Lecture I (Speaker : Miguel Moreira)

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Symplectic geometry GW (Reference: McDuff, Salamon: 'J-holomorphic curves and Quantum cohomolog

(C, j) manifold $\dim_{\mathbb{R}} = 2$ with complex structure j for curves

$\overset{\shortparallel}{\longleftarrow}$ almost complex structure

(X, J) manifold $\dim_{\mathbb{R}} = 2n$ with almost complex structure J

$$f : C \xrightarrow[\text{smooth}]{} X \text{ is } \underline{\text{holomorphic}} \text{ if } \bar\partial f = \underset{\substack{\uparrow \\ \text{Cauchy-Rieman equation}}}{0}$$

(0,1) forms $T_C \to f^* T_X$

$$\underset{\substack{\uparrow \\ \text{maps from} \\ C \text{ to } X}}{F := \bar\partial : M(C, X)} \longrightarrow \underset{\substack{\uparrow \\ \text{bundle over } M(C,X) \text{ with fiber over } f \text{ is } \Omega^{0,1}(f^* T_X)}}{\mathcal{E}}$$

$$f \longmapsto \bar\partial f \in \Gamma(\Omega^{0,1}(f^* T_X))$$

So $Z(\bar\partial) = \{\text{holomorphic maps } C \to X\}$

Thm

$F : \mathcal{A} \to \mathcal{B}$ smooth map between Banach manifolds, $q \in \mathcal{B}$ a regular value for F,

(i.e.
- $d_p F$ is surjective for all $p \in F^{-1}(q)$
- $\ker(d_p F)$ is finite dimensional $\forall\, p \in F^{-1}(q)$)

$\Rightarrow$ $F^{-1}(q)$ is smooth manifold of finite dimension $= \dim \operatorname{Ker}(d_p F)$ (at p)

Thm $F = \bar\partial$ is Fredholm

which means that $\operatorname{Ker}(d_p F)$ and $\operatorname{coker}(d_p F)$ are finite dimensional

$\Rightarrow$ we have the notion of index of F

$$\operatorname{Ind}(F) := \dim \operatorname{Ker}(d_p F) - \dim \operatorname{Coker}(d_p F)$$

$\uparrow$

independent of p in connected components and also remains the same perturbing F a little bit

The ideal scenario is $\operatorname{Coker}(d_pF) = 0$. $\circledast$

Def From this point of view

$$\operatorname{ind}(F) := \text{expected dimension of } Z(\bar\partial)$$

How do you perturb F? Perturbing J we can assume $\operatorname{coker}(d_pF)=0$.

~~Note that~~ Then we define this theory for such J.

g metric on X, then $\omega = g(J-,-)$. And two ~~of~~ between (g, J, ω) determine uniquely the third one

Rmk Given (X, ω) there are many J compatible with ω and the space of such J is contractible.
We can always find J compatible with ω s.t. $\circledast$ holds.

Basic model for Virtual fundamental class

(Rahul, Richard 13/2 ways of counting curves)

A smooth ambient space of dim N (in the previous discussion $A = M(C,X)$)

E bundle of rk r

$s : A \longrightarrow E$ section

$Z(s)$ smooth of $\dim = \underset{\substack{\| \\ \text{expected dimension}}}{N-r}$ if S is generic

In general $Z(s)$ might be singular and $\dim Z(s)$ is always $\geq N-r$.

Example to have in mind

Assume ① maps into some subbundle $E' \subset E$ of $rk = r'$

② E splits $E = E' \oplus E/E'$

Write $s = (s', 0) : A \longrightarrow E' \oplus E/E' = E$

③ s' is transverse to the zero section of E'

$\Rightarrow$ $Z(s')$ is smooth of dimension $= N - r' > N - r$

We would like to define

$$[Z(s)]^{vir} \in A_{N-r}(\underset{\substack{\| \\ X}}{\underline{Z(s)}})$$

Assume also

④ $\exists\ \varepsilon \in \Gamma(\underset{\uparrow\, rk = r-r'}{E/E'})$ transverse to 0 section

Then $\underset{\uparrow}{Z(s',\varepsilon)} \subseteq Z(s)$

is cut out from $Z(s)$ by the equation $\varepsilon = 0$

$\qquad\qquad\qquad\qquad \rotatebox{90}{$\in$}$

$\qquad\qquad\qquad\qquad \Gamma(Z(s), \underset{\uparrow\, rk = r-r'}{E/E'})$

and has codimension $= r-r' = rk(E/E')$

$$\Rightarrow \quad \underset{\substack{\| \\ [Z(s)]^{vir}}}{Z(S',\varepsilon)} = c_{r-r'}(\underset{\substack{\uparrow \\ \text{obstruction bundle}}}{\underline{E/E'}}) \in A^{r-r'}(Z(s)) \cong A_{\underset{\substack{\uparrow \\ N-r'-(r-r')=N-r}}{N-r}}(Z(s))$$

This implies that $[Z(s)]^{vir}$ is independent of the choice of s'.

■

In general given

$$\begin{tikzcd} & E \arrow[d] \\ & A \arrow[u, bend left, "s"] \end{tikzcd}$$

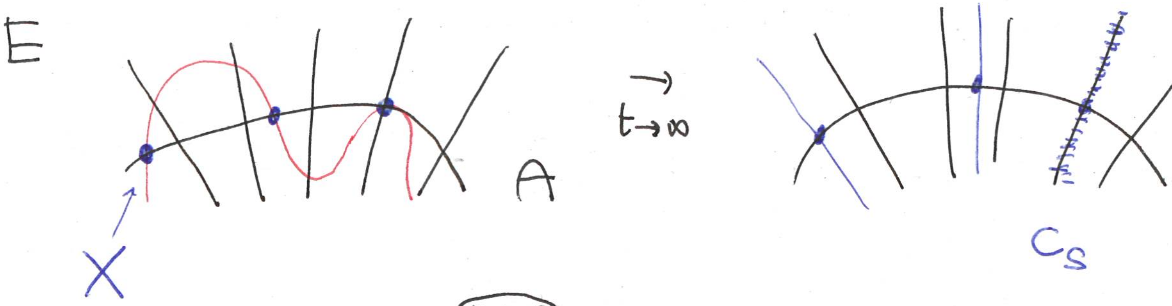
$A \supset X := Z(s)$

↑ smooth of pure dim N

how do we define $[X]^{vir}$?

<u>Idea</u>: let $C_s \subseteq E|X$ be the cone over X obtained by taking the graph of ts for $t \to \infty$

↑ pure dimension N



Then one define

$$[X]^{vir} := 0_E^! [C_s] \in A_{N-r}(X)$$

where $0_E : X \longrightarrow E|_X$

$$\begin{matrix} E|_X \\ \cup\!\!| \\ C_s \end{matrix}$$

■

In the case discussed in the example it is:

$$C_s = E' \subset E$$

and so

$$[X]^{vir} = 0^!_{X\to E}[E'] \underset{|}{=} \overbrace{0'^![E']}^{\overset{[X]}{\shortparallel}} . e\Big(\overset{\overset{E}{\shortparallel}}{\widetilde{N_{X|E}}} \Big/ \underset{\underset{E'}{\shortparallel}}{\widetilde{N_{X/E'}}}\Big)$$

$$\begin{tikzcd} X \arrow[r, hook, "0'"] \arrow[d, equal] \arrow[dr, phantom, "\square"] & E' \arrow[d, hook] \\ X \arrow[r, hook, "0"'] & E \end{tikzcd}$$

$0, 0'$ reg emb

+ excess intersection formula

$$= e\left(E/_{E'}\right)$$

■

Infinitesimal information :

consider $X \hookrightarrow A \longrightarrow E$, then we have an exact sequence

$$\begin{tikzcd} 0 \arrow[r] & T_pX \arrow[r] & T_pA \arrow[r, "ds_p"'] & E_p \arrow[r] & Ob_p \arrow[r] \arrow[from=l, bend left, phantom, "\text{definition}"] & 0 \end{tikzcd}$$

$$\underbrace{T_pA \longrightarrow E_p}_{\substack{\uparrow \\ \text{This is what [B-F]} \\ \text{call Obstruction theory}}}$$

ts always of X
s cutted out by
$_{1,..,} f_r$ then
$_pX$ is cutted out
by $d_pf_1,..,d_pf_r$

In the example $Ob_p = \left(E/_{E'}\right)_p$

Def Obstruction theory on a DM-stack X is

$$E^\bullet \in D(X)$$

together with a map

$$\phi : E^\bullet \longrightarrow \mathbb{L}^\bullet_X$$

s.t. $h^0(E^\bullet) \xrightarrow{\sim} h^0(\mathbb{L}^\bullet_X)$ is an isomorphism

$$\begin{matrix} h^0(\mathbb{L}^\bullet_X) \\ \shortparallel \\ \Omega_X \end{matrix} \longleftarrow \text{ if } X \subset \overset{\ulcorner \text{smooth DM}}{M}$$

$$\underset{-1}{I/I^2} \longrightarrow \underset{0}{\Omega_M} \longrightarrow \underset{\underset{h^0(\mathbb{L}^\bullet_X)}{\shortparallel}}{\Omega_X} \longrightarrow 0$$

and

$$h^{-1}(E^\bullet) \longrightarrow h^{-1}(\mathbb{L}^\bullet_X) \text{ is surjective}$$

$E^\bullet$ is said perfect if locally $E^\bullet = [E^{-1} \longrightarrow E^0]$, E^{-1}, E^0 vector bundle

Consider again

$$0 \longrightarrow T_pX \longrightarrow T_pA \longrightarrow E_p \longrightarrow Ob_p \longrightarrow 0$$

or better

$$0 \longrightarrow TX \longrightarrow TA|_X \xrightarrow{ds} E|_X \longrightarrow Obs \longrightarrow 0$$

$E^\bullet := \underset{\ \ -1 \qquad\qquad\qquad 0}{\left[E|_X^\vee \longrightarrow TA|_X^\vee \right]} \in D(X)$. Then we have

$$h^0(E^\bullet) = T_X^\vee = \Omega_X \quad \text{and} \quad h^{-1}(E^\bullet) = Obs^\vee$$

Cotangent complex

- If f is smooth this $\overset{\text{exact}}{\text{sequence}}$ ~~is also~~ extends to $0 \to f^*\Omega_{Y/Z} \to \Omega_{X/Z} \to \Omega_{X/Y} \to 0$ exact
- If f is a closed emb then $\Omega_{X/Y} = 0$ and we have $I/I^2 \to f^*\Omega_{Y/Z} \to \Omega_{X/Z} \to 0$ exact

If also $X \to Z$ is smooth then $0 \to I/I^2 \to f^*\Omega_{Y/Z} \to \Omega_{X/Z} \to 0$ is exac

consider a composition of maps

$$X \xrightarrow{f} Y \xrightarrow{g} Z$$

$$\rightsquigarrow \quad f^*\Omega_{Y/Z} \to \Omega_{X/Z} \longrightarrow \Omega_{X/Y} \longrightarrow 0$$

and we would like to complete it.

$\mathbb{L}^\bullet_{X/Y}$ = "left-derived functor" of $\Omega_{X/Y}$

↑

constructed by Illusie + Quillen + André : $\mathbb{L}^\bullet_{X/Y} \in D(X)$ satisfying

- $h^0(\mathbb{L}^\bullet_{X/Y}) = \Omega_{X/Y}$
- $h^{i>0}(\mathbb{L}^\bullet_{X/Y}) = 0$
- given $X \to Y \to Z$ we have

$$\underbrace{Lf^*}_{\text{left derived pull back}}\mathbb{L}^\bullet_{Y/Z} \to \mathbb{L}^\bullet_{X/Z} \longrightarrow \mathbb{L}^\bullet_{X/Y} \text{ is exact triangle in } D(X)$$

how does the left derived pull back work? represent $\mathbb{L}^\bullet_{Y/Z}$ as a complex $P^\bullet$ of proj modules on $D(X)$, then apply the usual pullback to $P^\bullet$ i.e. $Lf^*\mathbb{L}^\bullet_{Y/Z} = f^*P^\bullet$.

- if $X \underset{\text{cl.emb}}{\hookrightarrow} Y$ smooth then $[I/I^2 \to \Omega_Y|_X]$ is quasi-isomorphic to the truncated $\tau_{[-1,0]}\mathbb{L}^\bullet$

$$\rightsquigarrow \quad \cdots \to h^{-1}(\mathbb{L}_{X/Z}) \to h^{-1}(\mathbb{L}_{X/Y}) \to f^*\Omega_{Y/Z} \to \Omega_{X/Z} \to \Omega_{X/Y} \to 0$$

exact sequence of sheaves on X

Example (when $h^{-1}(\mathbb{L}_{X/Y}) \neq 0$)

$$X \xhookrightarrow[\text{closed emb}]{i} Y \longrightarrow Z$$

$$\longmapsto \quad \underset{\substack{\uparrow \\ \text{conormal sheaf}}}{I/I^2} \longrightarrow i^*\Omega_{Y/Z} \longrightarrow \Omega_{X/Z} \longrightarrow \Omega_{X/Y} = 0$$

conormal sheaf : $I/I^2 = h^{-1}(\mathbb{L}_{X/Y})$ ∎

Proposition : if $X \hookrightarrow Y$ is a reg emb, then

$$h^{-i}(\mathbb{L}_{X/Y}) = \begin{cases} I/I^2 & i=1 \\ 0 & \text{otherwise} \end{cases}$$

∎

Examples

- If $X \xrightarrow[\text{smooth}]{} Y$, then $\mathbb{L}_{X/Y} = \underset{\substack{\uparrow \\ \text{degree } 0}}{\Omega_{X/Y}}$

If X is l.c.i (i.e. $X \xhookrightarrow[\text{reg.emb}]{} \underset{\substack{\uparrow \\ \text{smooth}}}{A}$) then $\mathbb{L}^\bullet_X = [I/I^2 \to i^*\Omega_A]$ ∎

Rmk ‖ X l.c.i $\iff$ $\underset{\substack{\parallel \\ E^\bullet}}{\mathbb{L}^\bullet_X} \longrightarrow \mathbb{L}^{\bullet}_X$ is perfect

In this case we have

$$[X, \mathbb{L}^\bullet_X]^{vir} = [X]$$

How? $X \xhookrightarrow[\text{reg emb}]{} A$

Exercise : ~~show~~ check that if $E^\bullet$ is perfect of amplitude $[-1,0]$ then giving $E^\bullet \to \mathbb{L}_X$ is the same as $E^\bullet \to \mathbb{L}_X^{\geq -1}$.

Let $E^\bullet \longrightarrow \mathbb{L}_X^\bullet$ be a perfect obstruction theory

then we have

$$\begin{tikzcd} \mathfrak{C}_X \arrow[r, hook] \arrow[d, hook] & h^1/h^0(E^\vee) \\ N_X \arrow[ru] & \end{tikzcd}$$

and

$$[X, E^\bullet] = \underset{\substack{\uparrow \\ \text{Vertex in } h^1/h^0(E^\vee)}}{0}{}^! \, \mathfrak{C}_X$$

Exercise: Consider the case

$$\begin{tikzcd} & V \arrow[d] \\ Z(s)=X \arrow[r, hook] & A \arrow[u, bend right, "s"'] \end{tikzcd}$$

smooth

with $E_\bullet = [T_A|_X \to V|_X]$ i.e. $E^\bullet = [V|_X^\vee \to \Omega_A|_X]$ and

$$\begin{tikzcd} 0 \arrow[r] & V|_X^\vee \arrow[r, "ds"] \arrow[d, "s"'] & \Omega_A|_X \arrow[r] \arrow[d, equal] & 0 \arrow[r, Rightarrow] \arrow[d, dashed] & E^\bullet \arrow[d] \\ & I/I^2 \arrow[r, "d"] & \Omega_A|_X \arrow[r] & \Omega_X \arrow[r] & 0 \quad \tau_{[-1,0]}\mathbb{L}_X^\bullet \end{tikzcd}$$

Then one obtain exactly the construction you did above ■

The case of stable maps

Consider

$$X = M(C, Z) \ni f$$

Then $T_f X \underset{\substack{\uparrow \\ \text{Exercise}}}{=} H^0(C, f^* T_Z)$

$T_X = \mathrm{Maps}(k[\varepsilon]/_\varepsilon \longrightarrow X) = \mathrm{Spec}\, k[\varepsilon]/_{(\varepsilon)} \times C \longrightarrow Z$

If I want to write TX globally we have to use the universal map

$$\begin{tikzcd} C \times X \arrow[r, "f"] \arrow[d, "\pi"'] & Z \\ X & \end{tikzcd}$$

and $TX = \pi_* f^* T_Z$

Take $E_\bullet = R\pi_* f^* T_Z$ and $\mathcal{O}bs = h^1(E_\bullet) = H^1(C, f^* T_Z)$.

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Lecture II (Speaker: Alessio Cela)

Deformations of morphisms of curves (Reference: 'Rational curves on algebraic varieties' by J. Kollar)

Def $X/_S$, $Y/_S$ schemes over S, $B \subset X$ subscheme proper over S, $g : B \to Y$ a morphism.

$\mathrm{Hom}(X,Y,g)$ is the functor : $\mathrm{Sch}/_S \longrightarrow \mathrm{Sets}$

$$\mathrm{Hom}(X,Y,g)(T) = \left\{ \begin{array}{l} T\text{-morphisms } f : X \times_S T \longrightarrow Y \times_S T \\ \qquad \searrow T \swarrow \\ \text{s.t. } f|_{B \times_S T} = g \times_S \mathrm{id}_T \end{array} \right\}$$

Thm 1.10 in Kollar

$\exists\, Y, X \subseteq \mathbb{P}_S$ closed, embr

$$\begin{tikzcd} Y, X \arrow[r, phantom, "\subseteq"] \arrow[d] & \mathbb{P}_S \arrow[dl] \\ S & \end{tikzcd}$$

Thm 1 $X/_S$, $Y/_S$ projective schemes over S, X flat over S. Then

$$\mathrm{Hom}_S(X,Y) \underset{\text{open}}{\subset} \mathrm{Hilb}(X \times_S Y/_S)$$

is represented by an open subscheme

Proof (sketch)

Given $f : X \times_S T \longrightarrow Y \times_S T$ we have

$$\begin{tikzcd} \Gamma_f : X \times_S T \arrow[r, hook, "{(\mathrm{id},f)}", "\text{closed emb}"'] \arrow[rd, bend right, "\text{flat}"'] & X \times_S T \times_S Y = (X \times_S Y) \times_S T \arrow[d] \\ & T \end{tikzcd}$$

and thus we obtain a map $\mathrm{Hom}_S(X,Y) \longrightarrow \mathrm{Hilb}_S(X \times_S Y/_S)$

Consider the commutative diagram

universal family

$$\begin{tikzcd} t \arrow[rrr, mapsto] & & & (pr_X(t), [C]) \\ C \arrow[u, phantom, "\in"] \arrow[r, phantom, "\subset"] \arrow[d] & \mathrm{Univ} \arrow[rr, "\pi"] \arrow[d, "u"] & & X \times_S \mathrm{Hilb}(X \times_S Y/_S) \arrow[d, "p"] \\ {[C]} \arrow[r, phantom, "\in"] & \mathrm{Hilb}(X \times_S Y/_S) \arrow[rr, equal] & & \mathrm{Hilb}(X \times_S Y/_S) \end{tikzcd}$$

Then $C = \Gamma_f$ for some $f : X \times_S T \longrightarrow Y \times_S T \iff \pi_{[C]}$ is an isomorphism and we have to prove that

Thm $\Longleftrightarrow$ $\Big[$ If $[C] \in \mathrm{Hilb}(X \times_S Y/S)$ and $\pi_{[C]}$ is an iso $\Rightarrow$ $\exists$ open neighborhood U of $[C]$ in Hilb s.t. $\pi_{[C']}$ is an iso for all $[C'] \in U$ $\Big]$

■

↓ Prop 1.5 in Kollar.

