Fixed-curve counts in algebraic varieties

Alessio Cela

April 2024

Advisor: Prof. Rahul Pandharipande
Co-examiner: Prof. Andrew Kresch
Chair: Prof. Peter Bühlmann

Kontsevich's recursion

Enumerative Geometry aims to count geometric objects satisfying certain conditions.

Kontsevich's recursion

Enumerative Geometry aims to count geometric objects satisfying certain conditions.

Example

- How many lines in the plane pass through 2 points?

Kontsevich's recursion

Enumerative Geometry aims to count geometric objects satisfying certain conditions.

Example

- How many lines in the plane pass through 2 points?
- More generally one could aim to count the number N_d of degree d rational curves (genus 0) in the plane passing through $3d - 1$ general points.

Kontsevich's recursion

Mathematicians had computed the first few cases

$$N_1 = N_2 = 1, \quad N_3 = 12, \quad N_4 = 620$$

in the late 19th century. It took almost a century to compute $N_5 = 87304$. Remarkably, in 1994 Kontsevich, motivated by string theory, gave a recursive formula computing all N_d starting from $N_1 = 1$:

Kontsevich's recursion

Mathematicians had computed the first few cases

$$N_1 = N_2 = 1, \quad N_3 = 12, \quad N_4 = 620$$

in the late 19th century. It took almost a century to compute $N_5 = 87304$. Remarkably, in 1994 Kontsevich, motivated by string theory, gave a recursive formula computing all N_d starting from $N_1 = 1$:

$$N_d = \sum_{\substack{d_1+d_2=d \\ d_1,d_2>0}} \left(d_1^2 d_2^2 \binom{3d-4}{3d_1-2} - d_1^3 d_2 \binom{3d-4}{3d_1-1} \right) N_{d_1} N_{d_2}$$

for $d > 1$.

Stable maps

Let X be a smooth, proper and connected algebraic variety of dimension r over $\mathbb{C}$, $\beta \in H_2(X, \mathbb{Z})$ an effective curve class, $g \in \mathbb{Z}_{\geq 0}$ a genus and $n \in \mathbb{Z}_{\geq 0}$ an integer.

Stable maps

Let X be a smooth, proper and connected algebraic variety of dimension r over $\mathbb{C}$, $\beta \in H_2(X, \mathbb{Z})$ an effective curve class, $g \in \mathbb{Z}_{\geq 0}$ a genus and $n \in \mathbb{Z}_{\geq 0}$ an integer.

Define

$$\overline{\mathcal{M}}_{g,n}(X, \beta) = \left\{ f : (C, p_1, \ldots, p_n) \to X \;\middle|\; \begin{array}{l} \bullet C \text{ is a connected nodal} \\ \quad \text{genus } g \text{ curve;} \\ \bullet p_1, \ldots, p_n \in C \\ \quad \text{are distict smooth points;} \\ \bullet f_*[C] = \beta; \\ \bullet |\mathrm{Aut}(f)| < \infty. \end{array} \right\}$$

to be the moduli stack of n-pointed genus g stable maps in class β.

Stable maps

Let X be a smooth, proper and connected algebraic variety of dimension r over $\mathbb{C}$, $\beta \in H_2(X, \mathbb{Z})$ an effective curve class, $g \in \mathbb{Z}_{\geq 0}$ a genus and $n \in \mathbb{Z}_{\geq 0}$ an integer.

Define

$$\overline{\mathcal{M}}_{g,n}(X, \beta) = \left\{ f : (C, p_1, \ldots, p_n) \to X \;\middle|\; \begin{array}{l} \bullet C \text{ is a connected nodal} \\ \quad \text{genus } g \text{ curve;} \\ \bullet p_1, \ldots, p_n \in C \\ \quad \text{are distict smooth points;} \\ \bullet f_*[C] = \beta; \\ \bullet |\mathrm{Aut}(f)| < \infty. \end{array} \right\}$$

to be the moduli stack of n-pointed genus g stable maps in class β.
We will also denote by

$$\mathcal{M}_{g,n}(X, \beta) \subseteq \overline{\mathcal{M}}_{g,n}(X, \beta)$$

the open substack where the domain curve C is smooth.

Stable curves

When $X = \star$ is a point, we obtain the moduli stack of stable n-pointed genus g curves

$$\overline{\mathcal{M}}_{g,n} = \left\{ (C, p_1, \ldots, p_n) \;\middle|\; \begin{array}{l} \bullet C \text{ is a connected nodal} \\ \quad \text{genus } g \text{ curve;} \\ \bullet p_1, \ldots, p_n \in C \\ \quad \text{are distict smooth points;} \\ \bullet |\mathrm{Aut}(C, p_1, \ldots, p_n)| < \infty. \end{array} \right\}.$$

Properties

The moduli stacks of stable satisfy the following properties:

Properties

The moduli stacks of stable satisfy the following properties:

- there are evaluation maps

$$\mathrm{ev}_i : \overline{\mathcal{M}}_{g,n}(X, \beta) \to X$$

defined by $\mathrm{ev}_i([f]) = f(p_i)$ for $i = 1, \ldots, n$;

Properties

The moduli stacks of stable satisfy the following properties:

- there are evaluation maps

$$\mathrm{ev}_i : \overline{\mathcal{M}}_{g,n}(X, \beta) \to X$$

  defined by $\mathrm{ev}_i([f]) = f(p_i)$ for $i = 1, \ldots, n$;

- there is a forgetful morphism

$$\pi : \overline{\mathcal{M}}_{g,n}(X, \beta) \to \overline{\mathcal{M}}_{g,n}$$

  remembering the (stabilized) domain curve;

Properties

The moduli stacks of stable satisfy the following properties:

- there are evaluation maps

$$\mathrm{ev}_i : \overline{\mathcal{M}}_{g,n}(X,\beta) \to X$$

  defined by $\mathrm{ev}_i([f]) = f(p_i)$ for $i = 1, \ldots, n$;

- there is a forgetful morphism

$$\pi : \overline{\mathcal{M}}_{g,n}(X,\beta) \to \overline{\mathcal{M}}_{g,n}$$

  remembering the (stabilized) domain curve;

- The moduli stack $\overline{\mathcal{M}}_{g,n}(X,\beta)$ is proper. It can be non-reduced and have several components of different dimension. However, it carries a virtual fundamental class

$$[\overline{\mathcal{M}}_{g,n}(X,\beta)]^{\mathrm{vir}} \in H_{2\mathrm{vdim}}(\overline{\mathcal{M}}_{g,n}(X,\beta)).$$

Gromov-Witten invariants

Definition

For classes $\gamma_i \in H^*(X)$ and $\alpha \in H^*(\overline{\mathcal{M}}_{g,n})$ the integral

$$\int_{[\overline{\mathcal{M}}_{g,n}(X,\beta)]^{\mathrm{vir}}} \prod_{i=1}^{n} \mathrm{ev}_i^*(\gamma_i) \cdot \pi^*(\alpha)$$

is a **Gromov-Witten invariant** of X.

