used L'Hopital's rule and in the last inequality the fact that, since $j \le n-1-i$, in each product $\prod_{t \notin \{i_1,...,i_j\}} r_t(u_1)$ appear at least $i+1$ terms equal to $r_j(u_1)$ with $j \le n$ and, if $j \le n$, $r_j(u_1) = O(u_1)$ for $u_1 \to 0$. This completes the proof. □

Let k be a positive intenger and $n = (n_1, ..., n_k) \in \mathbb{N}_+^k$.

Notation 4.1.3. For $h \ge 1$ set $C_h = \binom{h+1}{2} - 3$. We will denote by $X(d,n)$ the subset of $C^\infty(\mathbb{R}^d \times S, \mathbb{R})$ of those functions f such that f_t is of finite type for all $t \in \mathbb{R}^d$ and satisfying the following property : if f_t has a level set $f_t^{-1}(a)$ containing k distinct critical points with $b_f(x_i) \ge n_i$ for all $i = 1, ..., k$, then $\sum_{i=1}^k (C_{n_i} + 1) \le d + 1$. Finally define $X'(d) = \bigcap_{k \in \mathbb{N}_+, n \in \mathbb{N}_+^k} X(d,n)$.

Proposition 4.1.14. *For every $d \in \mathbb{N}$ and $n \in \mathbb{N}_+^k$ the set $X(d,n)$ is residual in $C_S^\infty(\mathbb{R}^d \times S, \mathbb{R})$.*

Before proceeding with the proof of the proposition the reader should read the notational remark A.1.1 in the appendix A in which we give the definition of the spaces $S^{(k)}$ and $J(S,\mathbb{R})^{(k)}$ and of the map Ψ_f for f a smooth map $S \to \mathbb{R}$.

Before starting the proof of the proposition we also point out a very simple observation from Linear Algebra.

Lemma 4.1.15. *Let V and W be vector spaces and let $U \subseteq V \oplus W$ be a vector subspace. Call $\pi : V \oplus W \to V$ the projection. Then*

$$\operatorname{codim}(\pi(U), V) \ge \operatorname{codim}(U, V \oplus W) - \dim W.$$

Proof. Let $z_1, ..., z_j$ be a basis of $\pi(U) \subseteq V$ and $z_1, ..., z_j, v_1, ..., v_m$ a basis of V (thus $m = \mathrm{codim}(\pi(U), V)$). Let $w_1, ..., w_k$ be a basis of W. Thus $z_1, ..., z_j, v_1, ..., v_m, w_1, ..., w_k$ is a basis of $V \oplus W$. Finally let $u_1, ..., u_j \in U$ be such that $\pi(u_i) = z_i$ for all $i = 1, ..., j$. In particular $u_1, ..., u_j$ are linearly independent and $j \leq \dim U$. Then $u_1, ..., u_j, v_1, ..., v_m, w_1, ..., w_k$ is still a basis of $V \oplus W$ and thus

$$\mathrm{codim}(\pi(U), V) + \dim W = m + k = \dim V + \dim W - j \geq \\ \geq \dim V + \dim W - \dim U = \mathrm{codim}(U, V \oplus W).$$

□

Proof of Proposition 4.1.14. This is an application of Theorem A.1.1. Consider the map $\tau : J^n(S, \mathbb{R})^{(k)} \to \mathbb{R}^k$ defined by $\tau(J^{n_1-1}_{x_1} f_1, ..., J^{n_k-1}_{x_k} f_k) = (f_1(x_1), ..., f_k(x_k))$ and let $\Delta = \tau^{-1}(\Delta_{\mathbb{R}^k})$. Here $\Delta_{\mathbb{R}^k} = \{(y_1, ..., y_k) \in \mathbb{R}^k : y_1 = ... = y_k\}$. Then Δ is a closed submanifold of codimension $k-1$. Put $Z \subseteq J^n(S, \mathbb{R})^{(k)}$ the submanifold consisting of those points $(J^{n_1-1}_{x_1} f_1, ..., J^{n_k-1}_{x_k} f_k)$ where each $J^{n_i-1}_{x_i} f_i$ is the class of a constant function. Then Z is a closed submanifold of codimension $\binom{n_1+1}{2} - 1 + ... + \binom{n_k+1}{2} - 1 = \sum_{i=1}^k C_{n_i} + 2k$. What is more, $Z \pitchfork \Delta$, thus $Y = Z \cap \Delta$ is a closed submanifold of $J^n(S, \mathbb{R})^{(k)}$ of codimension $\sum_{i=1}^k C_{n_i} + 3k - 1$.
Now observe that if $f \in C^\infty(\mathbb{R}^d \times S, \mathbb{R})$ is such that f_t is of finite type for all $t \in \mathbb{R}^d$ and $\Psi_f \pitchfork Y$, then, if non-empty, $\Psi_f^{-1}(Y)$ is a submanifold of $\mathbb{R}^d \times S^{(k)}$ of codimension $\sum_{i=1}^k C_{n_i} + 3k - 1$. Note that if $\Psi_f^{-1}(Y) = \emptyset$ then, by Lemma 4.1.13, $f \in X(d, n)$. Suppose $\Psi_f^{-1}(Y) \neq \emptyset$ and let $(t, x) \in \Psi_f^{-1}(Y)$. Call $\pi : \Psi_f^{-1}(Y) \to \mathbb{R}^d$ the projection map. Then $d \geq \mathrm{codim}(\mathrm{Im}(d_{(t,x)}\pi), \mathbb{R}^d) \geq \sum_{i=1}^k C_{n_i} + 3k - 1 - 2k$, where in the second inequality we applied the previous lemma with $d_{(t,x)}\pi$ in place of π, and thus $f \in X(d, n)$. Hence the proposition follows from Theorem A.1.1 and Theorem A.2.17. □

Corollary 4.1.16. *For every $d \in \mathbb{N}$ the set $X'(d)$ is residual in $C^\infty_S(\mathbb{R}^d \times S, \mathbb{R})$.*

Note that, since $C^\infty_S(\mathbb{R}^d \times S, \mathbb{R})$ is a Baire space, residual $\Rightarrow$ dense in this case.

Proof of the density of $X(d)$

Lemma 4.1.17. *Let $h : S \to \mathbb{R}$ be a Morse function with critical points having all different image under h. Then h is non-degenerate.*

Proof. Since $S = S_g$ with $g \geq 2$, we have $\sum (-1)^{\mathrm{ind}_x(h)} = \chi(S_g) < 0$ (here $\mathrm{ind}_x(h)$ is the Morse index of h at x) and thus h has a critical point x of index $\mathrm{ind}_x(h) = 1$. Set $a = h(x)$ and let $\varepsilon > 0$ be such that a is the only critical value of h in $[a - \varepsilon, a + \varepsilon]$. Call N the component of $h^{-1}([a - \varepsilon, a + \varepsilon])$ containing x and A_0 the component of $h^{-1}(a)$ containing x and $A_\pm = h^{-1}(a \pm \varepsilon) \cap N$. By hypothesis, x is the only critical point of h in N; in particular, A_0 is connected with one only singular point at x which is a saddle for h. It follows that A_0 is a figure 8.

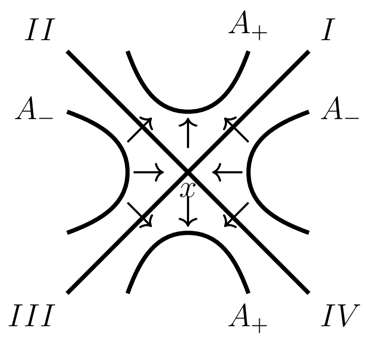


Figure 4.1: Levels and gradients at x.

One should ask how the four half-branches I, II, III and IV of A_0 are connected. We will do the case in which I connects to IV and II to III, the other cases are similar. Notice that A_+ is connected, for otherwise we could follow the top part of A_+ around A_0 and some gradient line would intersect twice A_+, but this is impossible.

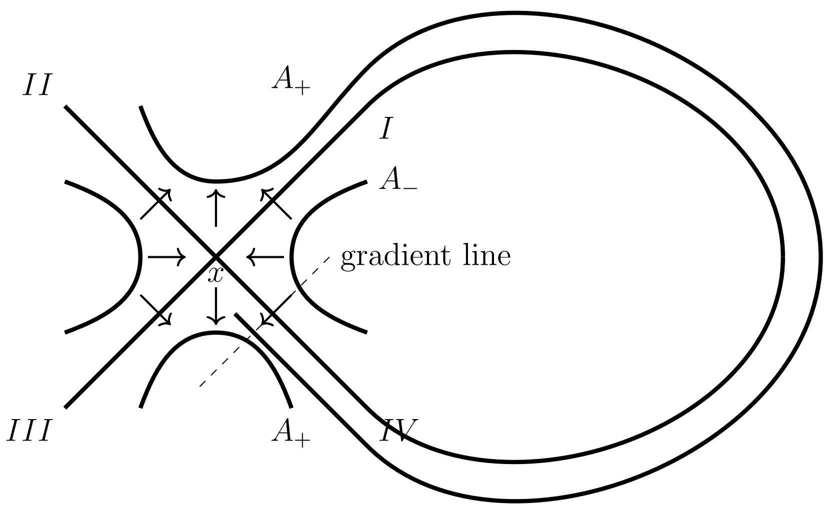


Figure 4.2: Impossible situation.

The same argument gives that both the right and the left branches of A_- must each close up. This describes how all the level sets around a critical point of odd index of h look like.
Now consider, in our case, a component of A_-. If it bounds a disk in S, then h has an extreme point in such a disk, in particular the index of h at this point is even. Since $\chi(S) < 0$ there must be a critical point x of odd index for h such that one component of A_- or A_+ does not bound a disk of S. In particular this component is a non-trivial circle of S.

□

Theorem 4.1.18. *For $d \leq e(S)+1 = -\chi(S)$, we have $X'(d) \subseteq X(d)$. In particular, $X(d)$ is dense in $C^\infty_S(\mathbb{R}^d \times S, \mathbb{R})$.*

Proof. Let $f \in X'(d)$. This means that for every $t \in \mathbb{R}^d$, the map f_t is of finite type and if $f_t^{-1}(a)$ contains k distinct critical points $x_1, ..., x_k$ such that for all $i = 1, ..., k$ $b_{f_t}(x_i) \geq n_i \geq 1$, then $\sum_{i=1}^k (C_{n_i} + 1) \leq d + 1$. We will call this property (P).
Fix t and call $g = f_t$. We will prove that g is non-degenerate. Suppose $g^{-1}(a)$ contains exactly k critical points $x_1, ..., x_k$ with $b_g(x_i) = n_i \geq 0$ for $i = 1, ..., k$. Then it is the disjoint union of a graph L with k vertices and $(2n_1+...+2n_k)/2 = n_1+...+n_k$ edges and some circles, in particular $\chi(g^{-1}(a)) = \chi(L) = k - n_1 - ... - n_k$. Let $\varepsilon > 0$ be such that a is the unique critical value of g in $[a-\varepsilon, a+\varepsilon]$. Since g is a proper map and the critical points of g are isolated, the critical value of g are also isolated and thus such an ε exists.
Let $L_\varepsilon \subseteq g^{-1}([a-\varepsilon, a+\varepsilon])$ be the union of those components of $g^{-1}([a-\varepsilon, a+\varepsilon])$ containing a component of L. For small ε, the space L_ε is a manifold with boundary that retracts by deformation onto L, in particular $\chi(L_\varepsilon) = k - n_1 - ... - n_k$. If some component of $\partial L_\varepsilon \subseteq g^{-1}(a-\varepsilon) \cup g^{-1}(a+\varepsilon)$ is a non-trivial circle of S then we are done. Suppose that every component of ∂L_ε is a trivial circle of S, equivalently that every component of ∂L_ε bounds a disk of S. We distinguish two cases.
Case 1: Suppose there is not a component of ∂L_ε that bounds a disk containing a component L. Then we can write $S = L_\varepsilon \cup \bigcup_{j=1}^m D^2$ where D^2 are disks of S bounded by the boundary components of L_ε. Moreover, $\chi(S) = \chi(L_\varepsilon) + \sum_{j=1}^m \chi(D^2) = \chi(L_\varepsilon) + m$. Observe that it must be $m \geq 2$, otherwise ∂L_ε is diffeomorphic to S^1 and thus $\chi(S) = -1$ while $\chi(S) < -1$. Therefore $-\chi(S) = n_1 + ... + n_k - k - m \leq n_1 + ... + n_k - k - 2$ and this inequality continue to hold if we discard those n_i equal to 0 or 1 and reduce k accordingly. We will assume $n_i \geq 2$ for all $i \leq k'$. Then $C_{n_i} - n_i + 2 \geq 0$ for all $i \leq k'$ and thus $0 \leq \sum_{i=1}^{k'} (C_{n_i} - n_i + 2)$ or equivalently $n_1 + ... + n_{k'} - k' - 2 \leq \sum_{i=1}^{k'} C_{n_i} + k' - 2$. We have finally obtained $d \leq -\chi(S) \leq n_1 + ... + n_{k'} - k' - 2 \leq \sum_{i=1}^{k'} C_{n_i} + k' - 2$, but this contradicts the properties of g.
Case 2: Suppose there is a component of ∂L_ε bounding a disk containing a compo-

nent L. In this case, consider a function that is equal to g outside this disk and has only one Morse critical point in the disk with image under h that is not a critical value of g. Note that in particular the branch number of h at this point is 0 and that h satisfy the property (P). Moreover, g has a level set with a non-singular component that is a non-trivial circle of S if and only if h has. This is because every non-singular component of any level set of h or g contained in the disk is trivial. Repeating the argument with h instead of g we see that every component of L is contained in a disk bounded by some component of ∂L_ε. Make a similar substitution for each of such disk. Finally repeat the argument for the other critical values of g obtaining a Morse function on S with critical point having all different image. From the previous lemma, this function is non-degenerate and thus g was non-degenerate, too. □

4.2 A remark on the action of $\mathbf{Mod}(S_g)[m]$ on $C(S_g)$

This last section is dedicated to prove a property of the action of Mod(S_g)[m] on $C(S_g)$.

Let $m \geq 3$ and $g \geq 2$. We want to prove that if $f \in$ Mod(S_g)[m] fixes a simplex of $C(S_g)$ then it fixes all the vertices of the simplex.

First a general result about graphs and automorphisms of graphs. For us, a graph can have loops, but multiple edges between two vertices are not allowed.

Lemma 4.2.1. *Let f be an automorphism of a finite connected graph X. If f fixes all the end vertices of X (that is edges from which only one edge emanates) and induces the identity on $H_1(|X|, \mathbb{Z}/m\mathbb{Z})$ then either f is the identity automorphism of X or $|X|$ is a circle and $|f|$ a rotation.*

Proof. First assume that X is a tree. Then each vertex of X is contained in a geodesic of X joining two end verteces, that is a path without return between two end vertices of X. But, if f fixes the end points of a geodesic then f fixes the entire geodesic. It follows that f is the identity.
Consider now the general case. Let T be a maximal tree in X and call $e_1, ..., e_n$ the edges of X that do not lie in T. Call g_i the unique geodesic of T joining the vertices of e_i and c_i the circle $g_i \cup e_i$ for $i = 1, ..., n$. If e_i is a loop then $g_i = \emptyset$. We choose an orientation for the circles c_i. Then $[c_1], ..., [c_n] \in H_1(|X|, \mathbb{Z}/m\mathbb{Z})$ is a basis of $H_1(|X|, \mathbb{Z}/m\mathbb{Z})$. Indeed, if we collapse an edge of $|X|$ with different end points we obtain a space that is homotopically equivalent through the quotient map, to $|X|$, so repeating this argument we see that the quotient map $p : |X| \to |X|/|T| = \bigvee_{i=1}^n |c_i|$ is an homotopy equivalence.

We now claim that $f(c_i) = c_i$ for all $i = 1, ..., n$.
Fix i. Let $e_{i_1}, ..., e_{i_k}$ be those edges between $e_1, ..., e_n$ that lie in $f(c_i)$. Then $H_1(|f|)([c_i]) = [c_i]$ must be linear combination with coefficients in $\mathbb{Z}/m\mathbb{Z}$ of $[c_{i_1}], ..., [c_{i_k}]$, it follows that $f(c_i)$ does not contain any e_j for $j \neq i$. Moreover, $f(c_i) \setminus e_i \subseteq T$ cannot be a circle, thus $e_i \in f(c_i)$ and $f(c_i) \setminus e_i$ is a geodesic of T joining the end points of e_i, by uniqueness of geodesics in a tree we obtain $f(c_i) = c_i$.
Consider now $\bigcup_{i=1}^{n} c_i$ and let $X_1, ..., X_k$ be its components. Since $f(c_i) = c_i$ for all i, it must be $f(X_i) = X_i$ for all i. Let X' be a new graph obtained from X by contracting each X_i to a point. Call f' the map induced by f. Since each end vertex of X' comes from an end vertex of X, the map f' fixes all the end vertices of X'.
Now we claim that X' is a tree and thus, from what we have seen at the beginning of the proof, f' must be the identity map.
The fact that X' is a tree can be seen as follows. Any circle C' in X' lifts to a circle C in X. If e_i occurs in C we can replace e_i with g_i and doing so for all e_i we obtain a closed curve $\tilde{C}$ that does not contain any e_i and thus is contained in T. Note that, orienting such curve, we obtain a non-trivial element of $H_1(|T|, \mathbb{Z}/m\mathbb{Z})$. This follows from the fact that $|\tilde{C}|$ is sent to the circle $|C'|$ in $|X'|$ under the qoutient map. But T is a tree and thus $H_1(|T|, \mathbb{Z}/m\mathbb{Z}) = 0$. This proves that X' cannot contain circles and must be a tree.
Now, since f' is the identity, f must fix all the edges of X not in $\bigcup_{i=1}^{n} c_i$.
Finally consider the action of f on c_i. Since $H_1(|f|) : H_1(|c_i|, \mathbb{Z}/m\mathbb{Z}) \to H_1(|c_i|, \mathbb{Z}/m\mathbb{Z})$ is the identity and $m \geq 3$ (thus $[c_i] \neq -[c_i]$)), f must be a rotation of c_i. If some edge not in $\bigcup_{i=1}^{n} c_i$ emanates from c_i then f must be the identity on c_i. If not, then, since X is connected, either $X = c_i$ or c_i intersect some c_j for $j \neq i$. If $X = c_i$, the lemma holds. Suppose that there exists $j \neq i$ such that $g_i \cap g_j = c_i \cap c_j \neq \emptyset$. Now $g_i \cap g_j$ is a geodesic of X preserved by f because $f(c_i \cap c_j) = f(c_i) \cap f(c_j) = c_i \cap c_j$, thus f is the identity on $g_i \cap g_j$ and thus f is the identity on c_i and on c_j.
The lemma follows. □

Theorem 4.2.2. *Let $\Delta \subseteq S_g$ be a 1-dimensional submanifold that is the union of finitely many non-trivial circles of S_g and let $\phi : S_g \to S_g$ be a diffeomorphism of S_g such that $\phi(\Delta) = \Delta$ and $H_1(\phi) : H_1(S_g, \mathbb{Z}/m\mathbb{Z}) \to H_1(S_g, \mathbb{Z}/m\mathbb{Z})$ is the identity. Then ϕ leaves each component of Δ invariant.*

Proof. Consider the following graph X: X has one vertex for each component of S_Δ and two (not necessarily distinct) verteces of X are connected by one edge iff the corresponding components of S_Δ have a boundary component that correspond to the same circle in Δ. Observe that X is connected. Clearly, ϕ induces an automorphism f of X. In addiction, there exists a continuous map $p : S_g \to |X|$ such that the inverse image of the midpoint of each edge is the component of Δ corresponding to this edge, $H_1(p) : H_1(S_g, \mathbb{Z}/m\mathbb{Z}) \to H_1(|X|, \mathbb{Z}/m\mathbb{Z})$ is surjective and the following diagram

$$\begin{array}{ccc} H_1(S_g, \mathbb{Z}/m\mathbb{Z}) & \xrightarrow{H_1(\phi)=1} & H_1(S_g, \mathbb{Z}/m\mathbb{Z}) \\ {\scriptstyle H_1(p)}\downarrow & & \downarrow{\scriptstyle H_1(p)} \\ H_1(|X|, \mathbb{Z}/m\mathbb{Z}) & \xrightarrow[H_1(|f|)]{} & H_1(|X|, \mathbb{Z}/m\mathbb{Z}) \end{array}$$

commutes. In particular, $H_1(|f|)$ must be the identity.
We claim that f fixes all the end vertices of X.
For, let v be an end vertex of X and call R the component of S_Δ corresponding to v. Since v is an end vertex, ∂R consist of a single component that is a non-trivial circle of S_g. Then R is not a disk and must have genus ≥ 1. Note that the map $H_1(R, \mathbb{Z}/m\mathbb{Z}) \to H_1(S_g, \mathbb{Z}/m\mathbb{Z})$ induced by the inclusion of R in S_g is injective. This is easily seen for R like in the picture below

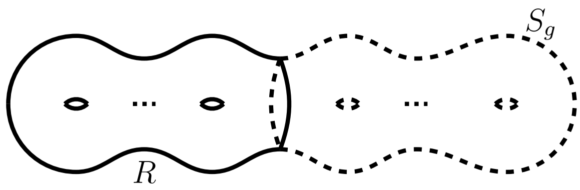


Figure 4.3 The easy case.

and the general case follows from Lemma 2.3.36.
The map $H_1(\phi)$ takes $H_1(R, \mathbb{Z}/m\mathbb{Z})$ (considered as a subset of $H_1(S_g, \mathbb{Z}/m\mathbb{Z})$) to the image of $H_1(\phi(R), \mathbb{Z}/m\mathbb{Z}) \hookrightarrow H_1(S_g, \mathbb{Z}/m\mathbb{Z})$. There are two possibilities: either $\phi(R) = R$ (and thus f fixes v) or $\phi(R) \neq R$. In the latter case, since $\phi(\partial R) \subseteq \Delta$, it must be $\phi(R) \cap R \subseteq \partial R$. But, then, $H_1(\phi(R), \mathbb{Z}/m\mathbb{Z}) \neq H_1(R, \mathbb{Z}/m\mathbb{Z})$ as subsets of $H_1(S_g)$ contradicting the fact that $H_1(\phi)$ is the identity. This completes the proof of our claim.
Now, applying the previous lemma, we obtain that f is the identity or X is a circle and f is a rotation. If f is the identity we are done. Suppose, instead, that X is a circle and f is a non-trivial rotation. In particular, the boundary of every component of S_Δ has two components each of which is a non-trivial circle of S_g. Now we see that in this case every component R of S_Δ must be an annulus, but this would mean that S_g is a torus while by assumption $g \geq 2$. In this way we will have proved the theorem. Suppose that R is not an annulus, i.e. that has genus ≥ 1 and thus $H_1(R, \mathbb{Z}/m\mathbb{Z}) \neq 0$. Since f is a non-trivial rotation we have $\phi(R) \neq R$, thus $\phi(R) \cap R \subseteq \partial R$ and, as above, this contradicts the fact that $H_1(\phi) = 1$. □

Finally, combining Theorem 4.2.2 and Corollary 2.3.39 we immediately obtain

Proposition 4.2.3. *Let $m \geq 3$ and $g \geq 2$. Let $f \in \mathrm{Mod}(S_g)[\mathrm{m}]$ and σ be a simplex of $C(S_g)$. If f fixes σ, then f fixes every vertex of σ.*

Chapter 5

Teichmüller space

This chapter introduces another protagonist of our story, namely the Teichmüller Space of a surface. It will be a smooth manifold diffeomorphic to some Euclidean Space. In particular, we will be interested in a submanifold with corners of the Teichmüller space obtained by discarding some pieces near infinity.

5.1 Definition of Teichmüller space

Let $S = S_g^b$ with $\chi(S) < 0$.

Definition 5.1.1. A **hyperbolic structure** on S is the data of a diffeomorphism $\phi : S \to X$ where X is a surface endowed with a hyperbolic metric.

We can record such a hyperbolic structure by the pair (X, ϕ).

Definition 5.1.2. Two hyperbolic structures (X_1, ϕ_1) and (X_2, ϕ_2) are said **homotopic** if there is an isometry $i : X_1 \to X_2$ so that $i \circ \phi_1$ and ϕ_2 are homotopic maps. Here homotopies are allowed to move points of the boundary of S, but must preserve ∂S setwise.

Definition 5.1.3. The **Teichmüller space** of S is

$$T_S = \{\text{hyperbolic structures on } S\}/\text{homotopy}$$

Equivalently, if $\mathrm{HypMet}(S)$ is the set of all hyperbolic metrics on S, we have a right action of $\mathrm{Diff}_0(\mathrm{S})$ on $\mathrm{HypMet}(S)$ given by the pullback and

$$T_S = \mathrm{HypMet}(S)/\mathrm{Diff}_0(S).$$

The equivalence of the two definitions follows from the results of chapter 2.

5.2 Fenchel-Nielsen coordinates

We will now introduce on T_S, where $S = S_g^b$, a structure of smooth manifold with respect to which it will be diffeomorphic to $\mathbb{R}^{6g-6+3b}$.

Length functions on S

Call $\mathcal{S}$ the set of all homotopy classes of non-trivial circles of S and let $\mathcal{X} \in T_S$.

Definition 5.2.1. The length function of $\mathcal{X}$ is the function

$$l_{\mathcal{X}} : \mathcal{S} \to \mathbb{R}_+$$

defined as as follows. Let C be a representative of an homotopy class and h a hyperbolic metric that is a representative of the equivalence class $\mathcal{X}$. Set

$l_{\mathcal{X}}([C]) =$ length of the unique geodesic circle (with respect to h) of S in the isotopy class of C

Note that this definition does not depend on the choice of the representatives.

5.2.1 Definition of the Fenchel-Nielsen coordinates

Fix once and for all a pants decomposition $\Delta = \{C_1, ..., C_{3g-3+b}\}$ of S. We will associate to Δ a bijection

$$\Psi = \Psi(\Delta) : \mathbb{R}_+^{3g-3+2b} \times \mathbb{R}^{3g-3+b} \to T_S.$$

in such a way that, for each two pants decomposition Δ_1 and Δ_2, the composition

$$\Psi(\Delta_1)^{-1} \circ \Psi(\Delta_2)$$

is a smooth map and thus we will have provided T_S with a smooth structure.

Actually, we will not prove that this is an atlas for T_S, but we will just define the functions Ψ.

Hexagons and pair of pants

Definition 5.2.2. By a **marked hexagon** (or a marked pentagon) we will mean a hexagon (or a pentagon) with one vertex distinguished.

Let $\mathcal{H}$ denote the set of all equivalence classes of marked right-angled geodesic hexagons in $\mathbb{H}^2$, where two marked hexagons are equivalent if there exists an orientation-preserving isometry of $\mathbb{H}^2$ taking one hexagon to the other and the marked point of the first to the marked point of the second.

Lemma 5.2.3. *The map $W : \mathcal{H} \to \mathbb{R}^3_+$ defined by taking the lengths of every other side of a hexagon, starting at the marked point and traveling counterclockwise, is a bijection.*

Before starting the proof of the lemma, it is better to make some general remarks.

Remark 5.2.4. It is a stardand fact from hyperbolic geometry that for two disjoint geodesic lines in $\mathbb{H}^2$ with four distinct endpoints in $\partial\mathbb{H}^2$ passes a unique third geodesic line that is perpendicular to both. A possible proof is the following. Using the half-plane model of $\mathbb{H}^2$, we may assume that one of the two geodesic line is $G_1 = \{(x_1, x_2) \in \mathbb{H}^2 : x_1 = 0\}$ and the other, G_2, is a semicircle that can be parametrized by $\gamma(t) = (x_0 + R\cos(t), \sin(t))$ for $t \in (0, \pi)$, where $0 < R < |x_0|$. The generic geodesic line perpendicular to G_1 can be parametrized by $\alpha(t) = (r\cos(t), r\sin(t))$ for $t \in (0, \pi)$. Imposing that this curve meets perpendicularly G_2 we obtain a unique solution for $r > 0$.

Remark 5.2.5. Let H and H' be two marked geodesic right-angled hexagons in $\mathbb{H}^2$ with sides (enumerated starting from the distinguished point and traveling counterclockwise) $a_1, b_1, a_2, b_2, a_3, b_3$ and $a_1', b_1', a_2', b_2', a_3', b_3'$ respectively. Suppose that $l(a_1) = l(a_1'), l(b_1) = l(b_1')$ and $l(a_2) = l(a_2')$, then H and H' define the same class in $\mathcal{H}$. This follows from the Remark 5.2.4 and the fact that the orientation-preserving isometries of $\mathbb{H}^2$ act transitively on the unit tangent bundle $UT(\mathbb{H}^2)$.
A similar conclusion holds for marked geodesic right-angles pentagons in $\mathbb{H}^2$. If two marked geodesic right-angled pentagons H and H' in $\mathbb{H}^2$ have sides (enumerated starting from the distinguished point and traveling counterclockwise) a_1, b_1, a_2, b_2, a_3 and $a_1', b_1', a_2', b_2', a_3'$ respectively and $l(a_1) = l(a_1'), l(b_1) = l(b_1')$, then there is an orientation-preserving isometry of $\mathbb{H}^2$ taking H to H' and making the marked points to correspond.

Proof of Lemma 5.2.3. We first prove that W is surjective. Let $(l_1, l_2, l_3) \in \mathbb{R}^3_+$, we want to construct a right-angled geodesic hexagon H in $\mathbb{H}^2$ such that the lengths of three pairwise non-consecutive sides of H are l_1, l_2 and l_3. We start by defining two functions $f, g : (0, \infty) \to (0, \infty)$ that will be useful in the proof. Let $l > 0$ and fix three geodesics G, G' and G'' in $\mathbb{H}^2$ as shown in the figure below.

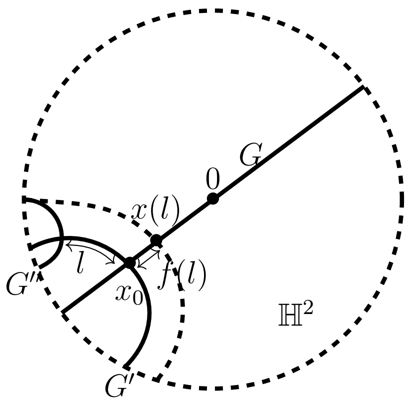


Figure 5.1: Definition of f.

Here we are using the Poincaré disk model of $\mathbb{H}^2$. The geodesics G, G' and G'' are pairwise orthogonal and the distance between $G'' \cap G'$ and $G' \cap G$ is l. The point $x(l)$ is the point of G closest to x_0 and lying on the right side of x_0 in G such that the geodesic line perpendicular to G at $x(l)$ does not intersect the geodesic G''. Finally, $f(l) = d_{\mathbb{H}^2}(x_0, x(l))$. Note that $f(l)$ does not depends on the choice of $x_0 \in G$. This is because for all $x \in G$ there is an isometry of $\mathbb{H}^2$ preserving G and its orientation, and sending x_0 to x. The function g is defined in the same manner as f, but now $g(l) = d_{\mathbb{H}^2}(x_0, y(l))$, where the point $y(l)$ is the point of G closest to x_0 and lying on the left side of x_0 in G such that the geodesic line perpendicular to G at $y(l)$ does not intersect the geodesic G''. Now for all $\lambda \geq 0$ we make the construction shown below

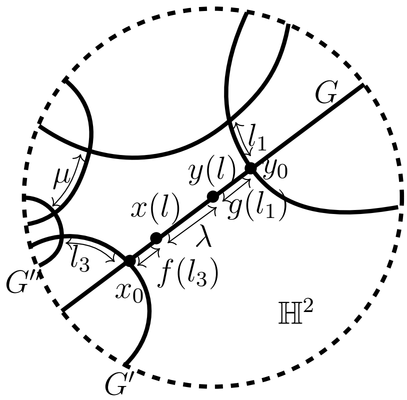


Figure 5.2: Construction of a right-angles hexagon H such that $W(H) = (l_1, l_2, l_3)$.