Corollary

Assume X/S and B/S are flat, X/S and Y/S are proj. Let

$$R: \mathrm{Hom}_S(X,Y) \longrightarrow \mathrm{Hom}_S(B,Y)$$

be the restriction morphism.

Let

$(T \to S) \longmapsto T \times_S B \to T \times_S Y$ given by

$$\begin{tikzcd} T \times_S B \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & B \arrow[r, "g"] \arrow[d] & Y \arrow[dl] \\ T \arrow[r] & S & \end{tikzcd}$$

$$\bar{g}: S \longrightarrow \mathrm{Hom}(B,Y)$$

be the section induced by $g: B \to Y$.

Then

$$\mathrm{Hom}(X,Y,g) = R^{-1}(\bar{g}(S)) \subset \mathrm{Hom}(X,Y)$$

represents $\mathrm{Hom}(X,Y,g)$.

■

↓ Thm 1.7 in Kollar

Thm2

Let C/S be a flat and proj curve without embedded points and Y/S a smooth ~~subscheme~~
proj scheme over S. Let $B/S \subset C/S$ be a closed subscheme finite and flat
over S. Assume that C/S is smooth along B/S. Let $G: B/S \to Y/S$ be a
morphism. Let $s \in S$ and $f_s \in \mathrm{Hom}(C_s, Y_s, G_s)(k(s))$. Then

$$T_{[f_s]} \mathrm{Hom}(C_s, Y_s, G_s) = H^0(C_s, f^* T_{Y_s} \otimes I_{B_s}^{\vee})$$

__Sketch of proof__ We may assume $S = \{s\}$.

__Step 1__ ‖ Assume $B = \emptyset$.

__Recall__ ‖ for $[Z] \in \mathrm{Hilb}(X/*)$ we have $T_{[Z]}\mathrm{Hilb}(X/*) = H^0(Z, N_{Z/X})$

Now identify

$$[f] = [\Gamma_f] \in \mathrm{Hilb}(C \times_S Y/S)$$

$$\underset{\text{Thm 1}}{\Rightarrow} \quad T_{[f]}\mathrm{Hom}(C,Y) = T_{[f]}\mathrm{Hilb}(C\times_S Y/S) = H^0(\Gamma_f, N_{\Gamma_f/C\times_S Y}) = H^0(C, f^*T_Y)$$

$\Downarrow$

$\Rightarrow$ γ is the data $\forall x \in C$ of $v \in T_{Y,f}$

__Claim__ ‖ Under the identification $C \underset{\gamma}{\xrightarrow{\sim}} \Gamma_f$ we have $N_{\Gamma_f/C\times_S Y} = f^*T_Y$

__proof of the claim__

We have $C \underset{\text{reg. emb.}}{\hookrightarrow} C \times_S Y \xrightarrow{pr_1} C$ and thus

] ← because Y/S is smooth $\Rightarrow C\times_S Y \to C$ smooth

$$\begin{tikzcd}
0 \arrow[r] & I/I^2 \arrow[r, "\sim"] \arrow[d, equal] & \Omega_{C\times_S Y/C}|_C \arrow[r] \arrow[d, equal] & \Omega_{C/C} = 0 \\
& N^\vee_{\Gamma/C\times_S Y} \arrow[r, phantom, "\cong"] & T^\vee_{C\times_S Y/C}|_C &
\end{tikzcd}$$

exact

Finally $\gamma^* T_{C\times_S Y/C}|_C \underset{\uparrow}{=} f^*T_Y$

in general given

$$\begin{tikzcd}
X \arrow[dr, "\mathrm{id}\times f = \gamma"] \arrow[drr, bend left, "\mathrm{id}"] \arrow[ddr, bend right, "f"'] & & \\
& X \times_S Y \arrow[r] \arrow[d, "u"'] \arrow[dr, phantom, "\square"] & X \arrow[d] \\
& Y \arrow[r] & S
\end{tikzcd}$$

we have

$$u^*\Omega_{Y/S} = \Omega_{X\times_S Y/X}$$

$$\Rightarrow f^*\Omega_{Y/S} = \underbrace{(\mathrm{id}\times f)^*}_{\overset{\shortparallel}{\gamma}}\Omega_{X\times_S Y/X}$$

∎

__Step 2__ ‖ Assume $\exists\, \sigma : \underset{\{s\}}{\overset{}{\underset{\shortparallel}{S}}} \to C$ section s.t. $B = \sigma(S) = \{\sigma\}$

Consider

$$\begin{tikzcd}
C=C_s \arrow[r, phantom, "\cong"] \arrow[dr, phantom, "\gamma_0\ \cong"'] & \Gamma_{f_s} \arrow[r, phantom, "\subset"] \arrow[d, phantom, "\shortparallel"] & C\times_S \mathcal{X} = C\times\mathcal{X} \arrow[d, phantom, "\shortparallel"] \\
& \Gamma_0 \arrow[r, phantom, "\subset"] \arrow[d, phantom, "\ni"] & X_0 \\
& \sigma &
\end{tikzcd}$$

Let $X_1 := Bl_\sigma X_0 \xrightarrow{\pi} X_0$ ~~where / p≠/s~~

$\cup$

$C \underset{\gamma_1}{\cong} \Gamma_1 :=$ proper transform of Γ_0

σ is Cartier divisor on C

reason:
$$\begin{tikzcd}
C = Bl_\sigma(C) \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & Bl_\sigma(C\times Y) \arrow[d] \\
C \arrow[r, hook] & C\times Y
\end{tikzcd}$$

Let I_0 be the ideal sheaf of $\Gamma_{f_s}=\Gamma_0$ in $C\times Y$ and I_1 the ideal sheaf of Γ_1 in X_1.

Then as before

$$\gamma_0^*\left(I_0/I_0^2\right) = f^*\Omega_Y$$

Claim $\gamma_1^*\left(I_1/I_1^2\right) \cong f^*\Omega_Y(+\sigma)$

proof of the claim

$$\begin{tikzcd}
\Gamma_1 \arrow[r, hook] \arrow[d, "\cong"'] \arrow[dr, phantom, "\circlearrowleft"] & X_1 \arrow[d, "\rho"] \\
\Gamma_0 \arrow[r, hook] & X_0
\end{tikzcd}$$

Consider

reg. embeddings being sections of $C\times Y \xrightarrow[\text{smooth}]{} C$, $Bl_\sigma(C\times Y) \xrightarrow{\text{smooth}} C$

$$\begin{tikzcd}
& 0 & 0 & \\
& \Omega_\rho|_{\Gamma_1} \arrow[u] & \Omega_\rho|_C \arrow[u] & \\
0 \arrow[r] & N^\vee_{\Gamma_1/X_1} \arrow[u] \arrow[r, "\sim"] & \Omega_{X_1/\Gamma_1}|_C \arrow[u] \arrow[r] & 0 \\
0 \arrow[r] & N^\vee_{\Gamma_0/X_0} \arrow[u] \arrow[r, "\sim"] & \Omega_{X_0/\Gamma_0}|_C \arrow[u] \arrow[r] & 0 \\
& 0 \arrow[u] & 0 \quad \circledast \arrow[u] &
\end{tikzcd}$$

see example 4.2.6 in Fulton, but to use that maybe you want X_0 and X_1 to be smooth

check that fibers are all smooth and for flatness around σ use miracle flatness (Assume $\sigma \in C^{sm}$) ~~[illegible]~~

$\llcorner$ usual exact sequence obtained from $X_1 \to X_0 \to \Gamma_0 = C$ restricted to C

$$\begin{tikzcd}
X_1 \arrow[r] \arrow[rr, bend left, "\text{smooth}"] & X_0 \arrow[r] & \Gamma_0 = C
\end{tikzcd}$$

Now we have
$$\begin{tikzcd}
E \cong \mathbb{P}^n \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & X_1 \arrow[d] \\
\sigma \arrow[r] & X_0
\end{tikzcd}$$
$\Rightarrow \Omega_{X_1/X_0}|_E = \Omega_{\mathbb{P}^n} \leftarrow$ locally free of $rk = n = \dim Y$

$\Rightarrow \Omega_\rho|_C = \mathbb{C}_\sigma^{\oplus n}$ and $\circledast$ becomes $0 \to \Omega_{\Gamma_0/X_0}|_C \to \Omega_{X_1/X_1}|_C \to \mathbb{C}_\sigma^{\oplus n} \to 0$

Now in general if you have $0 \to \mathcal{E}_1 \to \mathcal{E}_2 \to \mathbb{C}_\sigma^{\oplus n} \to 0 \rightsquigarrow 0 \to \mathcal{E}_1 \otimes \mathcal{E}_2^\vee \to \mathcal{O} \to \mathbb{C}_\sigma \to 0$

rk n vector bundles

$\Rightarrow \mathcal{E}_1 \otimes \mathcal{E}_2^\vee \cong \mathcal{O}(-\sigma) \quad \Rightarrow \mathcal{E}_1 \simeq \mathcal{E}_2 \otimes \mathcal{O}(+\sigma)$

Now take $\mathcal{E}_1 = \Omega_{\Gamma_1/X_1}|_C$ and $\mathcal{E}_2 = \Omega_{\Gamma_0/X_0}|_C$.

$$\Rightarrow \gamma_1^*\left(I_1/I_1^2\right) = \gamma_0^*\left(I_0/I_0^2\right) \otimes \mathcal{O}(\sigma)$$

■

Claim 2 ∃ open neighborhood

$$\begin{tikzcd}
{[\Gamma_1]} \arrow[r, phantom, "\in"] & U \arrow[r, phantom, "\overset{\text{open}}{\subset}"] \arrow[d, phantom, "\wr\|"] & \mathrm{Hilb}(X_1/k(s)) \\
{[f]} \arrow[u, mapsto] \arrow[r, phantom, "\in"] & \mathrm{Hom}(C,Y,G) &
\end{tikzcd}$$

Clearly, claim 1 + claim 2

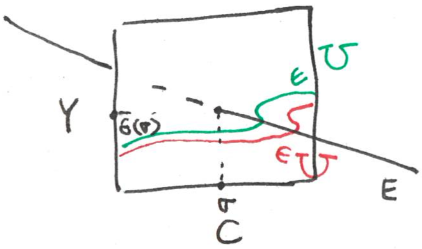
$$\Rightarrow T_{[f]} \mathrm{Hom}(C,Y,G) = T_{[\Gamma_1]} \mathrm{Hilb}(X_1/k(s)) = H^0(\Gamma_1, N_{\Gamma_1/X_1}) = H^0(C, f^*T_Y(-\sigma))$$

idea of the proof of the claim

Take $U := \{ D \underset{\substack{\text{closed}\\ \text{embs}}}{\subset} X_1 \mid$ ~~dim~~ $E \cap D \neq \emptyset$ and $D \subset X_1 \longrightarrow C$ is an iso $\}$ with $\overset{\cong}{\curvearrowright}$ from D to C

$$\begin{tikzcd}
U \arrow[d, phantom, "\cap \text{ open}"] & E \cap D \neq \emptyset \\
\mathrm{Hilb}(X_1/k(s)) & E := \pi^{-1}(\sigma) \subset X_1 \arrow[u]
\end{tikzcd}$$

picture



$= X_1$

Cones:

X = DM stack (separated + locally of finite type$/k$)

Def $S^\bullet = \bigoplus_{i\geq 0} S^i$ graded q.coherent sheaf on X

Assume : 1) $S^0 = \mathcal{O}_X$

2) $\mathrm{Sym}^\bullet S^1 \twoheadrightarrow S^\bullet$

3) S^1 is coherent

Then $\mathrm{Spec}(S^\bullet) \longrightarrow X$ is called a cone over X

Def A morphism of cones over X is induced by a graded morphism of graded $\mathcal{O}_X$-algebras

Def A closed subcone is the image of a closed emb of cones

Rmks :

① If
$$\begin{tikzcd} & C_2 \arrow[d] \\ C_1 \arrow[r] & C_3 \end{tikzcd}$$
is a diagram of cones over $X \Rightarrow C_2 \times_{C_3} C_1$ is a cone over X

② $\exists$ $0: X \longrightarrow C$ vertex induced by $S^\bullet \longrightarrow S^0 = \mathcal{O}_X$

③ $\exists$ $\mathbb{A}^1$ action $\curvearrowright C$

$$\mathbb{A}^1 \times_X C \longrightarrow C \quad \text{given by} \quad \begin{array}{ccc} S^\bullet[x] & \longleftarrow & S^\bullet \\ s_i x^i & \longleftarrow\!\shortmid & s_i \in S^i \end{array}$$

with some obvious properties

Abelian cones

Def $\mathcal{F} \in \mathrm{Coh}_X \longmapsto C(\mathcal{F}) := \mathrm{Spec}\left(\mathrm{Sym}^{\cdot}(\mathcal{F})\right)$ is an <u>abelian cone</u>.

Obs ‖ $C(\mathcal{F})$ is a group scheme over X

<u>proof</u>

we have to give $\mathrm{Hom}_X(-, C(\mathcal{F}))$ a group-structure, but if $T \xrightarrow{u} X$

$$\mathrm{Hom}_X(T, C(\mathcal{F})) = \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, u_*\mathcal{O}_T) \underset{|}{=} \mathrm{Hom}_{\mathcal{O}_T}(u^*\mathcal{F}, \mathcal{O}_T)$$

u_* and u^* are adjoint

A morphism $Y \to \mathrm{Spec}(A)$
is the same data as $A \to \Gamma(Y, \mathcal{O}_Y)$.

■

Rmks

① Fiber product of abelian cones is an abelian cone

② $C = \mathrm{Spec}(S^{\cdot}) \xhookrightarrow[\text{closed emb}]{} A(C) := \mathrm{Spec}(\mathrm{Sym}\, S^1)$

and ~~and surjective~~ ← ~~reason: $f \in S^+$ then $\{\text{primes of } ((S^{\cdot})_f)_0\} = \{\text{homogeneous primes of } (S^{\cdot})_f\}$~~

~~$\mathfrak{p} \cap ((S^{\cdot})_f)_0 \leftarrow\!\shortmid \mathfrak{p}$~~

~~and for $S^{\cdot}$ and $\mathrm{Sym}^{\cdot} S^1$ the degree 1 component is S^1 and generate all S^+.~~

FALSE.

<u>Example</u>

$$C = \mathrm{Spec}\, k[x,y]/(x\,y)$$
$$\downarrow f$$
$$A(C) = \mathrm{Spec}\, k[x,y] = \mathbb{A}^2$$

■

<u>Lemma</u>

$$\left\{\begin{matrix}\text{abelian cones}\\ \text{over } X\end{matrix}\right\} \hookrightarrow \{\text{cones over } X\}$$

has an adjoint

$$\left\{\begin{matrix}\text{abelian}\\ \text{cones over } X\end{matrix}\right\} \leftarrow \{\text{cones over } X\}$$

$$\begin{tikzcd} A(C) = \mathrm{Spec}(\mathrm{Sym}^{\cdot} S^1) & \mathrm{Spec}(S^{\cdot}) = C \arrow[l, mapsto] \\ C \arrow[u, hook] \end{tikzcd}$$

(i.e. if A is an abelian cone and C is a cone

$$\mathrm{Mor}_{\text{cones}}(A(C), A) \to \mathrm{Mor}_{\text{cones}}(C, A)$$

$$(A(C) \to A) \longmapsto (C \hookrightarrow A(C) \to A)$$

is a bijection)

Lemma 1.1

③ C cone over X

Then: [C is a vector bundle] $\Leftrightarrow$ [$C \to X$ is smooth]

proof

($\Leftarrow$) ok

($\Rightarrow$): $C = \mathcal{S}pec \bigoplus_{i \geq 0} S^i \longrightarrow X$ smooth $\overset{\text{of rel dim } r}{\rceil}$ Consider

$$X \xrightarrow{0} C \longrightarrow X$$

$\longmapsto$ $I/I^2 \xrightarrow{\sim} 0^* \Omega_{C/X}$ is a $\overset{\text{rk } r}{\rceil}$ vector bundle over X

But $I = S^1$ and so we have

$$\begin{tikzcd} C \arrow[r, "\text{smooth of rel dim } r"] \arrow[d, hook] & X \\ A(C) \arrow[ur, "\text{smooth of rel dim } r"'] & \end{tikzcd}$$

vector bundle $\to$ A(C)

$\Rightarrow$ $C \xrightarrow{\sim} A(C)$ is a vector bundle

$\uparrow$ consider $0 \to N^\vee_{C/A(C)} \to \Omega_{A(C)/X}|_C \xrightarrow{\sim} \Omega_{C/X} \to 0$ $\Rightarrow$ $N^\vee_{C/A(C)} = 0 \overset{!}{\Rightarrow} J/J^2 = 0 \Rightarrow J = 0$

J ideal of C in A(C)

! $J^n = 0$ for some n

∎

④ Example (to keep in mind)

If $Y \underset{\text{closed embr defined by } I}{\hookrightarrow} X$ $\Rightarrow$ $C_{Y/X} := \mathcal{S}pec \bigoplus_{n \geq 0} I^n/I^{n+1}$

$\rceil$ closed embr

$$N_{Y/X} = A(C_{Y/X}) = \mathcal{S}pec(\mathcal{S}ym^\circ I/I^2)$$

∎

Exact sequences of cones

Def A sequence of cone morphisms

$$0 \to E \xrightarrow{i} C \longrightarrow D \to 0$$

is exact if:

i) E is a vector bundle

ii) ~~∃ morphism of cones~~ locally on X ∃ morphism of cones $C \to E$ splitting i and inducing an isomorphism

$$C \xrightarrow{\sim} E \times_X D$$

Rmk Given $0 \to \mathcal{F}' \to \mathcal{F} \to \underset{\substack{\uparrow \\ \text{locally free}}}{\mathcal{E}} \to 0$ exact sequence on X $\Big\} \Rightarrow$ $0 \to C(\mathcal{E}) \to C(\mathcal{F}) \to C(\mathcal{F}') \to 0$ is exact sequence of cones

Equivalently

proof $\mathcal{E}$ locally free $\Rightarrow$ locally on X $\exists$ $\mathcal{E} \to \mathcal{F}$ splitting $\mathcal{F} \to \mathcal{E}$ ∎

Lemma 1.3

Consider

$$\begin{tikzcd} E \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & C \arrow[d, "\text{smooth and surj morphism of cones over } X"] \\ X \arrow[r, "0"'] & D \end{tikzcd}$$

Then E is ~~a cone~~ a vector bundle over X and

$$0 \to E \to C \to D \to 0$$

is an exact sequence of cones over X.

proof

Step 1 E is a vector bundle over X

proof

$C = \operatorname{Spec} S^{\bullet}$, $D = \operatorname{Spec} S'^{\bullet}$ then $E = \operatorname{Spec}(S^{\bullet} \otimes_{S'^{\bullet}} \mathcal{O}_X)$ ← cone

in degree 1 we have $S^1/_{\operatorname{Im}(S'^1 \to S^1)}$ $= \operatorname{coker}(S'^1 \to S^1)$

↑ graded $\mathcal{O}_X$-algebra

$$\begin{tikzcd} E & \arrow[l, Rightarrow] \arrow[d, "\text{smooth}"] \\ & X \end{tikzcd}$$

E is a vector bundle on X ∎

Step 2 We prove that

$$0 \to E \to A(C) \to A(D) \to 0$$

is exact sequence of cones over X

proof

Obs $C \to D$ smooth and surjective $\Rightarrow$ $S^1 \hookrightarrow S'^1$ is injective

proof

call $K = \ker(S^1 \to S'^1)$. Then $C \to D$ factors

$$\begin{tikzcd} C \arrow[r, "\text{smooth + surj}", "p"'] \arrow[dr, "p'"'] & D \\ & V = \operatorname{Spec}(S'^{\bullet}/(K)) \arrow[u, hook, "\text{closed emb}"'] \end{tikzcd}$$

$\Rightarrow V = D$ topologically.