Gromov-Witten invariants

Definition

For classes $\gamma_i \in H^*(X)$ and $\alpha \in H^*(\overline{\mathcal{M}}_{g,n})$ the integral

$$\int_{[\overline{\mathcal{M}}_{g,n}(X,\beta)]^{\mathrm{vir}}} \prod_{i=1}^{n} \mathrm{ev}_i^*(\gamma_i) \cdot \pi^*(\alpha)$$

is a **Gromov-Witten invariant** of X.

Example

We have

$$N_d = \int_{[\overline{\mathcal{M}}_{0,3d-1}(\mathbb{P}^2,d)]^{\mathrm{vir}}} \prod_{i=1}^{3d-1} \mathrm{ev}_i^*(\mathrm{P})$$

where $\mathrm{P} \in H^2(\mathbb{P}^2)$ is the point class.

Motivating question

Fix a general smooth genus g curve $\overline{C}$ and n general distinct points $p_1, \ldots, p_n \in \overline{C}$. Also fix n general points $x_1, \ldots, x_n \in X$.

Motivating question

Fix a general smooth genus g curve $\overline{C}$ and n general distinct points $p_1, \ldots, p_n \in \overline{C}$. Also fix n general points $x_1, \ldots, x_n \in X$.

Question

How many maps $f : (\overline{C}, p_1, \ldots, p_n) \to X$ are there such that $f(p_i) = x_i$ for all $i = 1, \ldots, n$ and $f_*[\overline{C}] = \beta$?

Goal of the presentation

Goal

We would like:

Goal of the presentation

Goal

We would like:

- to express the answer in the form of a closed formula;

Goal of the presentation

Goal

We would like:

- to express the answer in the form of a closed formula;
- or maybe to give a recursive formula for the answer (as done by Kontsevich);

Goal of the presentation

Goal

We would like:

- to express the answer in the form of a closed formula;
- or maybe to give a recursive formula for the answer (as done by Kontsevich);
- study the structure of such numbers.

Goal of the presentation

Goal

We would like:

- to express the answer in the form of a closed formula;
- or maybe to give a recursive formula for the answer (as done by Kontsevich);
- study the structure of such numbers.

Results

In this presentation we will study:

Goal of the presentation

Goal

We would like:

- to express the answer in the form of a closed formula;
- or maybe to give a recursive formula for the answer (as done by Kontsevich);
- study the structure of such numbers.

Results

In this presentation we will study:

- Projective spaces;

Goal of the presentation

Goal

We would like:

- to express the answer in the form of a closed formula;
- or maybe to give a recursive formula for the answer (as done by Kontsevich);
- study the structure of such numbers.

Results

In this presentation we will study:

- Projective spaces;
- Hypersurfaces in Projective spaces;

Goal of the presentation

Goal

We would like:

- to express the answer in the form of a closed formula;
- or maybe to give a recursive formula for the answer (as done by Kontsevich);
- study the structure of such numbers.

Results

In this presentation we will study:

- Projective spaces;
- Hypersurfaces in Projective spaces;
- Point Blow-ups of Projective spaces;

Goal of the presentation

Goal

We would like:

- to express the answer in the form of a closed formula;
- or maybe to give a recursive formula for the answer (as done by Kontsevich);
- study the structure of such numbers.

Results

In this presentation we will study:

- Projective spaces;
- Hypersurfaces in Projective spaces;
- Point Blow-ups of Projective spaces;
- Hirzebruch surfaces.

Virtual Tevelev degrees

Assume $\mathrm{vdim}(\overline{\mathcal{M}}_{g,n}(X,\beta)) = \dim(X^n \times \overline{\mathcal{M}}_{g,n})$.

Virtual Tevelev degrees

Assume $\mathrm{vdim}(\overline{\mathcal{M}}_{g,n}(X,\beta)) = \dim(X^n \times \overline{\mathcal{M}}_{g,n})$.

Definition

Define the **virtual Tevelev degree** $\mathrm{vTev}^X_{g,n,\beta} \in \mathbb{Q}$ as the Gromov-Witten invariant

$$\mathrm{vTev}^X_{g,n,\beta} = \int_{[\overline{\mathcal{M}}_{g,n}(X,\beta)]^{\mathrm{vir}}} \prod_{i=1}^{n} \mathrm{ev}_i^*(\mathsf{P}) \cdot \pi^*(\mathsf{P}_{\overline{\mathcal{M}}_{g,n}})$$

where $\mathsf{P} \in H^{2r}(X)$ and $\mathsf{P}_{\overline{\mathcal{M}}_{g,n}} \in H^{3g-3+n}(\overline{\mathcal{M}}_{g,n})$ are the point classes.

Virtual Tevelev degrees

Assume $\mathrm{vdim}(\overline{\mathcal{M}}_{g,n}(X,\beta)) = \dim(X^n \times \overline{\mathcal{M}}_{g,n})$.

Definition

Define the **virtual Tevelev degree** $\mathrm{vTev}^X_{g,n,\beta} \in \mathbb{Q}$ as the Gromov-Witten invariant

$$\mathrm{vTev}^X_{g,n,\beta} = \int_{[\overline{\mathcal{M}}_{g,n}(X,\beta)]^{\mathrm{vir}}} \prod_{i=1}^{n} \mathrm{ev}_i^*(\mathsf{P}) \cdot \pi^*(\mathsf{P}_{\overline{\mathcal{M}}_{g,n}})$$

where $\mathsf{P} \in H^{2r}(X)$ and $\mathsf{P}_{\overline{\mathcal{M}}_{g,n}} \in H^{3g-3+n}(\overline{\mathcal{M}}_{g,n})$ are the point classes.

Warning

This is a *virtual* count.

Geometric Tevelev degrees

Definition

The **geometric Tevelev degree** $\mathrm{Tev}^X_{g,n,\beta} \in \mathbb{Z}$ of X is defined under the assumption that the map

$$\tau = \pi \times \prod_{i=1}^{n} \mathrm{ev}_i : \mathcal{M}_{g,n}(X, \beta) \to \mathcal{M}_{g,n} \times X^n$$

has reduced and 0-dimensional general fiber, in which case its cardinality is by definition $\mathrm{Tev}^X_{g,n,\beta}$.

Geometric Tevelev degrees

Definition

The **geometric Tevelev degree** $\mathrm{Tev}^X_{g,n,\beta} \in \mathbb{Z}$ of X is defined under the assumption that the map

$$\tau = \pi \times \prod_{i=1}^{n} \mathrm{ev}_i : \mathcal{M}_{g,n}(X,\beta) \to \mathcal{M}_{g,n} \times X^n$$

has reduced and 0-dimensional general fiber, in which case its cardinality is by definition $\mathrm{Tev}^X_{g,n,\beta}$.

Fact (Lian-Pandharipande)

The geometric degree $\mathrm{Tev}^X_{g,n,\beta}$ is always defined for $n \geq g+1$.

