Tevelev degrees

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Introduction and main definitions

Let X be a smooth projective variety, $g \geq 0$ a genus, $n \geq 0$ and $\beta \in H_2(X,\mathbb{Z})$ a curve class. Fix $x_1,...,x_n \in X$ general points of X. We are interested in counting maps from C to X in class β and passing through $x_1,...,x_n$.

Assume $2g-2+n>0$, so that the moduli stack $\overline{\mathcal{M}}_{g,n}$ of stable curves is well-defined and let $\overline{\mathcal{M}}_{g,n}(X,\beta)$ be the moduli stack of n-pointed genus g stable maps in class β to X.

There is a map

$$\begin{aligned}\overline{\tau}: &\overline{\mathcal{M}}_{g,n}(X,\beta) \longrightarrow \overline{\mathcal{M}}_{g,n} \times X^{\times n}\\ &[f:(C,p_1,...,p_n) \to X] \mapsto ((\bar{C},\bar{p}_1,...,\bar{p}_n),(f(p_1),...,f(p_n)))\end{aligned}$$

recalling the stabilized domain curve and the image of the marked points under the morphism. One way to formulate our problem is by looking at the degree of $\overline{\tau}$.

Note that $\overline{\mathcal{M}}_{g,n}(X,\beta)$ has virtual dimension equal to the dimension of $\overline{\mathcal{M}}_{g,n} \times X^{\times n}$ if and only if

$$c_1(X).\beta = r(n+g-1). \tag{1}$$

Definition of the virtual count

Assume condition (1) is satisfied. Then the **virtual Tevelev degree** $\mathrm{vTev}^X_{g,n,\beta} \in \mathbb{Q}$ is defined by

$$\overline{\tau}_*([\overline{\mathcal{M}}_{g,n}(X,\beta)]^{\mathrm{vir}}) = \mathrm{vTev}^X_{g,n,\beta}[\overline{\mathcal{M}}_{g,n} \times X^n].$$

Here $[]^{\mathrm{vir}}$ and $[]$ denote the virtual and the usual fundamental classes.

One can also define the geometric count as follows. Let $\mathcal{M}_{g,n} \subset \overline{\mathcal{M}}_{g,n}$ and $\mathcal{M}_{g,n}(X,\beta) \subset \overline{\mathcal{M}}_{g,n}(X,\beta)$ be the loci where the curve C is smooth and let

$$\tau: \mathcal{M}_{g,n}(X,\beta) \to \mathcal{M}_{g,n} \times X^n$$

be the restriction of $\overline{\tau}$.

Definition of the geometric count

Assume condition (1) is satisfied. Assume further that for the general point $((C,p_1,...,p_n),(x_1,...,x_n)) \in \mathcal{M}_{g,n} \times X^n$ the fiber under τ consists of finitely many reduced (necessarily non-stacky) points. Then we define the **Geometric Tevelev degrees** $\mathrm{Tev}^X_{g,n,\beta} \in \mathbb{Z}$ by

$$\mathrm{Tev}^X_{g,n,\beta} = \#\text{general fiber of } \tau.$$

Projective line

Using a slightly different point of view, Tevelev [10] computed some Geometric Tevelev degrees of $\mathbb{P}^1$. The full description of Geometric Tevelev of $\mathbb{P}^1$ have been obtained in [5] via intersection theory on Hurwitz spaces, the case of $\mathbb{P}^r$ is instead treated in [6] via limit linear series. Building on these two approaches, these counts are generalized in [4] for $X=\mathbb{P}^1$ to the situation where the covers are constrained to have arbitrary ramification profiles. The following is [5, Theorem 6].

Explicit formulas for $\mathbb{P}^1$

Let $g \geq 0$, $\ell \in \mathbb{Z}$, and call

$$d[g,\ell] = g+1+\ell, \quad \text{and} \quad n[g,\ell] = g+3+2\ell.$$

Assume $n[g,\ell] \geq 3$ and $d[g,\ell] \geq 1$. Then we have:

$$\mathsf{Tev}^{\mathbb{P}^1}_{g,n[g,\ell],d[g,\ell]} = 2^g - 2\sum_{i=0}^{-\ell-2}\binom{g}{i} + (-\ell-2)\binom{g}{-\ell-1} + \ell\binom{g}{-\ell},$$

Sketch of Proof Let $\overline{\mathcal{H}}_{g,d[g,\ell],n[g,\ell]}$ be the moduli stack of degree $d[g,\ell]$ and $n[g,\ell]$ marked admissible covers [8] and

$$\overline{\tau}: \overline{\mathcal{H}}_{g,d[g,\ell],n[g,\ell]} \to \overline{\mathcal{M}}_{g,n} \times \overline{M}_{0,n[g,\ell]}$$

be the map recalling the marked domain curve (the ramification points are forgotten) and the marked target curve (the branch points are forgotten). The advantage of replacing $\overline{\mathcal{M}}_{g,n}(X,d[\mathbb{P}^1])$ with $\overline{\mathcal{H}}_{g,d,n}$ is that the boundary of the Hurwitz stack has a very nice stratification.