Here all the angles are right angles and the geodesic line in which lies the arc

of length μ is the unique geodesic line in $\mathbb{H}^2$ perpendicular to both the other two geodesic lines of $\mathbb{H}^2$ that pass through the vertices of the arc of length μ. The function $\mu : [0, \infty) \to [0, \infty)$ is a continuous function in λ with $\mu(0) = 0$ and $\mu(\infty) = \infty$. Thus there exists $\lambda \in [0, \infty)$ such that $\mu(\lambda) = l_2$. In this way, we have constructed a right-angles hexagon in $\mathbb{H}^2$ with marked point y_0 and such that $W(H) = (l_1, l_2, l_3)$.
Next we prove injectivity. Consider two marked geodesic right-angled hexagons H and H' in $\mathbb{H}^2$. Let $a_1, b_1, a_2, b_2, a_3, b_3$ and $a_1', b_1', a_2', b_2', a_3', b_3'$ be the sides (enumerated starting from the distinguished point and traveling counterclockwise) of H and H' respectively. Suppose that $l(a_i) = l(a_i')$ for all $i = 1, 2, 3$ and, by absurd, that H and H' do not define the same class in $\mathcal{H}$. Then, by Remark 5.2.5, it must be $l(b_1) \neq l(b_1)$. Suppose $l(b_1') > l(b_1)$. Consider the unique geodesic line L in H perpendicular to both the side b_1 and a_3. This line divides a_3 into two arcs of lengths γ and δ and b_1 into two arcs of lenghts α and β. Say $d_{\mathbb{H}^2}(L \cap a_3, b_3 \cap a_3) = \gamma$ and $d_{\mathbb{H}^2}(L \cap a_3, b_2 \cap a_3) = \delta$, $d_{\mathbb{H}^2}(L \cap b_1, a_1 \cap b_1) = \alpha$ and $d_{\mathbb{H}^2}(L \cap b_1, b_1 \cap a_2) = \beta$. Clearly $l(a_3) = \gamma + \delta$ and $l(b_1) = \alpha + \beta$. Now erect the geodesic line L_1 perpendicular to b_1' at distance α from $a_1' \cap b_1'$ and the geodesic line L_2 perpendicular to b_1' at distance β from $b_1' \cap a_2'$. See the picture below.

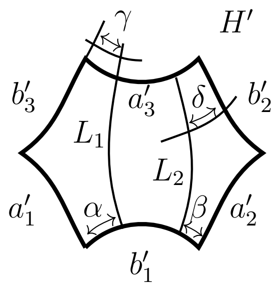


Figure 5.3: Proof of injectivity of W.

In this picture all the angles are right angles except for that formed by L_1 and a_3' and that formed by L_2 and a_3'.
Note that the geodesic lines L_1 and L_2 intersect $\partial H'$ in a point of a_3', other than in the point of b_1' from which they start out. To see that $L_2 \cap a_2' = \emptyset$ we can use the Gauss Bonnet Theorem (see Theorem 1.4.2 in chapter 1 of [27]), after that we know that there exist a (unique) right-angled pentagon with two consecutive edges of lengths β and $l(a_2')$, thus we can construct such a pentagon as shown in the figure inside H', in particular L_2 can intersect b_2' only above such a pentagon, but this is forbidden by the Gauss-Bonnet Theorem. This also proves that the lengths of certain edges in the picture are γ or δ. Note that, by the first variational formula (see Theorem 1.5 in chapter 1 of [8]), the distance in $\mathbb{H}^2$ between L_1 and b_3' is γ and

between L_2 and b'_2 is δ.
From this, we obtain that $l(a'_3) > \gamma + \delta = l(a_3)$ and this is a contradiction. $\square$

Lemma 5.2.6. *Let P be a pair of pants endowed with a hyperbolic metric. For $i = 1, 2, 3$, call ∂_i its boundary components. For all $1 \leq i \neq j \leq 3$ there exists a unique geodesic $\delta_{i,j}$ joining ∂_i and ∂_j and meeting them perpendicularly. Moreover, the $\delta_{i,j}$ are pairwise disjoint proper arcs of P.*

Proof. Consider the double $\bar{P}$ of P. Then $\bar{P}$ is a closed surface of genus $g = 2$ and, since P has geodesic boundary it can be endowed with a hyperbolic metric that restricted to $P \subseteq \bar{P}$ is exactly the metric of P. for $i = 1, 2$ let the C_i be the circle of $\bar{P}$ that goes around the i-th hole of $\bar{P}$. Note that they can be chosen to be disjoint. For each i replace C_i with the unique geodesic circle of $\bar{P}$ in its isotopy class. By symmetry each C_i must intersect the boundary of P perpendicularly. Take as δ_{ij} the pieces of the C_i that lie in P. This proves the existence of the δ_{ij}. Moreover, the C_i are disjoint by Corollary 2.3.23, thus the δ_{ij} are disjoint, too. Finally we prove the uniqueness. Any geodesic δ_{ij} joining ∂_i and ∂_j and meeting them perpendicularly is contained in a unique geodesic circle of $\bar{P}$ that goes around a hole of $\bar{P}$. Thus the uniqueness of the δ_{ij} follows from the uniqueness of the geodesic circle in any isotopic class of circles. $\square$

Proposition 5.2.7. *Let P be a pair of pants with boundary components $\partial_1, \partial_2, \partial_3$. Then, the map*

$$T_P \to \mathbb{R}^3_+ \quad \mathcal{X} \mapsto (l_{\mathcal{X}}(\partial_1), l_{\mathcal{X}}(\partial_2), l_{\mathcal{X}}(\partial_3))$$

is a bijection.

Proof. First we prove it is surjective. Let $(l_1, l_2, l_3) \in \mathbb{R}^3_+$. Consider $H \in W^{-1}(l_1/2, l_2/2, l_3/2)$ a marked hexagon in $\mathbb{H}^2$. The map W here is the map of the previous proposition. Create a second hexagon H' be reflecting H over the edge lying first in the clockwise direction from the marked point. Enumerate the edges of H from 1 to 6 starting from the marked point and traveling counterclockwise. Similarly, enumerate the edges of H' from 1 to 6 starting from the marked point and traveling clockwise. Finally identify the edge number i of H with the edge number i of H' for $i = 2, 4, 6$ in such a way to obtain a pair of pants endowed with a hyperbolic structure with respect to which the boundaries have lengths l_1, l_2 and l_3. Vice versa, let (X_1, ϕ_1) and (X_2, ϕ_2) be two hyperbolic surfaces diffeomorphic to S via the ϕ_k and such that the lengths of $\phi_k(\partial_1), \phi_k(\partial_2)$ and $\phi_k(\partial_3)$ are respectively l_1, l_2 and l_3. For each $k = 1, 2$ let $\delta^{(k)}_{i,j}$ be as in the previous lemma for X_k. Cut X_k along all the $\delta^{(k)}_{i,j}$, obtaining two hyperbolic hexagons H_k and H'_k. Choose in each of them the point in $\delta^{(k)}_{13} \cap \phi_k(\partial_1)$ as distinguished point. Then, Lemma 5.2.3 gives that H_k and H'_k are isometric by an orientation-preserving isometry taking the distinguished point of the first to the distinguished point of the second and making the $\delta^{(k)}_{i,j}$ correspond (note that any right-angled hyperbolic hexagon is isometric to a right-angles hexagon in $\mathbb{H}^2$ because it can first be isometrically embedded into a pair of pants as done above and then, being simply connected, lifted to the universal covering). In particular, this means that the $\delta_{i,j}^{(k)}$ cut each $\phi_k(\partial_i)$ in two pieces of equal length $l_i/2$. Again, Lemma 5.2.3 gives that H_1 and H_2 are isometric through an orientation-preserving isometry and the same is true for H_1' and H_2'. Moreover, such isometries make to correspond the distinguished points and the boundary arcs associated to the ∂_i. It follows that (X_1, ϕ_1) and (X_2, ϕ_2) are isometric through an orientation-preserving isometry i such that $\phi_2^{-1} \circ i \circ \phi_1$ fixes setwise the components of ∂P and thus is isotopic to 1_P (see Example 3.1.12). □

The surjection L_Δ

Recall that $\Delta = \{C_1, ..., C_{3g-3+b}\}$ is a fixed pants decomposition of S.

Corollary 5.2.8. *The map*

$$L_\Delta : T_S \to \mathbb{R}_+^{3g-3+2b} \quad L_\Delta(\mathcal{X}) = (l_\mathcal{X}(C_1), ..., l_\mathcal{X}(C_{3g-3+b}), l_\mathcal{X}(\partial_1), ..., l_\mathcal{X}(\partial_b))$$

is surjective.

Proof. Let $(l_1, ..., l_{3g-3+2b}) \in \mathbb{R}_+^{3g-3+2b}$. Consider the decomposition in components of S_Δ: $S_\Delta = P_1 \sqcup .... \sqcup P_n$. If $\partial_{i,j}$ are the boundary components of the P_i, using the previous result, we can endow each P_i with a hyperbolic metric such that if $\partial_{i,j}$ corresponds to the circle C_h in Δ then its length is l_h and if it corresponds the the boundary component ∂_h of S its length is $l_{h+3g-3+b}$. Finally, gluing back the pairs of pants, we obtain a hyperbolic metric on S whose class in T_S is sent, under L_Δ, to $(l_1, ..., l_{3g-3+2b})$. □

Construction of a section of L_Δ

The goal of this subsection is to find a (canonical) section of the surjection L_Δ.

Lemma 5.2.9. *There exists a collection Δ' of circles and proper arcs of S containing Δ and such that $S_{\Delta'}$ is union of disks with exactly 6 corners points.*

Proof. For each pair of pants P_i of S_Δ with boundary component $\partial_{i,1}, \partial_{i,2}$ and $\partial_{i,3}$ consider proper and disjoint arcs $\delta_{i,j}$ of P_i joining $\partial_{i,j}$ with $\partial_{i,j+1}$ (where $\partial_{i,4} = \partial_{i,1}$ for convention). Consider the boundary component $\partial_{i,j}$. If it correspond to a circle of Δ there is another boundary component $\partial_{i',j'}$ corresponding to the same circle. Glue $\partial_{i,j}$ and $\partial_{i',j'}$ in such a way that $\partial_{i,j} \cap \delta_{i,j}$ correspond to $\partial_{i',j'} \cap \delta_{i',j'}$ and $\partial_{i,j} \cap \delta_{i,j-1}$ to $\partial_{i',j'} \cap \delta_{i',j'-1}$ (where by convention $\partial_{i,0} = \partial_{i,3}$ for all i). Repeat this process for all $\partial_{i,j}$ corresponding to some circle of Δ, we obtain a surface S' that is diffeomorphic to S through a diffeomorphism that sends Δ to a pants decomposition of S' for which the lemma holds. Therefore the lemma holds for Δ, too. □

Fix a collection Δ' as in the lemma. Note that in the lemma we are just saying that $S_{\Delta'}$ is union of hexagons. Moreover each hexagon is naturally contained in some pair of paint of S_Δ and each pair of pants of S_Δ is obtained by suitably gluing two hexagons. To fix the notation, let $P_1, ..., P_k$ be the pair of pants in S_Δ and let $H_{i,k}$ for $k = 1, 2$ be the hexagons of $S_{\Delta'}$ inside P_i. For $k = 1, 2$, call $\iota_{i,k} : H_{i,k} \to P_i$ the natural embedding of $H_{i,k}$ in P_i. Note that the P_i and the $H_{i,k}$ come with a natural orientation induced by that of S. For $j = 1, 2, 3$, let $\partial_{i,j}$ be the boundary components of P_i and, for convenience, set $\partial_{i,4} = \partial_{i,1}$. For $k = 1, 2$, call $\partial_{i,j,k} = \partial_{i,j} \cap \iota_{i,k}(H_{i,k})$ and $\delta_{i,j,k}$ the arc of $\partial H_{i,k}$ joining $\iota_{i,k}^{-1}(\partial_{i,j,k})$ and $\iota_{i,k}(\partial_{i,j+1,k})$. Finally call $\delta_{i,j}$ the arc of P_i joining $\partial_{i,j}$ with $\partial_{i,j+1}$ given by $\iota_{i,1}(\delta_{i,j,1}) = \iota_{i,2}(\delta_{i,j,2}) = \delta_{i,j}$.
We can choose, for all i, k, orientation-preserving diffeomorphisms $\phi_{i,k} : H_{i,k} \to H'_i$ where $H'_i \subseteq \mathbb{H}^2$ are hyperbolic right-angled hexagons such that:

1. the length of $\phi_{i,k}(\partial_{i,j,k})$ is $l_h/2$ if $\partial_{i,j}$ comes from the circle C_h of Δ and length $l_{h+3g-3+b}$ if it comes from the boundary component ∂_h of S;
2. the identity $\iota_{i,1} \circ \phi_{i,1}^{-1} \circ \phi_{i,2}|_{\delta_{i,j,2}} = \iota_{i,j,1}|_{\delta_{i,j,2}}$ holds for all i, j;
3. whenever $\iota_{i,k}(\partial_{i,j,k}) = \iota_{i',k'}(\partial_{i',j',k'})$, we also have $\phi_{i,k}(\partial_{i,j,k}) = \phi_{i',k'}(\partial_{i',j',k'})$ and the identity of maps $\iota_{i,k} \circ \phi_{i,k}^{-1} \circ \phi_{i',k'}|_{\partial_{i',j',k'}} = \iota_{i',j'}|_{\partial_{i',j',k'}}$ holds.

Then S admits a (unique) hyperbolic metric h such that $(\phi_{i,k}^{-1})^* \iota_{i,k}^*(h)$ is exactly the hyperbolic metric of H'_i for all i, k and, from 1, we have $\Psi([h]) = (l_1, ..., l_{3g-3+2b})$. Thus we have defined a section

$$\sigma : \mathbb{R}_+^{3g-3+2b} \to T_S$$

of L_Δ. Note that $[h]$ does not depend on the choice of H'_i or of the diffeomorphisms $\phi_{i,k}$, but it depends on the choice of Δ'.

Definition of Ψ

We now introduce an action Θ of $\mathbb{R}^{3g-3+b}$ on T_S such that

$$\Psi : \mathbb{R}_+^{3g-3+2b} \times \mathbb{R}^{3g-3+b} \to T_S \quad \Psi(l, \theta) = \Theta_\theta(\sigma(l))$$

has the properties claimed at the beginning of this section.

We will just define Ψ. For the verifications of the claimed properties see chapter 2 of [1].

Let $\mathcal{X} = [h] \in T_S$. Replace the circles C_i of Δ with the corresponding geodesic circles (with respect to h). We will call these circles C_i again. For each $x \in C_i$ let

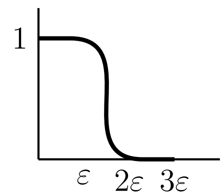
γ_x be the geodesic line starting at x, orthogonal to C_i and oriented in such a way that $(\alpha_i'(0), \gamma_x'(0))$ is a positive basis of T_xS. Here $\alpha_i : S^1 \to C_i$ is a parametrization of C_i (the definition of $\Theta_\theta(\mathcal{X})$ will be independent from the choice of the orientation of the C_i induced by the α_i, but will only depend on the orientation of S). We shall assume that both C_i and δ_x are parametrized by arc length.
Since C_i is compact, there is an $\varepsilon > 0$ such that

$$C_i \times [0, 3\varepsilon] \to S \quad (x, t) \mapsto \gamma_x(t)$$

is a diffeomorphism onto its image U_i. Clearly, we may assume that $U_i \cap U_j = \emptyset$ for $i \neq j$.
Now we use this diffeomorphism to define a diffeomorphism ϕ_i of U_i depending on $\theta \in \mathbb{R}^{3g-3+b}$. Consider the universal covering $\mathbb{R}$ of C_i chosen in such a way that the projection p is an orientation preserving local isometry. Note that if $a = l_{\mathcal{X}}(C_i)$ then $p(t) = p(s)$ if and only if $s = t + na$ for some $n \in \mathbb{Z}$. Consider the diffeomorphism

$$\widetilde{\phi_i} : \mathbb{R} \times [0, 3\varepsilon] \to \mathbb{R} \times [0, 3\varepsilon] \quad \widetilde{\phi_i}(t, s) = (t + \rho(s)\theta_i, s)$$

where ρ is the smooth map



Note that $\widetilde{\phi_i}(t_1 + t_2, s) = \widetilde{\phi_i}(t_1, s) + (t_2, 0)$ for all $t_1, t_2 \in \mathbb{R}$ and $s \in [0, 3\varepsilon]$ and thus $\widetilde{\phi_i}$ induces a diffeomorphism

$$\phi_i : C_i \times [0, 3\varepsilon] \to C_i \times [0, 3\varepsilon]$$

that is an isometry on a neighborhood of $C_i \times \{0\}$ and the identity out of $C_i \times [0, 2\varepsilon]$.
It follows that

$$h' = \begin{cases} h & \text{outside } \bigcup_{i=1}^{3g-3+b} U_i; \\ \phi_i^*(h) & \text{in } U_i, \end{cases}$$

is a new (smooth) hyperbolic metric on S. Observe that h' depends on many choices we have done: h, α_i, ε, ρ; but its class in T_S does not. Set $\Theta_\theta(\mathcal{X}) = [h'] \in T_S$.

5.3 Moduli Space

Now we will study the Moduli Space of a surface, defined as the quotient of the Teichmüller space under the action of the Mapping Class Group.

5.3.1 The action of Mod(S) on T_S

Let $S = S_g^b$ be a surface with $\chi(S) < 0$. Regarding the points of T_S as equivalence classes of pairs (X, ϕ) where X is a hyperbolic surface and $\phi : S \to X$ is a diffeomorphism, we can introduce a left action of $\mathrm{Mod}(S)$ on T_S as follows: for $f \in \mathrm{Mod}(S)$ and $\mathcal{X} = [(X, \phi)] \in T_S$ choose a representative $\psi \in \mathrm{Diff}^+(S)$ of f and set

$$f \cdot \mathcal{X} = [(X, \phi \circ \psi^{-1})].$$

Definition 5.3.1. The **Moduli Space** of S is the quotient $\mathcal{M}(S) = T_S/\mathrm{Mod}(S)$.

The Collar Lemma

In order to investigate the properties of this action, we first recall the so called Collar Lemma and derive some of its corollaries.

The existence of a right-angled geodesic pentagon in $\mathbb{H}^2$ is easily established.

Lemma 5.3.2. *For any right-angled geodesic pentagon in $\mathbb{H}^2$ with consecutive sides a, b, α, c, β (listed traveling counterclockwise) we have*

$$\cosh(l(c)) = \sinh(l(a))\sinh(l(b))$$

Proof. See Theorem 2.3.4 in chapter 2 of [7]. □

Proposition 5.3.3 (Collar Lemma)**.** *Let P be a pair of pants endowed with a hyperbolic metric. Let $\partial_1, \partial_2, \partial_3$ be the boundary components of P. Then, the sets*

$$\mathcal{C}_i = \{x \in P : \sinh(\mathrm{dist}(x, \partial_i))\sinh(\frac{l(\partial_i)}{2}) \leq 1\}$$

are disjoints.

Proof. Let δ_{ij} be as in Lemma 5.2.6. The closure of the two components of $P \setminus (\cup\delta_{ij})$ are hyperbolic right-angled hexagons H_1 and H_2. By Lemma 5.2.3, H_1 and H_2 are isometric. Fix $H = H_i$ for $i \in \{1, 2\}$. We now show that $\mathcal{C}_i \cap \mathcal{C}_j \cap H = \emptyset$ if $i \neq j$. Let k be such that $\{i, j, k\} = \{1, 2, 3\}$. Call h_k the shortest path in H from ∂_k to $\delta_{i,j}$. Such an h_k exists because $\partial_k \cap H$ and δ_{ij} are compact. Then, h_k must be a geodesic path in H and, by the first variational formula, it must be perpendicular to both ∂_k and $\delta_{i,j}$ (See the figure below). It follows that the distance between h_k and $\partial_i \cap H$ is the length of the arc of δ_{ij} between the two points $\delta_{ij} \cap \partial_i$ and $h_k \cap \delta_{ij}$; and, for the same reason, the distance between h_k and δ_{ik} is the length of the arc of $\partial_k \cap H$ between the two points $\delta_{i,k} \cap \partial_k$ and $\partial_k \cap h_k$. Thus the previous lemma gives

$$1 < \cosh(\mathrm{dist}(h_k, \delta_{i,k})) = \sinh(\frac{l(\partial_1)}{2})\sinh(\mathrm{dist}(h_k, \partial_i \cap H))$$

In particular, h_k does not meet $\mathcal{C}_i$ and the result is now obvious.

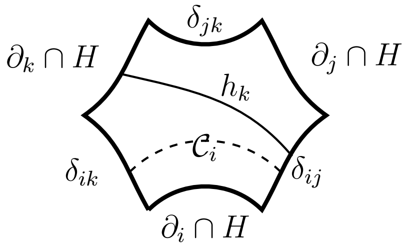


Figure 5.4: Explanation of the proof.

□

Corollary 5.3.4. *Let Δ be a pants decomposition of a hyperbolic surface S and for all $C_i \in \Delta \cup \{\text{boundary components of } S\}$ set*

$$\mathcal{C}_i = \{x \in S : \sinh(\text{dist}(x, C_i)) \sinh(\frac{l(C_i)}{2}) \leq 1\}.$$

Then the $\mathcal{C}_i$ are disjoint annuli in S.

Corollary 5.3.5. *Let S be a surface with $\chi(S) < 0$ and $0 < \varepsilon < 2\sinh^{-1}(1/2)$. Any hyperbolic metric on S has the following property: if C_1 and C_2 are two geodesic circles in S of length $\leq \varepsilon$ then either $C_1 = C_2$ or $C_1 \cap C_2 = \emptyset$.*

Proof. Indeed, suppose that $C_1 \neq C_2$ are two geodesic circles of S with $x_0 \in C_1 \cap C_2$ and $l(C_i) \leq \varepsilon$ for $i = 1, 2$. Note that they meet transversally at x_0. Let, for $i = 1, 2$, $\widetilde{C_i}$ be a lift of C_i in $\mathbb{H}^2$ with $z_0 \in \widetilde{C_1} \cap \widetilde{C_2}$ in the fiber of x_0. Since $\widetilde{C_1}$ and $\widetilde{C_2}$ are geodesics, the point z_0 is the only intersection point between them. Consider the subset of $\mathbb{H}^2$

$$\widetilde{\mathcal{C}} = \{z \in \mathbb{H}^2 : \sinh(\text{dist}(z, \widetilde{C_1})) \sinh(\frac{l(C_1)}{2}) \leq 1/2\}.$$

Then $\widetilde{C_2}$ must connect two boundary components of $\widetilde{\mathcal{C}}$, otherwise C_2 would not be a closed curve. Thus,

$$2\sinh^{-1}(1/2) > l(C_2) = l(\widetilde{C_2}) \geq 2\sinh^{-1}(\frac{1}{2\sinh(\frac{l(C_1)}{2})}) \geq 2\sinh^{-1}(1)$$

and this is a contradiction. □

Proper discontinuiy of the action

Recall that the action of a group G on a topological space X by homeomorphisms is **properly discontinuous** if, for any compact set $B \subseteq X$, the set

$$\{g \in G : g \cdot B \cap B \neq \emptyset\}$$

is finite.

Our goal is now to prove the following important result.

Theorem 5.3.6. *Let $g \geq 2$. The action of $\mathrm{Mod}(S_g)$ on T_{S_g} is properly discontinuous.*

Proof. Suppose, by contradiction, that that there is a compact subset $K \subseteq T_{S_g}$ and a sequence $\{f_n = [\phi_n]\}_{n\in\mathbb{N}}$ of distinct elements of $\mathrm{Mod}(S_g)$ such that $f_n \cdot K \cap K \neq \emptyset$ for all $n \in \mathbb{N}$. This means that there is a sequence $\{\mathcal{X}_n\}_{n\in\mathbb{N}} \subseteq K$ such that $f_n \cdot \mathcal{X}_n \in K$ for all $n \in \mathbb{N}$. Let C_1 and C_2 be two circles in S_g that fill S_g. Since the map $K \to \mathbb{R}_+$ $\mathcal{X} \mapsto l_{\mathcal{X}}([C_1]) + l_{\mathcal{X}}([C_2])$ is continuous and K is compact, there is $R > 0$ such that $l_{\mathcal{X}}([C_i]) \leq R$ for all $\mathcal{X} \in K$ and $i = 1, 2$. However, we will prove that, for some $i = 1, 2$, the set $\{l_{f_n \cdot \mathcal{X}_n}([C_i]) = l_{\mathcal{X}_n}(\phi_n^{-1}([C_i])\}_{n\in\mathbb{N}}$ is unbounded and thus we will have a contradiction. We proceed in steps.

Step 1 We first prove that at least one between the set of homotopy classes $\{[\phi_n^{-1}(C_1)]\}_{n\in\mathbb{N}}$ and the set $\{[\phi_n^{-1}(C_2)]\}_{n\in\mathbb{N}}$ is infinite.
Otherwise, for infinite n we would have $[\phi_n^{-1}(C_1)] = [C_1']$ and $[\phi_n^{-1}(C_2)] = [C_2']$ for some circles C_1' and C_2' that fill S_g. By Corollary 2.3.40, up to changing the representatives of the f_n, we may assume that $\phi_n^{-1}(C_1) = C_1'$ and $\phi_n^{-1}(C_2) = C_2'$ for infinite n. This implies that for all these n the mapping class f_n of the diffeomorphism ϕ_n is determined by the bijective correspondence between the disks of $S_g \smallsetminus (C_1 \cup C_2)$ and the disks of $S_g \smallsetminus (C_1' \cup C_2')$ that it induces. But the disks are finitely many and the $f_n = [\phi_n]$ are all distinct. This is impossible.

Step 2 Thus we may assume that the $\{[V_n] = [\phi_n^{-1}(C_1)]\}_{n\in\mathbb{N}}$ are all distinct.
We now claim that there is a pants decomposition Δ of S_g such that $\{i(V_n, C)\}_{n\in\mathbb{N}}$ is unbounded for some $C \in \Delta$. Such a Δ can be constructed as follows. Start with any pants decomposition Δ' of S_g and suppose that for all $C \in \Delta'$ and $n \in \mathbb{N}$ we have $i(C, V_n) \leq N$. Therefore, modulo a homotopy of the V_n, we may assume that the number of arcs of $V_n \smallsetminus (\cup_{C\in\Delta'} C)$ is bounded independently from n. What is more, there are infinite n such that the number of arcs of $V_n \smallsetminus (\cup_{C\in\Delta'} C)$ connecting the boundary components $\partial_{i,j}$ and $\partial_{i,k}$ of the component P_i of $(S_g)_\Delta$ is exactly m_{ijk} for all i, j, k. It follows that there exists a circle of S_g, that we call V, and mapping classes, that we call M_n,

belonging to the subgroup of $\mathrm{Mod}(S_g)$ generated by the Dehn twists around the circles $C \in \Delta'$, such that $[V_n] = [M_n(V)]$ are the same homotopy class for all of such n. Since the $[V_n]$ are all distinct, also the mapping classes M_n are all distinct. Note that, by Proposition 3.2.8, this subgroup is $\oplus_{C\in\Delta'}\mathbb{Z}T_C$. Up to extracting a subsequence, there is a $C' \in \Delta'$ such that $T_{C'}$ appears as a factor in all M_n and with arbitrary large power. Let C be a non-trivial circle of S_g non isotopic to any circle in $\Delta' \setminus C'$ and such that either $i(C', C) = 2$ if $S_{\Delta'\setminus C'}$ contains a sphere with four holes as a component or $i(C', C) = 1$ if $S_{\Delta'\setminus C'}$ has a component that is a torus with one hole. Setting $\Delta = (\Delta' \setminus C') \cup C$ we obtain the desired result.

Step 3 For all $\mathcal{X} \in T_{S_g}$ let $h(\mathcal{X})$ be a hyperbolic metric of S_g such that C is a geodesic circle for $h(\mathcal{X})$ and $[h(\mathcal{X})] = \mathcal{X}$ in T_{S_g}. From the continuity of length functions, the compactness of K and Corollary 5.3.4, we can find $\varepsilon > 0$ such that for all $\mathcal{X} \in K$ the set $\{x \in S_g : \mathrm{dist}_{h(\mathcal{X})}(x, C) \leq \varepsilon\}$ is an annulus around C. Thus, from the previous step, $\{l_{f_n\cdot\mathcal{X}_n}([C_1]) = l_{\mathcal{X}_n}([V_n]) \geq \varepsilon \cdot i(V_n, C)\}_{n\in\mathbb{N}}$ is unbounded as claimed.

□

Stabilizers of points

Let $g \geq 2$. We want to determine the stabilizer of a point $\mathcal{X} \in T_{S_g}$ with respect to the action of $\mathrm{Mod}(S_g)$.

Lemma 5.3.7. *For every $\mathcal{X} = [(X, \phi)] \in T_{S_g}$, there is a bijection*

$$\mathrm{Stab}(\mathcal{X}) \to \mathrm{Isom}^+(\mathrm{X}).$$

Proof. An element $f = [\psi] \in \mathrm{Mod}(S_g)$ stabilizes $\mathcal{X}$ if and only if $\phi \circ \psi^{-1} \circ \phi^{-1}$ is homotopic (and thus isotopic) to an isometry $\tau : X \to X$. Observe that since ψ is orientation-preserving, τ must be orientation-preserving, too. We claim that two isometries of X that are homotopic, are the same. The lemma then follows at once. We now prove our claim. Suppose given two homotopic isometries g_0 and g_1 of X and let g be an homotopy between g_0 and g_1. Then, g gives an homotopy $\widetilde{g}$ between two isometries $\widetilde{g_0}$ and $\widetilde{g_1}$ of $\mathbb{H}^2$. Moreover, $d_{\mathbb{H}^2}(\widetilde{g_1}^{-1}(\widetilde{g_0}(x)), x) = d_{\mathbb{H}^2}(\widetilde{g_0}(x), \widetilde{g_1}(x)) \leq \mathrm{diam}(X) < \infty$ for all $x \in \mathbb{H}^2$. It follows that $\widetilde{g_1}^{-1} \circ \widetilde{g_0} = 1_{\mathbb{H}^2}$ and thus $g_0 = g_1$. Indeed, for any geodesic line γ in $\mathbb{H}^2$, we have that $\mathrm{dist}(\gamma, \widetilde{g_1}^{-1} \circ \widetilde{g_0}(\gamma))$ is bounded and thus γ and $\widetilde{g_1}^{-1} \circ \widetilde{g_0}(\gamma)$ are two geodesic lines with the same end points in $\partial\mathbb{H}^2$. Hence, $\widetilde{g_1}^{-1} \circ \widetilde{g_0}$ is the identity on $\partial\mathbb{H}^2$ and thus on all $\mathbb{H}^2$.

□

Proposition 5.3.8. *Let X be a hyperbolic surface diffeomorphic to S_g, where $g \geq 2$. Then* $\mathrm{Isom}(X)$ *is finite.*

Proof. Consider $\text{Isom}(X)$ as a subspace of $C^0(X,X)$, endowed with the topology induced by the distance $d(g_0,g_1) = \max\{d_X(g_0(x),g_1(x)) : x \in X\}$. Note that, for every hyperbolic distance d_X, the distance d induces the compact-open topology on $C^0(X,X)$ (combine Proposition 2.6 in chapter 7 of [9] and Theorem 2.12 in chapter 7 of [4]). We will prove that, with respect to this topology, $\text{Isom}(X)$ is compact and discrete.
Clearly, it is closed in $C^0(X,X)$. Morover it is relatively compact by Ascoli-Arzelà Theorem. To apply Ascoli-Arzelà Theorem, we need to check that $\text{Isom}(X)$ is pointwise limited and equicontinuous. But $\text{diam}(X) < \infty$, thus the first check is immediate, while the equicontinuity follows from the fact that for all $x \in X$ and $g \in \text{Isom}(X)$ we have $d_X(g(x),g(y)) = d_X(x,y)$. This proves that $\text{Isom}(X)$ is compact.
Finally we prove that it is discrete. For every $g \in \text{Isom}(X)$ we have an homeomorphism, given by the composition with g^{-1}, between the path component of g and the path component of 1_X. Therefore, it is enough to observe that, as proved in the previous lemma, $C^0_0(X,X) \cap \text{Isom}(X) = 1_X$ (where $C^0_0(X,X)$ is the space of continuous maps $X \to X$ homotopic to 1_X) and that the path components in $C^0(X,X)$ are open.
This last statement can be seen as follows. Embed X in $\mathbb{R}^3$. The metric of $\mathbb{R}^3$ induces the same topology on X as that it already has, thus, using Theorem 2.12 in chapter 7 of [4] again, the compact-open topology on $C^0(X,X)$ is induced by the distance, that we call d again, $d(g_0,g_1) = \max\{d_{\mathbb{R}^3}(g_0(x),g_1(x)) : x \in X\}$ for $g_0,g_1 \in C^0(X,X)$. Let $X \subseteq U \subseteq \mathbb{R}^3$ be an ε- neighborhood of X in $\mathbb{R}^3$ and $r : U \to X$ a smooth retraction. If $d(g_0,g_1) < \varepsilon$, then $r \circ (tg_1 + (1-t)g_0)$ is an homotopy from g_0 to g_1. Hence, $C^0(X,X)$ is locally path-conected. □

Corollary 5.3.9. *For every $\mathcal{X} \in T_{S_g}$, $\text{Stab}(\mathcal{X})$ is finite.*

Corollary 5.3.10. *Let $m \geq 3$. Then the action of $\text{Mod}(S_g)[\text{m}]$ on T_{S_g} is free.*

Remark 5.3.11. In particular, the action of $\text{Mod}(S_g)[\text{m}]$ $(m \geq 3)$ on T_{S_g} is free and properly discontinuous and thus the quotient $T_{S_g}/\text{Mod}(S_g)[m]$ has a natural structure of smooth manifold.