Reformulation of the problem

Three questions emerge:

Reformulation of the problem

Three questions emerge:

- What is $\mathrm{vTev}^{X}_{g,\boldsymbol{n},\beta}$?

Reformulation of the problem

Three questions emerge:

- What is $\mathrm{vTev}^X_{g,\boldsymbol{n},\beta}$?
- What is $\mathrm{Tev}^X_{g,\boldsymbol{n},\beta}$?

Reformulation of the problem

Three questions emerge:

- What is $\mathrm{vTev}^X_{g,n,\beta}$?
- What is $\mathrm{Tev}^X_{g,n,\beta}$?
- Is $\mathrm{vTev}^X_{g,n,\beta} = \mathrm{Tev}^X_{g,n,\beta}$? (That is, is $\mathrm{vTev}^X_{g,n,\beta}$ enumerative?)

Reduction to genus 0 invariants

We can associate to X its **quantum cohomology ring**, denoted by $(\mathrm{QH}^*(X), \star)$.

Reduction to genus 0 invariants

We can associate to X its **quantum cohomology ring**, denoted by $(\mathrm{QH}^*(X), \star)$. This is defined using only 3-pointed genus 0 Gromov-Witen invariants of X.

Reduction to genus 0 invariants

We can associate to X its **quantum cohomology ring**, denoted by $(\mathrm{QH}^*(X), \star)$. This is defined using only 3-pointed genus 0 Gromov-Witen invariants of X.

Theorem (Buch-Pandharipande)

We have

$$\mathrm{vTev}^X_{g,n,\beta} = \mathrm{Coeff}(\mathsf{P}^{\star n} \star \mathsf{E}^{\star g}, q^\beta \mathsf{P})$$

Here $\mathsf{E} \in QH^*(X)$ *is the* **quantum Euler class** *of* X *and* P *is the point class on* X*.*

Reduction to genus 0 invariants

We can associate to X its **quantum cohomology ring**, denoted by $(\mathsf{QH}^*(X), \star)$. This is defined using only 3-pointed genus 0 Gromov-Witen invariants of X.

Theorem (Buch-Pandharipande)

We have

$$\mathsf{vTev}^X_{g,n,\beta} = \mathrm{Coeff}(\mathsf{P}^{\star n} \star \mathsf{E}^{\star g}, q^\beta \mathsf{P})$$

Here $\mathsf{E} \in QH^*(X)$ *is the* **quantum Euler class** *of* X *and* P *is the point class on* X*.*

The quantum Euler class of X is a quantum deformation of the Euler characteristic of X.

Virtual Tevelev degrees of Projective spaces

Example

We have $\mathrm{QH}^*(\mathbb{P}^r) = \frac{\mathbb{Q}[H,q]}{(\mathrm{H}^{\star r+1}-q)}$ where H is the hyperplane class. Then

$$\mathrm{E} = (r+1)\mathrm{H}^{\star r}$$

and

$$\mathrm{vTev}^{\mathbb{P}^r}_{g,n,d} = (r+1)^g.$$

Connection with Castelnuovo's classical count of g^1_d's

The only example where we know explicit closed formulas for all the geometric Tevelev degrees of X is when $X = \mathbb{P}^1$.

Connection with Castelnuovo's classical count of g^1_d's

The only example where we know explicit closed formulas for all the geometric Tevelev degrees of X is when $X = \mathbb{P}^1$.

Theorem (Castelnuovo, 1889)

We have

$$\mathsf{Tev}^{\mathbb{P}^1}_{g,3,\frac{g}{2}+1} = \frac{1}{1+\frac{1}{g}}\binom{g}{\frac{g}{2}}$$

Connection with Castelnuovo's classical count of g^1_d's

Of course Castelnuovo's perspective was different.

Connection with Castelnuovo's classical count of g^1_d's

Of course Castelnuovo's perspective was different.

Definition

A g^1_d on $\overline{C}$ is the datum of a 2 dimensional subspace of $H^0(\overline{C}, L)$ where $L \in Pic^d(\overline{C})$.

Connection with Castelnuovo's classical count of g_d^1's

Of course Castelnuovo's perspective was different.

Definition

A g_d^1 on $\overline{C}$ is the datum of a 2 dimensional subspace of $H^0(\overline{C}, L)$ where $L \in Pic^d(\overline{C})$.

Consider the Brill-Noether locus

$$G_d^1(C) = \{g_d^1\text{'s on } \overline{C}\}$$

which is smooth of dimension $\rho(d, 1, g) = g - 2(g - d + 1)$.

Connection with Castelnuovo's classical count of g_d^1's

Of course Castelnuovo's perspective was different.

Definition

A g_d^1 on $\overline{C}$ is the datum of a 2 dimensional subspace of $H^0(\overline{C}, L)$ where $L \in Pic^d(\overline{C})$.

Consider the Brill-Noether locus

$$G_d^1(C) = \{g_d^1\text{'s on } \overline{C}\}$$

which is smooth of dimension $\rho(d,1,g) = g - 2(g - d + 1)$.
Assuming $\rho = 0$ (i.e. $d = \frac{g}{2} + 1$), in one of his celebrated papers of 1889, Castelnuovo computed

$$\deg([G_d^1(\overline{C})]) = \frac{1}{1 + \frac{1}{g}} \binom{g}{\frac{g}{2}}$$

which agrees with $\mathrm{Tev}^{\mathbb{P}^1}_{g,3,\frac{g}{2}+1}$.

Geometric Tevelev degrees of the projective line

Theorem (J. Tevelev)

We have

$$\mathrm{Tev}^{\mathbb{P}^1}_{g,g+3,g+1} = 2^g.$$

Geometric Tevelev degrees of the projective line

Theorem (J. Tevelev)

We have

$$\mathrm{Tev}^{\mathbb{P}^1}_{g,g+3,g+1} = 2^g.$$

Theorem (C-Pandharipande-Schmitt)

We have

$$\mathrm{Tev}^{\mathbb{P}^1}_{g,n,d} = 2^g - 2\sum_{j=0}^{g-d-1}\binom{g}{j} + (g-d-1)\binom{g}{g-d} + (d-g-1)\binom{g}{g-d+1}.$$

Geometric Tevelev degrees of the projective line

Theorem (J. Tevelev)

We have

$$\mathrm{Tev}^{\mathbb{P}^1}_{g,g+3,g+1} = 2^g.$$

Theorem (C-Pandharipande-Schmitt)

We have

$$\mathrm{Tev}^{\mathbb{P}^1}_{g,n,d} = 2^g - 2\sum_{j=0}^{g-d-1}\binom{g}{j} + (g-d-1)\binom{g}{g-d} + (d-g-1)\binom{g}{g-d+1}.$$

This interpolates between Castelnuovo's and Tevelev's results.