We have $0 \to p'^{*}\Omega_{V/D} \to \Omega_{C/D} \xrightarrow{\sim} \Omega_{C/V} \to 0 \Rightarrow C \xrightarrow{p'} V$ is smooth

$\Rightarrow$ $V \to D$ is smooth + closed emb $\Rightarrow$ $V \xrightarrow{\sim} D$ is an isomorphism

An A-mod M is flat $\Leftrightarrow$ $\mathrm{Tor}_1^A(M, A/I) = 0$ $\forall I \subset A$ ideal

So if $J \subset \sqrt{0}$ and $A \twoheadrightarrow A/J$ is flat ~~Homeo~~ morphism, then

$$0 \to J \to A \to A/J \to 0 \qquad - \otimes_A A/J$$

$$\rightsquigarrow \quad J \otimes_A A/J \xhookrightarrow[0]{} A/J$$

$$\Rightarrow 0 = J \otimes_A A/J = J/J^2 \Rightarrow J = J^2 \underset{|}{\Rightarrow} J = 0$$

$$J \subset \sqrt{0_A} \Rightarrow \exists n : \underset{\overset{\|}{J}}{J^n} = 0$$

So we get an exact sequence

$$0 \to S^1 \to S'^1 \to \overset{\overset{\mathrm{coKer}(S^1 \to S'^1)}{\|}}{\mathcal{E}} \to 0$$

$\rightsquigarrow$ $0 \to E \to A(C) \to A(D) \to 0$ exact sequence of cones

Step 3 We prove that

$$\begin{tikzcd} C \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & D \arrow[d] \\ A(C) \arrow[r] & A(D) \end{tikzcd}$$

is cartesian

Proof

Consider

$$\begin{tikzcd} C \arrow[r, hook, "\text{closed emb + surj}"'] \arrow[rr, bend left, "\text{smooth and surjective}"] \arrow[dr, bend right, "\text{closed emb + surj}"'] & A(C) \times_{A(D)} D \arrow[r, "\text{smooth}"'] \arrow[d, "\text{closed emb + surj}"] \arrow[dr, phantom, "\square"] & D \arrow[d, "\text{closed emb + surj}"] \\ & A(C) \arrow[r, "\text{smooth}"'] & A(D) \end{tikzcd}$$

$\Rightarrow$ $C \xrightarrow{\sim} A(C) \times_{A(D)} D$ is an iso

as ~~flat~~ before: I ideal of C in $A(C) \times_{A(D)} D \Rightarrow I \subset \sqrt{0}$ and we have

$$0 \to I/I^2 \to \Omega_{A(C) \times_{A(D)} D / D}|_C \xrightarrow{\sim} \Omega_{C/D} \to 0$$

$$\Rightarrow I/I^2 = 0 \Rightarrow I = I^2 \Rightarrow I = 0.$$

E - cones

E vector bundle, $d : E \to C$ morphism of cones

Def C is said an E-cone if C is invariant under the action $E \curvearrowright A(C)$

Notation C E-cone, then
$$\begin{aligned} E \times_X C &\longrightarrow C \\ (\nu, \gamma) &\longmapsto d\nu + \gamma \end{aligned}$$

Def A morphism from a E-cone to an F-cone is

$$\begin{tikzcd} E \arrow[r, "d"] \arrow[d, "\phi"'] \arrow[dr, phantom, "\circlearrowleft"] & C \arrow[d, "\phi"] \\ F \arrow[r, "d"'] & D \end{tikzcd}$$

Def $\phi, \psi : (E, d, C) \longrightarrow (F, d, D)$ morphisms are called homotopic if

$\exists$ morphism of cones $k : C \longrightarrow F$ s.t.

i) $\underbrace{kd + \phi}_{\uparrow} = \psi$ This operation is on F which is a vector space

ii) $\underbrace{dk + \phi}_{\uparrow} = \psi$.

given by $d : F \to D \rightsquigarrow F \curvearrowright D$

Rmk Consider a sequence of cones

$$0 \to \underset{\substack{\uparrow \\ \text{vector bundle}}}{E} \to C \to D \to 0 . \quad ⊛$$

Then

[⊛ is exact] $\Longleftrightarrow$ [

(i) C is an E-cone $\Longleftrightarrow$ $E \to C \to D$ $\quad \begin{tikzcd} E \arrow[r] & C \arrow[r] & D \end{tikzcd}$ with 0 : $\begin{tikzcd} E \arrow[r] & C \arrow[r] \arrow[r, bend left, phantom] & D \end{tikzcd}$

(ii) $C \to D$ is surjective

(iii) the diagram

$$\begin{tikzcd} E \times_X C \arrow[r, "\text{action}"] \arrow[d, "\text{proj}"'] & C \arrow[d, "\phi"] \\ C \arrow[r, "\phi"'] & D \end{tikzcd}$$

is cartesian

$\Updownarrow$

$E = \mathrm{Ker}(C \to D)$]

Proposition

Let $(C,0,\gamma)$ and $(D,0,\gamma)$ be algebraic X-spaces with sections and $\mathbb{A}^1$-actions

(so
$$\begin{tikzcd} C \arrow[r] & X \arrow[l, bend left, "0"] \end{tikzcd}$$
and $\gamma : \mathbb{A}^1 \times_{X} \cancel{X} \to C$ with the obvious properties).

Let $\phi : C \xrightarrow[\substack{\text{smooth} \\ + \text{surjective}}]{} D$ be an $\mathbb{A}^1$-equivariant X-morphism.

Call
$$\begin{tikzcd} E \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & C \arrow[d] \\ X \arrow[r, "0"'] & D \end{tikzcd}$$
and assume $E \to X$ is a vector bundle.

Then

1) $\left[C \to X \text{ is an } E\text{-cone over } X\right] \Leftrightarrow \left[D \text{ is a cone over } X\right]$

2) $\left[C \text{ is abelian cone over } X\right] \Leftrightarrow \left[D \text{ is an abelian cone}\right]$

3) $\left[C \text{ is a vector bundle over } X\right] \Leftrightarrow \left[D \text{ is a vector bundle over } X\right]$

Lemma 3.2

An example of E-cones which will be useful later

Let $U \xrightarrow{f} M$ be a local immersion (locally closed embedding) of affine k-schemes of finite type, where M is smooth $/k$.

Then $C_{U/M} \hookrightarrow M$ is a $f^* T_M$-cone.

Rmk A priori we have a map $f^* T_M \to N_{U/M}$. We will prove that $C_{U/M}$ is $f^* T_M$-equivariant. Then

$$f^* T_M \xrightarrow[(\mathrm{id},0)]{} f^* T_M \times_X C_{U/M} \xrightarrow{\cdot} C_{U/M}$$

is the map making $C_{U/M}$ an $f^* T_M$-cone.

proof

Recall Given a commutative diagram

$$\begin{tikzcd} & M \arrow[d, "\rho"', "\text{smooth}"] \\ U \arrow[ur, "\text{loc. closed emb}"] \arrow[r, "\text{loc. closed emb}"'] & M' \end{tikzcd}$$

we have

$$\begin{tikzcd} \rho^{*}\mathcal{O}_{M'} \arrow[r, phantom, "\cong"] & \mathcal{O}_M \\ \cup & \cup \\ \rho^{*}I_{U/M'} \arrow[r] & I_{U/M} \end{tikzcd}$$

which induces

$$I_{U/M'}/I^2_{U/M'} \longrightarrow I_{U/M}/I^2_{U/M}$$

One proves that $0 \to I_{U/M'}/I^2_{U/M'} \to I_{U/M}/I^2_{U/M} \to \Omega_\rho|_U \to 0$ is exact

we thus obtain

$$\begin{tikzcd} 0 \arrow[r] & T_\rho|_U \arrow[r] & N_{U/M} \arrow[r] & N_{U/M'} \arrow[r] & 0 \\ & & \cup & \square \quad \cup & \\ & & C_{U/M} \arrow[r] & C_{U/M'} & \end{tikzcd}$$

exact.

low we want to apply this to

$$\begin{tikzcd} U \arrow[r, "\Delta f"] \arrow[dr, "f"'] & M\times M \arrow[d, "p_i"', "i=1,2"] \\ & M \end{tikzcd}$$

$$\begin{tikzcd} & & C_{U/M\times M} \arrow[r] & C_{U/M} & \\ & & \cap & \square \quad \cap & \\ 0 \arrow[r] & \underbrace{f^{*}T_M}_{L} \arrow[r, "j_i"] & N_{U/M\times M} \arrow[r, "p_{i*}"] & N_{U/M} \arrow[r] & 0 \end{tikzcd}$$

⟿

$$\begin{tikzcd} U \arrow[r, hook, "f"] & M \arrow[r, "\Delta"] \arrow[dr, equal] & M\times M \arrow[d, "p_i"] \\ & & M \end{tikzcd} \Rightarrow I_\Delta/I^2_\Delta \cong \Omega_{p_i}|_M \text{ and } N_{\Delta/M\times M} \cong T_M$$

The maps $N_{U/M\times M} \xrightarrow{p_{i*}} N_{U/M}$ have a common section $s: N_{U/M} \longrightarrow N_{U/M\times M}$

induced by the commutative diagram

$$\begin{tikzcd} U \arrow[r, "f"] \arrow[dr, "\Delta f"'] & M \arrow[d, "\Delta"] \arrow[dl, phantom, "\circlearrowright"] \\ & M\times M \end{tikzcd}$$

$$\rightsquigarrow$$

$$\begin{tikzcd} \Delta^*\mathcal{O}_{M\times M} \arrow[r, phantom, "\cong"] & \mathcal{O}_M \\ \Delta^* I_{\Delta f} \arrow[u, phantom, "\cup"] \arrow[r] & I_{U/M} \arrow[u, phantom, "\cup"] \end{tikzcd} \quad \rightsquigarrow \quad I_{\Delta f}/I^2_{\Delta f} \to I_{U/M}/I^2_{U/M}$$

Locally on U:

$$\begin{tikzcd} & & N_{U/M} & & \\ 0 \arrow[r] & f^*T_M \arrow[r, "j_1"] \arrow[dr] & N_{U/M\times M} \arrow[u, "p_{2*}"] \arrow[r, "p_{1*}"] \arrow[d, phantom, "\cong"] & N_{U/M} \arrow[r] & 0 \\ & & f^*T_M \times_U N_{U/M} \arrow[ur] \arrow[uu, bend right] & & \end{tikzcd}$$

This map is just the action of f^*T_M on $N_{U/M}$

Under the identification

$$\begin{tikzcd} N_{U/M\times M} & \cong & f^*T_M \times_U N_{U/M} \arrow[rr, "p_2"] & & N_{U/M} \\ \cup & \square & \cup & \circlearrowright & \cup \\ C_{U/M\times M} & \cong & f^*T_M \times_U C_{U/M} \arrow[rr, "p_2"] & & C_{U/M} \end{tikzcd}$$

Lecture III (Speaker: Alessio Cela)

Cone stacks

Def $\mathfrak{C} \to X$ an algebraic stack, $0 : X \longrightarrow \mathfrak{C}$ a section.
An $\mathbb{A}^1$-action on $(\mathfrak{C}, 0)$ is a morphism of stacks

$$\gamma : \mathbb{A}^1 \times \mathfrak{C} \longrightarrow \mathfrak{C}$$

which satisfies the expected compatibilities.

Def Let $(\mathfrak{C}, 0, \gamma)$ and $(\mathfrak{D}, 0, \gamma)$ be two X-stacks with section and $\mathbb{A}^1$-action.
Then a $\mathbb{A}^1$-equivariant morphism $\phi : \mathfrak{C} \to \mathfrak{D}$ is a triple $(\phi, \eta_0, \eta_\gamma)$
where ϕ is a morphism of stacks over X, η_0 and η_1 are 2-isomorphisms

$$\begin{tikzcd} X \arrow[r, "0"] \arrow[dr, "0"'] & \mathfrak{C} \arrow[d, "\phi"] \arrow[dl, phantom, "\eta_0" near start] \\ & \mathfrak{D} \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} \mathbb{A}^1 \times \mathfrak{C} \arrow[r, "id \times \phi"] \arrow[d, "\gamma"'] \arrow[dr, phantom, "\eta_\gamma"] & \mathbb{A}^1 \times \mathfrak{D} \arrow[d, "\gamma"] \\ \mathfrak{C} \arrow[r, "\phi"'] & \mathfrak{D} \end{tikzcd}$$

$$\eta_0 : 0 \to \phi \circ 0 \qquad\qquad \eta_\gamma : \phi \circ \gamma \longrightarrow \gamma \circ (id \times \phi)$$

this is a natural isomorphism such that for every object $(T \to X) \in X$ over T the map

$$\begin{tikzcd} 0(T\to X) = (T \to X \to \mathfrak{C}) \arrow[r, "\sim"] \arrow[d] & \phi \circ 0 (T \to X) = (T \to X \xrightarrow{0} \mathfrak{C} \xrightarrow{\Phi} \mathfrak{C}) \arrow[d] \\ T \arrow[r, equal] & T \end{tikzcd}$$

is over 1_T

Def Let $(\phi, \eta_0, \eta_\gamma)$, $(\psi, \eta_0', \eta_\gamma') : \mathfrak{C} \to \mathfrak{D}$ be two equivariant morphisms.
An $\mathbb{A}^1$-equivariant isomorphism $\xi : \phi \to \psi$ is a 2-isomorphism

s.t. the diagrams:

$$\begin{tikzcd} 0 \arrow[r, "\eta_0"] \arrow[dr, "\eta_0'"'] & \phi \circ 0 \arrow[d, "\xi"] \arrow[dl, phantom, "\circlearrowright" near start] \\ & \psi \circ 0 \end{tikzcd} \qquad \begin{tikzcd} \phi \circ \gamma \arrow[r, "\eta_\gamma"] \arrow[d, "\xi \circ \gamma"'] \arrow[dr, phantom, "\circlearrowright"] & \gamma \circ (id \times \phi) \arrow[d] \\ \psi \circ \gamma \arrow[r, "\eta_\gamma'"'] & \gamma \circ (id \times \psi) \end{tikzcd}$$

commute

Basic model of cone stacks

Let C be an E-cone $\longmapsto$ $[C/E]$ defined by :

objects = for $T \to X$ ~~m~~ i.e. $T \in Ob(X)$

$$[C/E](T) = \left\{ (P,f) \text{ where } \begin{tikzcd} P \arrow[r, "f"] \arrow[d] & C \\ T & \end{tikzcd} \right\}$$

E-torsor $\to$ $P \to T$; $\uparrow$ E-equivariant map f

There is a section

$$\begin{tikzcd} & C \arrow[dr] & \\ X \arrow[ur, "0"] \arrow[rr, "0" below] \arrow[r, phantom, "\circlearrowleft"] & & {[C/E]} \end{tikzcd}$$

$$(T \to X) \longmapsto \left(\begin{tikzcd} & X \arrow[dr, "0"] & \\ P = E \times_X T \arrow[d] \arrow[ur] \arrow[rr, "0"] & & C \\ T & & \end{tikzcd} \right)$$

and an $\mathbb{A}^1$-action

$$\gamma : \mathbb{A}^1 \times [C/E] \longrightarrow [C/E]$$

$$\Big(\underbrace{\alpha : T \to \mathbb{A}^1}_{\substack{\updownarrow \\ \alpha \in \mathcal{O}_T(T)}}, \underset{\substack{\cap \\ [C/E](T)}}{(P,f)} \Big) \longmapsto \alpha.(P,f) = (\alpha P, \alpha f) \text{ where}$$

$$\begin{tikzcd} \alpha P = P \times_{E,\alpha} E \arrow[d] \arrow[r, "\alpha f"] & C \\ T & \end{tikzcd}$$

$[p,v] \mapsto \alpha(t) f(p) + dv$

$[p,v] \mapsto t$

$E \circlearrowright P$ gives P the structure of $\mathbb{Z}E$-module

$E \circlearrowright P \otimes_{\mathbb{Z}E_T} E_T =: P \times_{E,\alpha} E$

where ~~$\mathbb{Z}E / \mathbb{A}^1 E$~~ ~~mse~~ E_T has a structure

of $\mathbb{Z}E_T$ module given by $E_T \longrightarrow E_T$

$v \longmapsto \alpha(pr_T(v))\, v$

Topologically : $|P \times_{E,\alpha} E| = P \times E / E$ where $E \circlearrowright P \times E$ by

$e.(p,v) = (e.p, -\alpha(t)e + v)$.

$(p,v) \mapsto t$, $T \ni t$

Observe that $(\alpha f)(e.p, v - \alpha(t)e) = \alpha(t) \underbrace{f(e.p)}_{\substack{\| \\ de + f(p)}} + dv - \alpha(t) de =$

$= \alpha(t) f(p) + dv.$

Rmks

① If $\phi : (E, C) \longrightarrow (F, D)$ is a morphism of vector bundle cones over X then we get an induced $\mathbb{A}^1$-equivariant map

$$\tilde{\phi} : [C/E] \longrightarrow [D/F]$$

topologically: $|P \times_{E,\phi} F| = P \times F/_E$ where $E \curvearrowright P \times F$ by $e.(p, \nu) := (e.p, \nu - d\phi(e))$

$$\begin{tikzcd} \left(\begin{matrix} P \xrightarrow{f} C \\ \downarrow \\ T \end{matrix} \right) \arrow[r, mapsto] & \left(\begin{matrix} P \times_{E,\phi} F \longrightarrow D \\ {[p,\nu]} \mapsto \phi f(p) + d\nu \\ \downarrow \\ T \end{matrix} \right) \end{tikzcd}$$

■

② An homotopy $k : \phi \longrightarrow \psi$ gives arise to an $\mathbb{A}^1$-equivariant 2-isomorphism $\tilde{k} : \tilde{\phi} \longrightarrow \tilde{\psi}$:

$$\begin{tikzcd} E \arrow[r, "d"] \arrow[d, "\phi,\psi"'] & C \arrow[d, "\phi,\psi"] \arrow[dl, "k"'] \\ F \arrow[r, "d"] & D \end{tikzcd} \rightsquigarrow$$

with:
- $kd + \phi = \psi$
- $dk + \phi = \psi$

$$\begin{tikzcd} & {[C/E]} \left(\begin{matrix} P \xrightarrow{f} C \\ \downarrow \\ T \end{matrix} \right) \arrow[dl, "\tilde{\phi}"'] \arrow[dr, "\tilde{\psi}"] \arrow[d, mapsto] \arrow[drr, mapsto] & & \\ {[D/F]} & \left(\begin{matrix} P \times_{E,\phi} F \xrightarrow{[p,\nu] \mapsto \phi f(p) + d\nu} D \\ \downarrow \\ T \end{matrix} \right) \arrow[r, phantom, "\cong"] & {[D/F]} & \left(\begin{matrix} P \times_{E,\psi} F \longrightarrow D \\ \downarrow \\ T \end{matrix} \right) \end{tikzcd}$$

given by:

$$\begin{tikzcd} P \times_{E,\phi} F \arrow[rr] \arrow[d] & & P \times_{E,\psi} F \arrow[d] \\ {[p,\nu]} \arrow[rr, mapsto] \arrow[d, mapsto] & & {[p, \nu - kf(p)]} \arrow[d, mapsto] \\ D \arrow[rr, phantom, "="] & & D \\ \phi f(p) + d\nu & & \psi f(p) + d\nu - dkf(p) \\ & & = \phi(f(p)) + d\nu \end{tikzcd}$$

Lemma 1.6

$\phi, \psi : (E, C) \longrightarrow (F, D)$ morphisms and $\zeta : \tilde{\phi} \longrightarrow \tilde{\psi}$ an $\mathbb{A}^1$-equivariant 2-iso between $\tilde{\phi}, \tilde{\psi} : [C/E] \longrightarrow [D/F]$.

$\Longrightarrow \exists!$ homotopy $k : \phi \longrightarrow \psi$ s.t. $\zeta = \tilde{k}$

proof of $\exists$

We want to construct $k : C \longrightarrow F$.