5.4 The submanifolds with corners $X_g(\delta)$

In this section we define a particular submanifold $X_g(\delta) \subseteq T_{S_g}$ and prove some of its properties.

Let S be a surface with $\chi(S) < 0$ and let $0 < \varepsilon < 2\sinh^{-1}(1/2)$.

Remark 5.4.1. Thanks to Corollary 5.3.5, if $C_1, ..., C_k$ are circles in S such that $l_{\mathcal{X}}([C_1]), ..., l_{\mathcal{X}}([C_k]) \leq \varepsilon$ for some $\mathcal{X} \in T_S$, then the set of homotopy classes $\{[C_1], ..., [C_k]\}$ is a simplex (not necessarily with k vertices) in $C(S)$. This also means that there cannot be more than $3g - 3 + b$ distinct isotopy classes of circles of length $\leq \varepsilon$ with respect to some $\mathcal{X}$.

Let $0 < \delta < \varepsilon$.

Definition 5.4.2. We define the spaces $T_S(\delta)$ and $X_S(\delta)$ as follows:

$$T_S(\delta) = \{\mathcal{X} \in T_S : l_{\mathcal{X}}(\partial_i) = \delta \text{ for all boundary components } \partial_i \text{ of } S\}$$

and

$$X_S(\delta) = \{\mathcal{X} \in T_S(\delta) : l_{\mathcal{X}}([C]) \geq \delta \text{ for all essential circles } [C] \text{ of } S\}$$

That is, $\mathcal{X} \in X_S(\delta)$ if and only if the length with respect to $\mathcal{X}$ of all boundary components is δ and of all essential circles is $\geq \delta$.

Observe that $T_S(\delta)$ is a smooth submanifold of T_S diffeomorphic to $\mathbb{R}_+^{3g-3+b} \times \mathbb{R}^{3g-3+b}$ by the the restriction of the Fenchel-Nielsen coordinates to $T_S(\delta)$.

To simplify the notation, if $S = S_g$ ($g \geq 2$), we will also use the symbol $X_g(\delta)$ for $X_{S_g}(\delta)$.

Proposition 5.4.3. *$X_S(\delta)$ is a smooth manifold with corners. Moreover, $\mathcal{X} \in \partial X_S(\delta)$ if and only if $l_{\mathcal{X}}([C]) = \delta$ for some essential circle C in S.*

The proof of this proposition relies on the following lemma.

Lemma 5.4.4. *Let $\eta < \varepsilon$. For any $\mathcal{X}_0 \in T_S(\delta)$ there is an open neighborhood U of $\mathcal{X}_0$ in $T_S(\delta)$ such that for all $\mathcal{X} \in U$ and essential circle C of S, $l_{\mathcal{X}}([C]) > \eta$ if $l_{\mathcal{X}_0}([C]) > \eta$.*

The proof of this lemma is not so easy. It involves the construction of a certain distance on T_S, the Teichmüller distance d_{Teich}. The reader can see Lemma 4.5 in [24] for the proof of this lemma and chapter 11 of [12] for the definition of d_{Teich}.

proof of the Proposition 5.4.3. Let $\mathcal{X}_0 \in X_S(\delta)$ and $\delta < \eta < \varepsilon$. Let U be an open set of $T_S(\delta)$ containing $\mathcal{X}_0$ as in the lemma. Call σ the set of the isotopy classes of the essential circles C in S with $l_{\mathcal{X}_0}([C]) \leq \eta$. Since $\eta < \varepsilon$, then σ is a simplex of $C(S)$ and in particular is finite. Moreover,

$$U \cap X_S(\delta) = \{\mathcal{X} \in U : l_{\mathcal{X}}([C]) \geq \delta \text{ for all } [C] \in \sigma\}.$$

Therefore it is enough to choose a pants decomposition Δ of S with $\sigma \subseteq \Delta$ and consider some Fenchel-Nielsen coordinates associated to Δ to see that $X_S(\delta)$ is a smooth manifold with corner and that $\mathcal{X}_0 \in \partial X_S(\delta)$ if and only if there exists an essential circle C in S such that $l_{\mathcal{X}_0}([C]) = \delta$. □

Remark 5.4.5. Obviously, $X_S(\delta)$ is preserved by the action of $\mathrm{Mod}(S)$ and, from the Proposition 5.4.3, the same is true for $\partial X_S(\delta)$.

The following theorem summarizes the main properties of $X_g(\delta)$.

Theorem 5.4.6. *The space $X_g(\delta)$ is a smooth manifold with corners of dimension $6g-6$ such that:*
(a) both $X_g(\delta)$ and $\partial X_g(\delta)$ are invariant under the action of $\mathrm{Mod}(S_g)$;
(b) $X_g(\delta)/\mathrm{Mod}(S_g)$ is compact;
(c) $X_g(\delta)$ is contractible;
(d) $\partial X_g(\delta)$ is homotopically equivalent to $|C(S_g)|$.

The rest of this chapter is dedicated to the proofs of parts (b), (c) and (d) of this theorem.

5.4.1 Proof of statement (b) of Theorem 5.4.6

The part (b) of the theorem is know as **Mumford's compactness Theorem**. In order to prove it, we will need the following Theorem of Bers.

Theorem 5.4.7 (Bers' constant)**.** *Let $S = S_g$ with $g \geq 2$. There exists a constant $L = L(g) > 0$ such that for any $\mathcal{X} \in T_S$ there is a pants decomposition $\Delta = \Delta(\mathcal{X})$ of S such that $l_{\mathcal{X}}([C]) \leq L$ for all $C \in \Delta$.*

Before proving Bers'theorem, we prove a lemma.

Lemma 5.4.8. *Let $S = S_g$ where $g \geq 2$ and let h be a fixed hyperbolic metric on S. There is a geodesic circle in S of length $\leq 2\log(4g-2)$.*

Proof. Let C be the shortest geodesic circle in S. We want to prove that $l(C) \leq 2\log(4g-2)$. Let $x \in C$. Then, by Lemma 2.3.27

$$D = \{y \in S : d(y,x) < r\}$$

is a hyperbolic disk in S for all $r < \frac{l(C)}{2}$. To conclude observe that

$$4\pi(g-1) = \mathrm{Area}(S) \geq \mathrm{Area}(D) = 2\pi \int_0^r \sinh(\rho)d\rho = 2\pi(\cosh(r) - 1)$$

where in the first equality we used the Gauss-Bonnet Formula and in the second equality Lemma 2.2.6. Hence

$$r \leq \cosh^{-1}(2g-1) \leq \log(4g-2)$$

and thus $l(C) \leq 2\log(4g-2)$. □

Proof of Bers's Theorem. Let $\mathcal{X} = [h] \in T_S$ and consider S endowed with the hyperbolic metric h. We will prove by induction on $k = 1, ..., 3g-3$ that there are circles $C_1, ..., C_k$ on S with the following two properties:

1. the C_j are non-trivial and pairwise non-isotopic circles;
2. $l(C_j) \leq 4j\log(\frac{8\pi}{j}(g-1))$ for all $j = 1, ..., k$;
3. call $S^0 = S$. For $k \geq 1$ the circle C_k is required to be a circle of S^{k-1} and, if we call $S^k = S^{k-1}_{C_k} \smallsetminus (Y_1 \cup ... \cup Y_m)$ where $Y_1, ..., Y_m$ are the components of $S^{k-1}_{C_k}$ homeomorphic to S^3_0, we require that $l(\partial S^k) \leq 4k\log(\frac{8\pi}{k}(g-1))$.

For $k = 1$ it is enough to choose C_1 to be the shortest geodesic circle in S. Suppose $2 \leq k \leq 3(g-1)$ and suppose that $C_1, .., C_{k-1}$ are already constructed. Call $\partial_1, ..., \partial_n$ the boundary components of S^{k-1}. Consider

$$Z(r) = \{x \in S^{k-1} : \text{dist}(x, \partial S^{k-1}) \leq r\}$$

for r such that the geodesic arcs of length r and emanating perpendicularly from ∂S^{k-1} are pairwise disjoint. Let $(\rho_i, t_i) \in [0, r] \times \mathbb{R}/[t_i \mapsto t_i + l(\partial_i)]$ be the Fermi coordinates on $\{x \in S^{k-1} : \text{dist}(x, \partial_i) \leq r\}$. By Lemma 2.2.4 we have

$$\text{Area}(Z(r)) = l(\partial S^{k-1}) \int_0^r \cosh(\rho) d\rho = l(\partial S^{k-1}) \sinh(r).$$

Define r_k to be the supremum of the r for which $Z(r)$ is defined. When $r = r_k$ we have at least two geodesic arcs of S^{k-1} of length r and emanating perpendicularly from ∂S^{k-1} that meet each other. Since the distance between any lift of ∂_i and any lift ∂_j (possibly $i = j$) in the universal covering space $\widetilde{S^{k-1}} \subseteq \mathbb{H}^2$ have distance at least $2r_k$, by the First Variational Formula, these two geodesic arcs meet in such a way to form a unique (simple) geodesic arc (in particular a smooth arc) δ of length $2r_k$ and meeting ∂S^{k-1} perpendicularly at both endpoints. We distinguish two cases.

<u>Case 1:</u> the arc δ joins two different boundary components ∂_1 and ∂_2 of S^{k-1}.

Let N be a closed neighborhood of $\delta \cup \partial_1 \cup \partial_2$ in the component of S^{k-1} containing ∂_1 and ∂_2. Call $C' = \partial N \smallsetminus (\partial_1 \cup \partial_2)$ and C the unique geodesic circle in the isotopy class of C'. Clearly C, ∂_1 and ∂_2 bound a pair of pants Y. We define $C_k = C$. Since no component of S^{k-1} is a pair of pants, C_k is not isotopic to any of the C_j for $j < k$. Call S^k_* the result of cutting S^{k-1} along C_k and giving away the component Y. Then we have $\partial S^k \subseteq \partial S^k_*$ and $C_k \subseteq \partial S^k_*$, thus to check properties 2 and 3 for C_k and S^k, it is enough to prove that

$$l(\partial S^k_*) \leq 4k\log(\frac{8\pi}{k}(g-1)).$$

To do this, consider $\zeta(r) = \partial Z(r) \setminus \partial S^{k-1}$ for $r \leq r_k$. By Lemma 2.2.4, we have

$$l(\zeta(r)) = l(\partial S^{k-1})\cosh(r).$$

As long as $r < r_k$, the set $\zeta(r)$ consists of n circles homotopic to the boundary components of S^{k-1}, when $r = r_k$ two components of $\zeta(r)$ meet each other and form a circle homotopic to C. Thus

$$l(\partial S^k_*) < l(\zeta(r_k))$$

and our claim is obvious if $l(\zeta(r_k)) \leq 4(k-1)\log(\frac{8\pi}{k-1}(g-1))$ (note that $\{1, ..., 3(g-1)\} \to \mathbb{R}$ defined by $k \mapsto 4k\log(\frac{8\pi}{k}(g-1))$ is increasing). Suppose instead that $l(\zeta(r_k)) > 4(k-1)\log(\frac{8\pi}{k-1}(g-1))$. Then there exists $r'_k \in (0, r_k)$ such that $l(\zeta(r'_k)) = 4(k-1)\log(\frac{8\pi}{k-1}(g-1))$. Let $d = r_k - r'_k$. Clearly we have

$$l(\partial S^k_*) \leq l(\zeta(r'_k)) + 4d.$$

Moreover

$$\text{Area}(Z(r_k)) - \text{Area}(Z(r'_k)) \leq \text{Area}(S) = 4\pi(g-1)$$

and

$$\begin{aligned}
\text{Area}(Z(r_k)) - \text{Area}(Z(r'_k)) =& l(\partial S^{k-1}) \int_{r'_k}^{r'_k+d} \cosh(\rho)d\rho = \\
=& l(\partial S^{k-1})\{\sinh(r'_k)\cosh(d) + \cosh(r'_k)\sinh(d) - \sinh(r'_k)\} \geq \\
\geq& l(\partial S^{k-1})\cosh(r'_k)\sinh(d) = \\
=& l(\zeta(r'_k))\sinh(d).
\end{aligned}$$

From these two inequalities we get

$$\begin{aligned}
d \leq& \sinh^{-1}(\frac{4\pi(g-1)}{l(\zeta(r'_k))}) \leq \sinh^{-1}(\frac{\pi(g-1)}{(k-1)\log(\frac{8\pi}{k-1}(g-1))}) \leq \\
\leq& \log(\frac{2\pi(g-1)}{(k-1)\log(\frac{8\pi}{k-1}(g-1))} + 1) \leq \log(\frac{3(g-1)}{k-1} + 1)
\end{aligned}$$

where we used the inequalities $\sinh^{-1}(x) \leq \log(2x+1)$ for $x > 0$, $k-1 < 3(g-1)$ and $\frac{2\pi/3}{\log(8\pi/3)} < 1$. To conclude observe that

$$\begin{aligned}
l(\partial S^k_*) \leq& l(\zeta(r'_k)) + 4d \leq \\
&\leq 4\{(k-1)\log(\frac{8\pi}{k-1}(g-1)) + \log(\frac{3(g-1)}{k-1} + 1)\} \leq \\
&\leq 4k\log(\frac{8\pi}{k}(g-1))
\end{aligned}$$

where the last inequality is equivalent to say that

$$k\log(\frac{k}{k-1}) + \log(\frac{3(g-1)}{k-1} + 1) \leq \log(\frac{8\pi}{k-1}(g-1))$$

and this inequality follows from the fact that

$$k\log(\frac{k}{k-1}) \leq \log(4)$$

and

$$\log(\frac{8\pi(g-1)}{3(g-1)+k-1}) \geq \log(\frac{4\pi}{3}) \geq \log(4)$$

for $k-1 < 3(g-1)$.

Case 2: the arc δ has its endpoints on one boundary component ∂ of S^{k-1}.

Define η_1 and η_2 as explained in the picture.

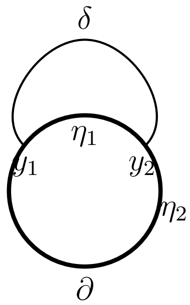


Figure 5.5: Definition of η_1 and η_2.

Then $\eta_1 \cup \delta$ and $\eta_2 \cup \delta$ are both homotopically non-trivial, otherwise they could be lift to a geodisic loops in $\mathbb{H}^2$. Call δ_1 and δ_2 the geodesic circles in the homotopy class of $\eta_1 \cup \delta$ and $\eta_2 \cup \delta$ respectively. Clearly ∂, δ_1 and δ_2 bound a pair of pants Y. In particular δ_1 and δ_2 cannot be both peripheral, otherwise the component X of S^{k-1} containing ∂ would have been a pair of pants and thus at least one between δ_1 and δ_2 is essential. Note that the possibility $\delta_1 = \delta_2$ is not excluded: if $\delta_1 = \delta_2$ then X is homeomorphic to S_1^1. In every case we set $C_k = \delta_i$ where δ_i is any essential circle in X chosen between δ_1 and δ_2. Now if $\delta_1 \neq \delta_2$ we set S_*^k to be $S_{C_k}^{k-1} \smallsetminus Y$ and proceed as in Case 1. If instead $\delta_1 = \delta_2$, then we only need to control $l(C_k)$. But clearly

$$l(C_k) \leq l(\zeta(r_k))$$

where $\zeta(r)$ is defined as above, thus we can proceed as above also in this case. $\square$

Theorem 5.4.9 (Mumford's compactness Theorem). *Let $g \geq 2$. Then $X_g(\delta)/\mathrm{Mod}(S_g)$ is compact.*

Proof. Since $X_g(\delta)$ has a countable basis, also the quotient $X_g(\delta)/\mathrm{Mod}(S_g)$ has a countable basis. Therefore, it is enough to check that it is sequentially compact. Let $\{[\mathcal{X}_n]\}_{n\in\mathbb{N}}$ be any sequence in $X_g(\delta)/\mathrm{Mod}(S_g)$. To prove that it has a convergent subsequence, we will show that, for some Fenchel-Nielsen coordinates, there is a compact K of $T_{S_g} = \mathbb{R}_+^{3g-3} \times \mathbb{R}^{3g-3}$ containing infinitely many representatives of the $[\mathcal{X}_n]$. Let $\mathcal{X}_n$ be lifts in $X_g(\delta)$ of the $[\mathcal{X}_n]$. From the previous theorem, for all n there is a pants decomposition Δ_n of S_g such that $l_{\mathcal{X}_n}([C]) \in [\delta, L]$ for all $C \in \Delta_n$. Since there are only a finite number of types of pants decomposition, we can find infinitely many n and $f_n = [\phi_n] \in \mathrm{Mod}(S_g)$ so that $\phi_n(\Delta_n) = \Delta$ is always the same. Now, using some Fenchel-Nielsen coordinates adapted to Δ, the corresponding $\mathcal{Y}_n = f_n \cdot \mathcal{X}_n$ have length parameters in $[\delta, L]$. Moreover, there are elements $h_n \in \mathrm{Mod}(S_g)$ that are compositions of Dehn twists about the circles in Δ such that the parameters θ_i of $h_n \cdot \mathcal{Y}_n$ lie in $[0, 2\pi]$. The proof is thus complete. □

5.4.2 Proof of statement (c) in Theorem 5.4.6

In this subsection we prove the following

Theorem 5.4.10. *$X_g(\delta)$ is contractible.*

Proof. We will contruct a continuous map $F : T_{S_g} \times [0, 1] \to T_{S_g}$ such that $F(\mathcal{X}, 1) \in X_g(\delta)$ for all $\mathcal{X} \in T_{S_g}$; $F(-, 0)$ is the identity of T_{S_g} and $F(x, t) \in X_g(\delta)$ for all $x \in X_g(\delta)$ and $t \in [0, 1]$. In particular, we will have that the inclusion $X_g(\delta) \hookrightarrow T_{S_g}$ is a homotopy equivalence and thus $X_g(\delta)$ is contractible.

<u>The strategy for the construction of F.</u>

Let $\Gamma = \mathrm{Mod}(S_g)[\mathrm{m}]$, with $m \geq 3$. We know that T_{S_g}/Γ has a structure of smooth manifold and that whenever a simplex σ of $C(S_g)$ is fixed by some $f \in \Gamma$ then f fixes all the vertices of σ. Note also that, since $X(\delta)/\Gamma$ is a covering space with finite sheets of the compact space $X_g(\eta)/\mathrm{Mod}(S_g)$, it is compact.
We will construct a smooth Γ-invariant vector field v on T_{S_g} such that $d_{\mathcal{X}} l_{[C]}(v(\mathcal{X})) = 1$ whenever $l_{\mathcal{X}}([C]) \leq \delta$ (here $l_{[C]}$ is the function $l_{[C]} : T_{S_g} \to \mathbb{R}$ defined by $l_{[C]}(\mathcal{X}) = l_{\mathcal{X}}([C])$). Such a vector field will descend to a vector field w on T_{S_g}/Γ. Moreover, for any $\eta \leq \delta$, on the points in $\partial X_g(\eta)/\Gamma$, w is directed inside $X_g(\eta)$ and, since $X_g(\eta)/\Gamma$ is compact, any trajectory of w starting in $X_g(\eta)/\Gamma$ is well-definite for all $t \in [0, \infty)$ and remains in $X_g(\eta)/\Gamma$ for positive times. Now note that $T_{S_g} = \bigcup_{0<\eta<\delta} X_g(\eta)$ and thus any trajectory of w is defined for all $t \geq 0$ and the corresponding half-flow leaves $X_g(\delta)/\Gamma$ invariant. Clearly, the corresponding statements hold for v and

$X_g(\delta)$. Finally, since $d_{\mathcal{X}} l_{[C]}(v(\mathcal{X})) = 1$ whenever $l_{\mathcal{X}}([C]) \leq \delta$, any trajectory of v starting outside $X_g(\delta)$ arrives in $X_g(\delta)$ in time most δ and remains inside it forever. This gives us our F.

<u>Construction of the vector field v</u>.

Fix $\delta < \eta < \epsilon$ and a smooth and monotone function $\varphi : \mathbb{R} \to [0, 1]$ such that $\varphi(t) = 1$ for $t \leq \delta$ and $\varphi(t) = 0$ for $t \geq \eta$. For any vertex $[C]$ of $C(S_g)$ let $\varphi_{[C]} = \varphi \circ l_{[C]}$ and for any simplex σ let $\varphi_\sigma = \prod_{[C]\in\sigma} \varphi_{[C]}$ and $\psi_\sigma = \varphi_\sigma \prod_{[C]\notin\sigma}(1 - \varphi_{[C]})$. Note that, from Remark 5.4.1, for any $\mathcal{X}$ there are a finite number of $\varphi_{[C]}$ that are different from 0 in $\mathcal{X}$ and thus, using Lemma 5.4.4 , on a neighborhood of every $\mathcal{X}$ almost all factors in the product defining ψ_σ are 1, so it is well-defined and is smooth.
Observe that for every $f \in \mathrm{Mod}(S_g)$ the following equalities hold: $l_{f\cdot[C]} = l_{[C]} \circ f^{-1}$, $\varphi_{f\cdot[C]} = \varphi_{[C]} \circ f^{-1}$, $\varphi_{f\cdot\sigma} = \varphi_\sigma \circ f^{-1}$ and $\psi_{f\cdot\sigma} = \psi \circ f^{-1}$.
We claim that for any simplex σ of $C(S_g)$ there is a vector field v_σ such that $dl_{[C]}(v_\sigma) = 1$ for all $[C] \in \sigma$ and $v_{f\cdot\sigma} = f_*(v_\sigma)$ for any $f \in \Gamma$.
Indeed, for any simplex $\sigma = \{[C_1], ..., [C_k]\}$ of $C(S_g)$ let Γ_σ be the stabilizer of σ under the action of Γ. By the properties of the action of Γ, the elements of Γ_σ stabilize each vertex of σ and thus the submersion

$$L_\sigma : T_{S_g} \to \mathbb{R}_+^{|\sigma|} \quad \mathcal{X} \mapsto (l_{\mathcal{X}}([C_1]), ..., l_{\mathcal{X}}([C_k]))$$

is Γ_σ-invariant and thus defines a smooth submersion $L'_\sigma : T_{S_g}/\Gamma_\sigma \to \mathbb{R}_+^{|\sigma|}$ by passing to the quotient. Let v'_σ be an arbitary lift of the constant vector field $(1, ..., 1)$. Such a lift can first be constructed locally and then the various pieces can be glued together with a partition of unity technique. Similarly, let v_σ be a lift of v'_σ to T_{S_g}. By construction, for all $f \in \Gamma_\sigma$ we have $v_\sigma = f_*(v_\sigma)$ and $dl_{[C]}(v_\sigma) = 1$ for all $[C] \in \sigma$. Therefore, we can first define the v_σ only for some representatives of the orbits in $C(S_g)$ under the action of Γ and then extend the definition to the remaining simplices by the formula $v_{f\cdot\sigma} = f_*(v_\sigma)$ for all $f \in \Gamma$. This proves the claim.
Now define the vector field v as

$$v = \sum_{\sigma\neq\emptyset} \psi_\sigma v_\sigma.$$

By Remark 5.4.1 and Lemma 5.4.4, this sum is locally finite and thus v is smooth. The proof that it is Γ-invariant is straightforward using the equalities we have listed above.
It remains to prove that $d_{\mathcal{X}} l_{[C]}(v(\mathcal{X})) = 1$ if $l_{[C]}(\mathcal{X}) \leq \delta$. Suppose that $l_{[C]}(\mathcal{X}) \leq \delta$. Observe that

$$d_{\mathcal{X}} l_{[C]}(v(\mathcal{X})) = \sum_{\sigma\neq\emptyset} \psi_\sigma(\mathcal{X}) d_{\mathcal{X}} l_{[C]}(v_\sigma(\mathcal{X}))$$

and that if $\psi_\sigma(\mathcal{X}) \neq 0$ then $[C] \in \sigma$ and thus $d_{\mathcal{X}} l_{[C]}(v_\sigma(\mathcal{X})) = 1$.
Therefore, to conclude the proof, we will show that if $l_{\mathcal{X}}([C]) \leq \delta$ for some $[C]$ then

$$\sum_{\sigma \neq \emptyset} \psi_\sigma(\mathcal{X}) = 1.$$

First, observe that if σ is any finite sets of vertices (not necessarily a simplex) of $C(S_g)$, we can define φ_σ and ψ_σ with the same formulas as before. However, if σ is not a simplex, $\psi_\sigma = \varphi_\sigma = 0$ identically. Hence, we may assume that the summation $\sum_{\sigma \neq \emptyset} \psi_\sigma$ is over all non empty finite set of vertices of $C(S_g)$. Now, note that

$$\psi_\sigma = \varphi_\sigma \prod_{[C] \notin \sigma} (1 - \varphi_{[C]}) = \sum_{\tau : \tau \cap \sigma = \emptyset} (-1)^{|\tau|} \varphi_\sigma \varphi_\tau$$

and thus

$$\sum_{\sigma \neq \emptyset} \psi_\sigma = \sum_{\tau, \sigma : \sigma \neq \emptyset \text{ and } \tau \cap \sigma = \emptyset} (-1)^{|\tau|} \varphi_\sigma \varphi_\tau = \sum_{\tau \subsetneq \rho} (-1)^{|\tau|} \varphi_\rho =$$
$$= \sum_{\rho \neq \emptyset} \varphi_\rho \sum_{\tau \subsetneq \rho} (-1)^{|\tau|} = \sum_{\rho \neq \emptyset} (-1)^{|\rho|+1} \varphi_\rho$$

where in the last equality we have used the fact that $\sum_{\tau \subsetneq \rho} (-1)^{|\tau|} = \sum_{k=0}^{|\rho|-1} \binom{|\rho|}{k} (-1)^k = (1-1)^{|\rho|} - (-1)^{|\rho|} = (-1)^{|\rho|+1}$.
Call σ the simplex of $C(S_g)$ defined as $\sigma = \{[V] : l_{[V]}(\mathcal{X}) < \eta\} = \{[C_1], ..., [C_k]\}$. Note that $[C] \in \sigma$. Then $\varphi_\rho(\mathcal{X}) \neq 0$ if and only if $\rho \subseteq \sigma$ and thus

$$\sum_{\sigma \neq \emptyset} \psi_\sigma(\mathcal{X}) = \sum_{\emptyset \neq \rho \subseteq \sigma} (-1)^{|\rho|+1} \varphi_\rho(\mathcal{X}) = \sum_{i=1}^{k} \varphi_{[C_i]}(\mathcal{X}) - \sum_{1 \leq i_1 < i_2 \leq k} \varphi_{[C_{i_1}]}(\mathcal{X}) \varphi_{[C_{i_2}]}(\mathcal{X}) + ...$$
$$+ ... + (-1)^{k+1} \prod_{i=1}^{k} \varphi_{[C_i]}(\mathcal{X}) = 1 - \prod_{i=1}^{k} (1 - \varphi_{[C_i]}(\mathcal{X})) = 1.$$

where the last equality holds because $[C] \in \sigma$. □

5.4.3 Proof of statement (d) of Theorem 5.4.6

We finally prove the following

Theorem 5.4.11. *Let $\chi(S) < 0$. The boundary $\partial X_S(\delta)$ is homotopically equivalent to the geometric realization of $C(S)$.*

To this porpose consider, for each vertex $[C]$ of $C(S)$ the closed sets $B_{[C]} = \{\mathcal{X} \in X_S(\delta) : l_{\mathcal{X}}([C]) = \delta\}$. By Proposition 5.4.3, $\partial X_S(\delta) = \bigcup_{[C]} B_{[C]}$.

Here is the main observation.

Remark 5.4.12. For any finite set σ of vertices of $C(S)$, set $B_\sigma = \bigcap_{[C]\in\sigma} B_{[C]}$ and note that $B_\sigma \neq \emptyset$ if and only if σ is a simplex of $C(S)$. In particular, $C(S)$ is the nerve of the covering of $\partial X_S(\delta)$ by the closed sets $B_{[C]}$.
The 'only if' part is an immediate consequence of Remark 5.4.1 while the 'if' part can be seen in the following way. Given a simplex σ of $C(S)$ we can find a pants decomposition Δ of S containing some disjoint representatives of the classes of circles in σ and thus a hyperbolic metric h on S with respect to which each circle in Δ and boundary component of S has length exactly δ. Again, Remark 5.4.1 gives that $[h] = \mathcal{X} \in X_S(\delta)$ and, thus, clearly belongs to B_σ.

Thus, the theorem will follow from the Nerve Theorem (see Theorem 10.6 in [2]) once we will have proved that:

1. whenever $B_\sigma \neq \emptyset$, it is contractible;
2. $\partial X_S(\delta)$ admits a tringulation and for every triangulation each $B_{[C]}$ is necessarily a subcomplex of $\partial X_S(\delta)$.