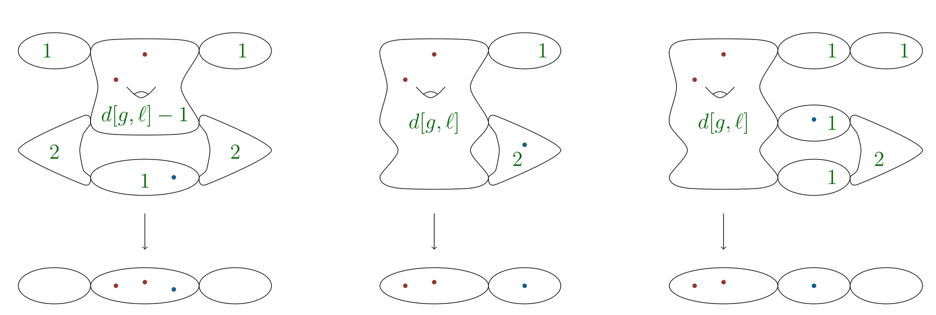
Up to a combinatorial factor, we want to find the degree of $\overline{\tau}$ and we do this by computing the degree of the zero cycle

$$\overline{\tau}^*[(C,D)] \in \mathsf{A}_0\left(\overline{\mathcal{H}}_{g,d[g,\ell],n[g,\ell]}\right).$$

where the point

$$(C,D) \in \overline{\mathcal{M}}_{g,n[g,\ell]} \times \overline{M}_{0,n[g,\ell]}$$

is chosen to have the following form: C is obtained by gluing at two points a smooth genus $g-1$ curve containing $n[g,\ell]-1$ marked points and a smooth genus 0 curve containing 1 marked point, D is a smooth $n[g,\ell]$-pointed genus 0 curve.
The actual fiber $\overline{\tau}^{-1}[(C,D)]$ will have excess dimension, so some care must be taken in the analysis. Fixed (C,D), the Hurwitz cover can degenerate only in one of the following three ways:



Type 1 Type 2 Type 3

Explanation of the picture: degrees of the map are written in green, the last marking is in blue and the first $n[g,\ell]$ markings are in red.

From this one deduces the following recursion:

$$\mathsf{Tev}_{g,n[g,\ell],d[g,\ell]} = \mathsf{Tev}_{g-1,n[g-1,\ell],d[g-1,\ell]} + \mathsf{Tev}_{g-1,n[g-1,\ell+1],d[g-1,\ell+1]}$$

reducing the problem to the genus 0 case. Finally the genus 0 case is treated by hand:

$$\mathsf{Tev}_{0,n[0,\ell],d[0,\ell]} = 1 \text{ for all } \ell \geq 0.$$

Application : Castelnuovo's classical count of g^1_d's

Let C be a general smooth genus g curve curve. Fix a degree $d \geq 1$ and consider the Brill-Noether locus

$$G^1_d(C) = \{g^1_d\text{'s on } C\}$$

which is smooth of dimension $\rho = g-2(g-d+1)$. Assume $\rho=0$. Then we can write $g=-2\ell$ and $d=g+\ell+1$ for some $\ell \in \mathbb{Z}$ and

$$G^1_d(C) = W^1_d(C) = \{L \in \mathrm{Pic}^d(C) \mid h^0(C,L) \geq 2\} \subseteq \mathrm{Pic}^d(C)$$

In his famous paper [2] Castelnuovo proved that

$$\deg([W^1_d(C)]) = \frac{1}{1+|\ell|}\binom{2|\ell|}{|\ell|}$$

which agrees (after some algebraic manipulations) with $\mathrm{Tev}^{\mathbb{P}^1}_{g,3,d}$.

The quantum Euler class

Denote by $(QH^*(X,\mathbb{Q}),\star)$ the small quantum cohomology ring of X (see [7] for an introduction) and let

$$H^*(X,\mathbb{Q}) \otimes H^*(X,\mathbb{Q}) \xrightarrow{\star} QH^*(X,\mathbb{Q}).$$

be the multiplication map.

Definition of the Quantum Euler class

The **quantum Euler class** E of X is the image of the diagonal class $[\Delta]$ under the multiplication map above (note that $[\Delta]$ lives naturally in $H^*(X,\mathbb{Q}) \otimes H^*(X,\mathbb{Q})$ via the Künneth isomorphism).