Given $(T \xrightarrow{c} C) \in Ob(C(T))$, we have

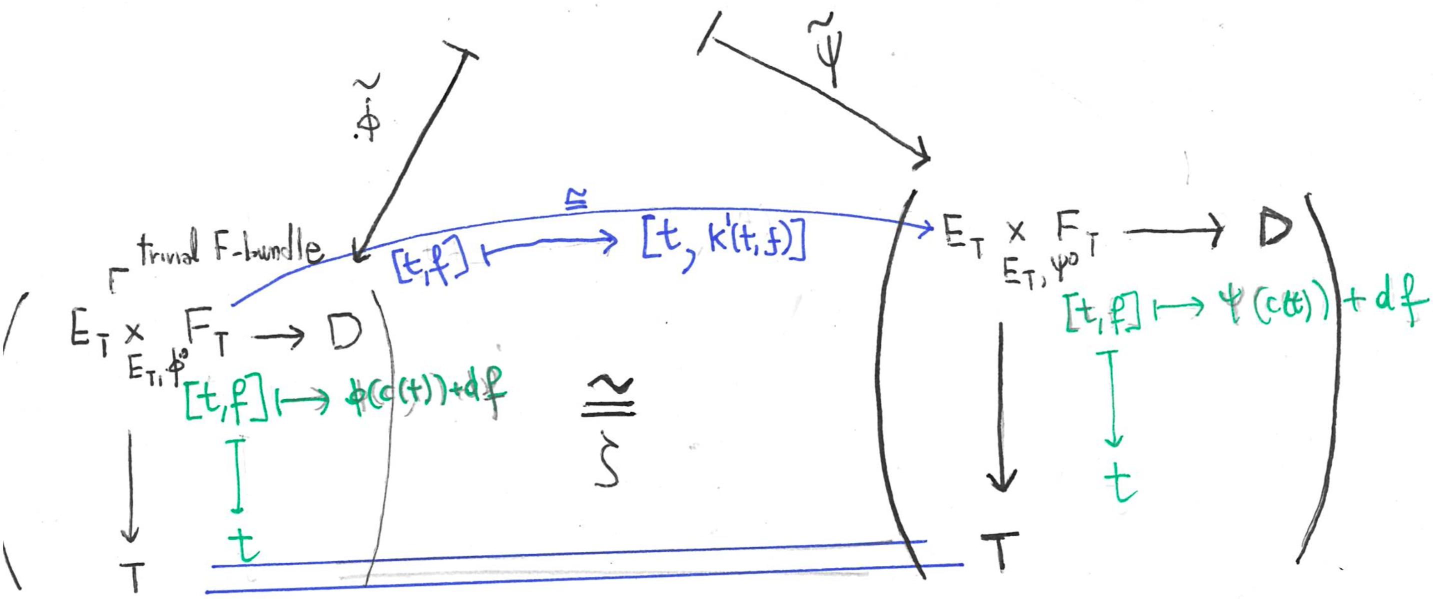
$$\begin{tikzcd} E_T = E \times_{\mathbb{C}} T \arrow[r, "c"] \arrow[d] & C \\ T & \end{tikzcd} \in Ob([C/E](T))$$

Trivial E-bundle

$[e,t] \mapsto de + c(t)$

This is basically obtained by composition:

$$\begin{tikzcd} T \times_C E \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & C \times_x E \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & C \arrow[d] \\ T \arrow[r] & C \arrow[r] & {[C/E]} \end{tikzcd}$$



From ~~$\psi(c(t))+df \neq$~~ $\phi(c(t)) + df = \psi(c(t)) + d\,\underbrace{k'(t,f)}_{\substack{\shortparallel \\ k(t,f)+f}} = \psi(c(t)) + df + dk(t,f)$

$\Rightarrow \phi(c(t)) = \psi(c(t)) + dk(t,f) \Rightarrow k = k(t)$

$k : T \longrightarrow F$ "$k(t) = \phi(c(t)) - \psi(c(t))$".

Prop 1.7

Let C be an E-cone and D an F-cone.

Let $\phi: (E,C) \to (F,D)$ be a morphism.

If the diagram

$$\begin{tikzcd}
E \arrow[r, "d"] \arrow[d, "\phi"'] \arrow[dr, phantom, "\square"] & C \arrow[d, "\phi"] \\
F \arrow[r, "d"'] & D
\end{tikzcd}$$

is cartesian and $F\times C \to D$, $(\mu,\gamma) \mapsto d\mu + \phi\gamma$ is surjective

$\Rightarrow \tilde{\phi}: [C/E] \xrightarrow{\sim} [D/F]$ is an isomorphism of alg. X-stacks with A^1-action.

proof

Step 1 Assume ϕ is an homotopy equivalence

Then we have

$$\begin{tikzcd}
F \arrow[r] \arrow[d, "\psi"'] & D \arrow[d, "\psi"] \arrow[dl, "K_1"'] \arrow[dd, bend left=60, "\sim id"] \\
E \arrow[r] \arrow[d, "\phi"'] \arrow[dd, bend right=0] & C \arrow[d, "\phi"] \arrow[dl, "K_2"'] \arrow[dd, bend left=60, "\sim id"] \\
F \arrow[r] \arrow[d, "\psi"'] & D \arrow[d, "\psi"] \\
E \arrow[r] & C
\end{tikzcd}$$

$\Rightarrow \tilde{K}_1: \tilde{\phi}\tilde{\psi} \xrightarrow{\simeq} id \qquad \tilde{K}_2: \tilde{\psi}\tilde{\phi} \xrightarrow{\simeq} id$ i.e.

$$\begin{tikzcd}
{[D/F]} \arrow[r, "\tilde{\psi}"'] \arrow[rr, bend right=40, "\cong"'] & {[C/E]} \arrow[r, "\tilde{\phi}"'] \arrow[rr, bend right=40, "\cong"'] & {[D/F]} \arrow[r, "\tilde{\psi}"'] & {[C/E]}
\end{tikzcd}$$

$\Rightarrow \tilde{\phi}$ is an isomorphism.

Step 2 Assume ϕ is surjective ~~and flat~~

Then $\tilde{\phi}$ is an iso $\Longleftrightarrow$

$$\begin{tikzcd}
E\times C \arrow[r, "\sigma"] \arrow[d, "pr"'] \arrow[dr, phantom, "\square"] & C \arrow[d] \\
C \arrow[r] & {[D/F]}
\end{tikzcd}$$

is cartesian

$(\Longleftarrow)$: because then we have a cartesian diagram

$$\begin{tikzcd}
C \arrow[r, equal] \arrow[d] & C \arrow[d, "\leftarrow surjective"] \arrow[dl] \\
{[C/E]} \arrow[r] \arrow[ur, phantom, "\square"] & {[D/F]}
\end{tikzcd}$$

$\Rightarrow [C/E] \xrightarrow{\simeq} [D/F]$

Reason: Given

$$\begin{tikzcd}
 & & E\times C \arrow[dr] & \\
Y_c \arrow[urr] \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & Y \arrow[r] \arrow[d] & C \arrow[d] \arrow[dl, "?"'] \\
c \arrow[r] & {[C/E]} \arrow[r] & {[D/F]}
\end{tikzcd}$$

$(\Longleftarrow)$:

$$\begin{tikzcd}
E\times C \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & E \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & C \arrow[d] \\
F\times C \arrow[r] & F \arrow[r] & D
\end{tikzcd}$$

$(\Longrightarrow)$:

$$\begin{tikzcd}
E \arrow[r, "{(id,0)}"] \arrow[d] \arrow[dr, phantom, "\square"] & E\times C \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & C \arrow[d] \\
F \arrow[r, "{(id,0)}"] & F \arrow[r] & D
\end{tikzcd}$$

$\Longleftrightarrow$

$$\begin{tikzcd}
E \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & C \arrow[d] \\
F \arrow[r] & D
\end{tikzcd}$$

$\Longleftrightarrow$

$$\begin{tikzcd}
E\times C \arrow[r, "\sigma"] \arrow[d] \arrow[dr, phantom, "\square"] & C \arrow[d] \\
F\times C \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & D \arrow[d] \\
C \arrow[r] & {[D/F]}
\end{tikzcd}$$

always cartesian because we have

$$\begin{tikzcd}
C\times F \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & D\times F \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & D \arrow[d] \\
C \arrow[r] & D \arrow[r] & {[D/F]}
\end{tikzcd}$$

Step 3 || We can alway factor $\phi = \overset{\text{flat}}{\text{epimorphism}} \circ$ homotopy equivalence

Consider

$$\phi : (E,C) \xrightarrow{\phi_1} (E\times F, C\times F) \xrightarrow{\phi_2} (F,D)$$

Where $\phi_1 : (E,C) \to (E\times F, C\times F)$ ← homotopy equivalence

$$v \longmapsto (v, 0)$$

and $\phi_2 : (E\times F, C\times F) \longrightarrow (F, D)$

$$(\gamma, \mu) \longmapsto d\mu + \phi\gamma$$

Def We call an algebraic stack $(\mathfrak{C}, 0, \gamma)$ over X with section and $\mathbb{A}^1$-action a <u>cone stack over</u> X if, étale locally on X there is a cone C over X and a $\mathbb{A}^1$-equivariant ~~morphism smooth morphism~~ smooth and surj morphism

$$C \xrightarrow[\substack{\text{smooth}\\ +\mathbb{A}^1\text{-equivariant}\\ +\text{surjective}}]{} \mathfrak{C}$$

s.t.

$$\begin{tikzcd} E \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & C \arrow[d, "p"] \\ X \arrow[r, "0"'] & \mathfrak{C} \end{tikzcd}$$

is a vector bundle over X

Rmk || It follows that $[C/E] \xrightarrow{\sim} \mathfrak{C}$ as stacks with $\mathbb{A}^1$-action

<u>proof</u>

consider

$$\begin{tikzcd} C\times E = C \times_{[C/E]} C \arrow[r, shift left, "p_1"] \arrow[r, shift right, "p_2"'] & C \arrow[r, "\pi"] \arrow[dr, "p"'] & {[C/E]} \arrow[d, dashed] \\ & & \mathfrak{C} \end{tikzcd}$$

$p \circ p_1 = p \circ p_2$ being p E-equivariant

$\rightarrow \exists\, h : [C/E] \longrightarrow \mathfrak{C}$. Moreover we have

$$\begin{tikzcd} C \arrow[r, equal] \arrow[d] \arrow[dr, phantom, "\square"] & C \arrow[d, "\text{smooth}+\text{surjective}"] \\ {[C/E]} \arrow[r] & \mathfrak{C} \end{tikzcd}$$

being a closed/open embr can be checked after a ~~[illegible]~~ faithfully flat base change

$\overset{!}{\Rightarrow} [C/E] \xrightarrow{\sim} \mathfrak{C}$.

Def Let $\mathfrak{C}, \mathfrak{D}$ be cone stacks over X. A morphism of cone stacks

$$\phi : \mathfrak{C} \to \mathfrak{D}$$

is an $\mathbb{A}^1$-equivariant morphism of algebraic X-stacks.

Def A 2-isomorphism of cone stacks is an $\mathbb{A}^1$-equivariant 2-isomorphism.

Rmks

1) $\phi : \mathfrak{C} \to \mathfrak{D}$ morphism of cone stacks, then locally on X

$$\phi : \mathfrak{C} = [C/E] \longrightarrow \mathfrak{D} = [D/F]$$

??

comes from

$$\begin{tikzcd} E \arrow[r] \arrow[d, "\phi"'] & C \arrow[d, "\phi"] \\ F \arrow[r] & D \end{tikzcd}$$

■

2) A 2-iso $\zeta : \phi \to \psi$ of cone stacks, where $\phi, \psi : \mathfrak{C} \to \mathfrak{D}$ is locally over X:

$$\mathfrak{C} = [C/E], \ \mathfrak{D} = [D/F]$$

$$\phi, \psi : (E, C) \longrightarrow (D, F)$$

and $\zeta : \tilde{\phi} \xrightarrow{\sim} \tilde{\psi}$

↑ This comes from an homotopy between ϕ and ψ by Lemma 1.6.

■

3) Let

$$\begin{tikzcd} & C_2 \arrow[d] \\ C_1 \arrow[r] & \mathfrak{C} \end{tikzcd}$$

be local presentations of $\mathfrak{C}$ $\Rightarrow$ $C_1 \times_{\mathfrak{C}} C_2 \longrightarrow \mathfrak{C}$ is again a local presentation of $\mathfrak{C}$

proof

$$\begin{tikzcd} C_1 \times_{\mathfrak{C}} C_2 \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & C_2 \arrow[d] \\ C_1 \arrow[r] & \mathfrak{C} \end{tikzcd}$$

Then we have

$$\begin{tikzcd} E_2 \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & C_1 \times_{\mathfrak{C}} C_2 \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & C_2 \arrow[d] \\ X \arrow[r, "0"'] & C_1 \arrow[r] & \mathfrak{C} \end{tikzcd}$$

vector bundle ⊣

smooth, surjective + $\mathbb{A}^1$-equivariant

↖ cone over X

Prop 1.4.

$\Rightarrow$ $C_1 \times_{\mathfrak{C}} C_2$ is an E_2-cone over X

$\Rightarrow C_1 \times_{\mathfrak{C}} C_2$ is an $E_1 \times_{\mathfrak{C}} E_2$-cone over X and

$$\begin{tikzcd} E_1 \times_X E_2 \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & C_1 \times_{\mathfrak{C}} C_2 \arrow[d] \\ X \arrow[r, "0"] & \mathfrak{C} \end{tikzcd}$$

■

Def A cone stack $\mathfrak{C}$ over X is called abelian / vector bundle stack if locally on X $\exists$ presentation

$$A(C) = C \longrightarrow \mathfrak{C} \quad / \quad \overset{}{E} \longrightarrow \mathfrak{C}$$

E: ↑ vector bundle over X

Rmk $\mathfrak{C}$ abelian cone stack / vector bundle stack

$\Rightarrow$ for any presentation $C \longrightarrow \mathfrak{C}$ C is an abelian cone / vector bundle on X

proof

- For vector bundles: consider

$$\begin{tikzcd}
C \times_{\mathfrak{C}} E \arrow[r, "\text{smooth + surj}"] \arrow[d, "\text{smooth}"'] \arrow[dr, phantom, "\square"] & C \arrow[d] \arrow[ddl] \\
E \arrow[r, "\text{smooth}"', "\text{+ surj}"'] \arrow[d, "\text{smooth}"'] & \mathfrak{C} \\
X &
\end{tikzcd}$$

← this is smooth by lemma 34.19 in the stack proj

⇓

$C \to X$ is a vector bundle.

- For abelian cones: consider

$$\begin{tikzcd}
E_A \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] \arrow[dr, phantom, "\ulcorner \text{ vector bundle}", near start] & C \times_{\mathfrak{C}} A \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & A \arrow[d] \\
X \arrow[r, "0"'] & C \arrow[r, "\text{smooth + surj}"'] & \mathfrak{C}
\end{tikzcd}$$

Then by prop 1.4

C is abelian cone over $X \iff C \times_{\mathfrak{C}} A$ is an abelian cone over X

$\iff$ A is an abelian cone over X

↑ symmetry

∎

Prop 1.10

Every cone stack is a closed subcone of an abelian cone stack, called abelian hull.

proof

Glue the ~~stack~~ abelian stacks coming from the local presentations

∎

Def

Let $\mathbb{E}$ be a vector bundle stack and $\mathbb{E} \to \mathfrak{C}$ a morphism of cone stacks. We say that $\mathfrak{C}$ is an $\mathbb{E}$-cone stack if $\mathbb{E} \to \mathfrak{C}$ is locally isomorphic to

$$\mathbb{E} = \left[E_1 / E_0 \right] \to \left[C/F \right] = \mathfrak{C}$$

coming from ~~the~~ a commutative diagram

$$\begin{tikzcd}
E_0 \arrow[r] \arrow[d] & F \arrow[d] \\
E_1 \arrow[r] & C
\end{tikzcd}$$

C ↑ both a E_1- and F- cone

In this case there is a natural morphism, called action of $\mathbb{E}$ on $\mathfrak{C}$,

$$\mathbb{E} \times \mathfrak{C} \xrightarrow{\sigma} \mathfrak{C}$$

given by

$$E_1 \times C \longrightarrow C$$

Def. Let $\mathbb{E} \to \mathfrak{C} \to \mathfrak{D}$ be a sequence of morphisms of cone stacks When $\mathfrak{C}$ is an $\mathbb{E}$-cone stack. If

1) $\mathfrak{C} \to \mathfrak{D}$ is smooth + surj

2)
$$\begin{tikzcd}
\mathbb{E} \times \mathfrak{C} \arrow[r, "\sigma"] \arrow[d, "\text{proj}"'] \arrow[dr, phantom, "\square"] & \mathfrak{C} \arrow[d] \\
\mathfrak{C} \arrow[r] & \mathfrak{D}
\end{tikzcd}$$
is cartesian

we call $0 \to \mathbb{E} \to \mathfrak{C} \to \mathfrak{D} \to 0$ exact sequence

Prop 1.13

[The sequence $\mathbb{E} \to \mathbb{C} \to \mathbb{D}$ is exact] $\Longleftrightarrow$ [$\exists$ commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & E_0 \arrow[r] \arrow[d] & F \arrow[r] \arrow[d] & G \arrow[r] \arrow[d] & 0 & \text{exact sequence of vector bundles} \\
0 \arrow[r] & E_1 \arrow[r] & C \arrow[r] & D \arrow[r] & 0 & \text{exact sequence of cones}
\end{tikzcd}$$

s.t.
$$\begin{tikzcd}
\mathbb{E} \arrow[r] \arrow[d, phantom, sloped, "\cong"] & \mathbb{C} \arrow[r] \arrow[d, phantom, sloped, "\cong"] & \mathbb{D} \arrow[d, phantom, sloped, "\cong"] \\
{[E_1/E_0]} \arrow[r] & {[C/F]} \arrow[r] & {[D/G]}
\end{tikzcd}$$
]

proof

($\Longleftarrow$): • $\mathbb{C}$ is an $\mathbb{E}$-cone stack: C is both an E_1- and F-cones

$\uparrow$ by def of exact sequence of cones

$\nwarrow$ of course: otherwise $[C/F]$ ~~does~~ doesn't make any sense.

•
$$\begin{tikzcd}
\mathbb{E} \times_X \mathbb{C} \arrow[r] \arrow[d] \arrow[dr, phantom, "\boxed{?}"] & \mathbb{C} \arrow[d] \\
\mathbb{C} \arrow[r] & \mathbb{D}
\end{tikzcd}$$

: locally on X $F = E_0 \times_X G$ and $C = E_1 \times_X D$ so

$$\mathbb{C} \cong [C/F] = [E_1/E_0] \times_X [D/G]$$

and
$$\begin{tikzcd}
\mathbb{E} \times_X \mathbb{C} \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \mathbb{E} \times_X \mathbb{D} \arrow[d] \\
\mathbb{C} \arrow[r] & \mathbb{D}
\end{tikzcd}$$
is clearly cartesian.

Finally cartesianity of diagrams can be checked étale locally.

$\Rightarrow$): I didn't check this, also because I didn't understand why locally $\mathbb{C} \to \mathbb{D}$ comes from $(E, C) \xrightarrow{\phi} (F, D)$.