Lemma 5.4.13. *Let σ be a simplex of $C(S)$ and suppose $B_\sigma \neq \emptyset$. Then, B_σ is contractible.*

Proof. First, consider a maximal simplex Δ of $C(S)$ containing σ and let h be any hyperbolic metric on S with respect to which each boundary component of S has length δ and such that the geodesics circles in each isotopy class contained in σ have length $< \varepsilon$. Call $C_1, ..., C_k$ $(k = 3g-3+b)$ the geodesic circles of S (with respect to h) such that $\Delta = \{[C_1], ..., [C_k]\}$ and say $\sigma = \{[C_1], ..., [C_j]\}$ where $j \leq k$. Consider the action Θ of $\mathbb{R}^k$ on T_S associated to Δ. Using the inclusion $\mathbb{R}^j = \mathbb{R}^j \times \{0\} \subseteq \mathbb{R}^k$, we get an induced action $\Theta|_\sigma$ of $\mathbb{R}^j$ on T_S.
Observe that B_σ is invariant with respect to $\Theta|_\sigma$.
Indeed, suppose that, for some $\theta \in \mathbb{R}^j$ and $\mathcal{X} \in B_\sigma$, $\mathcal{Y} = (\Theta|_\sigma)_\theta(\mathcal{X}) \notin B_\sigma$. This means that there is an essential circle C in S such that $l_{[C]}(\mathcal{Y}) < \delta$. Since $\Theta|_\sigma$ preserves the lengths $l_{[C_i]}$ for $i = 1, ..., j$, our choice of h and ε implies that the geodesic circle (with respect to h) in $[C]$ must be disjoint from C_i for $i = 1, ..., j$. But then $\Theta|_\sigma$ preserves $l_{[C]}$ and thus $\mathcal{X}$ was not in $X_S(\delta)$. This is a contradiction.
Now, let $A_\sigma = \{\mathcal{X} \in T_S(\delta) : l_{[C]}(\mathcal{X}) = \delta$ for all $[C] \in \sigma\}$. Clearly, $B_\sigma = A_\sigma \cap X_S(\delta)$ and A_σ is also invariant under the action of $\Theta|_\sigma$. The main observation here is that the quotient $A_\sigma/\mathbb{R}^j$ by the action of $\Theta|_\sigma$ coincides with $T_R(\delta) = T_{R_1}(\delta) \times ... \times T_{R_l}(\delta)$ where $R_1 \sqcup ... \sqcup R_l = S_\sigma$ is the decomposition in components of the result of cutting

S along $C_1, ..., C_j$. Here, it is important the fact that we are only considering hyperbolic metrics having boundary lengths forced to be equal to δ. Call π the projection $\pi : A_\sigma \to T_R(\delta)$. We claim that $\pi(B_\sigma) = X_R(\delta) = X_{R_1}(\delta) \times ... \times X_{R_l}(\delta)$. The inclusion $\subseteq$ is obvious. We will show that $\pi^{-1}(X_R(\delta)) \subseteq B_\sigma$. The argument is similar to that used to check that B_σ is invariant under the action of $\Theta|_\sigma$. We repeat it for completeness. Suppose $\mathcal{X} \in A_\sigma$ and $l_{[C]}(\mathcal{X}) < \delta$ for some essential circle C in S. Then $[C] \notin \sigma$ and $\sigma \cup \{[C]\}$ is still a simplex. But this means that the geodesic representative (with respect to h) of $[C]$ does not intersect any C_i for $i = 1, ..., j$ and thus the length of C considered as an essential circle in some T_{R_i} with respect to $\pi(\mathcal{X})$ is the same as its length in S with respect to $\mathcal{X}$ and thus is $< \delta$. This means that $\pi(\mathcal{X}) \notin X_R(\delta)$.
As a corollary of our claim, we obtain that $\pi(B_\sigma) = B_\sigma/\mathbb{R}^j$ is contractible. To conclude the proof, observe that $B_\sigma/\mathbb{R}^j$ is homeomorphic to the deformation retract of B_σ that under any Fenchel-Nielsen coordinates Ψ associated to Δ is given by $\Psi(B_\sigma) \cap \{\theta_i = 0 \text{ for all } i \leq j\}$. □

Lemma 5.4.14. *$\partial X_S(\delta)$ admits a triangulation and for any triangulation of $\partial X_S(\delta)$ each $B_{[C]}$ happens to be a subcomplex.*

In the proof of this lemma we will repeatedly uses the following standard result:

Lemma 5.4.15. *Let $|K|$ be a the realization of a simplicial complex and x, y two points in the same face of $|K|$. Then, there is an homeomorphism $|K| \to |K|$ sending x to y.*

Proof of Lemma 5.4.14. Note that $\partial X_S(\delta)$ is the boundary of the manifold with corners X_S. Triangulate $X_S(\delta) \cong |K|$. Then if a point of an open face of $|K|$ belongs to $\partial X_S(\delta)$, from the previous lemma, the entire open face must be in $\partial X_S(\delta)$ and, since $\partial X_S(\delta)$ is closed in $X_S(\delta)$, the entire closed face is in $\partial X_S(\delta)$. It follows that $\partial X_S(\delta)$ is a subcomplex of $|K|$ and in particular it admits a tringulation.
Now fix a triangulation $\partial X_S(\delta) \cong |L|$. Observe that $\partial X_S(\delta) = \bigcup_{[C]} B_{[C]}$ where each $B_{[C]}$ is a manifold with corners $\partial B_{[C]} = \bigcup_{[C'] \neq [C]} B_{[C']} \cap B_{[C]}$ (see the proof of Proposition 5.4.3). Call $M = \partial X_S(\delta)$ and $\partial M = \bigcup_{[C]} \partial B_{[C]}$.
From the previous lemma, if the interior of a face $|\tau|$ of $|L|$ intersects $B_{[C]} \setminus \partial B_{[C]}$, then the entire open face $|\mathring{\tau}|$ must be in $M \setminus \partial M$. Moreover if there is $\mathcal{Y} \in B_{[C']} \cap |\mathring{\tau}|$ for some $[C'] \neq [C]$, then there is a $\mathcal{X}$ in $\partial M \cap |\mathring{\tau}|$ and this is impossible. Therefore $|\mathring{\tau}| \subseteq B_{[C]}$ and, since each $B_{[C]}$ is closed in $\partial X_S(\delta)$, we have $|\tau| \subseteq B_{[C]}$.
Finally suppose that the open face $|\mathring{\tau}|$ intersects $\partial B_{[C]}$. We want to show that $|\tau| \subseteq B_{[C]}$. But there is a simplex of maximal dimesion containing $|\tau|$ and whose interior intersects $B_{[C]} \setminus \partial B_{[C]}$, thus this follows from the previous argument. □

This completes the proof of part (d).

Chapter 6

Computation of the virtual cohomological dimension of the Mapping Class Group

In this last chapter, we finally compute the virtual cohomological dimension of the Mapping Class Group, $\mathrm{Mod}(S)$. We will start with the case of closed surfaces and then obtain the general case using a completely topological argument.

For this last chapter the space $\mathrm{Diff}(S)$ will always be topologized with the C^∞-topology and the space $\mathrm{Homeo}(S)$ with the compact-open topology.

6.1 The $K(\mathrm{Mod}(S), 1)$-spaces $B\mathrm{Homeo}^+(S)$ and $B\mathrm{Diff}^+(S)$.

One of the main tool in computing $\mathrm{vcd}(\mathrm{Mod}(S))$ is the following theorem.

Theorem 6.1.1. *Let $\chi(S) < 0$. The classifying spaces $B\mathrm{Homeo}^+(S)$ and $B\mathrm{Diff}^+(S)$ are $K(\mathrm{Mod}(S), 1)$-spaces.*

In order to prove this result, observe that the long exact sequence of the universal bundles $E\mathrm{Homeo}^+(S) \to B\mathrm{Homeo}^+(S)$ and $E\mathrm{Diff}^+(S) \to B\mathrm{Diff}^+(S)$ show that

$$\pi_i(B\mathrm{Homeo}^+(S)) \cong \pi_{i-1}(\mathrm{Homeo}^+(S))$$

and

$$\pi_i(B\mathrm{Diff}^+(S)) \cong \pi_{i-1}(\mathrm{Diff}^+(S))$$

for all $i \geq 1$. In particular, $\pi_1(B\mathrm{Homeo}^+(S)) \cong \pi_1(B\mathrm{Diff}^+(S)) \cong \mathrm{Mod}(S)$. What remains to prove is the following theorem.

Theorem 6.1.2. *If $\chi(S) < 0$, then $\mathrm{Homeo}_0(S)$ and $\mathrm{Diff}_0(S)$ are weakly contractible.*

We will not prove this theorem. We refer the reader to the article [10] for the case of $\mathrm{Diff}_0(S_g)$ with $g \geq 2$, and to the article [13] for the general case of $\mathrm{Diff}_0(S)$ with $\chi(S) < 0$. Instead, as for homeomorphisms, in the series of articles [15], [16] and [17], Hamstrom has proved that the subspace $\mathrm{Homeo}_0(S, \partial S)$ of those homeomorphisms that restrict to the identity on ∂S is contractible. We will now prove that the spaces $\mathrm{Homeo}_0(S, \partial S)$ and $\mathrm{Homeo}_0(S)$ are weakly homotopy equivalent.

Let S be any surface with $\partial S = \partial_1 \sqcup ... \sqcup \partial_b \neq \emptyset$ and call $\mathrm{Homeo}^+(S, \{\partial S\})$ the space of orientation-preserving homeomorphisms of S fixing setwise each boundary component of S. Note that $\mathrm{Homeo}_0(S) \subseteq \mathrm{Homeo}^+(S, \{\partial S\})$ is the connected component of 1_S in $\mathrm{Homeo}^+(S, \{\partial S\})$.

Here is the main observation.

Proposition 6.1.3. *The restriction to* ∂S *gives a map*

$$\mathrm{Homeo}^+(S, \{\partial S\}) \xrightarrow{\mathcal{E}} \bigtimes_{i=1}^{b} \mathrm{Homeo}^+(\partial_i)$$

that is a fiber bundle map, with fiber $\mathrm{Homeo}^+(S, \partial S)$.

Proof. Endow S with an auxiliary complete and totally geodesic riemannnian metric with respect to which each boundary component of S has length 1. In particular, the compact open topology on $\mathrm{Homeo}_0(X)$ for $X = S$ or $X = \partial S$ is induced by the distance $d(h, g) = \max\{d(h(x), g(x)) : x \in X\}$ (where d is the distance on S induced by the riemannian metric).
Given $h = (h_1, ..., h_b) \in \bigtimes_{i=1}^{b} \mathrm{Homeo}^+(\partial_i)$, let U be the open neighborhood of h defined by $U = \{g = (g_1, ..., g_b) \in \bigtimes_{i=1}^{b} \mathrm{Homeo}^+(\partial_i) : d(h_i, g_i) < 1/2 \text{ for all } i = 1, ..., b\}$ and consider the continuous map

$$\begin{aligned} \Psi : &U \to \mathrm{Homeo}^+(\mathbb{R})^{\times b} \\ &g \mapsto (\widetilde{g_1}, ..., \widetilde{g_b}) \end{aligned}$$

defined as follows. For $i = 1, ..., b$, let $p_i : \mathbb{R} \to \partial_i$ be the universal covering map, chosen in such a way that p_i is 1-periodic and orientation-preserving and fix a lift $\widetilde{h_i} : \mathbb{R} \to \mathbb{R}$ of $h_i \circ p_i$. Then, set $\widetilde{g_i} : \mathbb{R} \to \mathbb{R}$ to be the unique lift of $g_i \circ p_i$ such that $\widetilde{g_i}(0) \in p_i^{-1}(g_i(p_i(0)) \cap (\widetilde{h_i}(0) - 1/2, \widetilde{h_i}(0) + 1/2)$. This defines Ψ.
Now, for $g \in U$, consider the straight-line homotopies $\widetilde{G_i}$ between $\widetilde{g_i}$ and $1_{\mathbb{R}}$. Actually, they are isotopies since for all fixed $t \in [0, 1]$ the function $\widetilde{G_i}(-, t)$ is monotonic strictly increasing. Moreover, since $\widetilde{g_i}(\theta + n) = \widetilde{g_i}(\theta) + n$ for all $n \in \mathbb{Z}$ and $\theta \in \mathbb{R}$,

these isotopies induce isotopies G_i between $G_i(-,0) = 1_{\partial_i}$ and $G_i(-,1) = g_i$. Using these isotopies, we obtain a continuous map

$$\begin{aligned} \Phi : \mathrm{Homeo}^+(S,\partial S) \times U &\to \mathcal{E}^{-1}(U) \\ (\phi, g) &\mapsto H(\phi, g) \end{aligned}$$

where $H(\phi, g)$ is defined as follows. For all $i = 1, ..., b$, let $(N_i, \partial_i) \cong (\partial_i \times [0,1], \partial_i \times \{1\})$ be disjoint closed neighborhoods of the ∂_i. Then, we define $H(\phi, g) = H'(g) \circ \phi$ where

$$H'(g)(x) = \begin{cases} x & \text{if } x \notin N_i \\ (G_i(x', t), t) & \text{if } x = (x', t) \in N_i = \partial_i \times [0,1]. \end{cases}$$

The map Φ is clearly continuous and bijective. The inverse map is the map that sends $H \in \mathcal{E}^{-1}(U)$ to (ϕ, g) where $g = H|_{\partial S}$ and $\phi = H'(g)^{-1} \circ H$. Clearly this is continuous, too. □

Lemma 6.1.4. $\mathrm{Homeo}_0(S^1)$ *contains* S^1 *as a deformation retract.*

Proof. Note that S^1 is naturally contained in $\mathrm{Homeo}^+(S^1)$ as the group of rotations. Let $\mathrm{Homeo}^+([0,1], \{0,1\})$ be the space of those homeomorphisms of $[0,1]$ that fix both 0 and 1. By Lemma 2.4.8, this space is contractible. Moreover, we have a homeomorphism

$$\Phi : S^1 \times \mathrm{Homeo}^+([0,1], \{0,1\}) \to \mathrm{Homeo}_0(S^1)$$

defined by sending $(e^{i\theta}, \phi)$ to $R_\theta \circ \bar{\phi}$, where R_θ is the counterclockwise rotation of angle θ and $\bar{\phi} : S^1 \to S^1$ is the map induced by ϕ in the following way. Let $p : \mathbb{R} \to S^1$ be the universal covering map, chosen in such a way that $p(0) = (1,0)$, p is 1-periodic and orientation-preserving. Then, $\bar{\phi}$ is defined by the condition $\bar{\phi} \circ p = p \circ \phi$. Observe that $R_\theta \circ \bar{\phi}$ is an orientation-preseving map and, since $\mathrm{Homeo}_0(S^1) = \mathrm{Homeo}^+(S^1)$, it belongs to $\mathrm{Homeo}_0(S^1)$. The inverse of Φ is the map that sends $\psi \in \mathrm{Homeo}_0(S^1)$ to (θ, ϕ) where $\theta = \psi((0,1))$ and $\phi : [0,1] \to [0,1]$ is the restriction to $[0,1]$ of the unique lift of $R_{-\theta} \circ \psi \circ p$ that sends 0 to 0. □

From these two results, considering the long exact sequence of the homotopy groups associated to the fiber bundle $\mathrm{Homeo}^+(S, \{\partial S\}) \xrightarrow{\mathcal{E}} \bigtimes_{i=1}^b \mathrm{Homeo}^+(\partial_i)$, we immediately obtain that $\pi_i(\mathrm{Homeo}_0(S, \partial S)) = \pi_i(\mathrm{Homeo}^+(S, \{\partial S\})) = 0$ for all $i > 1$. In addiction, this sequence ends with

$$0 \to \pi_1(\mathrm{Homeo}^+(S, \{\partial S\})) \to \pi_1(\bigtimes_{i=1}^b \mathrm{Homeo}^+(\partial_i)) \xrightarrow{\alpha}$$

$$\xrightarrow{\alpha} \pi_0(\mathrm{Homeo}^+(S, \partial S)) \xrightarrow{\eta} \pi_0(\mathrm{Homeo}^+(S, \{\partial S\})) \to \pi_0(\bigtimes_{i=1}^b \mathrm{Homeo}^+(\partial_i)) = 0$$

We will prove that α is injective and thus $\pi_1(\mathrm{Homeo}_0(S)) = \pi_1(\mathrm{Homeo}^+(S, \{\partial S\})) = 0$. Note that the map $\pi_0(\mathrm{Homeo}^+(S, \{\partial S\})) \to \pi_0(\mathrm{Homeo}^+(S))$ induced by the inclusion is injective and thus $\pi_0(\mathrm{Homeo}^+(S, \{\partial S\}))$ is naturally a subgroup of $\mathrm{Mod}(S)$. By the proof of Lemma 6.1.4, the injectivity of α is equivalent to the following proposition, in which the hypothesis $\chi(S) < 0$ is necessary. Observe that asking $\chi(S) < 0$ is the same as asking that S in not a disk or an annulus (since under our hypothesis $b \geq 1$).

Proposition 6.1.5. *Suppose that $\chi(S) < 0$. Let $C_1, ..., C_b$ be b circles in S with C_i isotopic to ∂_i for all $i = 1, ..., b$. Call T_i the Dehn twist about C_i. Then $\mathrm{Ker}(\eta) = \mathbb{Z}T_1 \oplus ... \oplus \mathbb{Z}T_b$.*

Proof. By the proof of Lemma 6.1.4, we have that $\mathrm{Ker}(\eta) = \mathrm{Im}(\alpha) = \langle T_1, ..., T_b \rangle$ is the subgroup generated by $T_1, ..., T_b$. Since S is neither a disk nor an annulus the $C_1, ..., C_b$ are non-trivial and non-isotopic circles in S, thus, by Proposition 3.2.8, we have $\mathrm{Ker}(\eta) = \mathbb{Z}T_1 \oplus ... \oplus \mathbb{Z}T_b$. □

In conclusion, $\mathrm{Ker}(\alpha) = 0$ and thus $\pi_1(\mathrm{Homeo}_0(S)) = \pi_1(\mathrm{Homeo}^+(S, \{\partial S\})) = 0$.

Observe that we have proved the following fact.

Proposition 6.1.6. *Let $\chi(S) < 0$. Then there is an exact sequence of groups*

$$0 \to \mathbb{Z}^b \to \mathrm{Mod}(S, \partial S) \to \pi_0(\mathrm{Homeo}^+(S, \{\partial S\}) \to 0$$

where $\pi_0(\mathrm{Homeo}^+(S, \{\partial S\})$ is naturally is subgroup of $\mathrm{Mod}(S)$ of index $b! < \infty$, where b is the number of boundary components of S.

6.2 Mess Subgroups B_g

In this section we introduce some subgroups of the Mapping Class Group $\mathrm{Mod}(S_g)$ called **Mess subgroups**, denoted by B_g, in name of G. Mess who first constructed them.

6.2.1 The definition of B_g

Let $g \geq 2$. The definition of B_g is recursive.

Definition 6.2.1. Define $B_2 \subseteq \mathrm{Mod}(S_2)$ to be the subgroup generated by the Dehn twists about the circles C_0, C_1 and C_2 shown in figure:

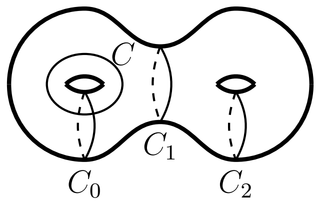


Figure 6.1: Definition of B_2.

Remark 6.2.2. By Proposition 3.2.8, B_2 is free and abelian with generators T_{C_0}, T_{C_1} and T_{C_2}. In particular $B_2 \cong \mathbb{Z}^3$.

In order to define B_{g+1} for $g \geq 2$, suppose that B_g is defined. Consider the surface S_g^1 obtained from S_g by removing an open disk. The extension of diffeomorphisms of S_g^1 that are the identity on ∂S_g^1 to diffeomorphisms of S_g that are the identity on $S_g \setminus S_g^1$ induces an homomorphism of groups

$$\mathrm{Mod}(S_g^1, \partial S_g^1) \to \mathrm{Mod}(S_g)$$

Definition 6.2.3. Define B_g^1 to be the preimage of B_g under this homomorphism.

Next, consider an embedding $S_g^1 \to S_{g+1}$. The extension of diffeomorphisms that are the identity on ∂S_g^1 to diffeomorphisms of S_{g+1} that are the identity on $S_{g+1} \setminus S_g^1$ gives another homomorphism

$$i : \mathrm{Mod}(S_g^1, \partial S_g^1) \to \mathrm{Mod}(S_{g+1}).$$

Observe that the closure in S_{g+1} of $S_{g+1} \setminus S_g^1$ is a torus with one hole. Let C_{g+1} be any essential circle in it and $T_{C_{g+1}}$ the correspondent Dehn twist in $\mathrm{Mod}(S_{g+1})$.

Definition 6.2.4. Define B_{g+1} as the subgroup of $\mathrm{Mod}(S_{g+1})$ generated by $i(B_g^1)$ and $T_{C_{g+1}}$.

A remark about B_g

We want to prove that $B_{g+1} \cong \mathbb{Z} \oplus B_g^1$. This will be an easy consequence of Proposition 6.2.6.

Lemma 6.2.5. *Let S be a surface and let $C_1, ..., C_k$ be a collection of homotopically distinct circles in S, each one not homotopic to a point. Let V_1 and V_2 be two other circles of S that are disjoint from $\cup_{i=1}^k C_i$, homotopically distinct from each C_i and not homotopic to a point. If V_1 and V_2 are isotopic in S then the isotopy can be chosen to takes values in $S \setminus \cup_{i=1}^k C_i$.*

Proof. First we perturb V_1 by isotopy in $S \setminus \cup_{i=1}^k C_i$ to be transverse to V_2. If $V_1 \cap V_2 = \emptyset$, then V_1 and V_2 bound an annulus in S and, by the hypothesis, none of the C_i can intersect this annulus. Thus we are done. Suppose, instead, that $V_1 \cap V_2 \neq \emptyset$. Then they form a bigon in S and, by the hypothesis, none of the C_i intersect this bigon. Thus we can push V_1 across this bigon by an isotopy in $S \setminus \cup_{i=1}^k C_i$. Repeating this process we reduce ourself to the case $V_1 \cap V_2 = \emptyset$. □

Proposition 6.2.6. *The homomorphism*

$$i : \mathrm{Mod}(S_g^1, \partial S_g^1) \to \mathrm{Mod}(S_{g+1})$$

induced by the inclusion $S_g^1 \hookrightarrow S_{g+1}$*, is injective.*

Proof. Let $f \in \mathrm{Mod}(S_g^1, \partial S_g^1)$ be in the kernel of i and $\phi \in \mathrm{Diff}^+(S_g^1, \partial S_g^1)$ be a representative of f. Call $\hat{\phi} \in \mathrm{Diff}^+(S_{g+1})$ the extension of ϕ to a diffeomorphism of S_{g+1} that is the identity on $S_{g+1} \setminus S_g^1$. Then $\hat{\phi}$ is a representative of $i(f)$ and thus is isotopic to the identity of S_{g+1}. Here, the key observation is the following: if C is an arbitrary essential circle in S_g^1, then $\phi(C) = \hat{\phi}(C)$ is isotopic to C in S_{g+1} and thus, by the previous lemma, in S_g^1. Note that ∂S_g^1 is not homotopic to a point in S_{g+1}. Now consider the circles $C_1, ..., C_k$ of S_g^1 in the figure below.

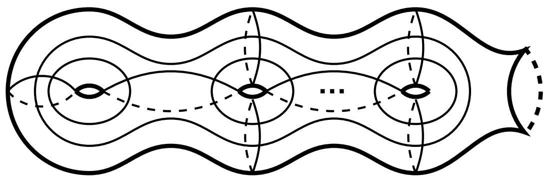


Figure 6.2 Subdivision of S_g^1 in disks and an annulus.

(This picture is the same as the one used to prove Lemma 2.4.12). By Proposition 2.3.38, we may assume that ϕ fixes setwise every C_i. In particular, ϕ fixes each point of the form $C_i \cap C_j$ with $i \neq j$. Note that $|C_i \cap C_j| \leq 1$ if $i \neq j$. Since for every i the set $C_i \setminus \cup_{j \neq i} C_j$ is a collection of intervals whose extreme points are fixed by ϕ and any two of such interval have at least one different extreme point, ϕ must fix each of these intervals setwise and acts as an orientation-preserving diffeomorphism on each of them. Since the space of orientation-preserving diffeomorphism of $[0, 1]$ that fix pointwise $\{0, 1\}$ is contractible, up to an isotopy of S, we may assume that $\phi|_{C_i} = 1_{C_i}$ for all i. Cutting S_g^1 along these circles we obtain a collection of disks and an annulus N corresponding to a closed neighborhood of ∂S_g^1. Thus, by Example 3.1.7, f has a representative that is supported in N and, by Example 3.1.9, f is a power of a Dehn twist about a circle that is isotopic to ∂S_g^1. Since ∂S_g^1 is not a trivial circle in S_{g+1} and $\hat{\phi}$ is isotopic to $1_{S_{g+1}}$, the power must be 0 and thus ϕ is isotopic to $1_{S_g^1}$. □

Corollary 6.2.7. *We have* $B_{g+1} \cong i(B_g^1) \oplus \mathbb{Z} \cong B_g^1 \oplus \mathbb{Z}$.

Proof. The twist $T_{C_{g+1}}$ obviously commutes with the elements of $i(B_g^1)$ thus we only need to prove that the sum of $i(B_g^1)$ and $\mathbb{Z}T_{C_{g+1}}$ inside $\mathrm{Mod}(S_{g+1})$ is direct. This follows from Proposition 3.2.6 and the fact that there exists a non-trivial circle C of S_{g+1} such that $[g(C)] = [C]$ for all $g \in i(B_g^1)$ and $i(C, C_{g+1}) > 0$. □

6.2.2 Computation of $\mathrm{cd}(B_g)$

We now compute the cohomological dimension of B_g for $g \geq 2$. This is done by constructing a $K(B_g, 1)$-space that satisfy the properties of Theorem 1.2.8.

Lemmas about $UT(S_g)$ and $\mathrm{Emb}^+(D^2, S_g)$

Let $g \geq 2$ and fix a Riemannian metric on S_g.

In order to compute $\mathrm{cd}(B_g)$ we will exploit some properties, that we now explain, of the unit tangent bundle $UT(S_g)$ of S_g and of the space of the orientation-preserving smooth embeddings in S_g of a fixed closed disk $D^2 \subseteq S_g$ contained in S_g, denoted by $\mathrm{Emb}^+(D^2, S_g)$. We will endow $\mathrm{Emb}^+(D^2, S_g)$ with the C^∞-topology.

The group $\mathrm{Diff}^+(S_g)$ acts in a continuous way on $\mathrm{Emb}^+(D^2, S_g)$ by $f \cdot i = f \circ i$ for all $f \in \mathrm{Diff}^+(S_g)$ and $i \in \mathrm{Emb}^+(D^2, S_g)$. In particular, we can consider the fiber bundle

$$E\mathrm{Emb}^+(D^2, S_g) = (E\mathrm{Diff}^+(S_g) \times \mathrm{Emb}^+(D^2, S_g))/\mathrm{Diff}^+(S_g) \to B\mathrm{Diff}^+(S_g)$$

See the Appendix B for the notation.
Similarly $\mathrm{Diff}^+(S_g)$ acts continuously on the unit tangent bundle $UT(S_g)$ of S_g by $f \cdot v_x = \frac{d_x f(v)}{|d_x f(v)|}$ for all $v \in T_x S_g$ (where $x \in S_g$ is a point in the surface) and $f \in \mathrm{Diff}^+(S_g)$. Thus we can also consider the fiber bundle

$$EUT(S_g) = (E\mathrm{Diff}^+(S_g) \times UT(S_g))/\mathrm{Diff}^+(S_g) \to B\mathrm{Diff}^+(S_g)$$

We start with observing that $UT(S_g)$ is a $K(\pi_1(UT(S_g)), 1)$-space.

Lemma 6.2.8. $UT(S_g)$ *is a* $K(\pi_1(UT(S_g)), 1)$*-space.*

Proof. There is a fiber bundle map

$$UT(S_g) \to S_g$$

with fiber S^1. Now use the associated exact sequence of homotopy groups recalling that S_g and S^1 are respectively $K(\pi_1(S_g), 1)$ and $K(\mathbb{Z}, 1)$-spaces being their universal cover respectively $\mathbb{R}^2$ and $\mathbb{R}$. $\square$

Next, we relate $UT(S_g)$ with $\text{Emb}^+(D^2, S_g)$.

Lemma 6.2.9. *Fix a unitary vector $v \in UT_{p_0}D^2$. Then the* $\text{Diff}^+(S_g)$*-map*

$$\begin{aligned} \text{Emb}^+(D^2,S_g) &\to UT(S_g) \\ i &\mapsto \frac{d_{p_0}i(v)}{|d_{p_0}i(v)|} \end{aligned}$$

is a weak homotopy equivalence.

Proof. This is proved in the Appendix C. See Corollary C.2.5 $\square$

Now let $i_0 \in \text{Emb}^+(D^2, S_g)$ be the inclusion map.

Lemma 6.2.10. *The* $\text{Diff}^+(S_g)$*-map*

$$\begin{aligned} \text{Diff}^+(S_g) &\to \text{Emb}^+(D^2, S_g) \\ f &\mapsto f \circ i_0 = f|_{D^2} \end{aligned}$$

is a fiber bundle map.

Proof. This is proved in the Appendix C. See Theorem C.2.1. $\square$

Call H the stabilizer of i_0 under the action of $\text{Diff}^+(S_g)$. Then, H consists of those those diffeomorphisms that restrict to the identity on $i_0(D^2) = D^2$.

Corollary 6.2.11. *There are exact sequences of groups*

$$0 \to \pi_1(UT(S_g)) \to \text{Mod}(S_g^1, \partial S_g^1) \to \text{Mod}(S_g) \to 0$$

and

$$0 \to \pi_1(UT(S_g)) \to B_g^1 \to B_g \to 1.$$

Proof. The second exact sequence follows from the first. The first exact sequence comes from the long exact sequence of homotopy groups associated to the fiber bundle $\text{Diff}^+(S_g) \to \text{Emb}^+(D^2, S_g)$ recalling that $\text{Emb}^+(D^2, S_g)$ is homotopy equivalent to $UT(S_g)$.

$\square$

Applying Lemma B.2.16 and Lemma B.2.18, we also obtain

Corollary 6.2.12. *There exist a commutative diagram*

$$\begin{array}{ccccc}
\pi_1(UT(S_g)) & \longrightarrow & \pi_1(EUT(S_g)) & \longrightarrow & \pi_1(B\mathrm{Diff}^+(S_g)) \\
\downarrow & & \downarrow & & \downarrow \\
\pi_1(\mathrm{Emb}^+(D^2, S_g)) & \longrightarrow & \pi_1(E\mathrm{Emb}^+(D^2, S_g)) & \longrightarrow & \pi_1(B\mathrm{Diff}^+(S_g)) \\
\| & & \downarrow & & \downarrow \\
\pi_1(\mathrm{Emb}^+(D^2, S_g)) & \longrightarrow & \pi_0(H) & \longrightarrow & \pi_0(\mathrm{Diff}^+(S_g))
\end{array}$$

where the vertical arrows are isomorphisms, the first horizontal sequence is part of the exact sequence of homotopy groups associated to the fiber bundle $EUT(S_g) \to B\mathrm{Diff}^+(S_g)$, the second is part of the exact sequence associated to fiber bundle $E\mathrm{Emb}^+(D^2, S_g) \to B\mathrm{Diff}^+(S_g)$ and the last one is part of the exact sequence of the fiber bundle $\mathrm{Diff}^+(S_g) \to \mathrm{Emb}^+(D^2, S_g)$.

Construction of a nice $K(B_g, 1)$-space

The main in step in computing $\mathrm{cd}(B_g)$ is the proof of the following result:

Theorem 6.2.13. *There exists a closed topological manifold K_g of dimensions $4g-5$ that is a $K(B_g, 1)$-complex. Similarly, there exists a closed manifold K_g^1 of dimension $4g-2$ which is a $K(B_g^1, 1)$-complex.*

Proof. We start with the 3-dimensional torus $(S^1)^{\times 3}$ which is a $K(B_2, 1)$-space and has dimension 3, and given K_g we costruct K_g^1. After that we can take $K_{g+1} = K_g^1 \times S^1$.
Suppose K_g has already been constructed. Consider the topological group $\mathrm{Diff}^+(S_g)$. Since $B\mathrm{Diff}^+(S_g)$ is a $K(\mathrm{Mod}(S_g), 1)$-space, using Propostion B.2.8, the inclusion $B_g \hookrightarrow \mathrm{Mod}(S_g)$ induces a continuous map $h : K_g \to B\mathrm{Diff}^+(S_g)$ with the property that $\pi_1(K_g) \xrightarrow{\pi_1(h)} \pi_1(B\mathrm{Diff}^+(S_g))$ is injective and the image of the composition $\pi_1(K_g) \xrightarrow{\pi_1(h)} \pi_1(B\mathrm{Diff}^+(S_g)) \to \pi_0(\mathrm{Diff}^+(S_g))$ (where the second map is the boundary map in the long exact sequence of homotopy groups associated to the bundle $E\mathrm{Diff}^+(S_g) \to B\mathrm{Diff}^+(S_g)$) is exactly B_g.
Consider the action of $\mathrm{Diff}^+(S_g)$ on the unit tangent bundle $UT(S_g)$ of S_g described in the previous subsection. Define K_g^1 to be the pullback under the map h of the fiber bundle $EUT(S_g) \to B\mathrm{Diff}^+(S_g)$. Thus we have a fiber bundle $K_g^1 \to K_g$ with fiber $UT(S_g)$. We claim that this definition of K_g^1 works.