Observation

Note that for $d > d[g]$ or $g = 0$, the formula

$$\mathrm{Tev}_{g,n,d}^{\mathbb{P}^1} = 2^g - 2\sum_{j=0}^{g-d-1}\binom{g}{j} + (g-d-1)\binom{g}{g-d} + (d-g-1)\binom{g}{g-d+1}.$$

reduces to

$$\mathrm{Tev}_{g,n,d}^{\mathbb{P}^1} = 2^g = \mathrm{vTev}_{g,n,d}^{\mathbb{P}^1}.$$

Observation

Note that for $d > d[g]$ or $g = 0$, the formula

$$\mathrm{Tev}_{g,n,d}^{\mathbb{P}^1} = 2^g - 2\sum_{j=0}^{g-d-1}\binom{g}{j} + (g-d-1)\binom{g}{g-d} + (d-g-1)\binom{g}{g-d+1}.$$

reduces to

$$\mathrm{Tev}_{g,n,d}^{\mathbb{P}^1} = 2^g = \mathrm{vTev}_{g,n,d}^{\mathbb{P}^1}.$$

Question

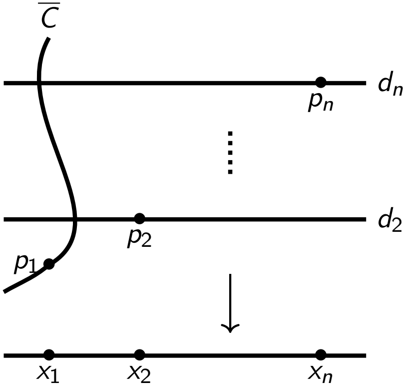
Is this a case?

Failure of enumerativity for small d

Why do we have $\mathrm{Tev}^{\mathbb{P}^1}_{g,n,d} \neq \mathrm{vTev}^{\mathbb{P}^1}_{g,n,d}$ for small d?

Failure of enumerativity for small d

Why do we have $\mathrm{Tev}^{\mathbb{P}^1}_{g,n,d} \neq \mathrm{vTev}^{\mathbb{P}^1}_{g,n,d}$ for small d?
For $n-1 \leq d$ (i.e. $d \leq g$) we have the following contribution from the boundary:



where $d_1 + \ldots + d_n = d$ and $d_i > 0$ for all $i = 2, \ldots, n$.

Generalities about hypersurfaces

Let $X \subseteq \mathbb{P}^{r+1}$ be a hypersurface of degree m. Assume $r \geq 3$ and that X is Fano (i.e. $m \leq r+1$).

Generalities about hypersurfaces

Let $X \subseteq \mathbb{P}^{r+1}$ be a hypersurface of degree m. Assume $r \geq 3$ and that X is Fano (i.e. $m \leq r+1$).
Then we have a canonical splitting

$$H^*(X) = H^*(X)^{\mathrm{prim}} \oplus H^*(\mathbb{P}^{r+1})$$

where $H^*(X)^{\mathrm{prim}} = Ker(\mathrm{H} \cup -) \subseteq H^r(X)$.

Generalities about hypersurfaces

Let $X \subseteq \mathbb{P}^{r+1}$ be a hypersurface of degree m. Assume $r \geq 3$ and that X is Fano (i.e. $m \leq r+1$).
Then we have a canonical splitting

$$H^*(X) = H^*(X)^{\mathrm{prim}} \oplus H^*(\mathbb{P}^{r+1})$$

where $H^*(X)^{\mathrm{prim}} = Ker(\mathsf{H} \cup -) \subseteq H^r(X)$.

In particular,

$$\beta = d\mathsf{L} \in H_2(X, \mathbb{Z})$$

where L is the class of a line in $\mathbb{P}^{r+1}$ and $d \in \mathbb{Z}_{\geq 0}$.

Quantum Euler class of hypersurfaces

Theorem (Cela)

The following equalities hold:

- *(conjectured by Buch-Pandharipande) if $m \leq r$ then*

$$\mathrm{E} = m^{-1}\chi(X)\mathrm{H}^{\star r} + (r + 2 - m - \chi(X))m^{m-1}q\mathrm{H}^{\star m-2},$$

Quantum Euler class of hypersurfaces

Theorem (Cela)

The following equalities hold:

- *(conjectured by Buch-Pandharipande) if $m \leq r$ then*

$$\mathrm{E} = m^{-1}\chi(X)\mathrm{H}^{\star r} + (r+2-m-\chi(X))m^{m-1}q\mathrm{H}^{\star m-2},$$

- *if $m = r+1$ then*

$$\mathrm{E} = m^{-1}\chi(X)\mathrm{H}^{\star r} + \sum_{j=1}^{r} m^{-1}(j-\chi(X))\binom{r}{j-1}(m!)^{j-1}\left[m^m - \frac{m!}{j}(r+1)\right]q^j\mathrm{H}^{\star r-j}.$$

Quantum Euler class of hypersurfaces

Theorem (Cela)

The following equalities hold:

- *(conjectured by Buch-Pandharipande) if $m \leq r$ then*

$$\mathrm{E} = m^{-1}\chi(X)\mathrm{H}^{\star r} + (r+2-m-\chi(X))m^{m-1}q\mathrm{H}^{\star m-2},$$

- *if $m = r+1$ then*

$$\begin{aligned}\mathrm{E} &= m^{-1}\chi(X)\mathrm{H}^{\star r} \\ &+ \sum_{j=1}^{r} m^{-1}(j-\chi(X))\binom{r}{j-1}(m!)^{j-1}\left[m^m - \frac{m!}{j}(r+1)\right]q^j\mathrm{H}^{\star r-j}.\end{aligned}$$

In particular, this expression does not involve the primitive cohomology of X !

Virtual Tevelev degrees of hypersurfaces

(Previous computation) + (Buch-Pandhariapande) $\rightsquigarrow$ Virtual Tevelev degrees of X:

Virtual Tevelev degrees of hypersurfaces

(Previous computation) + (Buch-Pandhariapande) $\rightsquigarrow$ Virtual Tevelev degrees of X:

Theorem (Buch-Pandharipande)

Let $m > 2$ and $r > 2m - 4$ and $g + n \geq 2$ then

$$\mathrm{vTev}^X_{g,n,d\mathsf{L}} = ((m-1)!)^n (r+2-m)^g m^{(d-n)m-g+1}.$$

Virtual Tevelev degrees of hypersurfaces

(Previous computation) + (Buch-Pandhariapande) $\rightsquigarrow$ Virtual Tevelev degrees of X:

Theorem (Buch-Pandharipande)

Let $m > 2$ and $r > 2m-4$ and $g+n \geq 2$ then

$$\mathrm{vTev}^X_{g,n,d\mathsf{L}} = ((m-1)!)^n (r+2-m)^g m^{(d-n)m-g+1}.$$

For higher degree Fano hypersurfaces, an algorithm computing $\mathrm{vTev}^X_{g,n,d\mathsf{L}}$ is known [Cela].

Enumerativity for hypersurfaces

What about the geometric degrees $\mathsf{Tev}^X_{g,n,d\mathsf{L}}$?