This class plays a central role in the computation of Virtual Tevelev degrees. Indeed, we have the following equality (see [1, Theorem 1.3]):

$$\mathbf{vTev}^X_{g,n,\beta} = \mathbf{Coeff}(\mathbf{P}^{\star n} \star \mathbf{E}^{\star g}, q^\beta \mathbf{P}) \tag{2}$$

where P is the point class.

Comparison between the virtual and the geometric count

Enumerativity results of Virtual Tevelev Degrees have been studied in [9]. To state their main result [9, Theorem 24] we require additional notation.
Assume X is a **Fano** variety of dimension r. Define $s(X)>0$ to be the smallest positive integer for which there exists an effective curve class $\beta \in H_2(X,\mathbb{Z})$ such that

$$s(X) = c_1(X).\beta$$

and such that the evaluation map $\mathrm{ev}_1: \overline{\mathcal{M}}_{0,1}(X,\beta) \to X$ is surjective.
Define $t(X)>0$ to be the smallest positive integer for which there exists an effective curve class $\beta \in H_2(X,\mathbb{Z})$ such that

$$t(X) = c_1(X).\beta.$$

Enumerativity

Fix a genus $g \geq 0$. Assume that:

- there exists $k>0$ such that for all β satisfying $c_1(X).\beta > k$ we have $c_1(X).\beta > (r-s(X))h^1(f^*T_X)$ for all $[f] \in \mathcal{M}_g(X,\beta)$;
- $s(X)+t(X) \geq r+1$.

Then there exist $d[g,X]>0$ such that for all β such that $c_1(X).\beta > d[g,X]$ and $n=n[g,X,\beta] \geq 0$ such that Equation (1) is satisfied, the Geometric Tevelev degree $\mathrm{Tev}^X_{g,n,\beta}$ is well-defined and coincides with the Virtual Tevelev degree $\mathrm{vTev}^X_{g,n,\beta}$.

Simple Example For $X=\mathbb{P}^r$ we have

$$\mathrm{vTev}^{\mathbb{P}^r}_{g,n,d\mathsf{L}} = (r+1)^g$$

where L is the class of a line (see [1, Example 2.2]). In particular, for $r=1$ and $\ell \geq 0$ we see that

$$\mathrm{vTev}^{\mathbb{P}^1}_{g,n[g,\ell],d[g,\ell]} = \mathrm{Tev}^{\mathbb{P}^1}_{g,n[g,\ell],d[g,\ell]} = 2^g.$$

Fano Hypersurfaces

Let $X \subset \mathbb{P}^{r+1}$ be a smooth Fano hypersurface of dimension $r \geq 3$ and degree $m \geq 2$. Note that X is Fano precisely when $m \leq r+1$. Also, by Lefschetz Hyperplane theorem we have

$$H_2(X,\mathbb{Z}) = \mathbb{Z}\mathsf{L}$$

where L is the class of a line in X. In particular

$$QH^*(X,\mathbb{Q}) = H^*(X,\mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{Q}[q]$$

as $\mathbb{Q}[q]$-module. Although the definition of E involves also the primitive cohomology of X, in [3, Theorem 5], we were able to obtain explicit simple formulas for E.

The Quantum Euler class for Fano Hypersurfaces

The following equalities hold:

- if $m \leq r$ then
$$\mathsf{E} = m^{-1}\chi(X)\mathsf{H}^{\star r} + (r+2-m-\chi(X))m^{m-1}q\mathsf{H}^{\star m-2},$$
- if $m=r+1$ then
$$\mathsf{E} = m^{-1}\chi(X)\mathsf{H}^{\star r} + \sum_{j=1}^{r} m^{-1}(j-\chi(X))\binom{r}{j-1}(m!)^{j-1}\left[m^m - \frac{m!}{j}(r+1)\right]q^j\mathsf{H}^{\star r-j}.$$

Using this and Equation (2) it is also possible to obtain formulas for $\mathrm{vTev}_{g,n,d\mathsf{L}}$ (in terms of P). In particular, for low degree hypersurfaces we have (see [1, Theorem 5.19] and [9, Theorem 11]):

Explicit formulas for low degree Fano Hypersurfaces

If $r > \max(2m-4,2)$ and X is not a quadric, then

$$\mathrm{vTev}^X_{g,n,d\mathsf{L}} = ((m-1)!)^n(r+2-m)^g m^{(d-n)m-g+1}$$

If in addition $r > (m+1)(m-2)$, then $\mathrm{Tev}_{g,n,d\mathsf{L}}$ are well-defined for $d \geq d[g,X]$ and coincide with $\mathrm{vTev}_{g,n,d\mathsf{L}}$.

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