<u>Properties of K_g^1.</u>

Clearly K_g^1 is a topological manifold of dimension $\dim K_g^1 = \dim K_g + 3 = 4g-5+3 = 4g-2$. It is compact because we have a closed map $p : K_g^1 \to K_g$ with compact fibers. The fact that p is a closed map is standard. The proof goes as follows. Cover K_g with open sets U trivializing K_g^1, then $p : K_g^1|_U \cong U \times UT(S_g) \xrightarrow{\mathrm{pr}_U} U$ is closed being $UT(S_g)$ compact. It follows that for ever closed $C \subseteq K_g^1$, the image $p(C) \cap U = \mathrm{pr}_U(C \cap K_g^1|_U)$ is closed in U and thus that $p(C)$ is closed in K_g. Moreover, according to a general result of Kirby and Siebanmann, every closed manifold of dimension greater than 4 admits a CW-structure (see [28]). In particular, K_g^1 admits a CW-complex structure. What remains to prove is that K_g^1 is a $K(B_g^1, 1)$-space. First of all, we prove that it is a $K(\pi_1(K_g^1), 1)$-space. Indeed, the long exact sequence of the fiber bundle $K_g^1 \to K_g$

$$... \to \pi_n(UT(S_g)) \to \pi_n(K_g^1) \to \pi_n(K_g) \to ...$$

shows that $\pi_n(K_g^1) = 0$ for $n \geq 2$ and thus K_g^1 is a $K(\pi_1(K_g^1), 1)$-space. In addition, this sequence ends with the short exact sequence of groups

$$0 \to \pi_1(UT(S_g)) \to \pi_1(K_g^1) \to \pi_1(K_g) \to 0.$$

From the commutativity of the diagram

$$\begin{array}{ccc} K_g^1 & \longrightarrow & EUT(S_g) \\ \downarrow & & \downarrow \\ K_g & \longrightarrow & B\mathrm{Diff}^+(S_g) \end{array}$$

we obtain a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \pi_1(UT(S_g)) & \longrightarrow & \pi_1(K_g^1) & \longrightarrow & \pi_1(K_g) & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \pi_1(UT(S_g)) & \longrightarrow & \pi_1(EUT(S_g)) & \longrightarrow & \pi_1(B\mathrm{Diff}^+(S_g)) & \longrightarrow & 0 \end{array}$$

where the horizontal sequences are exact and $\pi_1(K_g) \to \pi_1(B\mathrm{Diff}^+(S_g))$ is injective. It follows that also the map $\pi_1(K_g^1) \to \pi_1(EUT(S_g))$ is injective. As explained in Corollary 6.2.12, the lower horizontal sequence can be identified with the exact sequence of homotopy groups associated to the fibration $\mathrm{Diff}^+(S_g) \to \mathrm{Emb}^+(D^2, S_g)$

$$0 \to \pi_1(\mathrm{Emb}^+(D^2, S_g)) \to \pi_0(H) \to \pi_0(\mathrm{Diff}^+(S_g)) \to 0$$

where $H \subseteq \mathrm{Diff}^+(S_g)$ is the the stabilizer of the inclusion map $i_0 \in \mathrm{Emb}^+(D^2, S_g)$ and thus consists of those orientation-preserving diffeomorphisms of S_g that fix pointwise $i_0(D^2) = D^2$. In particular $\pi_0(H) \cong \mathrm{Mod}(S_g^1, \partial S_g^1)$ and $\pi_1(K_g^1)$ is isomorphic to the preimage of $B_g \subseteq \pi_0(\mathrm{Diff}^+(S_g))$ under the map $\pi_0(H) \to \pi_0(\mathrm{Diff}^+(S_g))$ that is the surjective map $\mathrm{Mod}(S_g^1, \partial S_g^1) \to \mathrm{Mod}(S_g)$ used to define B_g^1. In conclusion $\pi_1(K_g^1) \cong B_g^1$ and the proof is complete. □

Corollary 6.2.14. *We have* $\mathrm{cd}(B_g) = 4g - 5$ *and* $\mathrm{cd}(B_g^1) = 4g - 2$.

Proof. Apply Theorem 1.2.8 to the spaces K_g and K_g^1. □

6.3 The virtual cohomological dimension of Mod(S)

We can finally compute vcd(Mod(S)). We start with closed surfaces.

Theorem 6.3.1. *Let* $S = S_g$ *be a closed surface. Then the virtual cohomological dimension of* $\mathrm{Mod}(S_g)$ *is*

$$\mathrm{vcd}(\mathrm{Mod}(S_g)) = \begin{cases} 0 & \text{if } g = 0; \\ 1 & \text{if } g = 1; \\ 4g - 5 & \text{if } g \geq 2. \end{cases}$$

Proof. If $g = 0$, we have seen that $\mathrm{Mod}(S^2)$=0 and thus $\mathrm{vcd}(\mathrm{Mod}(S^2))$=0. If $g = 1$ we have seen that $\mathrm{Mod}(T^2) \cong \mathrm{SL}(2, \mathbb{Z})$ and that $\mathrm{vcd}(\mathrm{SL}(2, \mathbb{Z})) = 1$. Thus we may assume that $g \geq 2$. Consider a torsion free and finite index subgroup $\mathrm{Mod}(S_g)[\mathrm{m}]$ of $\mathrm{Mod}(S_g)$ where $m \geq 3$. It is enough to prove that $\mathrm{cd}(\mathrm{Mod}(S_g)[\mathrm{m}]) = 4g - 5$. Consider the smooth manifold with corners $X_g(\delta)$ of Definition 5.4.2, where $\delta < 2\sinh^{-1}(1/2)$. The action of $\mathrm{Mod}(S_g)[\mathrm{m}]$ on T_{S_g} is free and proper discontinuous, and preserves both $X_g(\delta)$ and $\partial X_g(\delta)$; thus the quotient $Y_g = X_g(\delta)/\mathrm{Mod}(S_g)[\mathrm{m}]$ is a smooth manifold with corners. Note that $\pi_1(Y_g) \cong \mathrm{Mod}(S_g)[\mathrm{m}]$. There is a finite sheet covering $Y_g \to X_g(\delta)/\mathrm{Mod}(S_g)$ and $X_g(\delta)/\mathrm{Mod}(S_g)$ is compact, hence Y_g is compact, too. It follows that Y_g is a compact and connected smooth manifold with corners and in particular it has a finite CW-complex structure. Furthermore, $X_g(\delta)$ is contractible and it is the universal covering of Y_g, thus Y_g is a $K(\mathrm{Mod}(S_g)[\mathrm{m}], 1)$-space. Applying Theorem 1.2.8, we obtain $\mathrm{cd}(\mathrm{Mod}(S_g)[\mathrm{m}]) = \dim Y_g - t - 1$ where $\dim Y_g = \dim X_g(\delta) = 6g - 6$ and $t = \min\{k : \widetilde{H}_k(\partial X_g(\delta)) \neq 0\} \geq -\chi(S_g) = 2g - 2$. Here we are using the fact that $\partial X_g(\delta)$ is homotopically equivalent to $C(S_g)$, that $C(S_g)$ is $\chi(S_g) - 1 = 2g - 3$ connected and Hurewicz Theorem. This proves that $\mathrm{cd}(\mathrm{Mod}(S_g)[\mathrm{m}]) \leq 6g - 6 - (2g - 2) - 1 = 4g - 5$. Next we prove the reverse inequality. Consider the intersection $\Gamma = \mathrm{Mod}(S_g)[\mathrm{m}] \cap B_g$. Then, Γ has finite index in B_g and, since $\mathrm{cd}(B_g) < \infty$, we have $\mathrm{cd}(\Gamma) = \mathrm{cd}(B_g) = 4g - 5$. But now Γ is a subgroup of $\mathrm{Mod}(S_g)[\mathrm{m}]$ and thus $\mathrm{cd}(\mathrm{Mod}(S_g)[\mathrm{m}]) \geq \mathrm{cd}(\Gamma) = 4g - 5$. □

Theorem 6.3.2. *Let $S = S_g^b$ with $g \geq 2$ and $b \geq 1$. Then* $\mathrm{vcd}(\mathrm{Mod}(S)) = 4g-4+b$.

In the proof of this result, we will need the following lemma.

Lemma 6.3.3. *Let $x \in S_g^b \setminus \partial S_g^b$. The map $p : \mathrm{Homeo}^+(S_g^b) \to S_g^b \setminus \partial S_g^b$ given by $\phi \mapsto \phi \cdot x = \phi(x)$ is a fiber bundle map, with fiber* $\mathrm{Homeo}^+(S_g, \{x\})$ *the subspace of those orientation-preserving homeomorphisms that fix x.*

Proof. For a standard result of Differential Topology, there are an open disk U around x in $S_g^b \setminus \partial S_g^b$ and a continuous map $\Phi : U \to \mathrm{Homeo}^+(S_g^b)$ such that $p \circ \Phi = 1_U$ (i.e. $\Phi(u)(x) = u$ for all $u \in U$). Thus we have a homeomorphism $U \times \mathrm{Homeo}^+(S_g, \{x\}) \to p^{-1}(U)$ defined by $(u, \phi) \mapsto \Phi(u) \circ \phi$. For any other point $y \in S_g^b \setminus \partial S_g^b$, we can choose a homeomorphism ξ of S_g^b taking x to y. Then there is a homeomorphism

$$\xi(U) \times \mathrm{Homeo}^+(S_g^b, \{x\}) \to U \times \mathrm{Homeo}^+(S_g, \{x\}) \to p^{-1}(U) \to p^{-1}(\xi(U))$$

given by $(\xi(u), \phi) \mapsto \xi \circ \Phi(u) \circ \phi$ and so we have verified the fiber bundle property. □

Proof of Theorem 6.3.2. We first explain the strategy of the proof. We will construct a finite-index subgroup $\Gamma_{g,b}$ of $\mathrm{Mod}(S_g^b)$ and a $K(\Gamma_{g,b}, 1)$-space $Y_{g,b}$ that is also a compact topological manifold with boundary and that has a structure of finite CW-complex. Then, using Theorem 1.2.8, we will compute $\mathrm{cd}(\Gamma_{g,b})$, that will necessarily be finite. In particular, $\Gamma_{g,b}$ must be torsion-free and $\mathrm{vcd}(\mathrm{Mod}(S_g^b))$=$\mathrm{cd}(\Gamma_{g,b})$.

Construction of $\Gamma_{g,b}$ and $Y_{g,b}$.

Fix $g \geq 2$. We construct $\Gamma_{g,b}$ and $Y_{g,b}$ by induction on b. Start with $\Gamma_{g,0} = \mathrm{Mod}(S_g)[\mathrm{m}]$ where $m \geq 3$ and $Y_{g,0} = X_g(\delta)/\mathrm{Mod}(S_g)[\mathrm{m}]$ where $\delta < 2\sinh^{-1}(1/2)$. Now suppose that $\Gamma_{g,b}$ and $Y_{g,b}$ are already constructed. Consider the action of $\mathrm{Homeo}^+(S_g^b)$ on $S_g^b \setminus \partial S_g^b$ given by $\phi \cdot y = \phi(y)$ for all $\phi \in \mathrm{Homeo}^+(S_g^b)$ and $y \in S_g^b \setminus \partial S_g^b$. Fix a point $x \in S_g^b \setminus \partial S_g^b$. Then, the map $\mathrm{Homeo}^+(S_g^b) \to S_g^b \setminus \partial S_g^b$ given by $\phi \mapsto \phi \cdot x = \phi(x)$ is a fiber bundle map, with fiber $\mathrm{Homeo}^+(S_g, \{x\})$ the subspace of those orientation-preserving homeomorphisms that fix x. Since $\chi(S_g^b) < 0$, the connected components of $\mathrm{Homeo}^+(S_g^b)$ are contractible and thus the homotopy sequence of this fiber bundle ends with

$$0 \to \pi_1(S_g^b \setminus \partial S_g^b) \to \pi_0(\mathrm{Homeo}^+(S_g, \{x\})) \to \pi_0(\mathrm{Homeo}^+(S_g^b)) \to 0.$$

Observe that, according to Remark 3.1.6, $\pi_0(\mathrm{Homeo}^+(S_g^b, \{x\})) = \mathrm{Mod}(S_g^b, \{x\})$ and thus the previous sequence is equal to the exact sequence of groups

$$0 \to \pi_1(S_g^b \setminus \partial S_g^b) \to \mathrm{Mod}(S_g^b, \{x\}) \to \mathrm{Mod}(S_g^b) \to 0.$$

Using Lemma B.2.18, this short exact sequence can also be identified with the short exact sequence of groups in the sequence of the fiber bundle $E(S_g^b \smallsetminus \partial S_g^b) = (E\mathrm{Homeo}^+(S_g^b) \times (S_g^b \smallsetminus \partial S_g^b))/\mathrm{Homeo}^+(S_g^b) \to B\mathrm{Homeo}^+(S_g^b)$ that follows

$$0 \to \pi_1(S_g^b \smallsetminus \partial S_g^b) \to \pi_1(E(S_g^b \smallsetminus \partial S_g^b)) \to \pi_1(B\mathrm{Homeo}^+(S_g^b)) \to 0.$$

Next, we replace $S_g^b \smallsetminus \partial S_g^b$ with S_g^b in the following way. The inclusion $S_g^b \smallsetminus \partial S_g^b \to S_g^b$ is clearly an homotopy equivalence and, if S_g^b is regarded as a $\mathrm{Homeo}^+(S_g^b)$-space by $\phi \cdot y = \phi(y)$ for all $\phi \in \mathrm{Homeo}^+(S_g^b)$ and $y \in S_g^b$, it is also a $\mathrm{Homeo}^+(S_g^b)$-map. Thus, by Lemma B.2.16, the last exact sequence can be identified with the exact sequence

$$0 \to \pi_1(S_g^b) \to \pi_1(ES_g^b) \to \pi_1(B\mathrm{Homeo}^+(S_g^b)) \to 0$$

where $ES_g^b = (E\mathrm{Homeo}^+(S_g^b) \times S_g^b)/\mathrm{Homeo}^+(S_g^b) \to B\mathrm{Homeo}^+(S_g^b)$ is a Serre fibration with fiber S_g^b.
Now, since $\chi(S_g^b) < 0$, the space $B\mathrm{Homeo}^+(S_g^b)$ is a $K(\mathrm{Mod}(S_g^b), 1)$-space and hence the inclusion $\Gamma_{g,b} \hookrightarrow \mathrm{Mod}(S_g^b)$ induces a continuous map $h : Y_{g,b} \to B\mathrm{Homeo}^+(S_g^b)$ such that $\pi_1(Y_{g,b}) \xrightarrow{\pi_1(h)} \pi_1(B\mathrm{Homeo}^+(S_g^b))$ is injective and the image of the composition $\pi_1(Y_{g,b}) \xrightarrow{\pi_1(h)} \pi_1(B\mathrm{Homeo}^+(S_g^b)) \to \pi_0(\mathrm{Homeo}^+(S_g^b))$ (where the second map is the boundary map in the long exact sequence of homotopy groups associated to the bundle $E\mathrm{Homeo}^+(S_g^b) \to B\mathrm{Homeo}^+(S_g^b)$) in exactly $\Gamma_{g,b}$. Define $Y_{g,b+1}$ to be the pullback under the map h of the fiber bundle $ES_g^b \to B\mathrm{Homeo}^+(S_g^b)$. Thus we have a fiber bundle $Y_{g,b+1} \to Y_{g,b}$ with fiber S_g^b. Since S_g^b is a $K(\pi_1(S_g^b), 1)$-space and $Y_{g,b}$ is a $K(\Gamma_{g,b}, 1)$-space, the long exact sequence of the fiber bundle $Y_{g,b+1} \to Y_{g,b}$ shows that $Y_{g,b+1}$ is a $K(\pi_1(Y_{g,b+1}), 1)$-space. Moreover, from the commutative diagram

$$\begin{array}{ccc} Y_{g,b+1} & \longrightarrow & ES_g^b \\ \downarrow & & \downarrow \\ Y_{g,b} & \longrightarrow & B\mathrm{Homeo}^+(S_g^b) \end{array}$$

we obtain a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \pi_1(S_g^b) & \longrightarrow & \pi_1(Y_{g,b+1}) & \longrightarrow & \pi_1(Y_{g,b}) & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \pi_1(S_g^b) & \longrightarrow & \pi_1(ES_g^b) & \longrightarrow & \pi_1(B\mathrm{Homeo}^+(S_g^b)) & \longrightarrow & 0 \end{array}$$

where the lower horizontal short exact sequence is exactly that contained in the homotopy sequence of the fiber bundle $ES_g^b \to B\mathrm{Homeo}^+(S_g^b)$ and $\pi_1(Y_g) \to \pi_1(B\mathrm{Homeo}^+(S_g))$

is injective. It follows that the map $\pi_1(Y_{g,b+1}) \to \pi_1(ES_g^b)$ is also injective and $\pi_1(Y_{g,b+1})$ is isomorphic to the preimage of $\Gamma_{g,b}$ under the map $\pi_1(ES_g^b) \to \pi_1(B\mathrm{Homeo}^+(S_g^b)) \cong \pi_0(\mathrm{Homeo}^+(S_g^b))$, that, as before explained, can be identified with the surjective homomorphism $\mathrm{Mod}(S_g^b, \{x\}) \to \mathrm{Mod}(S_g^b)$. In particular, $\Gamma_{g,b+1}$ is a finite-index subgroup of $\mathrm{Mod}(S_g^b, \{x\})$. Now observe that $\mathrm{Mod}(S_g^b, \{x\})$ is isomorphic to a subgroup of index $b+1 < \infty$ of $\mathrm{Mod}(S_g^{b+1})$, namely to the subgroup consisting of those homeomorphisms of S_g^{b+1} preserving setwise one boundary component. This follows as in the proof of Lemma 3.1.10. Hence, $\Gamma_{g,b+1}$ is a finite-index subgroup of $\mathrm{Mod}(S_g^{b+1})$ and $Y_{g,b+1}$ is a $K(\Gamma_{g,b+1}, 1)$-space. Now, since our $Y_{g,b}$ are constructed by consecutively taking the total spaces of fiber bundles with fiber a surface, we have that $Y_{g,b}$ is a topological manifold with boundary of dimension $\dim Y_{g,b} = \dim Y_{g,0} + 2b = 6g - 6 + 2b$. Moreover, $Y_{g,0}$ is compact and for all $b \geq 1$ the projection $Y_{g,b+1} \to Y_{g,b}$ is a closed map with compact fibers, thus, by induction on b, $Y_{g,b}$ is always compact. Finally, since $\dim Y_{g,b} \geq 5$, it always has a structure of finite CW-complex.

Study of the universal covering space $\widetilde{Y_{g,b}}$ of $Y_{g,b}$.

In order to apply Theorem 1.2.8, we need to study the universal covering space $\widetilde{Y_{g,b}}$ of $Y_{g,b}$. We already know that $\widetilde{Y_{g,0}} = X_g(\delta)$. Note that it is contractible. Suppose $\widetilde{Y_{g,b}}$ has already been constructed and is contractible, then $\widetilde{Y_{g,b+1}}$ can be constructed as follows. First, consider the pullback $M_{g,b+1} \to Y_{g,b+1}$ (as a fiber bundle) of the covering map $\widetilde{Y_{g,b}} \to Y_{g,b}$ under the projection map $Y_{g,b+1} \to Y_{g,b}$. Thus we have a commutative square

$$\begin{array}{ccc} M_{g,b+1} & \longrightarrow & \widetilde{Y_{g,b}} \\ \downarrow & & \downarrow \\ Y_{g,b+1} & \longrightarrow & Y_{g,b}. \end{array}$$

Note that $M_{g,b+1} \to Y_{g,b+1}$ is a covering map and $M_{g,b+1} \to \widetilde{Y_{g,b}}$ is a fiber bundle map with fiber $\widetilde{S_g^b}$. Since $\widetilde{Y_{g,b}}$ is contractible and paracompact, by Corollary B.1.8, the fiber bundle $M_{g,b+1} \to \widetilde{Y_{g,b}}$ is trivial and thus $M_{g,b+1}$ is homeomorphic to $\widetilde{S_g^b} \times \widetilde{Y_{g,b}}$. It follows that $\widetilde{Y_{g,b+1}} = \widetilde{S_g^b} \times \widetilde{Y_{g,b}}$ and thus it is contractible. Its boundary is $\partial\widetilde{S_g^b} \times \widetilde{Y_{g,b}} \cup \widetilde{S_g^b} \times \partial\widetilde{Y_{g,b}}$. We distinguish two cases.

First consider the case $b = 0$. Then, $\partial\widetilde{Y_{g,1}} = \mathbb{H}^2 \times \partial X_g(\delta)$ is homotopically equivalent to $\partial X_g(\delta)$. In particular $\widetilde{H}_0(\partial\widetilde{Y_{g,1}}) = \widetilde{H}_1(\partial\widetilde{Y_{g,1}}) = 0$.

Now consider the case $b \geq 1$. In this case $\partial\widetilde{S_g^b} \neq \emptyset$ and consists of countably (infinite) many components each homeomorphic to $\mathbb{R}$. What is more the pair $(\widetilde{S_g^b}, \partial\widetilde{S_g^b})$ is homotopically equivalent to $(C\mathbb{Z},\mathbb{Z})$, where $C\mathbb{Z}$ is the cone of $\mathbb{Z}$. In turn, $(C\mathbb{Z},\mathbb{Z})$ is homotopically equivalent to the pair $(\mathbb{R},\mathbb{Z})$, thus $\widetilde{\partial Y_{g,b+1}}$ is homotopy equivalent to $\mathbb{Z}\times\widetilde{Y_{g,b}}\cup\mathbb{R}\times\widetilde{\partial Y_{g,b}}$. Clearly, $\widetilde{H}_0(\mathbb{Z}\times\widetilde{Y_{g,b}}\cup\mathbb{R}\times\widetilde{\partial Y_{g,b}})=0$. To investigate $\widetilde{H}_m(\mathbb{Z}\times\widetilde{Y_{g,b}}\cup\mathbb{R}\times\widetilde{\partial Y_{g,b}})$ for $m\geq 1$, consider the good pair $(\mathbb{Z}\times\widetilde{Y_{g,b}}\cup\mathbb{R}\times\widetilde{\partial Y_{g,b}},\mathbb{Z}\times\widetilde{Y_{g,b}})$. The associated long exact sequence of the reduced homology groups shows that for all $m>1$ we have

$$\widetilde{H}_m(\mathbb{Z}\times\widetilde{Y_{g,b}}\cup\mathbb{R}\times\widetilde{\partial Y_{g,b}})\cong\widetilde{H}_m((\mathbb{Z}\times\widetilde{Y_{g,b}}\cup\mathbb{R}\times\widetilde{\partial Y_{g,b}})/(\mathbb{Z}\times\widetilde{Y_{g,b}}))$$

and for $m=1$ the natural map

$$\widetilde{H}_1(\mathbb{Z}\times\widetilde{Y_{g,b}}\cup\mathbb{R}\times\widetilde{\partial Y_{g,b}})\hookrightarrow\widetilde{H}_1((\mathbb{Z}\times\widetilde{Y_{g,b}}\cup\mathbb{R}\times\widetilde{\partial Y_{g,b}})/(\mathbb{Z}\times\widetilde{Y_{g,b}}))$$

is injective. Moreover the quotient $(\mathbb{Z}\times\widetilde{Y_{g,b}}\cup\mathbb{R}\times\widetilde{\partial Y_{g,b}})/(\mathbb{Z}\times\widetilde{Y_{g,b}})$ is homeomorphic to $(\mathbb{R}\times\widetilde{\partial Y_{g,b}})/(\mathbb{Z}\times\widetilde{\partial Y_{g,b}})$.

Let $m=1$. We have already observed that $\widetilde{H}_0(\widetilde{\partial Y_{g,b}})=0$ and now we prove, by induction on b, that $\widetilde{H}_1(\widetilde{\partial Y_{g,b}})=0$. If $b=1,0$ there is nothing to prove. For $b\geq 2$, it is enough to consider the commutative diagram

$$\begin{array}{ccccc}
 & & \widetilde{H}_1(\mathbb{R}\times\widetilde{\partial Y_{g,b}})=0 & & \\
 & & \downarrow & & \\
\widetilde{H}_1(\mathbb{Z}\times\widetilde{Y_{g,b}}\cup\mathbb{R}\times\widetilde{\partial Y_{g,b}}) & \hookrightarrow & \widetilde{H}_1(\mathbb{Z}\times\widetilde{Y_{g,b}}\cup\mathbb{R}\times\widetilde{\partial Y_{g,b}})/(\mathbb{Z}\times\widetilde{Y_{g,b}})) & \longrightarrow & \widetilde{H}_0(\mathbb{Z}\times\widetilde{Y_{g,b}}) \\
 & & \downarrow & \nearrow\cong & \\
 & & \widetilde{H}_0(\mathbb{Z}\times\widetilde{\partial Y_{g,b}}) & &
\end{array}$$

where the horizontal arrows come from the exact sequence of the pair $(\mathbb{Z}\times\widetilde{Y_{g,b}}\cup\mathbb{R}\times\widetilde{\partial Y_{g,b}},\mathbb{Z}\times\widetilde{Y_{g,b}})$, the vertical arrows come from the exact sequence of the pair $(\mathbb{R}\times\widetilde{\partial Y_{g,b}},\mathbb{Z}\times\widetilde{\partial Y_{g,b}})$ and the map $\widetilde{H}_0(\mathbb{Z}\times\widetilde{\partial Y_{g,b}})\to\widetilde{H}_0(\mathbb{Z}\times\widetilde{Y_{g,b}})$ is induced by the inclusion map. It follows that $\widetilde{H}_1(\widetilde{\partial Y_{g,b+1}})=\widetilde{H}_1(\mathbb{Z}\times\widetilde{Y_{g,b}}\cup\mathbb{R}\times\widetilde{\partial Y_{g,b}})=0$.

Now define $m_{g,b}=\min\{j\in\mathbb{N}_0:\widetilde{H}_j(\widetilde{\partial Y_{g,b}})\neq 0\}$. We know that $m_{g,b}\geq 2$. The long exact sequence of the reduced homology groups associated to the good pair $(\mathbb{R}\times\widetilde{\partial Y_{g,b}},\mathbb{Z}\times\widetilde{\partial Y_{g,b}})$ gives for $1\leq m<m_{g,b}$ injections

$$\widetilde{H}_m((\mathbb{R}\times\widetilde{\partial Y_{g,b}})/(\mathbb{Z}\times\widetilde{\partial Y_{g,b}}))\hookrightarrow\widetilde{H}_{m-1}(\mathbb{Z}\times\widetilde{\partial Y_{g,b}}).$$

Thus, if $1\leq m\leq m_{g,b}$ then $\widetilde{H}_{m-1}(\widetilde{\partial Y_{g,b}})=0$ and we have $\widetilde{H}_m(\widetilde{\partial Y_{g,b+1}})=0$.

What is more, the long exact sequence of the reduced homology groups associated to the pair $(\mathbb{R}\times\widetilde{\partial Y_{g,b}},\mathbb{Z}\times\widetilde{\partial Y_{g,b}})$ shows that if, for $m>1$, we have $\widetilde{H}_{m-1}(\widetilde{\partial Y_{g,b}})\neq 0$,

then also $\widetilde{H}_m(\widetilde{\partial Y_{g,b+1}}) \cong \widetilde{H}_m((\mathbb{R} \times \widetilde{\partial Y_{g,b}})/(\mathbb{Z} \times \widetilde{\partial Y_{g,b}})) \neq 0$.
In other words, we have proved that $m_{g,1} = m_{g,0}$ and $m_{g,b+1} = m_{g,b}+1$ for all $b \geq 1$. In conclusion, since $\dim Y_{g,b+1} = \dim Y_{g,b} + 2$ for all $b \geq 0$, applying Theorem 1.2.8, we obtain that $\mathrm{cd}(\Gamma_{g,1}) = \mathrm{cd}(\Gamma_{g,0}) + 2$ and $\mathrm{cd}(\Gamma_{g,b+1}) = \mathrm{cd}(\Gamma_{g,b}) + 1$. Since $\mathrm{cd}(\Gamma_{g,0}) = \mathrm{vcd}(\mathrm{Mod}(S_g)) = 4g-5$, we have $\mathrm{vcd}(\mathrm{Mod}(S_g^b)) = \mathrm{cd}(\Gamma_{g,b}) = 4g-4+b$. □

Remark 6.3.4. Note that the group $\Gamma_{g,b}$ constructed in the proof of Theorem 6.3.2 belongs to the subgroup $\pi_0(\mathrm{Homeo}^+(S_g^b, \{\partial S_g^b\}))$ of $\mathrm{Mod}(S_g^b)$.

Finally we compute the virtual cohomological dimension of the remaining Mapping Class Groups.

Theorem 6.3.5. $\mathrm{vcd}(\mathrm{Mod}(S_0^b)) = \min\{0, b-3\}$ *and* $\mathrm{vcd}(\mathrm{Mod}(S_1^b)) = \min\{1, b\}$.

Proof. For $g = 0$ and $b \leq 3$ the results follows from Examples 3.1.7, 3.1.8, 3.1.11 and 3.1.12. Note that $\chi(S_0^3) = -1 < 0$. Now the desired result is obtained proceeding as in the proof of the previous theorem starting with $Y_{0,3}$ equal to a point. Similarly, if $g = 1$ and $b \leq 1$, the result follows from Examples 3.1.13 and 3.1.14. Since $\chi(S_1^1) = -1 < 0$, we can proceed as above to complete the proof of the theorem. In this case start with $Y_{1,1}$ equal to a thickened figure 8. □

Remark 6.3.6. Also the group $\Gamma_{g,b}$ that one constructs in the proof of Theorem 6.3.5 belongs to the subgroup $\pi_0(\mathrm{Homeo}^+(S_g^b, \{\partial S_g^b\}))$ of $\mathrm{Mod}(S_g^b)$.

6.4 The virtual cohomological dimension of Mod$(S, \partial S)$

Using the computation of the previous section and the exact sequence of groups

$$0 \to \mathbb{Z}^b \to \mathrm{Mod}(S_g^b, \partial S_g^b) \to \pi_0(\mathrm{Homeo}^+(S_g^b, \{\partial S_g^b\})) \to 0$$

(see Proposition 6.1.6) we can also compute vcd $(\mathrm{Mod}(S_g^b, \partial S_g^b))$. The computation uses the same techniques used to compute cd B_g and $\mathrm{vcd}(\mathrm{Mod}(S_g^b))$.

Let $S = S_g^b$ be a surface with $b \geq 1$ boundary components $\partial_1, ..., \partial_b$.

Lemma 6.4.1. *Let* $(S^1)^{\times b} \subseteq \bigtimes_{i=1}^b \mathrm{Homeo}^+(\partial_i)$ *be the subgroup generated by the rotations around each boundary component and let* $\mathcal{E}$ *be the map of Proposition 6.1.3. We have a fiber bundle map*

$$\mathcal{E}^{-1}((S^1)^{\times b}) \xrightarrow{\mathcal{E}} (S^1)^{\times b}$$

with fiber $\mathrm{Homeo}^+(S, \partial S)$.

Proof. This follows from Proposition 6.1.3. $\square$

Notation 6.4.1. We will call $G_{g,b} = \mathcal{E}^{-1}((S^1)^{\times b})$.

Corollary 6.4.2. *The inclusion*

$$G_{g,b} \hookrightarrow \mathrm{Homeo}^+(S_g^b, \{\partial S_g^b\})$$

is a weak homotopy equivalence.

Proof. The commutative square

$$\begin{array}{ccc} G_{g,b} & \longrightarrow & \mathrm{Homeo}^+(S_g^b, \{\partial S_g^b\}) \\ \downarrow & & \downarrow \\ (S^1)^{\times b} & \longrightarrow & \bigtimes_{i=1}^b \mathrm{Homeo}^+(\partial_i) \end{array}$$

induces, for all $i \geq 0$, commutative diagrams

$$\begin{array}{ccccccccc} \pi_{i+1}((S^1)^{\times b}) & \to & \pi_i(F) & \to & \pi_i(G_{g,b}) & \to & \pi_i((S^1)^{\times b}) & \to & \pi_{i-1}(F) \\ \| & & \downarrow\cong & & \downarrow & & \downarrow\cong & & \| \\ \pi_{i+1}(\bigtimes_{i=1}^b \mathrm{Homeo}^+(\partial_i)) & \to & \pi_i(F) & \to & \pi_i(\mathrm{Homeo}^+(S_g^b, \{\partial S_g^b\})) & \to & \pi_i(\bigtimes_{i=1}^b \mathrm{Homeo}^+(\partial_i)) & \to & \pi_{i-1}(F) \end{array}$$

where $F = \mathrm{Homeo}^+(S_g^b, \partial S_g^b)$. The corollary follows from the Five Lemma. $\square$

Corollary 6.4.3. *If $\chi(S_g^b) < 0$, the space $G_{g,b}$ is weakly contractible.*

Proof. By the previous corollary, for $i \geq 1$, we have $\pi_i(G_{g,b}) = \pi_i(\mathrm{Homeo}^+(S_g^b, \{\partial S_g^b\}) = \pi_i(\mathrm{Homeo}_0(S_g^b))$ and, by Theorem 6.1.2, $\pi_i(\mathrm{Homeo}_0(S_g^b)) = 0$ for $i \geq 1$. $\square$

Observe that $G_{g,b}$ is a topological group. Clearly $G_{g,b}$ acts continuously on $\bigtimes_{i=1}^b \mathrm{Homeo}^+(\partial_i)$ by composition and $\mathcal{E}|_{G_{g,b}}$ is equal to $f \mapsto f \circ 1_{\partial S} = f|_{\partial S}$. Note that the stabilizer is $\mathrm{Homeo}^+(S_g^b, \partial S_g^b)$. Moreover, the subspace $(S^1)^{\times b} \subseteq \bigtimes_{i=1}^b \mathrm{Homeo}^+(\partial_i)$ is preserved under the action of $G_{g,b}$. It follows that the inclusion $(S^1)^{\times b} \hookrightarrow \bigtimes_{i=1}^b \mathrm{Homeo}^+(\partial_i)$ is a $G_{g,b}$-map and, by Lemma 6.1.4, an homotopy equivalence.