Appendix to Lecture III : Deformation theory of quotients (Speaker: Woonam Lim)

X projective scheme, $E \in \mathrm{Coh}(X)$

Then $\mathrm{Quot}_X(E)$ represents the functor:

pull back of E to $T\times X$, $E_T := E \otimes_{\mathcal{O}_X} \mathcal{O}_{X\times T} = E \otimes_k \mathcal{O}_T$

$$\mathrm{Sch}/k \longrightarrow \mathrm{Sets}$$

$$T \longmapsto \left\{ \begin{array}{c} \text{equivalent classes of quotients} \\ E_T \to Q \to 0 \quad \text{exact over } T\times X \end{array} \right\}$$

with Q flat over T

We say that $(E_T \to Q \to 0) \sim (E_T \to Q' \to 0)$ if $\exists$ commutative diagram

$$\begin{tikzcd} E_T \arrow[r] \arrow[d, equal] & Q \arrow[r] \arrow[d, "\simeq"] & 0 \\ E_T \arrow[r] & Q' \arrow[r] & 0 \end{tikzcd}$$

Goal: $p \in \mathrm{Quot}_X(E)$ we want to compute

$\quad\, \|$

$[\,0 \to S_0 \to E \to Q_0 \to 0\,]$

$T_p \mathrm{Quot}_X(E) = \mathrm{Hom}_{\mathcal{O}_X}(S_0, Q_0)$

$\mathrm{Obs}_p \mathrm{Quot}_X(E) = \mathrm{Ext}^1_{\mathcal{O}_X}(S_0, Q_0)$

Notation: $A_n = \mathrm{Spec}\, k[t]/(t^{n+1})$

Tangent space

Claim: there is a canonical identification

$$T_p \mathrm{Quot}_X(E) = \left\{ \begin{array}{l} 0 \to S_1 \to E_{A_1} \to Q_1 \to 0 \quad ⊛ \\ \text{exact on } X\times A_1 \\ \quad \text{i) } Q_1 \text{ flat over } A_1; \\ \quad \text{ii) } ⊛ \text{ restricts to } p \end{array} \right\} \xleftrightarrow{1:1} \mathrm{Hom}_{\mathcal{O}_X}(S, Q)$$

proof

($\mapsto$): consider the exact diagram over $X \times A_1$:

exactness here is due to the fact that $E_{A_1} = E \oplus Et$

exactness here is a completely formal: it follows from all the other arrows are exact.

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & S_0 \arrow[r] \arrow[dd, "xt"] & E \arrow[r] \arrow[d, "xt"] & Q_0 \arrow[r] \arrow[dd] & 0 \\
 & & E \oplus tE \arrow[d, equal] & & \\
0 \arrow[r] & S_1 \arrow[r] \arrow[d] \arrow[uur, "\varphi"] & E_{A_1} \arrow[r] \arrow[d] & Q_1 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & S_0 \arrow[r] \arrow[d] \arrow[uuurr, dashed, bend left] & E \arrow[r] \arrow[d] & Q_0 \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

$y \longmapsto \varphi(x) = \bar{y}$

$x + ty \mapsto x$ $\in$

well-defined up to S_0

injectivity here uses the fact that Q_1 is flat over A_1

what do you need it for ?

of sheaves over X. Here if $X = \mathrm{Spec}(R)$ if we consider $0 \to R \xrightarrow{\cdot t} R[t]/(t^2) \xrightarrow{t=0} R \to 0$ and take ~~$- \otimes_{k[t]/(t^2)} Q_1$~~ $- \otimes_{R[t]/t^2} Q_1$ we get $0 \to R \otimes_{k[t]/(t^2)} Q_1 \to Q_1 \to R \otimes_{k[t]/(t^2)} Q_1 \to 0$

No! $\to$ $R \otimes_{R[t]/(t^2)} Q_1 = Q_0$, $R \otimes_{k[t]/(t^2)} Q_1 = Q_0$

We want to define φ. Define φ as in green above. Since y is well-defined up to S_0, we get a well defined morphism

$$(\varphi : S_0 \longrightarrow Q_0) \in \mathrm{Hom}_{\mathcal{O}_X}(S_0, Q_0).$$

($\Leftarrow$): Suppose given $\varphi \in \mathrm{Hom}_{\mathcal{O}_X}(S_0, Q_0)$. Define

$$S_1 := \left\{ x + ty \;\middle|\; \begin{array}{l} x \in S_0 \\ y \in E \\ y \longmapsto \bar{y} = \varphi(x) \\ E \to Q_0 \end{array} \right\}$$

and then $Q_1 = E_{A_1} / S_1$.

Then check ~~that~~ all the properties.

∎

Obstruction class

Now suppose given

$$p = [0 \to S_0 \to E \to Q_0 \to 0]$$

and $\varphi_1 = [0 \to S_1 \to E_{A_1} \to Q_1 \to 0]$

Q) When does there exist a further deformation over $A_2 = \operatorname{Spec} \mathbb{C}[t]/(t^3)$?

Claim ∃ obstruction class $o(\varphi_1) \in \operatorname{Ext}^1(S_0, Q_0)$ s.t.

$$[\exists \text{ further def of } \varphi] \Leftrightarrow [o(\varphi_1) = 0]$$

Furthermore, in this case all further deformations form a torsor over $\operatorname{Hom}_{\mathcal{O}_X}(S_0, Q_0)$

proof of the claim

Suppose you have over X ~~[illegible]~~

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & S_0 \arrow[r] \arrow[d, "xt^2"] & E \arrow[r] \arrow[d, "xt^2"] & Q_0 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & S_2 \arrow[r] \arrow[d] \arrow[ru, "z"] & E_{A_2} \arrow[r] \arrow[d] & Q_2 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & S_1 \arrow[r] \arrow[d] \arrow[rruu, dashed, "\varphi_2"] & E_{A_1} \arrow[r] \arrow[d] & Q_1 \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

$z \mapsto \bar{z}$

$E_{A_1} \oplus t^2 E = E_{A_2}$

$x + ty + t^2 z \mapsto x + ty$

← uses that Q_2 is flat over A_2

exact because Q_2 is flat over $k[t]/(t^3)$

◁ Explanation: if $X = \operatorname{Spec}(R)$

Here: from $0 \to R \xrightarrow{\cdot t^2} R[t]/(t^3) \to R[t]/(t^2) \to$ take the tensor $- \otimes_{R[t]/(t^3)} Q_2$ we get

$$\begin{tikzcd}
 & 0 & 0 & 0 & \\
0 \arrow[r] & Q_0 \arrow[r] \arrow[u] & Q_2 \arrow[r] \arrow[u] & Q_1 \arrow[r] \arrow[u] & 0 \\
0 \arrow[r] & E \arrow[r] \arrow[u] & E_{A_2} \arrow[r] \arrow[u] & E_{A_1} \arrow[r] \arrow[u] & 0 \\
0 \arrow[r] & S_0 \arrow[r] \arrow[u] & S_1 \arrow[r] \arrow[u] & S_2 \arrow[r] \arrow[u] & 0 \\
 & 0 \arrow[u] & 0 \arrow[u] & 0 \arrow[u] &
\end{tikzcd}$$

exactness here is totally formal: it follows from the exactness of the others

exactness here is due to the fact that $E_{A_2} = E \otimes_R R[t]/(t^3) = E[t]/($

$= E \oplus Et \oplus Et^2$

So as before

$$\{\text{further extensions of } \varphi_1\} \xleftrightarrow{1:1} \operatorname{Hom}_{\mathcal{O}_X}(S_1, Q_0)$$

↯

Now consider again the exact sequence on $X \times A_1$

$$0 \to S_0 \to S_1 \to S_0 \to 0$$

$$\begin{tikzcd}[column sep=small]
 & \mathrm{Hom}_{\mathcal{O}_X}(S_0, Q_0) \arrow[d, equal] & & \mathrm{Hom}_{\mathcal{O}_X}(S_0, Q_0) \arrow[d, equal] & \mathrm{Hom}_{\mathcal{O}_X}(S_0, Q_0) \arrow[d, equal] \\
\overset{\Longrightarrow}{\mathrm{Hom}_{\mathcal{O}_{X\times A_1}}(-, Q_0)} & \mathrm{Hom}_{\mathcal{O}_{X\times A_1}}(S_0, Q_0) \arrow[r] & \mathrm{Hom}_{\mathcal{O}_{X\times A_1}}(S_1, Q_0) \arrow[r] \arrow[d, phantom, sloped, "\in"] & \mathrm{Hom}_{\mathcal{O}_{X\times A_1}}(S_0, Q_0) \arrow[r] \arrow[d, phantom, sloped, "\in"] & \mathrm{Ext}^1_{\mathcal{O}_{X\times A_1}}(S_0, Q_0) \arrow[d, phantom, sloped, "\in"] \\
 & & \varphi_2 \arrow[r, mapsto] & \varphi_1 \arrow[r, mapsto] & o(\varphi_1)
\end{tikzcd}$$

$\Downarrow$

this Explains what it means that

$\mathrm{Hom}_{\mathcal{O}_X}(S_0, Q_0) \curvearrowright \{\text{further extension of } \varphi_1\} = \mathrm{Hom}_{\mathcal{O}_X}(S_1, Q_0)$

is a $\mathrm{Hom}_{\mathcal{O}_X}(S_0, Q_0)$ torsor. and why φ_2 exists $\Leftrightarrow o(\varphi_1) = 0$

∎

<u>Example</u>

Consider $E = \mathcal{O}_X \Rightarrow \mathrm{Quot}_X(\mathcal{O}_X) = \mathrm{Hilb}(X)$ and so we have for $Z \subset X$ closed emb

$$T_{[Z]} \mathrm{Hilb}(X) = \mathrm{Hom}_{\mathcal{O}_X}(I_Z, i_* \mathcal{O}_Z) \underset{\substack{\uparrow \\ \text{adjunction}}}{\cong} \mathrm{Hom}_{\mathcal{O}_Z}(I_Z/I_Z^2, \mathcal{O}_Z) = \cancel{N_{Z/X}}$$

$$= \Gamma(Z, \underbrace{\mathcal{H}om_{\mathcal{O}_Z}(I_Z/I_Z^2, \mathcal{O}_Z)}_{\substack{\| \\ N_{Z/X}}}) = \Gamma(Z, N_{Z/X}).$$

∎

Lecture IV (Speaker: Miguel Moreira)

h^1/h^0 -stacks

X topos (X DM-stack with Zariski / Étale / fppf topology)

$$E^\bullet = [E^0 \xrightarrow{d} E^1] \rightsquigarrow h^1/h^0(E^\bullet) := [E^1/E^0]$$

$\uparrow\nearrow$ abelian sheaves

Special case: Suppose $E^\bullet = \left[A \oplus E \xrightarrow{\begin{bmatrix} id & 0 \\ 0 & 0 \end{bmatrix}} A \oplus C\right]$. Then $h^1/h^0(E^\bullet) = \left[\overset{\text{trivial action}}{\overset{\downarrow}{C}} / E\right] =$

$$\begin{matrix} \| \\ C \times BE \end{matrix}$$

$$= \left[h^1(E^\bullet) / h^0(E^\bullet) \right] \blacksquare$$

Goal: $h^1/h^0(E^\bullet)$ is really defined for $E^\bullet \in D(\mathcal{O}_X)$

Functoriality: Given

$$\begin{tikzcd} E^0 \arrow[r, "\varphi^0"] \arrow[d, "d_E"'] & F^0 \arrow[d, "d_F"] \\ E^1 \arrow[r, "\varphi^0"'] & F^1 \end{tikzcd} \rightsquigarrow h^1/h^0(E^\bullet) \xrightarrow{h^1/h^0(\varphi)} h^1/h^0(F^\bullet)$$

An homotopy $\varphi \overset{k}{\simeq} \psi$ from $\varphi, \psi : E^\bullet \to F^\bullet$ is $k : E^1 \to F^0$ s.t. $k d_E = \psi^0 - \varphi^0$, $d_F k = \psi^1 - \varphi$

Given $\varphi \underset{k}{\simeq} \psi$ we get an induced isomorphism

$$h^1/h^0(\varphi) \cong h^1/h^0(\psi)$$

We obtained ~~a~~ functors

$$\begin{tikzcd} \text{complexes in } 0 \to 1 \arrow[r] & C^{[0,1]}(X) \arrow[r, "h^1/h^0"] \arrow[d] & X\text{-PicStack} \\ \text{complexes up to homotopy} \arrow[r] & K^{[0,1]}(X) \arrow[ur] \arrow[d] & \\ \text{complexes in } D(X) \text{ s.t. } h^i = 0 \text{ unless } i = 0,1 \arrow[r] & D^{[0,1]}(X) \arrow[uur, dashed, bend right, "\leftarrow \text{we want this}"'] & \end{tikzcd}$$

Proposition

> If $\varphi : E^\bullet \longrightarrow F^\bullet$ is a quasi isomorphism
> $\Rightarrow \quad h^1/_{h^0}(E^\bullet) \xrightarrow[h^1/_{h^0}(\varphi)]{} h^1/_{h^0}(F^\bullet)$ is an isomorphism

proof

Consider the factorization:

$$\begin{tikzcd}
E^0 \arrow[r, "id\oplus 0"] \arrow[d, "d_E"] \arrow[rr, bend left, "\varphi^0"] & E^0\oplus F^0 \arrow[r, "\varphi^0+id"] \arrow[d, "d_E\oplus id"] & F^0 \arrow[d, "d_F"] \\
E^1 \arrow[r, "id\oplus 0"] \arrow[rr, bend right, "\varphi^1"'] & E^1\oplus F^0 \arrow[r, "\varphi^1+d_F"] & F^1
\end{tikzcd}$$

$\underbrace{\qquad\qquad}_{①}\underbrace{\qquad\qquad}_{②}$

This is surjective because $h^1(E^\cdot) \twoheadrightarrow h^1(F^\cdot)$ is surjective.

In ①: we have homotopy equivalence because we have the projections

$E^0\oplus F^0 \longrightarrow E^0$

$E^1\oplus F^0 \longrightarrow E^1$

and $k : E^1\oplus F^0 \xrightarrow[0\oplus id]{k} E^0\oplus F^0$

$$\begin{tikzcd}
E^0\oplus F^0 \arrow[r] \arrow[d, "d_E\oplus id"'] \arrow[rr, bend left, "(e_0{,}f)\mapsto(e_0{,}0)"] & E^0 \arrow[r] & E^0\oplus F^0 \arrow[d, "d_E\oplus id"] \\
E^1\oplus F^0 \arrow[r] \arrow[rru] \arrow[rr, bend right, "(e{,}f)\mapsto(e_1{,}0)"'] & E^1 \arrow[r] & E^1\oplus F^0
\end{tikzcd}$$

~~want~~

- $(e_0,f)-(e,0) \overset{?}{=} k(d_E\oplus id)\,(e,f$
  $= (0,f)$ ✓
- $(d_E\oplus id)\,k\,(e_1,f) \overset{?}{=} (e_1,f)-(e_1,0$
  ✓

$\Rightarrow$ ok by what we ~~say~~ said before

In ②: Assume that φ is epimorphism $\Rightarrow$ $E^1 \to F^1 \to [F^1/F^0]$ is epimorphism and

$h^1/_{h^0}(\varphi)$ is iso $\Leftrightarrow$

$$\begin{tikzcd}
E^0\times E^1 \arrow[r] \arrow[d] \arrow[rd, phantom, "\square"] & E^1 \arrow[d] \\
E^1 \arrow[r] & {[F^1/F^0]}
\end{tikzcd}$$

is cartesian

$\uparrow$ See the proof of prop 1.7

$\Leftrightarrow$

$$\begin{tikzcd}
E^0\times E^1 \arrow[r] \arrow[d] & E^1 \arrow[d] \\
F^0\times E^1 \arrow[r] \arrow[d] \arrow[rd, phantom, "\square"] & F^1 \arrow[d] \\
E^1 \arrow[r] & {[F^1/F^0]}
\end{tikzcd}$$

always cartesian

is cartesian $\Leftrightarrow$

$$\begin{tikzcd}
0 \arrow[r] & h^0(E) \arrow[r] \arrow[d, "\simeq"] & E^0 \arrow[r] \arrow[d] \arrow[rd, phantom, "\square"] & E^1 \arrow[r] \arrow[d] & h^1(E) \arrow[r] \arrow[d, "\simeq"] & 0 \\
0 \arrow[r] & h^0(F) \arrow[r] & F^0 \arrow[r] & F^1 \arrow[r] & h^1(F) \arrow[r] & 0
\end{tikzcd}$$

cartesian

$\uparrow$

given $f^0\in F^0$ and $e^1\in E^1$ s.t. $f^0 \mapsto f^1 \in F^1$, $e^1 \mapsto f^1$

$\Rightarrow \exists!\, e^0 \in E^0$ s.t. $e^0_0 \mapsto e^1$, $e^0_0 \mapsto f^0$

!): ie e^0 and $e^{0\prime}$ then $e^0-e^{0\prime} \mapsto 0$
i.e. $e^0-e^{0\prime} \in h^0(E) = h^0(F)$ and in F^0 is again 0 $\Rightarrow e^0-e^{0\prime}=0$.

$\exists$): in F^1 $e^1-f^0=0 \Rightarrow$ in $h^1(F)=h^1(E$
$0 = e^1-f^0 = e^1 \Rightarrow \exists\, e^0\in E^0$ s.t.
$e^0\mapsto e^1$. ∎

+

Derived Hom

$M^\bullet, L^\bullet \in D(X)$

Goal ‖ Define $R\mathrm{Hom}(M^\bullet, L^\bullet) \in D(X)$

We can do this in 2 ways:

1) $\mathrm{Hom}(M, -)$ left exact

2) $\mathrm{Hom}(-, L)$ right exact

Define $R\mathrm{Hom}^{inj}(M^\bullet, L^\bullet) := \mathrm{Hom}^\bullet(M^\bullet, I^\bullet)$ where $I^\bullet \overset{\sim}{\underset{iso}{\leftrightarrow}} L^\bullet$ ⌐ This always exists

↑ by injective objects

$R\mathrm{Hom}^{proj}(M^\bullet, L^\bullet) = \mathrm{Hom}^\bullet(P^\bullet, L^\bullet)$ when $\exists\, P^\bullet \xrightarrow{\sim} M^\bullet$ ⌐ by proj objects

$\mathrm{Hom}^i(P^\bullet, L^\bullet) = \bigoplus_j \mathrm{Hom}(P^j, L^{i+j})$

$$\begin{tikzcd}
\mathrm{Hom}(P_0, L_0) \arrow[r] \arrow[d] & \mathrm{Hom}(P_0, L_1) \\
\mathrm{Hom}(P_{-1}, L_0) &
\end{tikzcd}$$

$\rightsquigarrow \mathrm{Hom}^i(P^\bullet, L^\bullet) \to \mathrm{Hom}^{i+1}(P^\bullet, L^\bullet)$

Fact. ‖ $\mathrm{Hom}^\bullet(M^\bullet, I^\bullet) \cong \mathrm{Hom}^\bullet(P^\bullet, I^\bullet)$

and $\mathrm{Hom}^\bullet(P^\bullet, L^\bullet) = \mathrm{Hom}^\bullet(P^\bullet, I)$

$\Big\} \Rightarrow R\mathrm{Hom}^{inj}(M^\bullet, L^\bullet) = R\mathrm{Hom}^{proj}(M^\bullet, L^\bullet)$

Example $R\mathrm{Hom}(-, \mathcal{O}_X)$

⚠ ‖ $\mathcal{O}_X$ is not injective

Example $D \subset X$ Cartier divisor. Then we have

$$0 \to \mathcal{O}_X(-D) \to \mathcal{O}_X \to i_* \mathcal{O}_D \to 0 \qquad (\star)$$

Apply $\mathrm{Hom}_{\mathcal{O}_X}(-, \mathcal{O}_X)$ and we get

This is exactly $(\star)$ after $\otimes_{\mathcal{O}_X} \mathcal{O}_X(D)$

$$0 \to \underbrace{i_* \mathcal{O}_D^\vee}_{\substack{\| \\ 0}} \to \mathcal{O}_X \to \mathcal{O}_X(D) \to \underbrace{i_* \mathcal{O}_D(D)}_{\substack{\| \\ \mathrm{Ext}^1(\mathcal{O}_D, \mathcal{O}_X)}} \to 0 \qquad (\star\star)$$

Now Recall that I injective ~~Means~~ $\Leftrightarrow$ $[\, 0 \to A \to B \to C \to 0 \Rightarrow 0 \to \mathrm{Hom}(C, I) \to \mathrm{Hom}(B$

$\Leftrightarrow \mathrm{Ext}^1(A, I) = 0 \quad \forall A$

In other words ⊛ says

$$\begin{tikzcd}
0 \arrow[d] & & & \\
\mathcal{O}_X(-D) \arrow[d] & & 0 \arrow[d] & -1 \\
\mathcal{O}_X \arrow[d] \arrow[rr, "\sim"] & & i_*\mathcal{O}_D \arrow[d] & 0 \\
0 & & 0 &
\end{tikzcd}$$

$\uparrow$
proj resolution

$$\begin{tikzcd}
 & 0 & 0 \\
\mathcal{O}_X(-D) \arrow[d] & \mathcal{H}om(\mathcal{O}_X(-D),\mathcal{O}_X) \arrow[u] \arrow[r] & 0 \\
\mathcal{O}_X & \mathcal{O}_X \arrow[u] \arrow[r] & 0 \\
 & \mathcal{O}_X &
\end{tikzcd}$$

Now $R\mathcal{H}om(i_*\mathcal{O}_D,\mathcal{O}_X) \neq i_*\mathcal{O}_D^\vee$, but instead

$\mathcal{H}om^\bullet([\mathcal{O}_X(-D)\to\mathcal{O}_X],\mathcal{O}_X) = [\mathcal{O}_X \longrightarrow \mathcal{O}_X(D)]$

$$E_\bullet = R\mathcal{H}om(\mathcal{O}_D,\mathcal{O}_X) = [\underset{\substack{\uparrow \\ 0}}{\mathcal{O}_X} \longrightarrow \underset{\substack{\uparrow \\ 1}}{\mathcal{O}_X(D)}]$$

While ⊛⊛ is saying: $h^0(E_\bullet) = i_*\mathcal{O}_D^\vee$ and $h^1(E_\bullet) = i_*\mathcal{O}_D(D)$

h^1/h^0 stacks

Last time: $E_\bullet = [E_0 \to E_1]$ 2 term complex

$$h^1/h^0(E_\bullet) := [E_1/E_0]$$

Prop $\varphi: E_\bullet \longrightarrow F_\bullet$ quasi-isomorphic

$\Rightarrow h^1/h^0(E_\bullet) \longrightarrow h^1/h^0(F_\bullet)$ is an isomorphism of $\mathbb{A}^1$-equivariant stacks.