Enumerativity for hypersurfaces

What about the geometric degrees $\mathsf{Tev}^X_{g,n,d\mathsf{L}}$?

Theorem (Lian-Pandharipande)

Suppose $m \geq 3$ *and* $r > (m+1)(m-2)$ *then*

$$\mathsf{Tev}^X_{g,n,d\mathsf{L}} = \mathsf{vTev}^X_{g,n,d\mathsf{L}}$$

whenever $g = 0$ *or* $d > d[m,g]$.

Enumerativity for hypersurfaces

What about the geometric degrees $\mathrm{Tev}^X_{g,n,d\mathsf{L}}$?

Theorem (Lian-Pandharipande)

Suppose $m \geq 3$ and $r > (m+1)(m-2)$ then

$$\mathrm{Tev}^X_{g,n,d\mathsf{L}} = \mathrm{vTev}^X_{g,n,d\mathsf{L}}$$

whenever $g = 0$ or $d > d[m,g]$.

Comment

Maybe this is not a case!

Enumerativity for Blow-ups of Projective spaces

Let X be the Blow-up of $\mathbb{P}^r$ at ℓ general points.

Enumerativity for Blow-ups of Projective spaces

Let X be the Blow-up of $\mathbb{P}^r$ at ℓ general points.

Theorem (Cela-Lian)

The virtual degree $\mathrm{vTev}^X_{g,n,\beta}$ *is enumerative whenever* $g = 0$ *or* $\beta \cdot K_X^\vee > K[X, g]$ *in the following cases:*

Enumerativity for Blow-ups of Projective spaces

Let X be the Blow-up of $\mathbb{P}^r$ at ℓ general points.

Theorem (Cela-Lian)

The virtual degree $\mathrm{vTev}^X_{g,n,\beta}$ *is enumerative whenever* $g = 0$ *or* $\beta \cdot K_X^\vee > K[X, g]$ *in the following cases:*

- X *is a del Pezzo surface, i.e.* $r = 2$*, and the* $\ell \leq 8$ *points satisfy the property that no three lie on a line, no six lie on a conic, and, if* $\ell = 8$*, the points do not all lie on a cubic singular at one of the* q_j*;*

Enumerativity for Blow-ups of Projective spaces

Let X be the Blow-up of $\mathbb{P}^r$ at ℓ general points.

Theorem (Cela-Lian)

The virtual degree $\mathrm{vTev}^X_{g,n,\beta}$ *is enumerative whenever* $g = 0$ *or* $\beta \cdot K_X^\vee > K[X, g]$ *in the following cases:*

- X *is a del Pezzo surface, i.e.* $r = 2$*, and the* $\ell \leq 8$ *points satisfy the property that no three lie on a line, no six lie on a conic, and, if* $\ell = 8$*, the points do not all lie on a cubic singular at one of the* q_j*;*
- $r = 3, \ell \leq 4$*;*

Enumerativity for Blow-ups of Projective spaces

Let X be the Blow-up of $\mathbb{P}^r$ at ℓ general points.

Theorem (Cela-Lian)

The virtual degree $\mathrm{vTev}^X_{g,n,\beta}$ *is enumerative whenever* $g = 0$ *or* $\beta \cdot K_X^\vee > K[X, g]$ *in the following cases:*

- *X is a del Pezzo surface, i.e. $r = 2$, and the $\ell \leq 8$ points satisfy the property that no three lie on a line, no six lie on a conic, and, if $\ell = 8$, the points do not all lie on a cubic singular at one of the q_j;*
- $r = 3, \ell \leq 4$;
- *r is arbitrary and $\ell = 1$.*

Enumerativity for Blow-ups of Projective spaces

Let X be the Blow-up of $\mathbb{P}^r$ at ℓ general points.

Theorem (Cela-Lian)

The virtual degree $\mathrm{vTev}^X_{g,n,\beta}$ *is enumerative whenever* $g = 0$ *or* $\beta \cdot K_X^\vee > K[X, g]$ *in the following cases:*

- X *is a del Pezzo surface, i.e.* $r = 2$, *and the* $\ell \leq 8$ *points satisfy the property that no three lie on a line, no six lie on a conic, and, if* $\ell = 8$, *the points do not all lie on a cubic singular at one of the* q_j;
- $r = 3, \ell \leq 4$;
- r *is arbitrary and* $\ell = 1$.

In particular, in all the Fano cases $\mathrm{vTev}^X_{g,n,\beta}$ is enumerative whenever $g = 0$ or $\beta \cdot K_X^\vee > K[X, g]$.

Geometric Tevelev degrees of Blow-ups of Projective spaces

Assume $\ell \leq r+1$ and write

$$\beta = d\mathsf{H}^\vee + \sum_{i=1}^{\ell} k_i \mathsf{E}_i^\vee.$$

and assume that

$$d - \sum_{i \in I} k_i > 2g - 1 \text{ for all } I \subseteq \{1, \ldots, \ell\} \text{ with } |I| \leq r.$$

Geometric Tevelev degrees of Blow-ups of Projective spaces

Assume $\ell \leq r+1$ and write

$$\beta = d\mathrm{H}^\vee + \sum_{i=1}^{\ell} k_i \mathrm{E}_i^\vee .$$

and assume that

$$d - \sum_{i \in I} k_i > 2g - 1 \text{ for all } I \subseteq \{1, \ldots, \ell\} \text{ with } |I| \leq r.$$

Theorem (Cela-Lian)

Assume further that $n - d \geq 1$. Then,

$$\mathrm{Tev}^X_{0,n,\beta} = \sum_{m=0}^{\min(k_1,\ldots,k_{r+1},n)} (-1)^m \binom{n}{m} \prod_{i=1}^{r+1} \binom{n - d + \sum_{j \neq i} k_j - 1 - m}{k_i - m}$$

where we set $k_{\ell+1} = \cdots = k_{r+1} = 0$ when $\ell < r+1$.

One point Blow-ups of Projective spaces

Theorem (Cela-Lian)

Let $X = \mathrm{Bl}_q(\mathbb{P}^r)$ *and assume further that* $n - d \geq 1$. *Then,*

$$\mathrm{vTev}^X_{g,n,\beta} = \sum_{m=0}^{g} (2r)^{g-m}(1-r)^m \binom{g}{m} \binom{n-d+g-m-1}{k_1}.$$

If moreover $n - d \geq g + 1$, *then* $\mathrm{Tev}^X_{g,n,\beta}$ *is well-defined and coincides with* $\mathrm{vTev}^X_{g,n,\beta}$.