Applying Lemma B.2.16 and Lemma B.2.18, we obtain

Corollary 6.4.4. *There exist a commutative diagram*

$$\begin{array}{ccccc}
\mathbb{Z}^b = \pi_1((S^1)^{\times b}) & \longrightarrow & \pi_1(E((S^1)^{\times b})) & \longrightarrow & \pi_1(BG_{g,b}) \\
\downarrow & & \downarrow & & \| \\
\pi_1(\bigtimes_{i=1}^b \mathrm{Homeo}^+(\partial_i)) & \longrightarrow & \pi_1(E(\bigtimes_{i=1}^b \mathrm{Homeo}^+(\partial_i))) & \longrightarrow & \pi_1(BG_{g,b}) \\
\| & & \downarrow & & \downarrow \\
\pi_1(\bigtimes_{i=1}^b \mathrm{Homeo}^+(\partial_i)) & \longrightarrow & \pi_0(\mathrm{Homeo}^+(S_g^b, \partial S_g^b)) & \longrightarrow & \pi_0(G_{g,b})
\end{array}$$

where the vertical arrows are isomorphisms, the first horizontal sequence is part of the exact sequence of homotopy groups associated to the fiber bundle $E((S^1)^{\times b}) = (EG_{g,b} \times (S^1)^{\times b})/G_{g,b} \to BG_{g,b}$*, the second is part of the exact sequence associated to fiber bundle* $E(\bigtimes_{i=1}^b \mathrm{Homeo}^+(\partial_i)) = (EG_{g,b} \times \bigtimes_{i=1}^b \mathrm{Homeo}^+(\partial_i))/G_{g,b} \to BG_{g,b}$ *and the last one is part of the exact sequence of the Serre fibration* $G_{g,b} \to \bigtimes_{i=1}^b \mathrm{Homeo}^+(\partial_i)$*.*

Theorem 6.4.5. *Let* $S = S_g^b$ *be a surface with* $b \geq 1$*. Then*
(a) for $\chi(S) \geq 0$*, we have* $\mathrm{vcd}(\mathrm{Mod}(S, \partial S)) = b - 1$*;*
(b) for $\chi(S) < 0$*, we have* $\mathrm{vcd}(\mathrm{Mod}(S, \partial S)) = \mathrm{vcd}(\mathrm{Mod}(S)) + b$*.*

Proof. (a) follows from Examples 3.1.7 and 3.1.9. We prove (b).

Stretegy for the proof of (b)

We will construct construct a finite-index subgroup $\Gamma_{g,b}^\partial$ of $\mathrm{Mod}(S_g^b, \partial S_g^b)$ and a $K(\Gamma_{g,b}^\partial, 1)$-space $Y_{g,b}^\partial$ that is also a compact topological manifold with boundary and that has a structure of finite CW-complex. Then, using Theorem 1.2.8, we will compute $\mathrm{cd}(\Gamma_{g,b}^\partial)$, that will necessarily be finite. In particular, $\Gamma_{g,b}^\partial$ must be torsion-free and $\mathrm{vcd}(\mathrm{Mod}(S_g^b, \partial S_g^b))=\mathrm{cd}(\Gamma_{g,b}^\partial)$.

Construction of $\Gamma_{g,b}^\partial$ and $Y_{g,b}^\partial$.

The definition of $\Gamma_{g,b}^\partial$ is easy. Consider the exact sequence of groups obtained in Proposition 6.1.6

$$0 \to \mathbb{Z}^b \to \mathrm{Mod}(S_g^b, \partial S_g^b) \xrightarrow{\eta} \pi_0(\mathrm{Homeo}^+(S_g^b, \{\partial S_g^b\})) \to 0.$$

By Remark 6.3.4 or 6.3.6, the group $\Gamma_{g,b}$ constructed in Theorem 6.3.2 and in Theorem 6.3.5 is contained in $\pi_0(\mathrm{Homeo}^+(S, \{\partial S\}))$ and we define $\Gamma_{g,b}^\partial$ to be the preimage of $\Gamma_{g,b}$ under the homomorphism η. Since $\Gamma_{g,b}$ has finite index in $\pi_0(\mathrm{Homeo}^+(S_g^b, \{\partial S_g^b\}))$,

also $\Gamma^{\partial}_{g,b}$ has finite index in $\mathrm{Mod}(S^b_g, \partial S^b_g)$.
Next, we construct $Y^{\partial}_{g,b}$. Since $\chi(S^b_g) < 0$, the group $G_{g,b}$ is weakly contractible and thus the classifying space $BG_{g,b}$ is a $K(G_{g,b}, 1)$-space. Let $h : Y_{g,b} \to BG_{g,b}$ be a continuous map such that $\pi_1(h) : \pi_1(Y_{g,b}) \hookrightarrow \pi_1(BG_{g,b})$ is injective and the image of the composition $\pi_1(Y_{g,b}) \xrightarrow{\pi_1(h)} \pi_1(BG_{g,b}) \cong \pi_0(\mathrm{Homeo}^+(S^b_g, \{\partial S^b_g\}))$ is exactly $\Gamma_{g,b}$. Here the second map is the isomorphism obtained by compositions as $\pi_1(BG_{g,b}) \cong \pi_0(G_{g,b}) \cong \pi_0(\mathrm{Homeo}^+(S^b_g, \{\partial S^b_g\}))$ where the first isomorphism is the boundary map in the long exact sequence of homotopy groups associated to the bundle $EG_{g,b} \to BG_{g,b}$ and the second map is induced by the inclusion.
We define $Y^{\partial}_{g,b}$ to be the pullback under the map h of the fiber bundle $E((S^1)^{\times b}) \to BG_{g,b}$. Then $Y^{\partial}_{g,b}$ is a compact topological manifold with boundary of dimension $\dim Y^{\partial}_{g,b} = \dim Y_{g,b} + b$. Since $\dim Y_{g,b} \geq 5$, the manifold $Y^{\partial}_{g,b}$ also admits a CW-complex structure. Moreover, since $Y_{g,b}$ and $(S^1)^{\times b}$ are respectively a $K(\pi_1(Y_{g,b}), 1)$-space and a $K(\mathbb{Z}^b, 1)$-space, also $Y_{g,b+1}$ is a $K(\pi_1(Y_{g,b+1}), 1)$-space. Finally $\pi_1(Y^{\partial}_{g,b}) \cong \Gamma^{\partial}_{g,b}$. Indeed, the exact sequence of homotopy groups associated to the fiber bundle $Y^{\partial}_{g,b} \to Y_{g,b}$ contains the exact sequence of groups

$$0 \to \pi_1((S^1)^{\times b}) \to \pi_1(Y^{\partial}_{g,b}) \to \pi_1(Y_{g,b}) \to 0$$

and using the commutative diagram

$$\begin{array}{ccc} Y^{\partial}_{g,b} & \longrightarrow & E((S^1)^{\times b}) \\ \downarrow & & \downarrow \\ Y_{g,b} & \xrightarrow{h} & BG_{g,b} \end{array}$$

we obtain a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \pi_1((S^1)^{\times b}) & \longrightarrow & \pi_1(Y^{\partial}_{g,b}) & \longrightarrow & \pi_1(Y_{g,b}) & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow{\scriptstyle \pi_1(h)} & & \\ 0 & \longrightarrow & \pi_1((S^1)^{\times b}) & \longrightarrow & \pi_1(E(S^1)^{\times b}) & \longrightarrow & \pi_1(BG_{g,b}) & \longrightarrow & 0 \end{array}$$

where the lower horizontal short exact sequence is that contained in the homotopy sequence of the fiber bundle $E((S^1)^{\times b}) \to BG_{g,b}$. It follows that the map $\pi_1(Y^{\partial}_{g,b}) \to \pi_1(E((S^1)^{\times b}))$ is also injective and $\pi_1(Y^{\partial}_{g,b})$ is isomorphic to the preimage of $\mathrm{Im}(\pi_1(h))$ under the map $\pi_1(E(S^1)^{\times b}) \to \pi_1(BG_{g,b})$. Moreover, using Corollary 6.4.4, the last exact sequence can be identified with the exact sequence

$$0 \to \pi_1(\bigtimes_{i=1}^{b} \mathrm{Homeo}^+(\partial_i)) \to \pi_0(\mathrm{Homeo}^+(S^b_g, \partial S^b_g)) \to \pi_0(G_{g,b}) \to 0$$

and we can also identify $\pi_0(G_{g,b}) \cong \pi_0(\mathrm{Homeo}^+(S_g^b, \{\partial S_g^b\}))$. Under these identifications, $\mathrm{Im}(\pi_1(h))$ correspond to $\Gamma_{g,b}$ and thus we have $\pi_1(Y_{g,b}^\partial) \cong \Gamma_{g,b}^\partial$.

Study of the universal cover $\widetilde{Y_{g,b}^\partial}$ of $Y_{g,b}^\partial$.

Let $p : \widetilde{Y_{g,b}} \to Y_{g,b}$ be the universal covering map of $Y_{g,b}$ and let $M_{g,b}^\partial$ be the pullback under p of the fiber bundle $Y_{g,b}^\partial \to Y_{g,b}$. Thus we have a commutative diagram

$$\begin{array}{ccc} M_{g,b}^\partial & \longrightarrow & Y_{g,b}^\partial \\ \downarrow & & \downarrow \\ \widetilde{Y_{g,b}} & \xrightarrow{p} & Y_{g,b} \end{array}$$

where the map $M_{g,b}^\partial \to Y_{g,b}^\partial$ is a covering map and $M_{g,b}^\partial \to \widetilde{Y_{g,b}}$ is a fiber bundle map with fiber $(S^1)^{\times b}$. Since $\widetilde{Y_{g,b}}$ is contractible and paracompact, by Corollary B.1.8, the space $M_{g,b}^\partial$ is homeomorphic to the product $\widetilde{Y_{g,b}} \times (S^1)^{\times b}$ and hence the universal cover of $Y_{g,b}^\partial$ is $\widetilde{Y_{g,b}} \times \mathbb{R}^b$. In particular it is homotopy equivalent to $\widetilde{Y_{g,b}}$. Using Theorem 1.2.8, we obtain the desired equality $\mathrm{vcd}(\mathrm{Mod}(S, \partial S)) = \mathrm{vcd}(\mathrm{Mod}(S)) + b$. $\square$

Appendix A

Families of maps

We start with some notations.

Notation A.0.1. We will denote by C^∞_W and by C^∞_S the space of smooth maps from two manifolds endowed with the weak and the strong C^∞-topology respectively.

Notation A.0.2. When M and N are smooth manifolds, we will denote the manifold of n-jets of functions $M \to N$ with the symbol $J^n(M, N)$.

A.1 The Jet Transversality Theorem for families of maps

Let k be an integer and $\underline{n} = (n_1, ..., n_k)$ a k-upla of positive natural numbers. Let M be a manifold of dimension m without boundary.

Notation A.1.1. Denote by $M^{(k)}$ the open subset of $M^{\times k}$ consisting of the k-uple of points $(x_1, ..., x_k)$ such that $x_i \neq x_j$ for all $i \neq j$ and consider the map

$$\begin{aligned} \sigma :& J^{\underline{n}}(M, \mathbb{R}) = \bigtimes_{i=1}^{k} J^{n_i - 1}(M, \mathbb{R}) \to M \\ & (j^{n_1-1}_{x_1} f_1, ..., J^{n_k-1}_{x_k} f_k) \mapsto (x_1, ..., x_k) \end{aligned}$$

We will denote $\sigma^{-1}(M^{(k)})$ by $J^{\underline{n}}(M, \mathbb{R})^{(k)}$. Given a smooth function $f : \mathbb{R}^d \times M \to \mathbb{R}$ we will also denote by Ψ_f the map

$$\begin{aligned} \Psi_f :& \mathbb{R}^d \times M^{(k)} \to J^{\underline{n}}(M, \mathbb{R})^{(k)} \\ & (t, x_1, ..., x_k) \mapsto (J^{n_1-1}_{x_1} f_t, ..., J^{n_k-1}_{x_k} f_t) \end{aligned}$$

Theorem A.1.1. *Suppose* $Y \subseteq J^{\underline{n}}(M,\mathbb{R})^{(k)}$ *is a submanifold (possibly* $\partial Y \neq \emptyset$*). Then* $\{f \in C^\infty(\mathbb{R}^d \times M, \mathbb{R}) : \Psi_f \pitchfork Y\}$ *is residual in* $C^\infty_S(\mathbb{R}^d \times M, \mathbb{R})$*. If* Y *is closed, it is also open.*

Note that the case $k = 1$ is not excluded and that when $k = 1$ and $d = 0$ we obtain the statement of the Jet Transversality Theorem. Note also that, since $C^\infty_S(\mathbb{R}^d \times M, \mathbb{R})$ is a Baire space, residual implies dense.

The proof of this result relies on the so called **Globalization Theorem**. We will recall it in a moment.

Definition A.1.2. A **mapping class** on $(M, \mathbb{R})$ is a function Υ. The domain of Υ is the set of triples (L, U, V) where $U \subseteq M$ and $V \subseteq \mathbb{R}$ are open subsets and $L \subseteq M$ is a closed subset contained in U. To each triple Υ associates a subset $\Upsilon_L(U, V) \subseteq C^\infty(U, V)$. Moreover, Υ is required to satisfy the following property: if $f \in C^\infty(U, V)$ and there exist triples (L_i, U_i, V_i) and maps $f_i \in C^\infty(U_i, V_i)$ such that $L \subseteq \cup_i L_i$ and $f = f_i$ in a neghborhood of $L_i \cap U$ in U for all i, then $f \in \Upsilon_L(U, V)$.

The example to keep in mind for our purposes is

$$\Upsilon_L(U, V) = \{f \in C^\infty(U, V) : \Psi_f \pitchfork_{L'} (Y \cap J^n(U, \mathbb{R})^{(k)})\}$$

where $U \subseteq \mathbb{R}^d \times M$ is an open subset, $L \subset U$ is a closed subset of $\mathbb{R}^d \times M$ and L' is defined by

$$L' = \{(t, x_1, ..., x_k) \in \mathbb{R}^d \times M^{(k)} : (t, x_i) \in L \text{ for all } i = 1, ..., k\}.$$

Definition A.1.3. The mapping class Υ on $(M, \mathbb{R})$ is said **rich** if there are open covers $\mathcal{U}$ and $\mathcal{V}$ of M and $\mathbb{R}$ such that whenever open sets $U \subseteq M$ and $V \subseteq \mathbb{R}$ are respectively elements of $\mathcal{U}$ and $\mathcal{V}$ and $L \subseteq U$ is compact $\Upsilon_L(U, V)$ is dense and open in $C^\infty_W(U, V)$.

Theorem A.1.4. *If* Υ *is a rich mapping class on* $(M, \mathbb{R})$ *then* $\Upsilon_L(M, \mathbb{R})$ *is open and dense in* $C^\infty_S(M, \mathbb{R})$ *for every closed set* $L \subseteq M$*.*

Proof. See Theorem 2.2 in chapter 3 of [23]. □

Thus to prove Theorem A.1.1, it is enough to prove that $\Upsilon_L(U, V) = \{f \in C^\infty(U, V) : \Psi_f \pitchfork_{L'} (Y \cap (J^{\underline{n}}(U, \mathbb{R})^{(k)})\}$ is a rich mapping class when Y is closed. The case in which Y is not closed is then easily obtained as follows. Write $Y = \bigcup_k Y_k$ where each Y_k is a compact coordinate disk in Y. Then each $\{f \in C^\infty(\mathbb{R}^d \times M, \mathbb{R}) : \Psi_f \pitchfork Y_k\}$ is open and dense in $C^\infty_S(\mathbb{R}^d \times M, \mathbb{R})$ and thus their intersection is residual.

proof of the fact that Υ *is a rich mapping class (for* Y *closed).* The fact that it is a mapping class is trivial. We now check that it is rich, following the proof the Jet Transversality Theorem given in [23]. For $\mathcal{U}$ choose any atlas of $\mathbb{R}^d \times M$ with charts of the type $(\mathbb{R}^d \times U, 1_{\mathbb{R}^d} \times \varphi)$ with (U, φ) a chart for M; for $\mathcal{V}$ consider the trivial atlas of $\mathbb{R}$.
Fix a chart (U, φ) on M. Our claim is that

$$\{f \in C^\infty(\mathbb{R}^d \times U, \mathbb{R}) : \Psi_f \pitchfork_{L'} (Y \cap J^{\underline{n}}(U, \mathbb{R}))^{(k)}\} \subseteq C^\infty_W(\mathbb{R}^d \times U, \mathbb{R})$$

is open and dense for every compact subset L of $\mathbb{R}^d \times U$. Being L compact and Y closed, openess is clear. For density, we will prove that for $\max\{n_h - 1 : h = 1, ..., k\} << s < \infty$ big enough

$$\{f \in C^s(\mathbb{R}^d \times U, \mathbb{R}) : \Psi_f \pitchfork (Y \cap J^{\underline{n}}(U, \mathbb{R})^{(k)})\} \subseteq C^s_W(\mathbb{R}^d \times U, \mathbb{R})$$

is dense. Then the case $s = \infty$ will follow from the definition of the strong C^∞-topology, the fact

$$\{f \in C^s(\mathbb{R}^d \times U, \mathbb{R}) : \Psi_f \pitchfork_{L'} (Y \cap J^{\underline{n}}(U, \mathbb{R})^{(k)})\} \subseteq C^s_W(\mathbb{R}^d \times U, \mathbb{R})$$

is open for $L \subseteq \mathbb{R}^d \times U$ compact and the density of $C^\infty(\mathbb{R}^d \times M, \mathbb{R}) \subseteq C^s_W(\mathbb{R}^d \times M, \mathbb{R})$. Note that we are intentionally discarding L from now on.
First we explain the strategy. Fix $f \in C^s(\mathbb{R}^d \times M, \mathbb{R})$. We will find a smooth manifold Z and a continuous map $\alpha : Z \to C^s_W(\mathbb{R}^d \times U, \mathbb{R})$ such that $f \in \mathrm{Im}(\alpha)$ and the evaluation map associated to the composition

$$Z \xrightarrow{\alpha} C^s_W(\mathbb{R}^d \times U, \mathbb{R}) \xrightarrow{\Psi} C^{s-h}(\mathbb{R}^d \times U^{\times k}, J^{\underline{n}}(U, \mathbb{R})^{(k)})$$

where $h = \max\{n_i\} - 1$, is a smooth submersion. The evaluation map is just the map

$$\begin{aligned} \mathrm{ev} : & Z \times \mathbb{R}^d \times U^{(k)} \to J^{\underline{n}}(U, \mathbb{R})^{(k)} \\ & (z, t, x_1, ..., x_k) \mapsto \Psi_{\alpha(z)}(t, x_1, ..., x_k) \end{aligned}$$

We will have that $\Psi_{\alpha(z)} \pitchfork (Y \cap J^{\underline{n}}(U, \mathbb{R})^{(k)})$ for almost all $z \in Z$ and the claim will follow from the continuity of α.
Put $Z = J^s_0(\mathbb{R}^m, \mathbb{R})$. Every element of Z is thus an s-jet at 0 of a unique polynomial G in m variables of degree $\leq s$. We will identify the elements of Z with such polynomials. Define

$$\alpha : Z \to C^s_W(\mathbb{R}^d \times U, \mathbb{R}) \quad \alpha(G) = f + G$$

To compute the differential of ev make the natural identification

$$J^{\underline{n}}(U, \mathbb{R})^{(k)} \subseteq U \times J^{n_1 - 1}_0(U, \mathbb{R}) \times ... \times U \times J^{n_k - 1}_0(U, \mathbb{R}) \quad \text{(open subset)}$$

and

$$\begin{aligned}\text{ev} : Z \times \mathbb{R}^d \times U^{(k)} &\to U \times J_0^{n_1-1}(U,\mathbb{R}) \times ... \times U \times J_0^{n_k-1}(U,\mathbb{R}) \\ (G, t, x_1, ..., x_k) &\mapsto (x_1, J_{x_1}^{n_1-1}(f_t+G), ..., x_k, J_{x_k}^{n_k-1}(f_t+G))\end{aligned}$$

Thus ev is clearly smooth and, with our choice of s, for every fixed $(t, x_1, ..., x_k) \in \mathbb{R}^d \times U^{(k)}$, the restriction

$$\beta = \text{ev}|_{Z\times\{(t,x_1,...,x_k)\}} : Z\times\{(t, x_1, ..., x_k)\} \to \{x_1\}\times J_0^{n_1-1}(U,\mathbb{R})\times...\times\{x_k\}\times J_0^{n_k-1}(U,\mathbb{R})$$

is a submersion; in particular ev is a submersion as claimed. To see that β is a submersion, observe that the derivative of β is just

$$\begin{aligned}Z &\to J_0^{n_1-1}(U,\mathbb{R}) \times ... \times J_0^{n_k-1}(U,\mathbb{R}) \\ G &\mapsto (J_{x_1}^{n_1-1}G, ..., J_{x_k}^{n_k-1}G)\end{aligned}$$

(Z and $J_0^{n_1-1}(U,\mathbb{R})\times...\times J_0^{n_k-1}(U,\mathbb{R})$ are naturally vector spaces) and that for every $x_1, ..., x_k \in (\mathbb{R}^m)^{(k)}$ and collection of real numbers $\{a^j_{i_1...i_m}\}_{1\le j\le k,\ 0\le i_1+...+i_m\le n_j-1}$ such that for every permutation σ of $\{1, ..., m\}$ the equality $a^j_{i_1,...,i_m} = a^j_{i_{\sigma(1)},...,i_{\sigma(m)}}$ holds, there exists a polynomial G in m variables $y_1, ..., y_m$ and degree $\le s$ such that

$$\frac{\partial^{i_1+...+i_m}G}{\partial y_1^{i_1}...\partial y_m^{i_m}}(x_j) = a^j_{i_1,...,i_m} \text{ for all } 1 \le j \le k \text{ and } 0 \le i_1 + ... + i_m \le n_j - 1. \quad \text{(A.1)}$$

The existence of G can be proved by induction on k. When $k = 1$ the existence of G is obvious. Suppose $k > 1$ and write $x_i = (x_i^1, ..., x_i^m)$ for $i = 1, ..., k$. By the inductive step, there exists a polynomial G' that satisfies the equations in A.1 for $1 \le j \le k-1$. Consider the polynomial

$$G = G' + \prod_{i=1}^{k-1}(y_i - x_i^{e(i)})^{n_i} Q$$

where $x_i^{e(i)}$ is chosen in such a way that $x_i^{e(i)} \ne x_i^k$ for all $i = 1, ..., k-1$ and Q is a polynomial in $y_1, ..., y_m$ to be determined. Note that G still satisfies the equations in A.1 for $1 \le j \le k-1$. Imposing

$$a^k_{0,...,0} = G(x_k) = G'(x_k) + \prod_{i=1}^{k-1}(x_i^k - x_i^{e(i)})^{n_i} Q(x_k)$$

we obtain

$$Q(x_k) = \frac{a^k_{0,...,0} - G'(x_k)}{\prod_{i=1}^{k-1}(x_i^k - x_i^{e(i)})^{n_i}}.$$

Now assume we know what the value of $\frac{\partial^{i_1+...+i_m}}{\partial y_1^{i_1}...\partial y_m^{i_m}}Q(x_k)$ must be for all $i_1 \leq j_1$,...,$i_m \leq j_m$, then, as above, imposing for $h = 1, ..., m$

$$a^k_{j_1,...,j_h+1,...,j_m} = \frac{\partial^{j_1+....+j_m+1}}{\partial y_1^{j_1}...\partial y_h^{j_h+1}...\partial y_m^{j_m}}G(x_k)$$

we obtain an equation of the form

$$a^k_{j_1,...,j_h+1,...,j_m} = b^k_{j_1,...,j_h+1,...,j_m} + \prod_{i=1}^{k-1}(x_i^k - x_i^{e(i)})^{n_i}\frac{\partial^{j_1+...+j_m+1}}{\partial y_1^{j_1}...\partial y_h^{j_h+1}...\partial y_m^{j_m}}Q(x_k)$$

for some real number $b^k_{j_1,...,j_h+1,...,j_m}$ that depends also on the derivatives $\frac{\partial^{i_1+...+i_m}}{\partial y_1^{i_1}...\partial y_m^{i_m}}Q(x_k)$ for $i_1 \leq j_1$,...,$i_m \leq j_m$. From here we get the value of $\frac{\partial^{j_1+...+j_m+1}}{\partial y_1^{j_1}...\partial y_h^{j_h+1}...\partial y_m^{j_m}}Q(x_k)$. In this way we obtain that for G to satisfy the equations in A.1 is the same as the derivatives of Q in x_k of order $\leq n_k - 1$ to be equal to some real numbers and we can easily construct such a polynomial Q of degree $\leq n_k - 1$. In conlusion, this proves that we can find a polynomial G satisfying the equations in A.1 of degree $\leq n_1 + ... + n_k$ □

A.2 Families of maps of finite type

This section closely follows the book [5].

Notation A.2.1. Denote by $\mathcal{E}(m, p)$ the ring of the germs at 0 of smooth functions from $\mathbb{R}^m$ to $\mathbb{R}^p$. If $p = 1$, we will write $\mathcal{E}(m, 1) = \mathcal{E}(m)$.

Lemma A.2.1. *$\mathcal{E}(m)$ is a local ring, with unique maximal ideal $\mathfrak{m} = (x_1, ..., x_m)_{\mathcal{E}(m)}$.*

Proof. It is clear that $\mathcal{E}(m)$ is a local ring with unique maximal ideal $\{f \in \mathcal{E}(m) : f(0) = 0\}$. So we only need to prove that $\mathfrak{m} = \{f \in \mathcal{E}(m) : f(0) = 0\}$. The inclusion $\subseteq$ is obvious. For the other inclusion, given $f \in \mathcal{E}(m)$ with $f(0) = 0$, write

$$f(x) = f(0) + \int_0^1 \frac{d}{dt}f(tx)dt = \int_0^1 \sum_{i=1}^m x_i \frac{\partial f}{\partial x_i}(tx)dt = \sum_{i=1}^m x_i \int_0^1 \frac{\partial f}{\partial x_i}(tx)dt.$$

Thus $f \in \mathfrak{m}$.

□

Let $\mathcal{B}(m)$ be the set of germs h in $\mathcal{E}(m, m)$ such that the differential d_0h is invertible and $h(0) = 0$. Observe that $\mathcal{B}(m)$ is a group with respect to the composition. Moreover, $\mathcal{B}(m)$ acts on $\mathcal{E}(m)$ by precomposition.

Definition A.2.2. Two germs will be said **equivalent** if they are in the same orbit under the action of $\mathcal{B}(m)$. A germ f will be said k-**determined** if every germ with the same k-jet at 0 as f is equivalent to f. In particular, f must be equivalent to its Taylor expansion by k-th order at 0.

Another point of view is provided by the following result.

Theorem A.2.3. *f is finitely determinated if and only if $\mathcal{E}(m)/(\partial_{x_1} f, ..., \partial_{x_m} f)_{\mathcal{E}(m)}$ is a finite dimensional $\mathbb{R}$-vector space.*

Proof. It is Corollary 11.10 in chapter 11 of [5]. □

Definition A.2.4. A germ f is called a **singularity** if $d_0 f = 0$. The singularity f is said **isolated** if the set-germ $\{x \in \mathbb{R}^m : d_x f = 0\}$ is equal to $\{0\}$. The singularity f is called **algebraically isolated** if f is k-determined for some k.

Corollary A.2.5. *An algebraically isolated singularity is isolated.*

Proof. By hypothesis $A = \mathcal{E}(m)/(\partial_{x_1} f, ..., \partial_{x_m} f)_{\mathcal{E}(m)}$ is finite dimensional. Thus, by Nakayama's Lemma, there exists an intenger k such that $\mathfrak{m}^k A = 0$, equivalently such that $\mathfrak{m}^k \subseteq (\partial_{x_1} f, ..., \partial_{x_m} f)_{\mathcal{E}(m)}$. In particular, $x_i^m \in (\partial_{x_1} f, ..., \partial_{x_m} f)_{\mathcal{E}(m)}$ for all $i = 1, ..., m$ and thus if $\partial_{x_1} f(x) = ... = \partial_{x_m} f(x) = 0$ it must be $x = 0$. □

There is also a third description of finitely determined germs. First some notation.

Notation A.2.2. The space of k-jets of function at 0, $J_0^k(\mathbb{R}^m, \mathbb{R})$ can be identified with the quotient ring $\mathbb{R}[x_1, ..., x_m]/(x_1, ..., x_m)^{k+1}$. We will denote this ring by $\hat{\mathcal{E}}_k(m)$ and think to a k-jet $j_0^k f$ as a polynomial of degree $\leq k$.

Note that $\hat{\mathcal{E}}_k(m)$ has both a structure of smooth manifold and a structure of $\mathbb{R}$-algebra.

Lemma A.2.6. *For every $(f_1, ..., f_p) \in \mathcal{E}(m)^{\times p}$ and $k \geq 0$, the composition*

$$\hat{\mathcal{E}}_k(m) \hookrightarrow \mathcal{E}(m) \to \mathcal{E}(m)/((f_1, ..., f_p)_{\mathcal{E}(m)} + \mathfrak{m}^{k+1})$$

induces, by passing to the quotient, an isomorphism

$$\hat{\mathcal{E}}_k(m)/(j_0^k f_1, ..., j_0^k f_p)_{\hat{\mathcal{E}}_k(m)} \cong \mathcal{E}(m)/((f_1, ..., f_p)_{\mathcal{E}(m)} + \mathfrak{m}^{k+1}).$$

Proof. Applying repeatedly the argument in the proof of Lemma A.2.1, we immediately get that for all $f \in \mathcal{E}(m)$, $f - j_0^k f \in \mathfrak{m}^{k+1}$. Thus the composition $\hat{\mathcal{E}}_k(m) \hookrightarrow \mathcal{E}(m) \to \mathcal{E}(m)/((f_1, ..., f_p) + \mathfrak{m}^{k+1})$ is surjective and $((f_1, ..., f_p)_{\mathcal{E}(m)} + \mathfrak{m}^{k+1})) \cap \hat{\mathcal{E}}_k(m) = (j_0^k f_1, ..., j_0^k f_p)_{\hat{\mathcal{E}}_k(m)}$. □

We can now easily prove the following

Theorem A.2.7. *Let $f_1, ...f_p \in \mathcal{E}(m)$ and $k \geq 0$. Then*

$$\dim_{\mathbb{R}}(\mathcal{E}(m)/(f_1, ..., f_p)_{\mathcal{E}(m)}) \leq k$$

if and only if

$$\dim_{\mathbb{R}}(\hat{\mathcal{E}}_k(m)/(j_0^k f_1, ..., j_0^k f_p)_{\hat{\mathcal{E}}(m)}) \leq k.$$

Proof. Suppose $\dim_{\mathbb{R}}(\mathcal{E}(m)/(f_1, ..., f_p)_{\mathcal{E}(m)}) \leq k < \infty$. Then

$$\dim_{\mathbb{R}}(\hat{\mathcal{E}}_k(m)/(j_0^k f_1, ..., j_0^k f_p)_{\hat{\mathcal{E}}_k(m)}) = \dim_{\mathbb{R}}(\mathcal{E}(m)/((f_1, ..., f_p)_{\mathcal{E}(m)} + \mathfrak{m}^{k+1})) \leq k.$$

Vice versa, suppose the previous inequality holds. By Nakayama's Lemma, $\mathfrak{m}^k \subseteq (f_1, ..., f_p)_{\mathcal{E}(m)} + \mathfrak{m}^{k+1}$ and, applying again Nakayama's Lemma (here we use that $k > 0$), we obtain $\mathfrak{m}^{k+1} \subseteq \mathfrak{m}^k \subseteq (f_1, ..., f_p)_{\mathcal{E}(m)}$ and thus $\dim_{\mathbb{R}}(\mathcal{E}(m)/(f_1, ..., f_p)_{\mathcal{E}(m)}) \leq k$. □

Corollary A.2.8. *A singularity f is finitely determined if and only if*

$$\dim_{\mathbb{R}}(\hat{\mathcal{E}}_k(m)/(j_0^k(\partial_{x_1} f), ..., j_0^k(\partial_{x_m} f)_{\hat{\mathcal{E}}(m)}) \leq k$$

for some $k \geq 1$.