Def $E_\bullet \in D(\mathcal{O}_X)$

$$h^1/h^0(E_\bullet) := h^1/h^0(\tau_{[0,1]}E_\bullet)$$

$\uparrow$

Given

$$\begin{tikzcd}
{[\ \ldots} \arrow[r] & E_{i-1} \arrow[r, "d_{i-1}"] \arrow[d] & E_i \arrow[r] \arrow[d] & E_{i+1} \arrow[r] \arrow[d] & {\ldots\ ]} \\
{\tau_{[i,\infty]}E_\bullet = [\ \ldots} \arrow[r] & 0 \arrow[r] & E_i/\mathrm{Im}(d_{i-1}) \arrow[r] & E_{i+1} \arrow[r] & {\ldots\ ]}
\end{tikzcd}$$

$E_\bullet \to \tau_{[i,\infty]}E_\bullet$ induces isomorphism $h^j(E_\bullet) \to h^j(\tau_{[i,\infty]}E_\bullet)$ $\forall j \geq i$.

And in our case we get

$$\begin{tikzcd} E_\bullet \arrow[r] & \tau_{[0,\infty]} E_\bullet \\ & \tau_{[0,1]} E_\bullet \arrow[u] \end{tikzcd}$$

induces iso on h^j for $j \geq 0$

← induces iso on h^0 and h^1

$$\tau_{[0,1]} E_\bullet = \left[\ldots \to 0 \to \operatorname{coker}(d_0) \to \ker(d_1) \to 0 \to \ldots \right]$$

This way we defined

$$D(\mathcal{O}_X) \longrightarrow \text{Picard Stacks over } X$$

$$E_\bullet \longmapsto h^1/h^0 \left(\tau_{[0,1]} E_\bullet\right) = h^1/h^0 \left(E_\bullet\right)$$

Application to DM stacks

$X =$ DM stack

$M^\bullet \in D(\mathcal{O}_{X_{\text{ét}}})$

We want to define $c(M^\bullet)$ a Picard Stack over X

Example $M^\bullet = \mathbb{L}^\bullet_X$ or $M^\bullet = E^\bullet$ obstruction theory

Step 1 || Pull back to flat topology

We have

$$v : X_{fl} \longrightarrow X_{\text{ét}}$$

A wrong but correct way to think about it is the following: v is the identity but on the domain the topology is finer than that on the target.

$$\rightsquigarrow \quad v^* : \operatorname{Mod}(\mathcal{O}_{X_{\text{ét}}}) \to \operatorname{Mod}(\mathcal{O}_{X_{fl}})$$

right-exact

$$M \longmapsto \mathcal{O}_{X_{fl}} \otimes_{\mathcal{O}_{X_{\text{ét}}}} v^{-1} M$$

This is defined similarly to the ~~pullback~~ inverse image:

$f : X \to Y$, $\mathcal{G}$ sheaf on Y

$$\Rightarrow f^{-1}\mathcal{G}(U) := \varinjlim_{V \supset f(U)} \mathcal{G}(V)$$

In particular for example you have in this way

$$\operatorname{Mod}(\mathcal{O}_{X_{Zar}}) \to \operatorname{Mod}(\mathcal{O}_{X_{\text{ét}}})$$

$$M \longmapsto f^{-1} M \otimes_{\mathcal{O}_X} \mathcal{O}_{X_{\text{ét}}}$$

When $X_{\text{ét}} \rightsquigarrow X_{fl}$

$$v^{-1}M(U) := \varinjlim_{U \to U' \xrightarrow{\text{ét}} X} M(U')$$

$$\begin{tikzcd} U \arrow[d, "\text{flat}"] \\ X \end{tikzcd}$$

derived functor of v^*

$$\rightsquigarrow \quad Lv^* : D(\mathcal{O}_{X_{\text{ét}}}) \to D(\mathcal{O}_{X_{fl}})$$

$$M^\bullet \longmapsto M^\bullet_{fl}$$

If $P^\bullet \xrightarrow{\sim} M^\bullet$ by proj modules (this always exists), then

because $\operatorname{Mod}(\mathcal{O}_{X_{\text{ét}}})$ has enough projectives

$$M^\bullet_{fl} = v^* P^\bullet$$

Step 2 : Consider

$$M_{fl}^{\vee} = R\mathcal{H}om^{\bullet}(M_{fl}^{\bullet}, \mathcal{O}_{X_{fl}})$$

Step 3 : Define

$$c(M^{\bullet}) := h^{1}/h^{0}\,(M_{fl}^{\vee})$$

⌈ because all the constructions are functorial

This construction is functorial :

$$\phi : E^{\bullet} \longrightarrow F^{\bullet} \quad \longmapsto \quad \phi^{\vee} : c(F^{\bullet}) \longrightarrow c(E^{\bullet})$$

Condition ⊛ : $\overset{\overset{D(\mathcal{O}_{X_{ét}})}{\Downarrow}}{M^{\bullet}}$ satisfies ⊛ if :

(i) $h^{i}(M^{\bullet}) = 0 \quad \forall i > 0$

(ii) $h^{0}(M^{\bullet}), h^{-1}(M^{\bullet})$ are coherent.

Def $M^{\bullet} \in D(\mathcal{O}_{X_{ét}})$ is perfect of amplitude $[-1,0]$ if étale locally

$$M^{\bullet} = [\underbrace{M^{-1} \to M^{0}}_{\text{free of fine rk}}]$$

Example ‖ L_X always satisfies ⊛ and is perfect of amplitude $[-1,0]$ iff X is l.c.i. ∎

Prop

i) If ⊛ $\Longrightarrow c(M^{\bullet})$ is abelian cone stack

ii) If $M^{\bullet}$ perfect of amplitude in $[-1,0] \Longrightarrow c(M^{\bullet})$ is a vector bundle stack

proof

⌈ Take a representative of $M^{\bullet}$ by projective modules $P^{\bullet}$. Then $\tau_{\leq 0}P^{\bullet}$ is still by proj-modules

i) : $M^{\bullet} \overset{\text{étale locally}}{=} [\; \to M^{-2} \to M^{-1} \to M^{0} \to 0 \to 0 \; \ldots]$

where M^{i} are ~~free~~ proj and M^{-1}, M^{0} are finite rk locally free

⌊ finitely generated + proj ⟹ locally free

$\Rightarrow M_{fl}^{0}$ doesn't do anything

$$M_{fl}^{\vee} = [0 \to M_{0} \to M_{1} \to M_{2} \to \ldots] \quad \text{where } M_{i} = (M^{-i})^{\vee}$$

So

$$C(M^\bullet)|_U = \Big[\overset{\ker(M_1\to M_2)}{Z_1(M_\bullet)} \Big/ \underset{\text{vector bundle}}{\underset{\uparrow}{M_0}}\Big]$$

ⓘⓘ then we can assume ~~Z_1~~ $M^{-i}=0$ for $i<-1$ and so $Z_1(M_\bullet) = \underset{\text{vector bundle}}{\underset{\uparrow}{M_1}}$

■

Proposition

$\phi : E^\bullet \to L^\bullet$, $E^\bullet, L^\bullet$ satisfy ⊛

i) $\phi^\vee : C(L^\bullet) \to C(E^\bullet)$ is a map of cone stacks;

ii) $[\phi^\vee$ representable$] \Longleftrightarrow [h^0(\phi)$ surjective$]$

iii) $[\phi^\vee$ closed emb$] \Longleftrightarrow [h^0(\phi)$ iso ~~surj~~ $+ h^{-1}(\phi)$ surjective$]$

iv) $[\phi^\vee$ is an iso$] \Longleftrightarrow [h^0(\phi)$ and $h^{-1}(\phi)$ is an iso.$]$

Conclusion : $E^\bullet \xrightarrow{\phi} L_X^\bullet$ obstruction theory, i.e.

- $E_\bullet$ perfect
- $\phi^\vee$ closed embedding

$\Rightarrow$ $C(L_X^\bullet) \subset C(E^\bullet)$

$\quad \| $

$\quad \mathfrak{N}_X \leftarrow$ The intrinsic normal sheaf

proof of the proposition (partial ; we prove $(\Leftarrow)$ in ⓘⓘ and $(\Rightarrow)$ in ⓘⓘⓘ)

All the statement are étale locale so as before

$$C(E^\bullet) = \Big[\underset{E^{-1}/\mathrm{Im}(E^{-2}\to E^{-1})}{Z_1(E_\bullet)} \Big/ \underset{\text{locally free}}{\underset{\uparrow}{E_0}}\Big] \longrightarrow C(L^\bullet) = \Big[Z_1(L_\bullet) \Big/ \underset{\text{locally free}}{\underset{\uparrow}{L_0}}\Big]$$

and Write

$$\begin{tikzcd} C^{-1}(E^\bullet) \arrow[r] \arrow[d, "\phi"'] \arrow[dr, phantom, "\circlearrowleft"] & E^0 \arrow[d, "\phi"] \\ C^{-1}(L^\bullet) \arrow[r] & L^0 \end{tikzcd}$$

. Consider the fiber product

which induces isomorphisms

$$\begin{tikzcd} 0 \arrow[r] & \cdot \arrow[r] \arrow[d, "\cong"'] & F \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & E^0 \arrow[r] \arrow[d] & \cdot \arrow[r] \arrow[d, "\cong"] & 0 \\ 0 \arrow[r] & h^{-1}(L^\bullet) \arrow[r] & C^{-1}(L^\bullet) \arrow[r] & L^0 \arrow[r] & h^0(L) \arrow[r] & 0 \end{tikzcd}$$

and so we get an exact sequence

$$\begin{tikzcd}
 & & 0 \arrow[r] & C^{-1}(L) \arrow[d] & \\
0 \arrow[r] & F \arrow[r] & E^0 \oplus C^{-1}(L^0) \arrow[r, "(a,b)\mapsto\phi(a)-d_E(b)"] \arrow[rr, bend left, "0"] \arrow[rrd, bend right, "0"'] & L^0 \arrow[r] \arrow[rd] & \mathrm{Coker}(h^0(\phi)) \\
 & & & & h^0(L) \arrow[u] \\
 & & & & h^0(E) \arrow[u] \\
 & & & & 0 \arrow[u]
\end{tikzcd}$$

So for ii)(⟸): if $h^0(\phi)$ is surjective, then $\mathrm{coker}(h^0(\phi))=0$

$$0\to F\to E^0\oplus C^{-1}(L^\bullet)\to \underset{\uparrow\text{locally free}}{L^0}\to 0$$

is exact of sheave

$$C^{-1}(L^\bullet)^\vee=\mathrm{Hom}\left(L^{-1}/\mathrm{Im}(L^{-2}\to L^{-1}),\mathcal{O}\right)=\{\varphi:L^{-1}\to\mathcal{O}\mid \underbrace{L^{-2}\to L^{-1}\to\mathcal{O}}_{0}\}=\ker(L_1\to L_2)$$

$$\Rightarrow\quad 0\to L_0\to E_0\oplus \underset{\substack{\parallel\\ \ker(L_1\to L_2)}}{Z_1(L_0)}\to C(F)\to 0$$

exact sequence of cones

$$\begin{tikzcd}
L_0 \arrow[r] \arrow[d] \arrow[rd, phantom, "\square"] & L_0\times E_0 \arrow[r] \arrow[d] \arrow[rd, phantom, "\square"] & E_0 \arrow[d] \\
L_0\times Z_1(L) \arrow[r] \arrow[d] \arrow[rd, phantom, "\square"] & L_0\times(E_0\oplus Z_1) \arrow[r] \arrow[d] \arrow[rd, phantom, "\square"] & E_0\oplus Z_1(L) \arrow[d] \\
Z_1(L) \arrow[r] & E_0\oplus Z_1(L) \arrow[r] & C(F)
\end{tikzcd}$$

$$\Rightarrow\quad c(L^\bullet)=\left[Z_1(L_0)/L_0\right]=\left[C(F)/E_0\right]$$

+ Prop 1.7 $\Rightarrow$ $[Z_1(L)/L_0]=[C(F)/E_0]$

And so

$$\begin{tikzcd}
\overset{\text{scheme}}{C(F^\bullet)} \arrow[r] \arrow[d] \arrow[rd, phantom, "\square"] & Z_1(E_0) \arrow[d] \\
\left[C(F^\bullet)/E_0\right]=C(L^\bullet) \arrow[r, "\phi^\vee"'] & C(E^\bullet)=\left[Z_1(E_0)/E_0\right]
\end{tikzcd}$$

Now we prove ⓘⓘ;

$$[\phi^\vee\text{ closed emb}]\iff\left[C(F^\bullet)\underset{\substack{\text{closed}\\\text{emb}}}{\hookrightarrow}Z_1(E_0)\right]\iff\left[\underset{\text{surjective}}{C^{-1}(E^\bullet)\to F^\bullet}\right]$$

Consider

$$\begin{tikzcd}
0 \arrow[r] & h^{-1}(E^\bullet) \arrow[r] \arrow[d] & C^{-1}(F^\bullet) \arrow[r] \arrow[d] & E^0 \arrow[r] \arrow[d, equal] & h^0(E^\bullet) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & h^{-1}(E^\bullet) \arrow[r] & F \arrow[r] & E^0 \arrow[r] & h^0(L^\bullet) \arrow[r] & 0
\end{tikzcd}$$

and now its a diagram chasing.

Proposition

Let $$\overset{\text{satisfy } \circledast}{E^\bullet \longrightarrow F^\bullet} \longrightarrow \overset{\text{perfect}}{G^\bullet} \longrightarrow E^\bullet[1]$$

be a distinguished triangle in $D(\mathcal{O}_{X_{et}})$

then $0 \to C(G^\bullet) \to C(F^\bullet) \to C(E^\bullet) \to 0$ perfect of abelian cones

Lecture V (Speaker: Alessio Cela)

The intrinsic normal cone

X separated DM stack

Def $\mathfrak{N}_X := C(L^\bullet_X) = h^1/h^0\left(\left((L^\bullet_X)_{fl}\right)^\vee\right)$

is the intrinsic normal sheaf of X

Def A local embedding of X is a diagram

$$\begin{tikzcd}
\text{affine k-scheme of finite type} \arrow[r] & U \arrow[r, hook, "i"] \arrow[r, hook, "\text{closed emb}"'] \arrow[d, "f"] \arrow[d, "\text{étale}"'] & M & \text{smooth and affine} \arrow[l] & =: (U, M) \\
& X & & &
\end{tikzcd}$$

A morphism of local embeddings is $\phi : (U, M) \to (U', M')$

$$\begin{tikzcd}
\text{étale} \arrow[dr] & & & & \\
& U' \arrow[r, "\phi_U"] \arrow[rrr, "G"] \arrow[dr] & U \arrow[rr] \arrow[d] & & M \\
& & X & & M' \arrow[u, "\phi_M"] & \text{smooth} \arrow[l]
\end{tikzcd}$$

Rmk 1 (U,M), (U',M') local embeddings $\Rightarrow$ $(U \times_X U', M \times M')$ local emb.

proof

$$\begin{tikzcd}
\text{affine scheme} \arrow[r] & U' \times_X U \arrow[r] \arrow[d] & M' \times M & \text{smooth + affine scheme} \arrow[l] \\
\text{étale} \arrow[r] & X & \text{closed emb because :} \arrow[ul] &
\end{tikzcd}$$

$$\begin{tikzcd}
\text{scheme} \arrow[d] & \Leftarrow & \text{scheme} \arrow[d] & \\
U' \times_X U \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\square"] & U' \times U \arrow[rr, hook] \arrow[d] & & M \times M' \\
X \arrow[r, "\Delta"'] & X \times X & & \\
& \text{closed emb} \arrow[u] & &
\end{tikzcd}$$

Rmk 2 $L^\bullet_X |_U = L^\bullet_U$

proof

We will only prove that if

$f : U \xrightarrow{\text{étale}} X$ ($\nwarrow \nearrow$ schemes) $\Rightarrow f^* \tau_{[-1,0]} L_X = \tau_{[-1,0]} L_U$ in $D(\mathcal{O}_{U_{ét}})$

This pullback ~~is~~ should be computed by taking a proj res of $\tau_{[-1,0]} L^\bullet_X$ and the applying the pullback term by term but since f is étale $\Rightarrow f$ flat you can directly apply f^* term by term.