Sketch of proof

We divide the proof in steps:

Step 1: Thinking of $f : \overline{C} \to X$ as a map $f : \overline{C} \to \mathbb{P}^r$, which maps divisors D_i of degree k_i to the i-th blown-up point, provides a parametrization of the space of maps $f : \overline{C} \to X$ in class β as the 'interior' of a projective bundle

$$\mathbb{P}^\circ \subseteq \mathbb{P}(\mathcal{E}) \to \mathrm{Pic}^d(\overline{C}) \times \prod_{i=1}^{\ell} \mathrm{Sym}^{k_i}(\overline{C})$$

Sketch of proof

Step 2: The class of the closure $V(x_i)$ in $\mathbb{P}(\mathcal{E})$ of the locus in $\mathbb{P}^\circ$ where $f(p_i) = x_i$

$$[V(x_i)] = \widetilde{\mathrm{H}}^r + \sigma_1(\eta_1, \ldots, \eta_\ell)\widetilde{\mathrm{H}}^{r-1} + \ldots + \sigma_r(\eta_1, \ldots, \eta_\ell)$$

where

- $\eta_i \in H^*(\mathrm{Sym}^{k_i}(\overline{C}))$ is the class of divisors containing a fixed point of $\overline{C}$;

Sketch of proof

Step 2: The class of the closure $V(x_i)$ in $\mathbb{P}(\mathcal{E})$ of the locus in $\mathbb{P}^\circ$ where $f(p_i) = x_i$

$$[V(x_i)] = \widetilde{\mathsf{H}}^r + \sigma_1(\eta_1, \dots, \eta_\ell)\widetilde{\mathsf{H}}^{r-1} + \dots + \sigma_r(\eta_1, \dots, \eta_\ell)$$

where

- $\eta_i \in H^*(\mathrm{Sym}^{k_i}(\overline{C}))$ is the class of divisors containing a fixed point of $\overline{C}$;
- $\widetilde{\mathsf{H}} = c_1(\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)) - \eta_1 - \dots - \eta_\ell$;

Sketch of proof

Step 2: The class of the closure $V(x_i)$ in $\mathbb{P}(\mathcal{E})$ of the locus in $\mathbb{P}^\circ$ where $f(p_i) = x_i$

$$[V(x_i)] = \widetilde{\mathrm{H}}^r + \sigma_1(\eta_1, \dots, \eta_\ell)\widetilde{\mathrm{H}}^{r-1} + \dots + \sigma_r(\eta_1, \dots, \eta_\ell)$$

where

- $\eta_i \in H^*(\mathrm{Sym}^{k_i}(\overline{C}))$ is the class of divisors containing a fixed point of $\overline{C}$;
- $\widetilde{\mathrm{H}} = c_1(\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)) - \eta_1 - \dots - \eta_\ell$;
- the σ_i are the symmetric functions in $\eta_1, \dots, \eta_\ell$.

Sketch of proof

Step 3: (Transversality) The intersection

$$\bigcap_{i=1}^{n} V(x_i)$$

is transverse (i.e. consists of finitely many reduced points) and lies in $\mathbb{P}^\circ$.

Sketch of proof

Step 4: By the previous steps

$$\mathrm{Tev}^X_{g,n,\beta} = \int_{\mathbb{P}} \left(\widetilde{\mathrm{H}}^r + \sigma_1(\eta_1, \ldots, \eta_\ell)\widetilde{\mathrm{H}}^{r-1} + \ldots + \sigma_r(\eta_1, \ldots, \eta_\ell) \right)^n$$

which we computed explicitely in the two stated cases.

Enumerativity in general

Let X be a Fano variety.

Enumerativity in general

Let X be a Fano variety.

Speculation

The virtual count $\mathrm{vTev}^X_{g,n,\beta}$ is always enumerative whenever $g = 0$ or $\beta . K_X^\vee > K[X, g]$.

Enumerativity in general

Let X be a Fano variety.

Speculation

The virtual count $\mathrm{vTev}^X_{g,n,\beta}$ is always enumerative whenever $g = 0$ or $\beta.K_X^\vee > K[X,g]$.

This is true in all the above examples, but NOT in general.

Enumerativity in general

Let X be a Fano variety.

Speculation

The virtual count $\mathrm{vTev}^{X}_{g,n,\beta}$ is always enumerative whenever $g = 0$ or $\beta.K_X^\vee > K[X, g]$.

This is true in all the above examples, but NOT in general.

Example (Beheshti, Lehmann, Lian, Riedl, Starr, Tanimoto)

Enumerativity in general

Let X be a Fano variety.

Speculation

The virtual count $\mathrm{vTev}^X_{g,n,\beta}$ is always enumerative whenever $g = 0$ or $\beta.K_X^\vee > K[X, g]$.

This is true in all the above examples, but NOT in general.

Example (Beheshti, Lehmann, Lian, Riedl, Starr, Tanimoto)

- (Certain) Fano splitting projective bundles over $\mathbb{P}^k$ for $k > 1$;

Enumerativity in general

Let X be a Fano variety.

Speculation

The virtual count $\mathrm{vTev}^X_{g,n,\beta}$ is always enumerative whenever $g = 0$ or $\beta . K_X^\vee > K[X, g]$.

This is true in all the above examples, but NOT in general.

Example (Beheshti, Lehmann, Lian, Riedl, Starr, Tanimoto)

- (Certain) Fano splitting projective bundles over $\mathbb{P}^k$ for $k > 1$;
- Fermat hypersurfaces $X \subseteq \mathbb{P}^{r+1}$ ($r \geq 3$) of degree m such that either $\frac{r+4}{2} < m < r + 1$ or $m = r + 1 > 4$.

Failure of enumerativity for Fermat hypersurfaces

For simplicity, assume $\frac{r+4}{2} < m < r+1$ and let

$$X = \{-X_0^m + X_1^m + \ldots + X_{r+1}^m = 0\}.$$

Failure of enumerativity for Fermat hypersurfaces

For simplicity, assume $\frac{r+4}{2} < m < r+1$ and let

$$X = \{-X_0^m + X_1^m + \ldots + X_{r+1}^m = 0\}.$$

Then X contains a point and a divisor

$$p = [1 : 1 : 0 : \ldots : 0] \in D = \{-X_0 + X_1 = 0\} \cap X.$$

with the property that for every $q \in D$ the line $\overline{pq}$ lies in D.

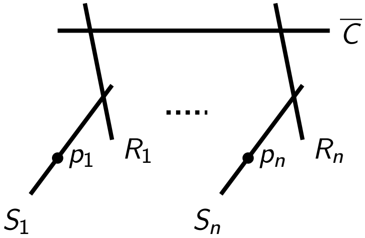
Failure of enumerativity for Fermat hypersurfaces

Given $x_1, \ldots, x_n \in X \setminus D$, the map $[f : (C, p_1, \ldots, p_n) \to X]$ is as follows.

Failure of enumerativity for Fermat hypersurfaces

Given $x_1, \ldots, x_n \in X \setminus D$, the map $[f : (C, p_1, \ldots, p_n) \to X]$ is as follows.

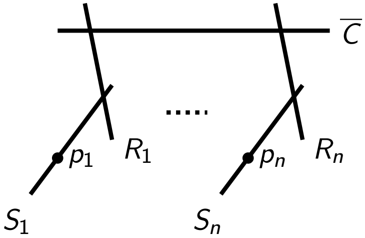
The domain curve C has the following shape:



Failure of enumerativity for Fermat hypersurfaces

Given $x_1, \dots, x_n \in X \setminus D$, the map $[f : (C, p_1, \dots, p_n) \to X]$ is as follows.