Our problem is: how many are the non-algebraic singularities? This problem can be formulated in mathematical terms as follows.

For $k \geq 2$, consider the subsets $A_k \subseteq \hat{\mathcal{E}}_k(m)$,

$$A_k = \left\{ j_0^k f \;\middle|\; \begin{array}{l} f \in \mathcal{E}(m),\ Df(0) = 0 \\ \text{and } \dim_{\mathbb{R}}(\hat{\mathcal{E}}_{k-1}(m)/(j_0^{k-1}(\partial_{x_1} f), ..., j_0^{k-1}(\partial_{x_m} f))_{\hat{\mathcal{E}}_{k-1}(m)}) > k-1 \end{array} \right\}$$

and set $A_1 = \{j_0^1 f : f \in \mathcal{E}(m) \text{ and } Df(0) = 0\}$.
Then f is a non-algebraic singularity iff $j_0^k f \in A_k$ for all $k \geq 1$ and the set of non-algebraic singularities is small in the sense specified by the following theorem:

Theorem A.2.9. *A_k is the union of finitely many submanifolds A_k^i ($i = 1, ..., n(k)$) of $\hat{\mathcal{E}}_k(m)$. Moreover, if $C(k) = \min\{\operatorname{codim}(A_k^i, \hat{\mathcal{E}}_k(m)) : i = 1, ..., n(k)\}$, then $C(k)$ diverges for $k \to \infty$.*

In order to prove this theorem we will use some real algebraic geometry. First we introduce some notation.

Notation A.2.3. For $l > k$ denote by $\pi^l_k : \hat{\mathcal{E}}_l(m) \to \hat{\mathcal{E}}_k(m)$ the natural projection. It is both a smooth map and a morphism of $\mathbb{R}$-algebras. Define $\hat{\mathcal{E}}(m)$ to be the inverse limit of

$$... \xrightarrow{\pi^{k+2}_{k+1}} \hat{\mathcal{E}}_{k+1}(m) \xrightarrow{\pi^{k+1}_{k}} \hat{\mathcal{E}}_k(m) \xrightarrow{\pi^{k}_{k-1}} ...$$

Thus $\hat{\mathcal{E}}(m)$ is both a smooth manifold, diffeomorphic to the space of ∞-jets at 0, and an $\mathbb{R}$-algebra, isomorphic to $\mathbb{R}[x_1, ..., x_m]$. Moreover the natural projection $\pi_k : \hat{\mathcal{E}}(m) \to \hat{\mathcal{E}}_k(m)$ is a smooth map and a morphism of $\mathbb{R}$-algebras.

Definition A.2.10. A subset $Y \subseteq \hat{\mathcal{E}}(m)$ will be said **proalgebraic** if there are algebraic sets $Y_k \subseteq \hat{\mathcal{E}}_k(m)$ such that $Y = \bigcap_{k=1}^{\infty} \pi_k^{-1}(Y_k)$.

The reason why we have introduced this notion is that the A_k are algebraic subsets of $\hat{\mathcal{E}}(m)$ and thus $A = \bigcap_{k=1}^{\infty} \pi_k^{-1}(A_k)$ is proalgebraic.

Lemma A.2.11. *The A_k are algebraic subsets of $\hat{\mathcal{E}}(m)$.*

Proof. For $k = 1$ the lemma is clear. Suppose $k \geq 2$. The condition

$$\dim_{\mathbb{R}}(\hat{\mathcal{E}}_{k-1}(m)/(j_0^{k-1}(\partial_{x_1} f), ..., j_0^{k-1}(\partial_{x_m} f))_{\hat{\mathcal{E}}_{k-1}(m)}) > k - 1$$

can be expressed by asking that, if $\{\phi_j\}$ are all the monomial of degree $\leq k-1$, the linear map

$$\Phi_f : \mathbb{R}^m \otimes < \phi_j >_{\mathbb{R}} \to \hat{\mathcal{E}}_{k-1}(m)$$
$$e_i \otimes \phi_j \mapsto j_0^{k-1}(j_0^{k-1}(\partial_{x_i} f) \cdot \phi_j) = j_0^{k-1}(\partial_{x_i}(j_0^k f) \cdot \phi_j)$$

has rank $< \dim_{\mathbb{R}} \hat{\mathcal{E}}_k(m) - k + 1$ and this condition is the same as imposing that some determinants, which are polynomials in the coefficients of $j_0^k f$, are zero.

□

Remark A.2.12. It is clear that the definition of a proalgebraic set remains invariant if we require that $\pi^{k+1}_k(Y_{k+1}) \subseteq Y_k$ for all k.

Definition A.2.13. If the algebraic sets Y_k are such that $\pi^{k+1}_k(Y_{k+1}) \subseteq Y_k$ for all k, the **codimension** of $Y = \bigcap_{k=1}^{\infty} \pi_k^{-1}(Y_k)$ in $\hat{\mathcal{E}}(m)$ is the supremum of the codimensions (as algebraic varieties) of Y_k in $\hat{\mathcal{E}}_k(m)$.

Lemma A.2.14. *Suppose Y_k are algebraic subsets of $\hat{\mathcal{E}}_k(m)$ such that $\pi^{k+1}_k(Y_{k+1}) \subseteq Y_k$. If for all $k \geq 1$ and polynomial p in $\hat{\mathcal{E}}_k(m)$ there is $l > k$ and a polynomial $q \in \hat{\mathcal{E}}_l(m) \setminus Y_l$ such that $\pi^l_k(q) = p$, then the codimension of Y in $\hat{\mathcal{E}}(m)$ is infinite.*

Proof. Call d_i the codimension of Y_i in $\hat{\mathcal{E}}_i(m)$. Fix $k \geq 1$. Since $Y_{k+1} \subseteq (\pi_k^{k+1})^{-1}(Y_k)$ and $(\pi_k^{k+1})^{-1}(Y_k)$ has codimension d_k in $\hat{\mathcal{E}}_{k+1}(m)$, we have $d_k \leq d_{k+1}$. Suppose that from a certain k we have $d_k = d_{k+1} = d_{k+2} = ...$ We will prove that in this case $\bigcup_{l>k'} \pi_{k'}^l(\hat{\mathcal{E}}_l(m) \smallsetminus Y_l) \subsetneq \hat{\mathcal{E}}_{k'}(m)$ for some $k' > k$. Consider an irreducible component X_k of Y_k with highest dimension. Then $(\pi_k^l)^{-1}(X_k)$ is irreducible for all $l > k$ and thus either $Y_l \cap (\pi_k^l)^{-1}(X_k) = (\pi_k^l)^{-1}(X_k)$ or $Y_l \cap (\pi_k^l)^{-1}(X_k)$ has codimension $> d_k = d_l$ in $\hat{\mathcal{E}}_l(m)$.
We claim that, up to increasing k and l, the second possibility cannot hold. To see this, call b_k the number of irreducible components of Y_k with highest dimension. Since $Y_{k+1} \subseteq (\pi_k^{k+1})^{-1}(Y_k)$ and the lift of every component of Y_k is irreducible, any irreducible component of Y_{k+1} is contained in the lift of some irreducible component of Y_k. Moreover, since $d_k = d_{k+1}$, every irreducible component of Y_{k+1} of highest dimension is equal to some lift of a component and thus $b_k \geq b_{k+1}$. This proves that $b_k \geq b_{k+1} \geq ....$ Now the b_k are finite numbers and thus this sequence must stabilize. Say that $b_{k'} = b_{k'+1} = ...$ But then, $Y_l \cap (\pi_{k'}^l)^{-1}(X_{k'}) = (\pi_{k'}^l)^{-1}(X_{k'})$ for all $l > k'$ and $X_{k'} \subseteq Y_{k'}$ irreducible component of maximal dimension. This implies that $\bigcup_{l>k'} \pi_{k'}^l(\hat{\mathcal{E}}_l(m) \smallsetminus Y_l) \subsetneq \hat{\mathcal{E}}_{k'}(m)$ and the proof is complete. □

Theorem A.2.15. *For all $k \geq 1$ we have $\pi_k^{k+1}(A_{k+1}) \subseteq A_k$ and $A = \bigcap_{k=1}^{\infty} \pi_k^{-1}(A_k)$ has infinite codimension in $\hat{\mathcal{E}}(m)$.*

Proof. The fact that $\pi_k^{k+1}(A_{k+1}) \subseteq A_k$ for all $k \geq 1$ follows from Theorem A.2.7. Now we prove that A has infinite codimension in $\mathcal{E}(m)$. By the previous lemma, it is sufficient to show that if $p \in \hat{\mathcal{E}}_k(m)$ there exist $l > k$ and $q \in \hat{\mathcal{E}}_l(m) \smallsetminus A_l$ such that $\pi_k^l(q) = p$. Let $h = \frac{1}{k+2}(x_1^{k+2} + ... + x_m^{k+2})$ and $h_t = (1-t)p + th$ for $t \in \mathbb{R}$. For $l > k+2$, call $I_l = \{t \in \mathbb{R} : h_t \in A_l\}$. Then I_l is an algebraic subset of $\mathbb{R}$. This is because I_l is the preimage of A_l under the algebraic map $\mathbb{R} \to \hat{\mathcal{E}}_l(m)$ defined by $t \mapsto (1-t)p + th$. Moreover $1 \notin I_l$ for large l and thus I_l is finite for large l. Pick $t \notin I_l$, $t \neq 1$. Then $q = \frac{h_t}{1-t} \notin A_l$ and $\pi_k^l(q) = p$. □

We can finally prove the Theorem A.2.9.

Proof of Theorem A.2.9. We have shown that the A_k are algebraic varieties in $\hat{\mathcal{E}}_k(m)$ and that $\text{codim}(A_k, \hat{\mathcal{E}}_k(m)) \to \infty$ for $k \to \infty$. It follows that A_k is the union of finitely many smooth submanifold of $\hat{\mathcal{E}}_k(m)$ each having codimension $\geq \text{codim}(A_k, \hat{\mathcal{E}}_k(m))$. Namely $A_k = \bigcup_i A_k^i$ (finite union) where A_k^i can be defined recursively as follows. Call $B_k^1 = A_k$ and, for $i > 1$, let $B_k^i = \text{Sing}(B_k^{i-1})$ be the singular locus of B_k^{i-1}, then $A_k^i = B_k^i \smallsetminus B_k^{i+1}$. This complete the proof. □

Our last step consists of formulating the global counterpart of this local theory.

Definition A.2.16. Let M be a manifold of dimension m and without boundary. A smooth map $f : M \to \mathbb{R}$ is said to be of **finite type** if the induced germ at every singular point of f is an algebraic singularity.

Theorem A.2.17. $\{f \in C^\infty(\mathbb{R}^d \times M, \mathbb{R}) : f_t$ is of finite type for all $t \in \mathbb{R}^d\}$ *is residual (thus dense) in* $C^\infty_S(\mathbb{R}^d \times M, \mathbb{R})$.

Proof. The submanifold A^i_k determines a submanifold Y^i_k of $J^k(M, \mathbb{R})$ with

$$C(k) \leq \text{codim}(A^i_k, \hat{\mathcal{E}}_k(m)) = \text{codim}(Y^i_k, J^k(M, \mathbb{R}))$$

and such that $f : M \to \mathbb{R}$ is not of finite type if and only if $\text{Im}(j^k f) \cap (\cup_i Y^i_k) \neq \emptyset$ for all $k \geq 1$.
But if k is such that $C(k) > \text{dim} M + d$ then for

$$\begin{aligned} \Psi_f : &\mathbb{R}^d \times M \to J^k(M, \mathbb{R}) \\ &(t, x) \mapsto J^k_x f_t \end{aligned}$$

to be transverse to A^i_k means not to meet it for all i. Thus the theorem follows from Theorem A.1.1. □

Appendix B

Fiber bundles and classifying spaces

In this Appendix we collect some results about fiber bundles and classifying spaces that are used throughtout the thesis. For a concise, but complete enough for our purposes exposition of this subject we recommend [32] and [33].

B.1 Fiber bundles

We stat recalling the definition of fiber bundles.

Definition B.1.1. A **fiber bundle** structure on a topological space E, with fiber F, consists of a continuous projection $p : E \to B$ such that each point of B has a neighborhood U for which there is a homeomorphism $h : p^{-1}(U) \to U \times F$ such that $\mathrm{pr}_U \circ h = p$. Given two fiber bundles $p_1 : E_1 \to B$ and $p_2 : E_2 \to B$ over the same space B, a **morphism of fiber bundles** from E_1 to E_2 is a continuous map $f : E_1 \to E_2$ such that $p_2 \circ f = p_1$.

This defines the category of fiber bundles over a fixed space B.

Definition B.1.2. A fiber bundle E with fiber F over B will be said **trivial** if it is isomorphic to the fiber bundle $\mathrm{pr}_B : B \times F \to B$.

One of the main characteristics of fiber bundles is that they are Serre fibrations. This is because being a Serre fibartion is a local property.

Proposition B.1.3. *Let $p : E \to B$ be a continuous map. If for all $b \in B$ there exists an open neighborhood U of b such that the map $p|_{p^{-1}(U)} : p^{-1}(U) \to U$ is a Serre fibration, then p is a Serre fibration.*

Proof. This is Theorem 6.11 in chapter 7 of [4]. □

Corollary B.1.4. *A fiber bundle is a Serre fibration.*

We recall that any Serre fibration induces a long exact sequence as below explained.

Proposition B.1.5. *Let $p : E \to B$ be a Serre fibration with fiber F over $b_0 \in B$. Let $e_0 \in E$ such that $p(e_0) = b_0$. Then there is a long exact sequence of pointed sets*

$$\dots \xrightarrow{\partial_n} \pi_n(F, e_0) \xrightarrow{\pi_n(i)} \pi_n(E, e_0) \xrightarrow{\pi_n(p)} \pi_n(B, b_0) \xrightarrow{\partial_{n-1}} \pi_{n-1}(F, e_0) \xrightarrow{\pi_{n-1}(i)} \dots$$
$$\dots \xrightarrow{\pi_1(i)} \pi_1(E, e_0) \xrightarrow{\pi_1(p)} \pi_1(B, b_0) \xrightarrow{\partial_0} \pi_0(F, e_0) \xrightarrow{\pi_0(i)} \pi_0(E, e_0) \xrightarrow{\pi_0(p)} \pi_0(B, b_0).$$

where the distinguished point of $\pi_n(E, e_0)$ is the identity element for $n \geq 1$ and the path component of e_0 for $n = 0$. Similarly for $\pi_n(F, e_0)$ and $\pi_n(B, b_0)$.

Proof. See Theorem 4.1 in [32]. □

Note that this means that this is an exact sequence of groups as far as $\pi_1(E, e_0)$.

Another important fact about fiber bundles is that they behave well under pullbacks.

Definition B.1.6. Given a fiber bundle $p : E \to B$ and a continuous map $f : B' \to B$, the **pullback** of E under f is the fiber bundle $f^*E = E \times_B B' \to B'$ defined by $(e, b') \mapsto b'$.

Proposition B.1.7. *Let $E \to B$ be a fiber bundle and let $f, g : B' \to B$ be two maps from a paracompact space. If f and g are homotopic, then the two fiber bundles f^*E and g^*E over B' are isomorphic.*

Proof. See [14]. □

Corollary B.1.8. *If B is a contractible paracompact space then every fiber bundle over B is trivial.*

B.2 Principal and universal bundles

Let G be a topological group.

Definition B.2.1. A G**-space** is a topological space X with a continuous action $G \times X \to X$. Given two G-spaces X and Y, a G**-map** between X and Y is a continuous map $\phi : X \to Y$ such that for all $g \in G$ and $x \in X$, $\phi(g \cdot x) = g \cdot \phi(x)$.

This defines the category of G-spaces.

Definition B.2.2. A **principal G-bundle** is the data of a G-map $\pi : P \to B$ where the action of G on B is trivial and such that there is a open covering $\{U_i\}$ of B and G-isomorphisms $\phi_i : \pi^{-1}(U_i) \to U_i \times G$ such that $\mathrm{pr}_{U_i} \circ \phi_i = \pi|_{\pi^{-1}(U_i)}$. Here the action of G on $U_i \times G$ is given by $g \cdot (u, g') = (u, gg')$ for all $g, g' \in G$ and $u \in U$. Let $\pi_P : P \to B$ and $\pi_Q : Q \to B$ be two principal G-bundles. A **morphism of principal G-bundles** is a G-map $\phi : P \to Q$ such that $\pi_Q \circ \phi = \pi_P$.

This defines a category.

Remark B.2.3. It is easy to see that any morphism of principal G-bundle is actually an isomorphism.

Definition B.2.4. A principal G-bundle $P \to B$ is said a **universal G-bundle** if P is weakly contractible. In this case B is said to be a **classifying G-space**.

The case in which G has the discrete topology is not excluded.

Definition B.2.5. Let G be a group endowed with the discrete topology and $P \to B$ a universal G-bundle. Then B is called a **$K(G, 1)$-space**.

The following important theorem is due to Milnor.

Theorem B.2.6. *Let G be a topological group. Then there exists a classifying G-space.*

Proof. Recall that a point of the infinite join $A = A_1 * A_2 * ...$ is specified by

1. $n \geq 0$ real numbers $t_{i_1}, ..., t_{i_n} \in [0, 1]$ satisfying $t_{i_1} + ... + t_{i_n} = 1$;
2. for every i such that $t_i \neq 0$, a point $a_i \in A_i$.

Such a point is denoted by $t_{i_1}a_{i_1} \oplus t_{i_2}a_{i_2} \oplus ... \oplus t_{i_n}a_{i_n}$. The topology on $A_1 \circ A_2 \circ ...$ is the weakest topology that makes the coordinate functions $t_i : A \to [0, 1]$ and $a_i : t_i^{-1}(0, 1] \to A_i$ continuous.
In [30], Milnor has proved that the infinite join $A = A_1 * A_2 * A_3 * ...$ of infinitely many non-empty spaces is always weakly contractible. Assuming this result the construction of a universal G-bundle is easy.

Construction of a universal G-bundle.

Let $E = G * G * .... = G^{*\infty}$. The group G acts continuously on E by $g \cdot (t_{i_1}g_{i_1} \oplus ... \oplus t_{i_n}g_{i_n}) = t_{i_1}gg_{i_1} \oplus .... \oplus t_{i_n}gg_{i_n}$ for all $g \in G$ and $t_{i_1}g_{i_1} \oplus ... \oplus t_{i_n}g_{i_n} \in E$. Let $p : E \to E/G = B$ be the quotient map. We want to show that it is a universal G-bundle. We already know that E is weakly contractible, thus we only need to find local trivializations. Since $t_i(g \cdot e) = t_i(e)$ for all $i \in \mathbb{N}$, $g \in G$ and $e \in E$, we have induced continuous maps $u_i : B \to [0,1]$ such that $u_i \circ p = t_i$. Call $U_i = u_i^{-1}((0,1]) = p(t_i^{-1}((0,1]))$. The maps $s'_i : t_i^{-1}((0,1]) \to t_i^{-1}((0,1])$ defined by $s'_i(e) = a_i(e)^{-1} \cdot e$ have the property that

$$s'_i(g \cdot e) = a_i(g \cdot e)^{-1} \cdot (g \cdot e) = a_i(e)^{-1} \cdot e = s'_i(e)$$

for all $g \in G$ and $e \in t_i^{-1}((0,1])$; thus we have induced continuous maps $s_i : U_i \to p^{-1}(U_i) = t_i^{-1}((0,1])$ such that $p \circ s_i = 1_{U_i}$, i.e. s_i are local section of p. Using s_i, it is immediate to find local trivializations. Indeed, the G-map $U_i \times G \to p^{-1}(U_i)$ defined by $(b,g) \mapsto g \cdot s_i(b)$ is a homeomorphism with inverse the map sending $e \in p^{-1}(U_i)$ to $(p(e), a_i(e)) \in U_i \times G$. $\square$

Remark B.2.7. Suppose G has the discrete topology. Then the space E constructed in the previous proposition has a natural simplicial structure with 0-simplices the elements of the form $1_i \cdot g$ where $g \in G$ and, more generally, k-simplices the subsets of the form $\{t_{i_1} g_{i_1} \oplus .... \oplus t_{i_{k+1}} g_{i_{k+1}} : t_{i_1}, ..., t_{i_{k+1}} \geq 0 \text{ and } t_{i_1} + ... + t_{i_{k+1}} = 1\}$ where $g_{i_1}, ..., g_{i_{k+1}}$ are fixed elements of G. What is more, the action of G on E takes simplices to simplices and if $g \in G$ fixes any simplex, then $g = 1$. Thus, the quotient B comes endowed with a natural simplicial structure.

Proposition B.2.8. *Let X be a connected CW-complex and B a $K(G,1)$-space. Then every homomorphism of groups $\varphi : \pi_1(X, x_0) \to \pi_1(B, b_0)$ is induced by a continuous map $f : (X, x_0) \to (B, b_0)$.*

Proof. We may assume that $x_0 \in X^{(0)}$. Let $T \subseteq X^{(1)}$ be a maximal tree. This means that T is a contractible subcomplex of $X^{(1)}$ maximal with respect to the inclusion. Note that $X^{(0)} \subseteq T$. Set $f(T) = \{b_0\}$. Consider the quotient map $p : X^{(1)} \to X^{(1)}/T$ and let $\overline{x_0}$ be equal to $p(x_0)$. Note that $\pi_1(p)$ is an isomorphism of groups and thus we have a homomorphism of groups

$$\overline{\varphi} : \pi_1(X^{(1)}/T, \overline{x_0}) \xrightarrow{\pi_1(p)^{-1}} \pi_1(X^{(1)}, x_0) \xrightarrow{\pi_1(i)} \pi_1(X, x_0) \xrightarrow{\varphi} \pi_1(B, b_0)$$

where $i : X^{(1)} \hookrightarrow X$ is just the inclusion map.
The image under p of the closure in X of each edge e^1_α in $X^{(1)} \setminus T$ is a closed path throught the point $\overline{x_0}$ and thus determines an element $[e^1_\alpha] \in \pi_1(X^{(1)}/T, \overline{x_0})$. Define f on e^1_α in such a way that $f(e^1_\alpha)$ represents $\overline{\varphi}([e^1_\alpha])$. In this way, since $\pi_1(X^{(1)}/T, \overline{x_0})$ is generated by the elements $[e^1_\alpha]$, we have $\overline{\varphi} = \pi_1(\overline{f})$ where $\bar{f}$ is the map $X^{(1)}/T \to B$ induced by f. Equivalently, we have $\varphi \circ \pi_1(i) = \pi_1(f)$. Next, consider a 2-cell e^2_β with attaching map $\psi_\beta : S^1 \to X^{(1)}$. To extend f over e^2_β, all we need to check is that $f \circ \psi_\beta$ is nullhomotopic. Choosing a basepoint $s_0 \in S^1$ and a path γ_β in $X^{(1)}$ from $\psi_\beta(s_0)$ to x_0, we have an element $[\psi_\beta] \in \pi_1(X^{(1)}, x_0)$ represented by $\gamma_\beta^{-1}\psi_\beta\gamma_\beta$. To check that $f \circ \psi_\beta$ is nullhomotopic is equivalent to check that $\pi_1(f)([\psi_\beta])$ is 0

in $\pi_1(B, b_0)$. But $\pi_1(i)([\psi_\beta]) = 0$, because e^2_β provide a homotopy between $\gamma_\beta^{-1}\psi_\beta\gamma_\beta$ and a constant in X and thus $\pi_1(f)([\psi_\beta]) = \varphi \circ \pi_1(i)([\psi_\beta]) = 0$. We proceed by defining f inductively on cells e^n_γ with $n > 2$. This is possible since the attaching maps $\psi_\gamma : S^{n-1} \to X^{(n-1)}$ have nullhomotopic compositions $f \circ \psi_\gamma : S^{n-1} \to Y$. This is because $f \circ \psi_\gamma$ lifts to the universal cover of Y for $n > 2$ and this cover is weakly contractible by hypothesis. The conclusion of the proposition follows by observing that the inclusion $X^{(2)} \hookrightarrow X$ gives an isomorphism $\pi_1(X^{(2)}, x_0) \to \pi_1(X, x_0)$ and the inclusion $X^{(1)} \hookrightarrow X^{(2)}$ gives a surjection $\pi_1(X^{(1)}, x_0) \to \pi_1(X^{(2)}, x_0)$. □

Next we explain the reason why classifying spaces are important.

Remark B.2.9. Given a principal G-bundle $\pi : P \to B$ and a continuous map $f : B' \to B$, the pullback of P under f is still a principal G-bundle.

Lemma B.2.10. *Let $f, g : B' \to B$ be homotopic maps from a paracompact space and $P \to B$ be a principal G-bundle. Then f^*P and g^*P are isomorphic principal G-bundles.*

Proof. This is essentially a restatement of Lemma B.1.7 for principal bundles. One can check that the same proof works in this setting, too. For a different proof in the case in which B' admits a structure of CW-complex see Proposition 7.1 of [33]. Note that a CW-complex is a paracompact space (See Proposition 1.20 in [21]). □

Notation B.2.1. When B and B' are two topological spaces, we will denote by $[B', B]$ the set of the homotopy classes of continuous map $B' \to B$.

Notation B.2.2. We will denote by $\mathcal{P}_G(B)$ the set of all isomorphism classes of principal G-bundles over B.

Theorem B.2.11. *Let $P \to B$ be a universal G-bundle. For every CW-complex B', the map*

$$\begin{aligned}[B', B] &\to \mathcal{P}_G(B') \\ f &\mapsto f^*P\end{aligned}$$

is a bijection.

Proof. See Theorem 7.4 of [33]. □

A classifying G-space is essentially unique:

Proposition B.2.12. *There exists a universal G-bundle $P \to B$ where B is CW-complex. Moreover, if $P' \to B'$ is another of such universal G-bundle, then B' is homotopically equivalent to B and the same is true for P' and P.*

The proof of this proposition uses the following extension result.

Proposition B.2.13. *Let (A, B) be a CW-pair and assume that the inclusion $A \hookrightarrow B$ is a weak homotopy equivalence. Let P and Q be two principal G-bundles. Then any morphism of principal G-bundles $P|_A \to Q|A$ can be extended to a morphism $P \to B$.*

Proof. This is Corollary 6.4 of [33]. □

Proof of Proposition B.2.12. By Proposition 4.13 in chapter 4 of [20], the space B has a CW approximation $f : B' \to B$. This means that B' is a CW complex and f is a weak homotopy equivalence. Let $P' = f^*P$ be the pullback of P under f. The P' is weakly contractible. To see this it is enough to apply the Five Lemma to the commutative diagram

$$\begin{array}{ccccccccc}
\pi_{i+1}(B') & \longrightarrow & \pi_i(G) & \longrightarrow & \pi_i(P') & \longrightarrow & \pi_i(B') & \longrightarrow & \pi_{i-1}(G) \\
\cong\downarrow & & \cong\downarrow & & \downarrow & & \cong\downarrow & & \cong\downarrow \\
\pi_{i+1}(B) & \longrightarrow & \pi_i(G) & \longrightarrow & \pi_i(P) & \longrightarrow & \pi_i(B) & \longrightarrow & \pi_{i-1}(G)
\end{array}$$

where the upper horizontal sequence is part of the long exact sequence of the homotopy groups associated to $P' \to B'$, the lower horizontal sequence is part of the long exact sequence of the homotopy groups associated to $P \to B$ and the vertical maps are induced by the commutuative square

$$\begin{array}{ccc}
P' & \longrightarrow & P \\
\downarrow & & \downarrow \\
B' & \longrightarrow & B.
\end{array}$$

This proves the first part of the proposition. Now suppose that $P \to B$ and $P' \to B'$ are two universal G-bundles. By Theorem B.2.11, there are continuous maps $f : B' \to B$ and $g : B \to B'$ such that $f^*P \cong P'$ and $g^*P' \cong P$. Then, $(f \circ g)^*P = g^*f^*P \cong P$ and $(g \circ f)^*P' \cong P'$. By uniqueness in Theorem B.2.11, we have that $f \circ g$ is homotopic to 1_B and $g \circ f$ is homotopic to $1_{B'}$. Thus B and B' are homotopically equivalent. Now, as above, we immediately get that P and P' are weakly homotopically equivalent. To prove that they are actually homotopically equivalent we proceed as follows. The maps $g : B \to B'$ and $f : B' \to B$ induce maps $\alpha : P \cong g^*P' \to P'$ and $\beta : P' \cong f^*P \to P$ respectively. We will prove that $\alpha \circ \beta$ is homotopic to $1_{P'}$. Similarly $\beta \circ \alpha$ is homotopic to 1_P and this will conclude the proof. Let F be a homotopy between $F_0 = 1_{B'}$ and $F_1 = g \circ f$. Then we have a commutative diagram of pullbacks

$$\begin{array}{ccccc} P' \times \{0,1\} & \longrightarrow & F^*P' & \longrightarrow & P' \\ \downarrow & & \downarrow & & \downarrow \\ B' \times \{0,1\} & \underset{i_0 \sqcup i_1}{\hookrightarrow} & B' \times [0,1] & \underset{F}{\longrightarrow} & B'. \end{array}$$

where $i_t : B' \times \{t\} \to B' \times [0,1]$ is the inclusion for $t = 0, 1$ and $P' \times \{0,1\} = i_0^*(F^*P') \sqcup i_1^*(F^*P')$. Now extend the map $P' \times \{0,1\} \to F^*P'$ to a map $P' \times [0,1] \to F^*P'$ using the previous proposition. The composition $P' \times [0,1] \to F^*P' \to P'$ gives the required homotopy. □

Notation B.2.3. Usually the (unique) CW-complex B of the proposition is denoted by BG, while the (unique) space P is denoted by EG.

Remark B.2.14. Observe that if B is a $K(G,1)$-complex, then also P is a CW-complex and, being weakly contractible, by Whitehead's Theorem, it must be contractible.

Finally, the property in Theorem B.2.11 characterizes the universal G-bundles, as the following proposition specifies.

Proposition B.2.15. *Let $P \to B$ be a principal G-bundle with B is a CW-complex. Suppose that for every CW-complex B' the map*

$$[B', B] \to \mathcal{P}_G(B')$$
$$f \mapsto f^*P$$

is a bijection. Then P is weakly contractible.

Proof. Let $P' \to B'$ be a universal G-bundle with B' a CW complex. Then, by Theorem B.2.11, there is a continuous map $g : B \to B'$ such that $P \cong g^*P'$. Moreover, by hypothesis, there is a continuous map $f : B' \to B$ such that $P' \cong f^*P$. Now, by Proposition B.2.12, we have that $f \circ g$ is homotopic to 1_B and $g \circ f$ is homotopic to $1_{B'}$. Therefore P and P' are weakly homotopic (actually by Proposition B.2.12 they are homotopic).

□

Construction of fiber bundles and Serre fibrations from universal bundles

Given two G-spaces X and Y, the product $X \times Y$ has a natural structure of G-space given by $g \cdot (x, y) = (g \cdot x, g \cdot y)$ for all $g \in G$, $x \in X$ and $y \in Y$.