Recall Consider a commutative diagram of schemes

$$\begin{tikzcd} X \arrow[r, hook, "i"] \arrow[d, "f"'] & X' \arrow[dl, "f'"] \\ Y & \end{tikzcd}$$

closed emb defined by I

smooth

$$\Rightarrow \tau_{[-1,0]} L^{\bullet}_{X/Y} = \left[0 \to \underset{-1}{I/I^2} \to \underset{0}{i^* \Omega_{X'/Y}} \to 0 \right]$$

Thus if

$$\begin{tikzcd} U = \operatorname{Spec} \mathbb{C}[y_1,..,y_n,x_1,...x_t]/I+(g_1,..,g_t) \arrow[d, "f"', "\text{étale}"] \arrow[r, hook] \arrow[dr, phantom, "\circlearrowright"] & \mathbb{A}^{t+m} \arrow[d] \\ X = \operatorname{Spec} \mathbb{C}[y_1,..,y_n]/I \arrow[r, hook] & \mathbb{A}^m \end{tikzcd}$$

$$(y_1,..,y_n,x_1,..x_t) \mapsto (y_1,..,y_n)$$

Then we want to check that

$$\begin{tikzcd} := f^* I/I^2 = I/I^2 \otimes_{\mathbb{C}[\underline{y}]/I} \mathbb{C}[\underline{y},\underline{x}]/I+(g_1..g_t) \arrow[r] \arrow[d] \arrow[dr, phantom, "\circlearrowright"] & f^*\Omega_{\mathbb{A}^m}|_X = \bigoplus_{i=1}^{m} \mathbb{C}[\underline{y},\underline{x}]/I+(g_1..g_t) \, dy_i \arrow[d] \\ F^{\bullet} = I+(g_1,..,g_t)/(I+(g_1..g_t))^2 \arrow[r] & \bigoplus_{i=1}^{m} \mathbb{C}[\underline{y},\underline{x}]/I+(g_1,.g_t) \, dy_i \oplus \bigoplus_{j=1}^{t} \mathbb{C}[\underline{y},\underline{x}]/I+(g_1..g_t) \, dx_j \end{tikzcd}$$

$i \longmapsto di$

$g_i \longmapsto dg_i$

is a quasi-isomorphism

But

$$h^{-1}(F^{\bullet}) \underset{\substack{\uparrow \\ \text{since } \langle \underset{\substack{\uparrow \\ \text{basis}}}{dg_i} \rangle = \langle dx_i \rangle}}{=} \ker\left(I/I^2 \to \bigoplus \mathbb{C}[\underline{y}]/I \, dy_i \right) \otimes_{\mathbb{C}[\underline{y}]/I} \underset{\substack{| \\ \text{flat } \mathbb{C}[\underline{y}]/I\text{-module}}}{\mathbb{C}[\underline{y},\underline{x}]/I+(g_1..g_t)} = \underset{\substack{\| \\ h^{-1}(E^{\bullet})}}{\ker\left(f^* I/I^2 \to f^* \Omega_{\mathbb{A}^n}|_X\right)}$$

and

$$h^0(F^{\bullet}) \cong h^0(E^{\bullet})$$

∎

∴ Therefore we have a natural morphism

$$L^\circ_X|_U = L^\circ_U \longrightarrow \tau_{[-1,0]} L^\circ_U = [\ I/I^2 \to \Omega_M|_U]$$

which induces isomorphism on h^0 and h^{-1}.

$$\Rightarrow \left\| \ \eta_X|_U \overset{\uparrow}{\cong} \left[N_{U/M} / T_M|_U \right] \right.$$

This is clear when $U \hookrightarrow M$ is regular because then I/I^2 is locally free and thus $[I/I^2 \to \Omega_M|_U]_{fl}^{\vee} = [\ T_M|_U \to I/I^2{}^{\vee}]$

In general: if

proj res of I/I^2

$$\begin{tikzcd} \vdots \arrow[d] & \\ P_{-2} \arrow[d] & \\ P_{-1} \arrow[d] & \\ I/I^2 \arrow[r] \arrow[d] & \Omega_M|_U \\ 0 & \end{tikzcd}$$

$$\Rightarrow \begin{tikzcd} \cdots \arrow[r] & P_{-2} \arrow[r] \arrow[d] & P_{-1} \arrow[r] \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & \Omega_M|_U \arrow[r] \arrow[d] & 0 \\ \cdots \arrow[r] & 0 \arrow[r] & I/I^2 \arrow[r] & \Omega_M|_U \arrow[r] & 0 \\ & -2 & -1 & 0 & \end{tikzcd}$$

} this is a quasi-iso on h^{-1} and h^0

$$\Rightarrow c([I/I^2 \to \Omega_M|_U]) = c([P_\bullet \to \Omega_M|_U \to 0]) =$$

$$= h^1/h^0 ([T_M|_U \to P_{-1}^{\vee} = P^1 \to P_{-2}^{\vee} = P^2 \to \ldots]) =$$

$$= h^1/h^0 (\tau_{[0,1]} [T_M|_U \to P^\bullet]) \underset{|}{=} h^1/h^0 ([T_M|_U \to I/I^2{}^{\vee} = N_{U/M}])$$

$$\mathrm{Ker}(P^1 \to P^2) = \{ \varphi : P_{-1} \to \mathcal{O} \mid P_{-2} \to P_{-1} \overset{=0}{\curvearrowright} \to$$

$$= \mathrm{Hom}\left(P_1 / \mathrm{Im}(P_2 \to P_1), \mathcal{O} \right) =$$

$$= \left(P_1 / \mathrm{Im}(P_2, P_1) \right)^{\vee} = I/I^2{}^{\vee}$$

Rmk3 If $x : (U, M) \longrightarrow (U', M')$ morphism

Then we have

$$\begin{tikzcd} & U' \arrow[rr, hook, "I'"] \arrow[dd] & & M' \\ U \arrow[ur, "x_U"] \arrow[rr, hook, "I"] \arrow[dr] & & M \arrow[ur, "x_M"] & \\ & X & & \end{tikzcd}$$

$$\begin{tikzcd} x_M^* \mathcal{O}_{M'} \arrow[r, phantom, "\cong"] & \mathcal{O}_M \\ x_M^* I' \arrow[u, phantom, "\cup"] \arrow[r] & I \arrow[u, phantom, "\cup"] \end{tikzcd} \rightsquigarrow \quad x_{M'}^* I'/I'^2 \longrightarrow I/I^2$$

and

$$\begin{tikzcd} x_{M'}^* I'/I'^2 \arrow[r] \arrow[d, "x"'] \arrow[dr, phantom, "\circlearrowright"] & x_{M'}^* \Omega_{M'}|_{U'} \arrow[d, "x"] \\ I/I^2 \arrow[r] & \Omega_M|_U \end{tikzcd}$$ commutes

$$\rightsquigarrow \quad \tilde{x} : [I'/I'^2 \longrightarrow \Omega_{M'}|_{U'}]\Big|_U \longrightarrow [I/I^2 \longrightarrow \Omega_M|_U] \text{ homo in } D(\mathcal{O}_{M_{\acute{e}t}})$$

↑

Again here we are using $x_U^* L^\bullet_X$ = apply x_U^* to each

↑ term of $L^\bullet_X$

x_U is étale

$$\Rightarrow \boxed{\tilde{x}^\vee : [N_{U/M}/T_M|_U] \xrightarrow{\sim} [N_{U'/M'}/T_{M'}|_{U'}]|_U}$$

↑

isomorphism of cone stacks being

$$\begin{tikzcd} {[N_{U/M}/T_M|_U]} \arrow[r, "\cong"] \arrow[dr] \arrow[r, phantom, shift right=6, "\circlearrowright"] & \eta_X|_U \\ & {[N_{U'/M'}/T_{M'}|_{U'}]|_U} \arrow[u, "\cong"'] \end{tikzcd}$$

∎

Recall $T_M|_U \circlearrowright N_{U/M}$ preserves $C_{U/M}$

Rmk 4 Given

$$\begin{tikzcd} X \arrow[r, hook, "i"] \arrow[dr, bend right, "i'"'] & M \arrow[d, "g"', "\text{smooth}"] \arrow[dl, phantom, "\circlearrowleft"] \\ & M' \end{tikzcd} \rightsquigarrow \begin{tikzcd} 0 \arrow[r] & T_{M/M'}|_X \arrow[r] & N_{X/M'} \arrow[r] \arrow[d, phantom, "\cup"] & N_{X/M} \arrow[r] \arrow[d, phantom, "\cup"] \arrow[dl, phantom, "\square"] & 0 \\ & & C_{X/M'} \arrow[r] & C_{X/M} \end{tikzcd}$$

exact sequence

More. generally I would say that given

$$\begin{tikzcd} U \arrow[r, hook] \arrow[d, "\text{étale}"'] \arrow[dr, phantom, "\circlearrowleft"] & M \arrow[d, "\text{smooth}"] \\ U' \arrow[r, hook] & M' \end{tikzcd}$$

Then we have

$$\begin{tikzcd} & & N_{U/M} \arrow[r] \arrow[d, phantom, "\cup"] \arrow[dr, phantom, "\square"] & N_{U'/M'}|_U \arrow[r] \arrow[d, phantom, "\cup"] & 0 \\ 0 \arrow[r] & T_{M/M'}|_U \arrow[r] \arrow[ur] & C_{U/M} \arrow[r] & C_{U'/M'}|_U \arrow[r] & 0 & \leftarrow \text{exact} \\ 0 \arrow[r] & T_{M/M'}|_U \arrow[r] \arrow[u, equal] & T_M|_U \arrow[r] \arrow[u] \arrow[ur, phantom, "\circlearrowleft"] & T_{M'}|_U \arrow[r] \arrow[u] & 0 & \leftarrow \text{exact} \end{tikzcd}$$

⊛

dual of the exact sequence

exactness here is due to the fact that $M \to M'$ is smooth

$0 \to \Omega_{M'} \to \Omega_M \to \Omega_{M/M'} \to 0$

$\rightsquigarrow 0 \to T_{M/M'} \to T_M \to T_{M'} \to 0$ exact being all locally free

all of them are locally free ⇒ restriction to U is still exact

$$\Rightarrow \left[C_{U'/M'} / T_{M'}|_{U'} \right]|_U \overset{\tilde{\chi}^\vee}{\cong} \left[C_{U/M} / T_M|_U \right]$$

$\uparrow$

⊛ $\Longleftrightarrow$ $\begin{tikzcd} T_M|_U \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & T_{M'}|_U \arrow[d] \\ C_{U/M} \arrow[r] & C_{U'/M'}|_U \end{tikzcd}$ is cartesian

and $C_{U/M} \times T_{M'}|_U \to C_{U'/M'}|_U$ is surjective

Now apply Prop 1.7.

~~Corollary~~

we can by Rmk 1 and Rmk 2

Def Applying descent for closed substacks to

$$\mathfrak{C}_X|_U := \left[{C_{U/M}}/{T_M|_U} \right] \hookrightarrow \left[{N_{U/M}}/{T_M|_U} \right]$$

we obtain a unique closed substack $\mathfrak{C}_X \hookrightarrow \mathfrak{N}_X$ with $\mathfrak{C}_X|_U \underset{\substack{\uparrow \\ \text{Canonically}}}{=} \left[{C_{U/M}}/{T_M|_U} \right]$

$\mathfrak{C}_X$ is called the intrinsic normal cone of X

Thm $\boxed{\mathfrak{C}_X \text{ has pure dim} = 0}$

Recall $\underset{\substack{\uparrow \\ \text{pure dim } d}}{X \hookrightarrow Y} \Rightarrow C_X(Y)$ has pure dimension d

This is not an example of what we were looking for, but I want to keep it here because it explains eq (3.6) at page 180 of [ACG] in an easy way.

⚠ This is false for $N_{X/Y}$

Example

$X = \{xy=0\} \hookrightarrow \underset{\substack{\uparrow \\ \text{smooth}}}{\mathbb{A}^2} \Rightarrow$ you have exact sequence

$$\begin{tikzcd}
 & & & \mathcal{O}_{\mathbb{A}^2}(X)|_X \arrow[d, phantom, "\parallel"] & \\
0 \arrow[r] & T_X \arrow[r] & T_{\mathbb{A}^2}|_X \arrow[r] & N_{X/\mathbb{A}^2} \arrow[r] & T^1_X \arrow[r] \arrow[d, phantom, "\parallel"] & 0 \\
 & & & & \mathrm{Ex}_{c}(\mathcal{O}_X, \mathcal{O}_X) \arrow[d, phantom, "\parallel"] & \\
 & & & & \mathrm{Ext}^1_{\mathcal{O}_X}(\Omega_X, \mathcal{O}_X) &
\end{tikzcd}$$

$\mathcal{O}_X$ locally free + X reduced and smooth on $\underset{\text{open dense}}{X - \{0\} \subset X}$ $\longrightarrow$ $\parallel$

$\uparrow$ for instance when X is smooth

$\Rightarrow \mathcal{E}xt^1_{\mathcal{O}_X}(\underset{\substack{\uparrow \\ \text{locally free}}}{\Omega_X}, \mathcal{O}_X) = 0$

And this also proves that

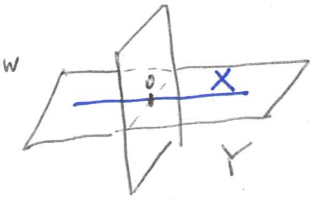
$$\mathcal{E}xt^1_{\mathcal{O}_X}(\Omega_X, \mathcal{O}_X)\Big|_{X - \{0\}} = 0$$

Restricting ~~Localizing~~ at $p=0$ we get

$$\left\| \; N_{X/\mathbb{A}^2}|_p \cong T^1_X|_p = \overset{\mathcal{E}xt^1_{\mathcal{O}_X}(\Omega_X, \mathcal{O}_X)|_p}{\parallel} \; \cancel{\mathcal{E}xt}\ \mathrm{Ext}^1_{\mathcal{O}_{X,0}}(\Omega_{X,0}, \mathcal{O}_{X,0}) \otimes_{\mathcal{O}_{X,0}} \mathbb{C} \right.$$

Example

$$\mathbb{A}^1 \cong X = \{x=0\} \hookrightarrow \underset{\underset{Y \leftarrow \text{pure dim}=2}{\parallel}}{\begin{Bmatrix} zw=0 \\ xy=0 \end{Bmatrix}} \subset \overset{\ulcorner \text{coordinates } x,y,z,w}{\mathbb{A}^4}$$



◀ picture

Then $\quad 0 \to T_X \to T_Y|_X \to N_{X/Y} \to 0 \quad$ is exact being X smooth

We have $\quad N_{X/Y}|_x = \begin{cases} \mathbb{C}^3 & x=0 \\ \mathbb{C} & x\neq 0 \end{cases}$

Let $\pi \; N_{X/Y} \longrightarrow X \cong \mathbb{A}^1$

Claim $\forall v \in N_{X/Y}$ over 0 $\dim_v(N_{X/Y}) \geq 3$
while if $v \in N_{X/Y}$ $\pi(v) \neq 0 \Rightarrow \dim_v(N_{X/Y}) = 2$

proof

~~dim~~ $v \in \pi^{-1}(0) \Rightarrow \dim_v(N_{X/Y}) \geq \dim_v(\underset{\underset{\mathbb{C}^3}{\parallel}}{N_{X/Y}|_0}) = 3$

~~ve~~ and $N_{X/Y}|_{X-\{0\}} \longrightarrow X-\{0\}$ is smooth of dim 1.

∎

proof of the theorem

Let $\begin{matrix} U \to M \\ \downarrow \\ X \end{matrix}$ local emb. Then we have

$$\begin{tikzcd} C_{U/M} \times_X T_M|_U \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & C_{U/M} \arrow[d] \\ C_{U/M} \arrow[r] & {[C_{U/M}/T_M|_U]} \end{tikzcd}$$

$\leftarrow$ pure dim = dim M

$\leftarrow$ smooth of rel dim = dim M

∎

Basic properties

Property I (l.c.i):

TFAE:

i) X l.c.i;

ii) $\mathfrak{C}_X$ is a vector bundle stack;

iii) $\mathfrak{C}_X = \mathfrak{N}_X$

∎

Property II (Products)

$$\mathfrak{N}_{X\times Y} = \mathfrak{N}_X \times \mathfrak{N}_Y \quad \text{and} \quad \mathfrak{C}_{X\times Y} = \mathfrak{C}_X \times \mathfrak{C}_Y$$

∎

Property III (Pull back)

$f: X \longrightarrow Y$ l.c.i. Then we have an exact sequence of cone stacks over X

$$h^1/h^0(T^{\bullet}_{X/Y}) = \mathfrak{N}_{X/Y} \to \mathfrak{C}_X \longrightarrow f^{*}\mathfrak{C}_Y$$

∎

proof of Property II:

$$\begin{tikzcd} U \arrow[r, "I"] \arrow[d] & M \\ X & \end{tikzcd} , \quad \begin{tikzcd} V \arrow[r, "J"] \arrow[d] & N \\ Y & \end{tikzcd} \rightsquigarrow \begin{tikzcd} U\times V \arrow[r, "I\otimes J"] \arrow[d] & M\times N \\ X\times Y & \end{tikzcd} \rightsquigarrow C_{U\times V/M\times N} = C_{U/M} \times C_{V/N}$$

$$N_{U\times V/M\times N} = N_{U/M} \times N_{V/N}$$

$$T_{M\times N}\big|_{U\times V} = T_M\big|_U \boxplus T_N\big|_V$$

Finally if $\underset{E\text{-cone}}{E \to C}$ and $\underset{F\text{-cone}}{F \to D}$ $\Rightarrow$ $E\times F \longrightarrow C\times D$ is an $E\times F$-cone and

$$[C/E] \times [D/F] \xrightarrow{\sim} [C\times D/E\times F]$$

$$\left(\begin{tikzcd} P \arrow[r] \arrow[d] & C \\ T & \end{tikzcd} , \begin{tikzcd} Q \arrow[r] \arrow[d] & F \\ T & \end{tikzcd} \right) \longmapsto \left(\begin{tikzcd} P\times Q \arrow[r] \arrow[d] & C\times F \\ T & \end{tikzcd} \right)$$

∎

Lecture VI (Speaker: Miguel Moreira)

Obstruction theory

Excursus

Rmk X smooth $E_\bullet = [E_0 \to E_1]$ perfect obstruction theory

$$\Rightarrow \underset{\overset{\|}{\underbrace{\mathrm{rk}(\underbrace{h^0(E_\bullet)}_{\overset{\|}{T_X}})}_{\text{L constant}} - \mathrm{rk}(h^1(E_\bullet))}}{\mathrm{rk}(E_\bullet)} = \mathrm{rk}(E_0) - \mathrm{rk}(E_1) \text{ is locally constant}$$

X smooth $\Rightarrow$ $h^1(E_\bullet)$ locally free and $[X, E^\bullet] \overset{\text{next lecture}}{=} e_{top}(h^1(E_\bullet))$.
$+ \mathrm{rk}\, h^1(E_\bullet)$ constant

Obstruction theories

Def $E^\bullet \in D(\mathcal{O}_{X_{et}})$

$E^\bullet \to L^\bullet_X$ is an obstruction theory if $E^\bullet$ satisfies $\circledast$ and $h^0(\phi)$ is an isomorphism and $h^{-1}(\phi)$ is surjective.

meaning that $c(E)$ is a cone stack over X

$\circledast \leadsto$ makes sense to consider

$$c(E^\bullet) := h^1/h^0\left(\left(E^\bullet_{fl}\right)^\vee\right) =: \mathfrak{E}$$

and $\left[h^0(\phi) \text{ iso} + h^{-1}(\phi) \text{ surj}\right] \Longleftrightarrow \left[\mathfrak{N}_X \xhookrightarrow{\phi^\vee} \mathfrak{E} \text{ is closed embedding}\right]$

Goal ‖ To explain why $E^\bullet \to L^\bullet_X$ is called an obstruction theory.

Square-zero extensions

Def A closed emb $T \hookrightarrow \bar{T}$ ~~[illegible]~~ is called square-zero extension if the corresponding sheaf J satisfies $J^2=0$

Rmk $J^2=0 \implies J/J^2 = J$ on T.

Example A ring, M an A-module. Then we can give a **ring** structure for $A \oplus \varepsilon M$ where $\varepsilon^2=0$.

The multiplication is $(a_1+\varepsilon m_1)(a_2+\varepsilon m_2) = a_1 a_2 + \varepsilon(a_1 m_2 + a_2 m_1)$

The projection map $A \cancel{\oplus} \varepsilon M \longrightarrow A$ gives an inclusion

$$\mathrm{Spec}(A) \hookrightarrow \mathrm{Spec}(A \oplus \varepsilon M)$$

which is a square-zero obstruction. ∎

Suppose now we have

$$\begin{tikzcd}
T \arrow[r, "g"] \arrow[d, hook, "j"'] & X \\
\bar{T} \arrow[ur, dashed, "\bar{g}"'] &
\end{tikzcd}$$

Q) When can we extend g to $\bar{T} \xrightarrow{\bar{g}} X$?