The domain curve C has the following shape:



and the map f contracts $\overline{C}$ to p, sends each rational tail S_i to lines in X such that $f(p_i) = x_i$ and each rational tail R_i to the line in D through p and $q_i \in f(S_i) \cap D$.

Logarithmic Tevelev degrees

Let X be a toric variety.

Logarithmic Tevelev degrees

Let X be a toric variety.
We can refine the notion of Tevelev degree by requiring that the map $f : (\overline{C}, p_1, \ldots, p_n) \to X$ meets the toric boundary ∂X of X with prescribed multiplicities.

Logarithmic Tevelev degrees

Let X be a toric variety.
We can refine the notion of Tevelev degree by requiring that the map $f : (\overline{C}, p_1, \ldots, p_n) \to X$ meets the toric boundary ∂X of X with prescribed multiplicities.

Using the moduli spaces of logarithmic stable maps $\overline{\mathcal{M}}_\Gamma(X)$, we obtain the notion of **virtual** and **geometric Logarithmic Tevelev degree**, respectively denoted by vTev_Γ^X and Tev_Γ^X.

Logarithmic Tevelev degrees

Let X be a toric variety.
We can refine the notion of Tevelev degree by requiring that the map $f : (\overline{C}, p_1, \ldots, p_n) \to X$ meets the toric boundary ∂X of X with prescribed multiplicities.

Using the moduli spaces of logarithmic stable maps $\overline{\mathcal{M}}_\Gamma(X)$, we obtain the notion of **virtual** and **geometric Logarithmic Tevelev degree**, respectively denoted by vTev_Γ^X and Tev_Γ^X.
Here Γ encodes the data of the genus, number of markings and incidence conditions with ∂X.

Logarithmic Tevelev degrees

Let X be a toric variety.
We can refine the notion of Tevelev degree by requiring that the map $f : (\overline{C}, p_1, \ldots, p_n) \to X$ meets the toric boundary ∂X of X with prescribed multiplicities.

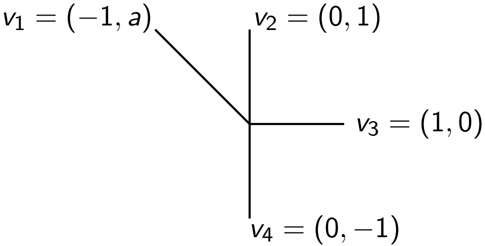
Using the moduli spaces of logarithmic stable maps $\overline{\mathcal{M}}_\Gamma(X)$, we obtain the notion of **virtual** and **geometric Logarithmic Tevelev degree**, respectively denoted by vTev_Γ^X and Tev_Γ^X.
Here Γ encodes the data of the genus, number of markings and incidence conditions with ∂X.

Fact

When $g = 0$, we always have $\mathrm{vTev}_\Gamma^X = \mathrm{Tev}_\Gamma^X$.

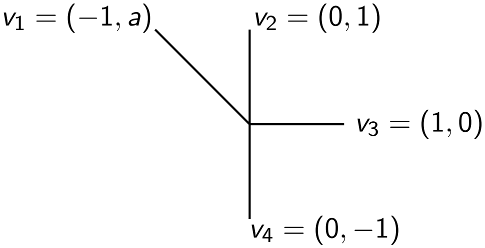
Hirzebruch surfaces

Let $\mathcal{H}_a = \mathbb{P}_{\mathbb{P}^1}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-a))$ be the Hirzebruch surface with fan



Hirzebruch surfaces

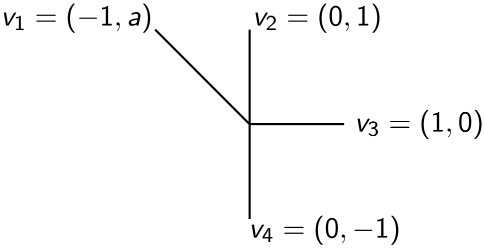
Let $\mathcal{H}_a = \mathbb{P}_{\mathbb{P}^1}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-a))$ be the Hirzebruch surface with fan



and let D_i be the toric divisor corresponding to v_i for $i = 1, \ldots, 4$.

Hirzebruch surfaces

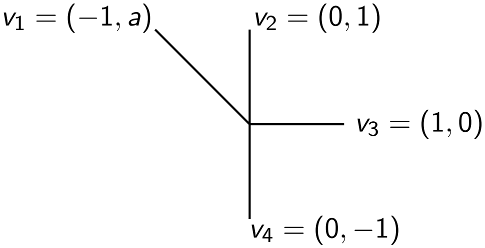
Let $\mathcal{H}_a = \mathbb{P}_{\mathbb{P}^1}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-a))$ be the Hirzebruch surface with fan



and let D_i be the toric divisor corresponding to v_i for $i = 1, \dots, 4$.
Then $\Gamma = (g, n, (\mu_1, \mu_2, \mu_3, \mu_4))$ where $\mu_i \in \mathbb{Z}_{\geq 0}^{|\mu_i|}$.

Hirzebruch surfaces

Let $\mathcal{H}_a = \mathbb{P}_{\mathbb{P}^1}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-a))$ be the Hirzebruch surface with fan



and let D_i be the toric divisor corresponding to v_i for $i = 1, \ldots, 4$.
Then $\Gamma = (g, n, (\mu_1, \mu_2, \mu_3, \mu_4))$ where $\mu_i \in \mathbb{Z}_{\geq 0}^{|\mu_i|}$.

With this notation, we are requiring that $[f]$ meets the toric divisor D_i with multiplicities $\mu_{i,j}$ for $i = 1, \ldots, 4$ and $j = 1, \ldots, |\mu_i|$.

Genus 0 Log Tevelev degrees of Hirzebruch surfaces

Fix $g = 0$ and assume the dimensional constraint

$$|\mu_1| + |\mu_2| + |\mu_3| + |\mu_4| = 2(n-1).$$

Genus 0 Log Tevelev degrees of Hirzebruch surfaces

Fix $g = 0$ and assume the dimensional constraint

$$|\mu_1| + |\mu_2| + |\mu_3| + |\mu_4| = 2(n-1).$$

Theorem (Cela-Lopez)

We have:

- *if either* $|\mu_1| > n-1$ *or* $|\mu_3| > n-1$, *then*

$$\mathrm{Tev}_{\Gamma}^{\mathcal{H}_a} = 0,$$

Genus 0 Log Tevelev degrees of Hirzebruch surfaces

Fix $g = 0$ and assume the dimensional constraint

$$|\mu_1| + |\mu_2| + |\mu_3| + |\mu_4| = 2(n-1).$$

Theorem (Cela-Lopez)

We have:

- *if either* $|\mu_1| > n-1$ *or* $|\mu_3| > n-1$, *then*

$$\mathrm{Tev}_{\Gamma}^{\mathcal{H}_a} = 0,$$

- *otherwise*

$$\mathrm{Tev}_{\Gamma}^{\mathcal{H}_a} = \left(\prod_{i=1}^{4} \frac{|\mu_i|! \prod_{j=1}^{|\mu_i|} \mu_{i,j}}{\prod_{u \geq 1} |\{v \mid \mu_{i,v} = u\}|!} \right) a^{n-1-|\mu_2|-|\mu_4|} \binom{n-1-|\mu_4|}{|\mu_2|}$$

Application

The above formula is zero whenever $|\mu_2| > n - 1 - |\mu_4|$.