Now suppose are given a principal G-bundle $\pi : E \to B$ and a G-space F. We have an induced map

$$(P \times F)/G \to B$$

that is easily seen to be a fiber bundle over B with fiber F. In the case $\pi : EG \to BG$ is the universal G-bundle, we will denote by EF the space $(E \times F)/G$. Thus $EF \to BG$ is a fiber bundle with fiber F.

Suppose F' is another G-space and $\phi : F \to F'$ is a G-map. Then, we have a continuous map $EF \to EF'$ that sends $[(e, x)]$ to $[(e, \phi(x))]$. Note that the triangle

$$\begin{array}{ccc} EF & \longrightarrow & EF' \\ & \searrow \quad \swarrow & \\ & BG & \end{array}$$

commutes.

Lemma B.2.16. *If the G-map $F \to F'$ is a weak homotopy equivalence, then $EF \to EF'$ is a weak homotopy equivalence and there is a commutative diagram*

$$\begin{array}{ccccccccccccc} \ldots & \longrightarrow & \pi_{i+1}(BG) & \longrightarrow & \pi_i(F) & \longrightarrow & \pi_i(EF) & \longrightarrow & \pi_i(BG) & \longrightarrow & \pi_{i-1}(F) & \longrightarrow & \ldots \\ & & \| & & \cong\downarrow & & \cong\downarrow & & \cong\downarrow & & \| & & \\ \ldots & \longrightarrow & \pi_{i+1}(BG) & \longrightarrow & \pi_i(F') & \longrightarrow & \pi_i(EF') & \longrightarrow & \pi_i(BG) & \longrightarrow & \pi_{i-1}(F') & \longrightarrow & \ldots \end{array}$$

Proof. The existence of the commutative diagram is obvious: the upper horizontal sequence is the exact sequence associated to the fiber bundle $EF \to BG$, the lower horizonatal sequence is the exact sequence associated to the fiber bundle $EF' \to BG$ and the vertical arrows are induced by the map $EF \to EF'$. Except for the maps $\pi_i(EF) \to \pi_i(EF')$, all the vertical arrows are isomorphisms and thus, by the Five Lemma, also the maps $\pi_i(EF) \to \pi_i(EF')$ are isomorphisms. This means that $EF \to EF'$ is a weak equivalence. □

Fix now a basepoint $x \in F$. We have a G-map $G \to F$ defined by $g \mapsto g \cdot x$ and thus a continuous map $EG \to EF$ defined by $e \mapsto [(e, x)]$.

Lemma B.2.17. *If $G \to F$ is a Serre fibration, then $EG \to EF$ is a Serre fibration.*

Proof. BG is covered by open sets U such that $p : EG|_U = p^{-1}(U) \cong U \times G \xrightarrow{\mathrm{pr_U}} U$. For each of such U, we have a commutative diagram

$$\begin{array}{ccc} EG|_U & \xrightarrow{\cong} & U \times G \\ \downarrow & & \downarrow \\ EF|_U & \xrightarrow{\cong} & U \times F \end{array}$$

where $U \times G \to U \times F$ is given by $(b, g) \mapsto (b, g \cdot x)$. This last map is a Serre fibration, thus the lemma follows from Proposition B.1.3. $\square$

Lemma B.2.18. *Let $H = \mathrm{Stab}_G(x)$ be the stabilizer of the base point x in G. Suppose that $G \to F$ is a Serre fibration. Then there is a commutative diagram*

$$\begin{array}{ccccc} \pi_1(F) & \longrightarrow & \pi_1(EF) & \longrightarrow & \pi_1(BG) \\ \| & & \downarrow & & \downarrow \\ \pi_1(F) & \longrightarrow & \pi_0(H) & \longrightarrow & \pi_0(G) \end{array}$$

where the vertical arrows are isomorphisms, the upper horizontal sequence is the exact sequence of the fiber bundle $EF \to BG$ and the lower horizontal sequence is the exact sequence associated to the Serre fibration $G \to F$.

Proof. By the previous Lemma, the map $EG \to EF$ is a Serre fibration. Clearly, its fiber is H. Moreover, since EG is weakly contractible, the boundary map $\pi_1(EF) \to \pi_0(H)$ associated to this Serre fibration is an isomorphism. Similarly, since EG is weakly contractile, the boundary map $\pi_1(BG) \to \pi_0(G)$ associated to the Serre fibration $EG \to BG$ is an isomorphism. A direct check shows that the above diagram commutes with these choices for the vertical arrows. $\square$

Appendix C

Orientation-preserving embeddings of disks

C.1 The Disk Theorem

We start with proving a useful result, sometimes called the **Disk Theorem** that states that, except for orientation, there is essentially one only way to embed a disk in a connected manifold.

Notation C.1.1. As done in previous occasions, in this Appendix, we will use the symbol C^∞_S to denote the space of smooth maps between two manifolds endowed with the strong C^∞-topology.

Lemma C.1.1. *Let $f : D^m \to \mathbb{R}$ be a smooth map with $f(0) = 0$. Then there are smooth maps $g_1, ..., g_m : D^m \to \mathbb{R}$ such that $f(x) = \sum_{i=1}^m x_i g_i(x)$ for all $x \in D^m$. In particular, $g_i(0) = \frac{\partial f}{\partial x_i}(0)$ for all $i = 1, ..., m$.*

Proof. Write

$$f(x) = f(x) - f(0) = \int_0^1 \frac{d}{dt} f(tx)dt = \int_0^1 \sum_{i=1}^m x_i \frac{\partial f(tx)}{\partial x_i} dt = \sum_{i=1}^m x_i \int_0^1 \frac{\partial f(tx)}{\partial x_i} dt$$

and take $g_i(x) = \int_0^1 \frac{\partial f(tx)}{\partial x_i} dt$. □

Remark C.1.2. The map defined by

$$C^\infty_S(D^m, \mathbb{R}) \to C^\infty_S(D^m, \mathbb{R})^{\times m}$$
$$f \mapsto (g_1, ..., g_m)$$

is continuous.

Theorem C.1.3 (Disk theorem)**.** *Let M be a smooth, connected and orientable manifold of dimension m possibly with boundary. Let $f_0, f_1 : D^m \to M \smallsetminus \partial M$ be two orientation-preserving embeddings of the m-disk. Then f_0 and f_1 are smoothly isotopic.*

Proof. Using a first smooth diffeotopy of M, we reduce ourself to consider the case in which $f_0(0) = f_1(0)$. Let (φ, U) be a chart on M such that $\varphi(U) = \mathbb{R}^m$ and $\varphi(f_0(0)) = 0$. We can isotope f_0 and f_1 in U using the isotopies

$$(x,t) \mapsto f_i((1-t+t\varepsilon)x)$$

for $x \in D^m$, $t \in [0,1]$, $i = 0,1$ and $\varepsilon > 0$ sufficiently small. Now the theorem follows from the following observation. Given any orientation-preserving embedding $f : D^m \to \mathbb{R}^m$ such that $f(0) = 0$, we can first isotope it to become the restriction of a linear and invertible map using the isotopy

$$(x,t) \mapsto \begin{cases} \frac{f(tx)}{t} & \text{if } t \neq 0; \\ Df(0)x & \text{if } t = 0 \end{cases}$$

and then isotope $Df(0)$ to become the inclusion map via a smooth path in $\mathrm{GL}^+(n,\mathbb{R})$ between $Df(0)$ and the identity. To see that the first isotopy is smooth observe that, thanks to the previous lemma, for $i = 1, ..., m$, we can write $t^{-1}f_i(tx) = \sum_{j=1}^m x_j g_{ij}(tx)$ for some $g_{i1}, ..., g_{im} \in C^\infty(D^m, \mathbb{R})$ such that $g_{ij}(0) = \frac{\partial f_i}{\partial x_j}(0)$. □

C.2 A homotopy equivalence

For this section, let M be a connected and oriented manifold of dimension m and assume $\partial M = \emptyset$. Fix an subset $X \subseteq M$ diffeomorphic to the closed disk D^m. We will identify $X = D^m$.

Notation C.2.1. Denote by $\mathrm{Emb}(D^m, M)$ and by $\mathrm{Emb}^+(D^m, M)$ respectively the space of the smooth embeddings of D^m in M and the subspace of the orientation-preserving embeddings. Endow both of them with the C^∞_S-topology.

Definitions and statement of the theorem

Let $\mathrm{Fr}^{\mathrm{GL}^+}(TM)$ be the principal $\mathrm{GL}^+(m,\mathbb{R})$-bundle over M associated to the tangent bundle TM. Explictly, $\mathrm{Fr}^{\mathrm{GL}^+}(TM)$ is the subbundle of $TM^{\oplus m}$ consisting of those m-uple $(v_1, ..., v_m) \in T_xM$ such that $v_1, ..., v_m$ is a positive basis of T_xM. Endow $\mathrm{Fr}^{\mathrm{GL}^+}(TM)$ with the subspace topology of $TM^{\oplus m}$. The (left) action of $\mathrm{GL}^+(m,\mathbb{R})$ on $\mathrm{Fr}^{\mathrm{GL}^+}(TM)$ is given by $A \cdot (v_1, ..., v_n) = (\sum_{j=1}^n a_{1j}v_j, ..., \sum_{j=1}^n a_{mj}v_j)$

for all $A = (a_{ij})_{i,j} \in \mathrm{GL}^+(m, \mathbb{R})$ and $(v_1, ..., v_n) \in \mathrm{Fr}^{\mathrm{GL}^+}(TM)$. The local trivializations are obtained from the local trivialization of TM as follows. If (φ, U) is an orientation-preserving chart on M, then we have the trivialization $\mathrm{Fr}^{\mathrm{GL}^+}(TM)|_U \to U \times \mathrm{GL}^+(m, \mathbb{R})$ given by associating to the positive basis $(v_1, ..., v_m)$ of T_xM $(x \in U)$ the pair (x, B) where B in the matrix whose rows are the $d_x\varphi(v_i) \in \mathbb{R}^m$ for $i = 1, ..., m$. It is easy to check that this is a $\mathrm{GL}^+(m, \mathbb{R})$-map.
There is also a further description of $\mathrm{Fr}^{\mathrm{GL}^+}(TM)$. Consider the vector bundle $\underline{\mathrm{Hom}}(\varepsilon^m, TM)$, where ε^m denotes the product bundle $M \times \mathbb{R}^m$. Call $\underline{\mathrm{Iso}}^+(\varepsilon, TM)$ the subbundle of $\underline{\mathrm{Hom}}(\varepsilon^m, TM)$ for which the fiber $\underline{\mathrm{Iso}}^+_x(\varepsilon, TM)$ over $x \in M$ is the space of all orientation-preserving isomorphisms $\mathbb{R}^m \to T_xM$. The group $\mathrm{GL}^+(m, \mathbb{R})$ acts on the right on $\mathbb{R}^m$ by $v \cdot A = A^T v$ and this action induces a left action of $\mathrm{GL}^+(m, \mathbb{R})$ on $\underline{\mathrm{Iso}}^+(\varepsilon^m, TM)$. Then $\underline{\mathrm{Iso}}^+(\varepsilon, TM)$ is a $\mathrm{GL}^+(m, \mathrm{R})$-bundle with trivialization maps defined as follows. If (φ, U) is an orientation-preserving chart on M, then $\underline{\mathrm{Iso}}^+(\varepsilon, TM)|_U \to U \times \mathrm{GL}^+(n, \mathbb{R})$ sending the element $f \in \underline{\mathrm{Iso}}^+_x(\varepsilon, TM)$ to the pair (x, B) where B is the matrix whose rows are the vectors $d_x\varphi(f(e_i))$ for $i = 1, ..., n$. Here $e_1, ..., e_n$ is any fixed positive basis of $\mathbb{R}^n$. It is easy to check that this is a $\mathrm{GL}^+(m, \mathbb{R})$-map. Moreover, we have an isomorphism of principal $\mathrm{GL}^+(m, \mathbb{R})$-bundles $\underline{\mathrm{Iso}}^+(\varepsilon, TM) \to \mathrm{Fr}^{\mathrm{GL}^+}(TM)$ given by $\underline{\mathrm{Iso}}^+_x(\varepsilon, TM) \ni f \mapsto (f(e_1), ..., f(e_n)) \in \mathrm{Fr}^{\mathrm{GL}^+}_x(TM)$.

The goal of this section is to prove the following result.

Theorem C.2.1. *The map*

$$\begin{gathered} \mathrm{Emb}^+(D^m, M) \to \mathrm{Fr}^{GL^+}(TM) \\ f \mapsto d_0 f \end{gathered}$$

is a weak homotopy equivalence.

Proof of the theorem

We start with studying the case $M = \mathbb{R}^m$

Lemma C.2.2. *The space* $\mathrm{GL}^+(m, \mathbb{R})$ *is a deformation retract of* $\mathrm{Emb}^+(D^m, \mathbb{R}^m)$.

Proof. Firstly, by translation, we may deformation retract $\mathrm{Emb}^+(D^m, \mathbb{R}^m)$ onto the subspace $\mathrm{Emb}^+(D^m, \mathbb{R}^m; \{0\})$ of embeddings that fix the origin. Note that $\mathrm{GL}^+(m, \mathbb{R})$ remains pointwise fixed during this deformation. Then

$$F(f, t)(x) = \begin{cases} \frac{1}{t} f(tx) & \text{if } t \neq 0; \\ Df(0)x & \text{if } t = 0 \end{cases}$$

defined for $f \in \mathrm{Emb}^+(D^m, \mathbb{R}^m; \{0\})$ and $t \in [0,1]$ is a deformation retraction of $\mathrm{Emb}^+(D^m, \mathbb{R}^m; \{0\})$ onto $\mathrm{GL}^+(m, \mathbb{R})$. To see that F is continuous use Lemma C.1.1 and Remark C.1.2. $\square$

Let U be an open subset of M containing $D^m = X$ and such that the pair (U, X) is diffeomorphic to the pair $(\mathbb{R}^m, D^m)$. We will identify these two pairs.

Notation C.2.2. Define $\mathrm{Emb}^+(D^m, M; \{0\})$ to be the suspaces of $\mathrm{Emb}^+(D^m, M)$ consisting of those embeddings that fix 0 and similarly and $\mathrm{Emb}^+(D^m, \mathbb{R}^m; \{0\})$ to be the subspace of those embedding with image in $\mathbb{R}^m$ and fixing 0.

Lemma C.2.3. *The space* $\mathrm{Emb}^+(D^m, \mathbb{R}^m; \{0\})$ *is a weak deformation retract of* $\mathrm{Emb}^+(D^m, M; \{0\})$. *In particular the inclusion*

$$\mathrm{Emb}^+(D^m, \mathbb{R}^m; \{0\}) \hookrightarrow \mathrm{Emb}^+(D^m, M; \{0\})$$

is a homotopy equivalence.

Proof. First a consideration. Note that, since D^m is compact, the weak and the strong C^∞-topology on $\mathrm{Emb}^+(D^m, M; \{0\})$ coincide and thus $\mathrm{Emb}^+(D^m, M; \{0\})$ is a metric space (see Theorem 4.4 in chapter 4 of [23]). In particular it is paracompact and Hausdorff and hence every open cover of $\mathrm{Emb}^+(D^m, M; \{0\})$ admits a subordinated partition of unity.
Now consider a single map $f \in \mathrm{Emb}^+(D^m, M; \{0\})$. Clearly there is an $\varepsilon > 0$ such that $f(\varepsilon x) \in \mathbb{R}^m$ for all $x \in D^m$. What is more, this ε works for all g in a neighborhood of f and thus, using a partition of unity, we can construct a continuous map $\varepsilon : \mathrm{Emb}^+(D^m, M; \{0\}) \to \mathbb{R}_+$ such that $f(\varepsilon(f)x) \in \mathbb{R}^m$ for all $f \in \mathrm{Emb}^+(D^m, M; \{0\})$ and $x \in D^m$. Finally $[0,1] \times \mathrm{Emb}^+(D^m, M; \{0\}) \ni (t, f) \mapsto f((1 - t + t\varepsilon(f))-) \in \mathrm{Emb}^+(D^m, M; \{0\})$ is a weak deformation retraction onto $\mathrm{Emb}^+(D^m, \mathbb{R}^m; \{0\})$. $\square$

Proof of Theorem C.2.1. We have a commutative square

$$\begin{array}{ccc} \mathrm{Emb}^+(D^m, M) & \longrightarrow & \mathrm{Fr}^{\mathrm{GL}^+}(TM) \\ \downarrow{\scriptstyle ev_0} & & \downarrow{\scriptstyle \pi} \\ M & = & M \end{array}$$

where $ev_0(f) = f(0)$ for all $f \in \mathrm{Emb}^+(D^m, M)$ and π is the projection map. Then ev_0 is a fiber bundle map with fiber $\mathrm{Emb}^+(D^m, M; \{0\})$ the subset of $\mathrm{Emb}^+(D^m, M)$ consisting of those embeddings that fix 0 (recall that $0 \in D^m \subseteq M$). This follows as for Lemma 6.3.3.
To conclude the proof, it suffices to prove that our map on fibers over $0 \in M$ is a weak equivalence. Using the previous lemma, it is enough to prove that $\mathrm{Emb}^+(D^m, \mathbb{R}^m; \{0\}) \to \mathrm{GL}^+(m, \mathbb{R})$ that sends f to $Df(0)$ is a weak homotopy equivalence, but this is the content of the proof of Lemma C.2.2. $\square$

Application to surfaces

Suppose now $M = S_g$ where $g \geq 0$ and fix a Riemannian metric h on S_g, so that it makes sense to consider the unit tangent bundle $UT(S_g)$. Let $D^2 \subseteq S_g$ be an embedded disk.

Lemma C.2.4. *The map*

$$\Phi : \mathrm{Fr}^{GL^+}(TS_g) \to UT(S_g)$$
$$(v_1, v_2) \mapsto \frac{v_1}{|v_1|}$$

is a homotopy equivalence.

Proof. The inverse map Ψ (up to homotopy) is given by sending to $v \in UT_x(S_g)$ to the pair (v, w) where w is the unique h-unitary vector in T_xS_g that is h-orthogonal to v and such that (v, w) is a positive basis of T_xS_g. The homotopy between $1_{\mathrm{Fr}^{GL^+}(TS_g)}$ and $\Psi \circ \Phi$ is provided by Gram-Schmidt. □

Putting all together, we obtain the following result.

Corollary C.2.5. *Fix a unitary vector $v \in UT_0D^2$. Then the map*

$$\mathrm{Emb}^+(D^2, S_g) \to UT(S_g)$$
$$f \mapsto \frac{d_0f(v)}{|d_0f(v)|}$$

is a weak homotopy equivalence.

C.3 A fiber bundle

For semplicity, in this section we assume that M is also compact. Thus M is a compact, connected and oriented manifold of dimension m with $\partial M = \emptyset$

In this section, we will adapt the content of [36] to our purposes.

Fix any (necessarily complete) Riemannian metric on M.

Notation C.3.1. Call $\mathrm{Exp} : TM \to M$ the associated exponential map and denote by $\mathfrak{X}(M)$ the set of all smooth vector fields on M, topologized as a subspace of $C^\infty_S(M, TM)$.

Lemma C.3.1. *The map*

$$E : \mathfrak{X}(M) \to C^\infty_S(M, M)$$

defined by sending $X \in \mathfrak{X}(M)$ *to* $E(X)$ *where* $E(X)(x) = \mathrm{Exp}(X_x)$ *is continuous.*

Proof. It is sufficient to note that this map is obtained by composing $\mathfrak{X}(M) \ni X \mapsto (X, \mathrm{Exp}) \in \mathfrak{X}(M) \times C^\infty_S(TM, M)$ with $\mathfrak{X}(M) \times C^\infty_S(TM, M) \ni (X, H) \mapsto H \circ X \in C^\infty_S(M, M)$. The second map is continuous since M is compact (see Proposition 8.3.4 of [34]).
Alternatively one can apply Exercise 8.7 on page 243 of [34]. □

We now recal a result from Point-Set Topology.

Lemma C.3.2. *Let* X *and* Y *be connected topological manifolds and* $f : X \to Y$ *a local homeomorphism and a proper map. Then* f *is a finite covering map.*

Proof. We prove the results in two steps.

Step 1 First we note that if $f : X \to Y$ is a local homeomorphism and there is an integer n such that $f^{-1}(y)$ has exactly n points for each $y \in Y$, then f is an n-sheeted covering map. Indeed, fixed $y \in Y$, let $f^{-1}(y) = \{x_1, ..., x_n\}$ and let W_i be disjoint open neighborhoods of x_i such that $f|_{W_i} : W_i \to f(W_i)$ is a homeomorphism onto the open set $f(W_i)$ for each i. Let V be the connected component of $\cap_{i=1}^n f(W_i)$ containing y and U_i the component of $f^{-1}(V) \cap W_i$ containing x_i. It is clear that $f^{-1}(V) = U_1 \cup ... \cup U_n$ and $f|_{U_i} : U_i \to V$ is a homeomorphism for all i.

Step 2 Let Y_n be the subset of Y such that $f^{-1}(y)$ contains at least n points for all $y \in Y_n$. Note that, since f is a local homeomorphism, the fibers of f are discrete and, since f is proper, they are also compact. Therefore they are finite. Moreover, since f is a local homeomorphism, each Y_n is open in Y. If we prove that they are also closed, then each of them is either empty or equal to Y and thus there exists a unique n such that $Y_m = Y$ for all $m \leq n$ and $Y_m = \emptyset$ if $m > n$. The lemma then follows from Step 1. To conclude the proof, we now prove that each Y_n is closed in Y. Let $y \in Y$ and $\{y_k\}_k \subseteq Y_n$ a sequence convergent to y. Say $\{x_{k,1}, ..., x_{k,n}\} \subseteq f^{-1}(y_k)$. Up to extracting a subsequence from $\{y_k\}_k$, we may assume that each sequence $\{x_{k,i}\}_k$ converges to some x_i in X. Indeed the union $K = \{y_k : k \in \mathbb{N}\} \cup \{y\}$ is a compact set of Y and thus $f^{-1}(K)$ is compact, too. Clearly $f(x_i) = y$ for all $i = 1, ..., n$, thus to conclude the proof we only need to prove that the x_i are distinct. Suppose $x_i = x_j$ for some $i \neq j$. Then there would be a neighborhood U of x_i on which f is injective and to which belong two points $x_{k,i}$ and $x_{k,j}$ for large k. This is a contradiction.

□

Proposition C.3.3. *There exists an open neighborhood U of 1_M in $C^\infty_S(M,M)$ such that $U \subseteq \mathrm{Diff}(M)$.*

Proof. We give the proof in Steps.

Step 1 First we find an open neighborhood U' of 1_M in $C^\infty_S(M,M)$ such that every $f \in U'$ is a finite covering map. For every $x \in M$ there exist an open neighborhood V_x of x in M and an open neighborhood U_x of 1_M in $C^\infty_S(M,M)$ such that for all $f \in U_x$ and $y \in V_x$ we have that $d_y f$ is an isomorphism. Since M is compact, there are $x_1, ..., x_k \in M$ such that $M = \cup_{i=1}^k V_{x_i}$. Set $U' = \cap_{i=1}^k U_{x_i}$. If $f \in U'$, then $d_x f$ is an isomorphism for all $x \in M$ and thus f is a local diffeomorphism. Since M is compact, f is automatically a proper map and thus it is a finite covering map.

Step 2 Now we find an open neighborhood U of 1_M contained in U' such that every $f \in U$ is injective and thus is a diffeomorphism of M. Here is the key observation: if $f : M \to M$ is a covering map and $x, y \in M$ are such that $f(x) = f(y)$ and for some path γ in M from x to y the image $f(\gamma)$ is contained in some open disk of M, then $x = y$. This suggest to proceed as follows. For every $x \in M$ there exists an $\varepsilon_x > 0$ such that the exponential map $\mathrm{Exp}_x : T_x M \to M$ is a diffeomorphism from the open ball of radius $4\varepsilon_x$ (with respect to the fixed Riemannian metric) onto an open set $B(x, 4\varepsilon_x)$ of M containing x. Since M is compact, there are $x_1, ..., x_k \in M$ such that $M = \cup_{i=1}^k B(x_i, \varepsilon_{x_i})$. Call $U = \{f \in U' : d(f(x), x) < \varepsilon \text{ for all } x \in M\}$ where $0 < \varepsilon < \min\{\varepsilon_{x_i} : i = 1, ..., k\}$. Clearly U is open in $C^\infty_S(M,M)$ and contains 1_M. Moreover, if $f \in U$ and $f(x) = f(y)$, then $d(x,y) \leq d(x, f(x)) + d(f(y), y) < 2\varepsilon$ and thus there exists an index i such that x and y belongs to $B(x_i, 3\varepsilon_{x_i})$. Let γ be a path in $B(x_i, 3\varepsilon_{x_i})$ be a path between x and y, then $f(\gamma)$ is a path in $B(x_i, 4\varepsilon_{x_i})$. It follows that $x = y$ and thus f is injective.

The proof is complete. □

Corollary C.3.4. *There exists an open neighborhood U of the zero vector field in $\mathfrak{X}(M)$ such that $E(U) \subseteq \mathrm{Diff}_0(M)$*

Proof. Call Z the zero vector field on M. From the proposition, there is an open neighborhood of Z in $\mathfrak{X}(M)$ mapped in $\mathrm{Diff}(M)$ by E. Since $\mathfrak{X}(M)$ is locally convex we may assume that U is convex. Therefore if $X \in U$, the map $[0,1] \ni t \mapsto E(tX) \in \mathrm{Diff}(M)$ is an arc from 1_M to $E(X)$. Thus $E(X) \in \mathrm{Diff}_0(M)$. □

Recall that D^m is a closed disk in M.

Notation C.3.2. We use the notation $\mathfrak{X}(TM|_{D^m})$ to refer to the space of smooth sections of $TM|_{D^m} \to D^m$.

Lemma C.3.5. *There is a continuous linear map*

$$k : \mathfrak{X}(TM|_{D^m}) \to \mathfrak{X}(M)$$

such that $k(X)|_{D^m} = X$ for all $X \in \mathfrak{X}(TM|_{D^m})$.

Proof. We may assume that $D^m \subseteq \mathbb{R}^m \subseteq M$ is contained in an open subset diffeomorphic to $\mathbb{R}^m$. Let $\rho : \mathbb{R}^m \to [0, \infty)$ be a smooth map such that $\rho(x) = 1$ for all $x \in D^m$ and $\rho(x) = 0$ if $|x| > 2$. Then

$$k(X) = \begin{cases} X_x & \text{if } x \in \mathrm{D}^m; \\ \rho(x)X_{x/|x|} & \text{if } x \in \mathbb{R}^m \text{ (the norm is that of } \mathbb{R}^m); \\ 0 & \text{otherwise;} \end{cases}$$

for $X \in \mathfrak{X}(TM|_{D^m})$ is a possible solution.

□

Proposition C.3.6. *Call $i_{D^m} : D^m \to M$ the inclusion map. There is an open neighborhood U of i_{D^m} in $\mathrm{Emb}(D^m, M)$ and a continuous map $X : U \to \mathfrak{X}(TM|_{D^m})$ such that $X(i_{D^m}) = Z$ is the zero vector field and $\mathrm{Exp}(X(f)_x) = f(x)$ for all $f \in U$ and $x \in D^m$.*

Proof. Call $|Z| = \mathrm{Im}(Z) \subseteq TM$ the image of Z. Consider the map

$$\begin{aligned} \Phi : TM|_{D^m} &\to D^m \times M \\ v &\mapsto (p(v), \mathrm{Exp}(v)) \end{aligned}$$

where $p : TM \to M$ is the projection. Note that Φ embeds $|Z|$ into $D^m \times D^m \subseteq D^m \times M$ and the differential of Φ is non-singular at every point of $|Z|$. Thus Φ is a diffeomorphism from a neighborhood $\mathcal{V}$ of $|Z|$ in $TM|_{D^m}$ to an open set $\mathcal{W}$ of $D^m \times M$. Let $j : \mathcal{W} \to \mathcal{V}$ be the inverse map and $U = \{f \in \mathrm{Emb}(D^m, M) : (x, f(x)) \in \mathcal{W} \text{ for all } x \in D^m\}$. It is clear that U is open in $\mathrm{Emb}(D^m, M)$. Finally, define $X : U \to \mathfrak{X}(TM|_{D^m})$ by $X(f)_x = j(x, f(x))$ for all $f \in U$ and $x \in D^m$. We need to check that X is continuous, that $X(i_{D^m}) = Z$ and that $\mathrm{Exp}(X(f)_x) = f(x)$ for $f \in U$ and $x \in D^m$. To see that X is continuous note that it is obtained as the composition of $U \ni f \mapsto 1_{D^m} \times f \in C^\infty_S(D^m, \mathcal{W})$ with $C^\infty_S(D^m, \mathcal{W}) \ni u \mapsto j \circ u \in C^\infty_S(D^m, TM|_{D^m})$. The first map is immediately checked to be continuous and the second is continuous again for Proposition 8.3.4 or, more simply, for Exercise 8.7 in [34]. We now check that $X(i_{D^m}) = Z$. Indeed, $X(i_{D^m})(x) = j(x, x) = Z_x$ for all $x \in D^m$. Finally, suppose $f \in U$ and $x \in D^m$. Writing $(x, f(x)) = (p(v), \mathrm{Exp}(v))$ for some $v \in T_xM$, we have $\mathrm{Exp}(X(f)_x) = \mathrm{Exp}(j(x, f(x))) = \mathrm{Exp}(j((p(v), \mathrm{Exp}(v))) = \mathrm{Exp}(v) = f(x)$ and we are done. □

Corollary C.3.7. *Let $f_0 \in \mathrm{Emb}(D^m, M)$. There exists an open neighborhood U of f_0 in $\mathrm{Emb}(D^m, M)$ and a continuous map $\chi : U \to \mathrm{Diff}_0(M)$ such that $f = \chi(f) \circ f_0$ for all $f \in U$.*

Proof. If $f_0 = i_{D^m}$, then the map $\chi = E \circ k \circ X$ defined in a sufficiently small neighborhood of i_{D^m} solves the problem. If f_0 is not i_{D^m}, consider $D' = f_0(D^m)$. Then there is a neighborhood U' of $i_{D'}$ in $\mathrm{Emb}(D', M)$ and a continuous map $\chi' : U' \to \mathrm{Diff}_0(M)$ such that $f = \chi'(f) \circ i_{D'}$ for all $f \in U'$. Let $\xi : \mathrm{Emb}(D^m, M) \to \mathrm{Emb}(D', M)$ be the homeomorphism $\xi(f) = f \circ f_0^{-1}$, then we can take $U = \xi^{-1}(U')$ and $\chi = \chi' \circ \xi$. □

We can finally prove the main result of this section.

Theorem C.3.8. *The restriction map*

$$\pi : \mathrm{Diff}^+(M) \to \mathrm{Emb}^+(D^m, M)$$

is a fiber bundle map.

Note that in particular this theorem implies that π is surjective. Actually the proof we give of Theorem C.3.8, uses this fact.

Theorem C.3.9. *The restriction map $\pi : \mathrm{Diff}^+(M) \to \mathrm{Emb}^+(D^m, M)$ is surjective.*

Proof. This is Theorem C in [35]. □

Proof of Theorem C.3.8. Call $\mathrm{Diff}^+(M; D^m)$ the subspace of $\mathrm{Diff}^+(M)$ consisting of those diffeomorphisms that fix pointwise D^m. Let $f_0 \in \mathrm{Emb}^+(D^m, M)$ and let $\xi : \mathrm{Emb}^+(D^m, M) \to \mathrm{Emb}^+(D^m, M)$ be the diffeomorphism defined by $f \mapsto \widetilde{f_0} \circ f$ where $\widetilde{f_0}$ is an extension of f_0 to a diffeomorphism of M. Note that $\xi(i_{D^m}) = f_0$. Let U be a neighborhood of i_{D^m} and let $\chi : U \to \mathrm{Diff}_0(M) \subseteq \mathrm{Diff}^+(M)$ be such that $f = \chi(f)|_{D^m}$ for all $f \in U$. Then

$$\xi(U) \times \mathrm{Diff}^+(M; D^m) \to U \times \mathrm{Diff}^+(M; D^m) \to \pi^{-1}(U) \to \pi^{-1}(\xi(U))$$

given by $(\xi(f), g) \mapsto \widetilde{f_0} \circ \chi(f) \circ g$ is a local trivialization of π.

□

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