Application

The above formula is zero whenever $|\mu_2| > n - 1 - |\mu_4|$.
In particular, this is the case when $\mu_{i,j} = 1$ for all i, j, that $a \geq 2$ and that $|\mu_2| \geq 1$.

Application

The above formula is zero whenever $|\mu_2| > n - 1 - |\mu_4|$.
In particular, this is the case when $\mu_{i,j} = 1$ for all i, j, that $a \geq 2$ and that $|\mu_2| \geq 1$.
Thus

$$\mathrm{vTev}_{\Gamma}^{\mathcal{H}_a} = \mathrm{Tev}_{\Gamma}^{\mathcal{H}_a} = \mathrm{Tev}_{0,n,\beta}^{\mathcal{H}_a} = 0$$

where β is defined by $\beta.D_i = |\mu_i|$ for $i = 1, \ldots, 4$.

Application

The above formula is zero whenever $|\mu_2| > n - 1 - |\mu_4|$.
In particular, this is the case when $\mu_{i,j} = 1$ for all i, j, that $a \geq 2$ and that $|\mu_2| \geq 1$.
Thus

$$\mathrm{vTev}_\Gamma^{\mathcal{H}_a} = \mathrm{Tev}_\Gamma^{\mathcal{H}_a} = \mathrm{Tev}_{0,n,\beta}^{\mathcal{H}_a} = 0$$

where β is defined by $\beta.D_i = |\mu_i|$ for $i = 1, \ldots, 4$.

Application

Let

$$\alpha : \overline{\mathcal{M}}_\Gamma(\mathcal{H}_a) \to \overline{\mathcal{M}}_{0,n}(\mathcal{H}_a, \beta)$$

be the natural morphism forgetting the log-structure. Then, in general,

$$\alpha_*[\overline{\mathcal{M}}_\Gamma(\mathcal{H}_a)]^{\mathrm{vir}} \neq [\overline{\mathcal{M}}_{0,n}(\mathcal{H}_a, \beta)]^{\mathrm{vir}}$$

Proof.

The virtual count $\mathrm{vTev}_{g,n,\beta}^{\mathcal{H}_a} \neq 0$ (by deformation invariance). □

Correspondence theorem

There exists a moduli space $M^{\text{trop}}_{g,n}(\mathbb{R}^r, \Gamma)$ parametrizing tropical maps

$$h : (C, p_1, \ldots, p_n) \to \mathbb{R}^r$$

where:

Correspondence theorem

There exists a moduli space $M_{g,n}^{\mathrm{trop}}(\mathbb{R}^r, \Gamma)$ parametrizing tropical maps

$$h : (\mathsf{C}, p_1, \ldots, p_n) \to \mathbb{R}^r$$

where:

- C is a genus g graph with n marked ends;

Correspondence theorem

There exists a moduli space $M^{\text{trop}}_{g,n}(\mathbb{R}^r, \Gamma)$ parametrizing tropical maps

$$h : (\mathsf{C}, p_1, \ldots, p_n) \to \mathbb{R}^r$$

where:

- C is a genus g graph with n marked ends;
- the map h is continuous and is affine linear on the edges of C;

Correspondence theorem

There exists a moduli space $M_{g,n}^{\text{trop}}(\mathbb{R}^r, \Gamma)$ parametrizing tropical maps

$$h : (C, p_1, \dots, p_n) \to \mathbb{R}^r$$

where:

- C is a genus g graph with n marked ends;
- the map h is continuous and is affine linear on the edges of C;
- the marked ends of C are contracted to points by h

Correspondence theorem

There exists a moduli space $M^{\text{trop}}_{g,n}(\mathbb{R}^r, \Gamma)$ parametrizing tropical maps

$$h : (C, p_1, \ldots, p_n) \to \mathbb{R}^r$$

where:

- C is a genus g graph with n marked ends;
- the map h is continuous and is affine linear on the edges of C;
- the marked ends of C are contracted to points by h and the directions of the non-marked ends of C are prescribed by incidence conditions in Γ.

Correspondence theorem

There exists a moduli space $M_{g,n}^{\text{trop}}(\mathbb{R}^r, \Gamma)$ parametrizing tropical maps

$$h : (C, p_1, \ldots, p_n) \to \mathbb{R}^r$$

where:

- C is a genus g graph with n marked ends;
- the map h is continuous and is affine linear on the edges of C;
- the marked ends of C are contracted to points by h and the directions of the non-marked ends of C are prescribed by incidence conditions in Γ.

This leads to the definition, in genus $g = 0$, of **tropical Tevelev degrees**, denoted by

$$\text{tropTev}_{\Gamma}^{X}.$$

Correspondence theorem

The following correspondence theorem holds.

Correspondence theorem

The following correspondence theorem holds.

Theorem (Cela-Lopez)

In genus $g = 0$ (and for any Γ), we have the following equality:

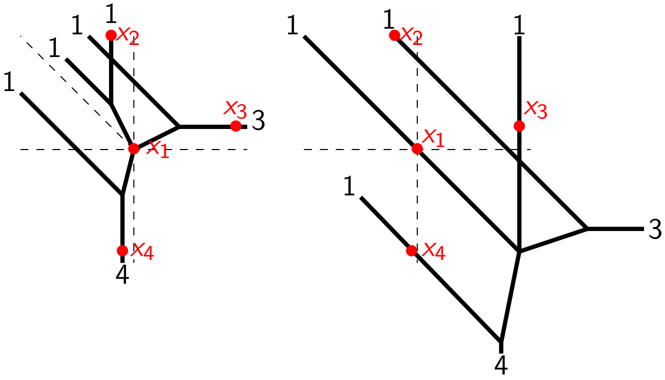
$$\mathrm{trop}\mathrm{Tev}_{\Gamma}^{X} = \mathrm{Tev}_{\Gamma}^{X}.$$

A tropical example

Suppose $a = 1$, $n = 4$ and $\mu_1 = (1,1,1)$, $\mu_2 = (1)$, $\mu_3 = (3)$ and $\mu_4 = (4)$. Then

$$\mathrm{Tev}_\Gamma^X = (3 \cdot 4) \cdot \binom{2}{1} = 24.$$

Below the two contributing curves each with multiplicity $3 \cdot 4 = 12$:



Thank you for the attention!