We have

$$w(g) : \underbrace{g^* L^\circ_X \to L^\circ_T \longrightarrow L^\circ_{T/\bar{T}} \to \tau_{[-1,0]} L^\circ_{T/\bar{T}} = J[1]}_{\in \mathrm{Hom}(g^* L^\circ_X,\, J[1]) = \mathrm{Ext}^1(g^* L^\circ_X,\, J)}$$

$\uparrow$

Called obstruction class

Obs If $\bar{g}$ $\exists$ then we have

$$\begin{tikzcd} T \arrow[r, "g"] \arrow[d, "G"'] & X \\ \bar{T} \arrow[ur, "\bar{g}"'] & \end{tikzcd} \quad \rightsquigarrow \quad \begin{tikzcd} j^*\bar{g}^*L_X^{\bullet} \arrow[d] & g^*L_X^{\bullet} \arrow[l, "\cong"'] \arrow[dl] \arrow[d, dash] \arrow[rrrd, "w(g)=0"] & & & \\ j^*L_{\bar{T}} \arrow[r] \arrow[rr, bend right, "0"'] & L_T^{\bullet} \arrow[r] & L_{T/\bar{T}} \arrow[r] & \tau_{[-1,0]}L_{T/\bar{T}} = J[1] & \end{tikzcd}$$

■

┌ We will give a proof later

Fact An extension $\bar{g}$ $\exists$ $\Longleftrightarrow$ $w(g)=0$

In that case extensions are a torsor over $\mathrm{Hom}(g^*L_X^{\bullet}, J) = \mathrm{Hom}_{\mathcal{O}_T}(g^*\Omega_X, J)$

■

Example

$$\begin{tikzcd} \mathrm{Spec}(A) \arrow[r] \arrow[d] & \mathrm{Spec}(B) = X \\ \mathrm{Spec}(A\oplus\varepsilon M) \arrow[u, dashed, bend right] & \end{tikzcd}$$

$\exists$ section given by $A \to A\oplus\varepsilon M$

$\Longrightarrow$ extension always exists

Now an extension is

$$\begin{tikzcd} A & B \arrow[l, "\phi"'] \arrow[dl, dashed, "\bar{\phi}"] \\ A\oplus\varepsilon M \arrow[u] & \end{tikzcd} \longleftrightarrow$$

$\bar{\phi}(b_1b_2) = \phi(b_1)\phi(b_2) + \varepsilon\,\delta(b_1b_2)$

$\bar{\phi}(b_1b_2) = (\phi(b_1)+\varepsilon\delta(b_1))(\phi(b_2)+\varepsilon\delta(b_2)) = \phi(b_1)\phi(b_2) + \varepsilon(\phi(b_1)\delta(b_2) + \phi(b_2)\delta(b_1))$

$\Longleftrightarrow \delta(b_1b_2) = \phi(b_1)\delta(b_2) + \phi(b_2)\delta(b_1)$

$$\bar{\phi} : B \longrightarrow A\oplus\varepsilon M$$
$$b \longmapsto \phi(b) + \varepsilon\delta(b)$$

and $\bar{\phi}$ is a ring homomorphism

$\Longleftrightarrow$ $\delta : B \longrightarrow M$ is a derivation of B-modules

$$\begin{tikzcd} & \Omega_B \arrow[dr, "B \text{ linear}"] & \\ B \arrow[ur] \arrow[rr] & & M \end{tikzcd}$$

Therefore the extensions are in 1:1 correspondence with

$$\mathrm{Der}_{\phi}(B, M) = \mathrm{Hom}_B(\Omega_B, M_B) = \mathrm{Hom}_A(\Omega_B \otimes_B A, M)$$

■

Rmk X smooth $\Rightarrow$ $L_X^{\bullet}$ is locally free and so extensions always exist.

$$L_X^{\bullet} = \left[\cdots \to 0 \to \Omega_X \to 0\right]$$

$$w(g) \in Ext^1(g^* L_X^{\bullet}, J) \underset{\uparrow}{=} Ext^1(\overset{\text{locally free}}{\overbrace{g^*\Omega_X}}, J) = 0$$

$\mathcal{A} = Coh_X$ has enough projective + Corollary 10.7.5 in Weibel.
Note that Ω_X loc. free $\Rightarrow g^* L_X^{\bullet}$ is just the complex obtained by applying g^* term by term.

Interpretation in terms of cones:

$$w(g) : g^* L_X^{\bullet} \longrightarrow J[1]$$

$$\rightsquigarrow \underset{\substack{\| \\ ob(g)}}{w(g)^{\vee}} : \underset{\substack{\| \\ C(J)}}{C(J[1])} \longrightarrow g^*\eta_X$$

We also have

$$\begin{tikzcd} 0 : C(J) \arrow[rr] \arrow[dr] & & g^*\eta_X \\ & T \arrow[ur, "0"'] & \end{tikzcd}$$

Def of 2 Sheves on $T_{\acute{e}t}$:

$$\underline{Ext}(g, \bar{T})(U) := \left\{ \begin{tikzcd} U \arrow[r, "g|_U"] \arrow[d] & X \\ \bar{T} \times_T U \arrow[ur, "\bar{g}"'] & \end{tikzcd} \right\}$$

$$\underline{Iso}(ob(g), 0)(U) = \left\{ \begin{array}{l} \text{2-iso of cone stacks} \\ \quad ob(g)|_U \text{ and } 0|_U \end{array} \right\}$$

Prop $\boxed{\begin{array}{l} \text{There is a canonical isomorphism} \\ \qquad \underline{Ext}(g, \bar{T}) \xrightarrow{\sim} \underline{Iso}(ob(g), 0) \end{array}}$

proof (sketch)

Consider locally on X,

$$\begin{tikzcd} X \arrow[r, hook, "I", "\text{closed emb}"'] & W \end{tikzcd} \ \ulcorner \text{smooth}$$

Consider

$$\begin{tikzcd} T \arrow[r, hook, "J"] \arrow[d, "g"'] \arrow[dr, phantom, "\circlearrowleft"] & \bar{T} \arrow[d, dashed, "h"] \\ X \arrow[r, hook, "I"] & W \end{tikzcd}$$

h ← $\exists$ by the previous remark

In this case we get a homo $\overset{\swarrow \text{ identified with } \omega(g)}{h^\#} : g^*\left[I/I^2 \to \Omega_W|_X\right] \longrightarrow J[1] = \tau_{[-1,0]}\left(L^{\bullet}_{T/\bar{T}}\right)$

obtained ~~as follows~~ as follows: let $P^\bullet \to I/I^2 \to 0$ be a projective resolution. Then

$$\begin{tikzcd} \arrow[r] & P_2 \arrow[r] \arrow[d] & P_1 \arrow[r] \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & P_0 \arrow[r] \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & \Omega_W|_X \arrow[r] \arrow[d, equal] & 0 \\ \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & I/I^2 \arrow[r] & \Omega_W|_X \arrow[r] & 0 \end{tikzcd}$$

is a quasi iso and we have

$$\begin{tikzcd} \arrow[r] & g^*P_2 \arrow[r] \arrow[dd] & g^*P_1 \arrow[r] \arrow[dd] \arrow[ddr, phantom, "\circlearrowleft"] & g^*P_0 \arrow[r] \arrow[d] & g^*\Omega_W|_X \arrow[r] \arrow[d, equal] & 0 \\ & & & g^*I/I^2 \arrow[r] \arrow[d] & g^*\Omega_W|_X \arrow[d] & \\ & 0 \arrow[r] & 0 \arrow[r] & J \arrow[r] & 0 & \end{tikzcd}$$

<u>Obs</u>

If h and $\tilde{h}$ are different extensions $\Rightarrow$ $h^\#$ and $\tilde{h}^\#$ are homotopic

$\Rightarrow$ They define th~~e same~~ maps $c(J) \longrightarrow g^*\eta_X$ which are ~~isomorphic~~ isotopic ∎

Suppose $\bar{g}$ is given ~~If $\bar{g}$ exists~~, then we can take

$$\begin{tikzcd} T \arrow[r, hook, "j"] \arrow[d] & \bar{T} \arrow[dl, "\bar{g}"'] \arrow[d, "h_0 = j \circ \bar{g}"] \\ X \arrow[r, "i"'] & W \end{tikzcd}$$

and for this particular h_0

we get for every

$h_0^\# = 0 \Rightarrow \exists$ 2-iso $h^\# \cong h_0^\# = 0$. and since

↑

reason: $h_0^\# = (j \circ \bar{g})^\# =$ [illegible]

$\Rightarrow$ $g^* I/I^2 \xrightarrow[0]{} J$ [a function $W \to \mathbb{A}^1$ which is 0 on X is also 0 on $\bar{T}$]

$$ob(g)\Big|_X = (h^\#)^\vee : c(J) \longrightarrow g^*\eta_X$$

↑ étale neighborhood of your space

we obtain a 2-iso $ob(g) \cong 0$ on X.

This define our map. ∎

Corollary

$\bar{g}$ exists globally $\Longleftrightarrow$ $\emptyset \neq H^0(\underline{Ext}(g,\bar{T})) = H^0(\underline{Iso}(ob(g), \mathcal{O}))$

$\Longleftrightarrow$ $ob(g) \cong \mathcal{O}$ $\Longleftrightarrow$ $\omega(g) = 0$

($\Longleftarrow$) is clear

($\Longrightarrow$) because $ob(g) \cong \mathcal{O} \Longleftrightarrow \exists\, \bar{g}$ extension $\Longrightarrow$ $\omega(g) = 0$

a previous obs

∎

Prop There is a canonical isomorphism

$$\underline{Aut}(\mathcal{O}\quad) \xrightarrow{\sim} \mathcal{H}om(g^*\Omega_X, J)$$

∎

Corollary

If $\mathcal{O} \cong ob(g)$ $\Longrightarrow$ $\underline{Iso}(\mathcal{O}, ob(g))$ is a torsor over $\underline{Aut}(\mathcal{O}(g)) \cong \mathcal{H}om(g^*\Omega_X, J)$

$$\begin{array}{c}\underline{Iso}(\mathcal{O}, ob(g)) \\ \wr\!\wr \\ \underline{Ext}(g,\bar{T})\end{array}$$

$\Longrightarrow$ $\{\text{Extension } \bar{g}\} = \Gamma(T, \underline{Ext}(g,T))$ is a torsor over $\mathcal{H}om(g^*\Omega_X, J)$

∎

Meaning of obstruction theory

Suppose given $\phi : \underset{\substack{\uparrow \\ \text{satisfying } ⊛}}{E^\bullet} \longrightarrow L_X^\bullet$ and

$$\begin{tikzcd} T \arrow[r, "g"] \arrow[d, hook] & X \\ \overline{T} & \end{tikzcd}$$

$\rightsquigarrow$ $\omega(g) \in \mathrm{Hom}(g^* L_X^\bullet, J[1]) = \mathrm{Ext}^1(g^* L_X^\bullet, J)$

Then we have

$$\phi^* \omega(g) \in \mathrm{Ext}^1(g^* E^\bullet, J)$$

Thm

TFAE:

1) $\phi : E^\bullet \longrightarrow L_X^\bullet$ is an obstruction theory;

2) $\phi^\vee : \mathfrak{N}_X \hookrightarrow \mathfrak{E}$ is a closed embedding of cone stacks over X;

3) Given $(g, T, \overline{T})$ then
$[\exists \text{ extension } \overline{g}] \Longleftrightarrow [\phi^* \omega(g) = 0]$
If that's the case $\Longrightarrow$ extensions form a torsor over $\mathrm{Hom}(g^* E^\bullet, J)$

4) $\underline{\mathrm{Ext}}(g, \overline{T}) \cong \underline{\mathrm{Hom}}(\phi^\vee \circ ob(g), 0)$

proof of ① $\Longrightarrow$ ③

Consider
$$g^* E^\bullet \to g^* L_X^\bullet \to C^\bullet \to g^* E^\bullet[1]$$
an exact triangle in $D(\mathcal{O}_{T\,et})$

Applying the $\mathrm{Hom}_{D(\mathcal{O}_{T\,et})}(-, J)$ functor we obtain the long exact sequence

$$\begin{tikzcd}
 & & & 0 \arrow[d, equal] & & \\
0 & \mathrm{Ext}^0(g^* E^\bullet, J) \arrow[l] & \mathrm{Ext}^0(g^* L_X^\bullet, J) \arrow[l, "\cong"'] & \mathrm{Ext}^0(C^\bullet, J) \arrow[l] & \mathrm{Hom}_{D(\mathcal{O}_T)}(g^* E^\bullet[-1], J) \arrow[l] \arrow[d, equal] & \arrow[l] \\
 & & & & \mathrm{Hom}_{D(\mathcal{O}_T)}(g^* E^\bullet, J[+1]) \arrow[d, equal] & \\
\arrow[r, leftarrow] & \mathrm{Ext}^1(g^* L_X^\bullet, J) & \mathrm{Ext}^1(C^\bullet, J) \arrow[l] \arrow[d, equal] & \mathrm{Ext}^2(g^* E; J) \arrow[l] & \mathrm{Ext}^1(g^* E^\bullet, J) & \\
 & & 0 & & &
\end{tikzcd}$$

If we prove that $\mathrm{Ext}^0(C^\bullet, J) = 0 \implies \mathrm{Hom}(g^*E^\bullet, J) \cong \mathrm{Hom}(g^*L_x^\bullet, J)$

and $\underbrace{\mathrm{Ext}^1(C^\bullet, J) = 0}_{\circledast} \implies \mathrm{Ext}^1(g^*E^\bullet, J) \hookrightarrow \mathrm{Ext}^1(g^*L_x, J)$

and so

$$[\exists \text{ extension } \bar{g}] \iff \left[\begin{array}{ccc} \omega(g) = 0 \in \mathrm{Ext}^1(g^*L_x^\bullet, J) & \hookrightarrow & \mathrm{Ext}^0(g^*E^\bullet, J) \\ \overset{\psi}{\omega(g)} & \longmapsto & \phi^*\omega(g) \end{array}\right]$$

$$\iff \left[\phi^*\omega(g) = 0 \text{ in } \mathrm{Hom}(g^*E^\bullet, J[1]) = \mathrm{Ext}^1(g^*E^\bullet, J)\right]$$

and moreover in this case

$\{\underset{\bar{g}}{\text{extesions}}\}$ is a torsor under $\mathrm{Hom}(g^*L_x^\bullet, J) = \mathrm{Hom}(g^*E^\bullet, J)$

proof of $\circledast$

Apply $h^i(-)$ to $0 \to g^*E \to g^*L_x \to C^\bullet \to g^*E[-1]$ we get

$$\begin{tikzcd} \cdots \arrow[r] & h^{-1}(g^*E) \arrow[r, two heads] \arrow[d, equal] & h^{-1}(g^*L_x^\bullet) \arrow[r, "0"] & h^{-1}(C^\bullet) \arrow[r] \arrow[d, equal] & h^0(g^*E) \arrow[r] \arrow[d, equal] & h^0(g^*L_x^\bullet) \arrow[r, "0"] \arrow[d, equal] & h^0(C^\bullet) \arrow[r] \arrow[d, equal] & h^1(E^\bullet) = 0 \\ & g^*h^{-1}(E^\bullet) \arrow[r, two heads] & g^*h^0(L_x^\bullet) & 0 & g^*h^0(E) \arrow[r, "\sim"] & g^*h^0(L_x) & 0 & \end{tikzcd}$$

$\implies h^i(C^\bullet) = 0 \quad \forall i \geq -1$. Representing

$$C^\bullet = [\ \to C_{-3} \to C_{-2} \to 0 \to 0\ ]$$

we see that

here we are using $C_i = 0 \ \forall i > 0$

$$\mathrm{Ext}^0(C^\bullet, J) = \mathrm{Hom}(\underset{\overset{\|}{0}}{h^0(C^\bullet)}, J) = 0$$

$$\mathrm{Ext}^1(C^\bullet, J) = \mathrm{Hom}(C^\bullet, J[1]) = \mathrm{Hom}(\underset{\overset{\|}{0}}{\underbrace{h^{-1}(C^\bullet)}}, J) = 0$$

here we are using $C_i = 0 \ \forall i > -1$

Lecture VII (Speaker: Alessio Cela)

·Virtual fundamental class

The construction

X sep. DM stack

$E^\bullet \longrightarrow L_X^\bullet$ perfect obstruction theory $\longleftrightarrow$ $\mathfrak{C}_X \subset \mathfrak{N}_X \hookrightarrow \mathfrak{E}$.

Def $\underset{\substack{\uparrow \\ \text{locally constant on } X}}{rk(E^\bullet)} = rk\, E^0 - rk\, E^1$ is the <u>virtual dimension of X</u>

Assume $rk(E^\bullet) =: n$ is constant on X

Goal Construct $[X, E^\bullet] \in A_n(X)$

Idea:

$$\begin{tikzcd} \overset{\text{dim} = -rk(E^\bullet)}{\mathfrak{E}} \arrow[d] & \supset \mathfrak{C}_X \\ X \arrow[u, bend left, "0"] & \end{tikzcd}$$

$[X, E^\bullet] := 0^![\mathfrak{C}_X] \in A_{-(-rk E) = n}(X)$

Pb: $\mathfrak{E}$ is <u>not</u> DM, but it is an Artin stack.

Def The data of $F^\bullet = [F^{-1} \to F^0] \in D(\mathcal{O}_{X_{ét}})$ (vector bundles on X)

$$\begin{tikzcd} F^\bullet \arrow[d, "\text{iso}"', "\cong"] \\ E^\bullet \end{tikzcd}$$

is called <u>global resolution of $E^\bullet$</u>

So we have $\mathfrak{E} = [F_1 / F_0]$ where $F_i = (F^{-i})^\vee$

We have

$$\begin{tikzcd} C(F^\bullet) \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & F_1 \arrow[d] \\ \mathfrak{C}_X \arrow[r, hook] & {[F_1/F_0] = \mathfrak{E}} \end{tikzcd}$$

this is a closed subcone (it is a scheme + cone stack over X.

Locally $C(F^\bullet) = [C / E]$ (C: cone over X; E: vector bundle)

$C(F^\bullet)$ scheme $\Leftrightarrow$ $E = 0$

otherwise points of $C(F^\bullet)$ would have non-trivial automorphisms)

Def $[X, E^\cdot] := 0_{F_1}^! [C(F^\cdot)] \in A_{\underbrace{n}} (X)$

$$\begin{tikzcd} C(F^\cdot) \arrow[r] & \mathfrak{C}_X \end{tikzcd}$$

rel ~~dim~~ $\dim = \cancel{\dim F_0}\ \mathrm{rk}\, F_0$

$\Rightarrow \dim C(F^\cdot) = \cancel{\dim F_0}\ \mathrm{rk} F_0$

and $X \rightarrow F_1$ has codim $\mathrm{rk}\, F_1$

Prop $\boxed{[X, E^\cdot] \text{ is independent of the choice of } F^\cdot}$

Proof

Suppose $H^\cdot \xrightarrow{\phi} E^\cdot$ and $F^\cdot \xrightarrow{\psi} E^\cdot$ are actually morphisms of complexes. Form

$$H^0 \oplus F^0 \longrightarrow E^0$$

$\{(e_{-1}, h_0, f_0) \mid d_E(e_{-1}) = \phi(h_0) + \psi(f_0)\} \hookleftarrow F^{-1}$

$(\psi(f_{-1}), 0, d_F(f_{-1})) \leftarrow f_{-1}$

$\| $

$K^{-1} \subset E^{-1} \oplus H^0 \oplus F^0$

and consider the cartesian diagram

$$\begin{tikzcd}
 & & F^{-1} \arrow[d] \arrow[r] & F^0 \arrow[d] & & \\
0 \arrow[r] & \ker \arrow[r, hook] \arrow[d, equal, "\text{locally free}"] & K^{-1} \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & H^0 \oplus F^0 =: K^0 \arrow[r] \arrow[d] & \mathrm{Coker} \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \ker \arrow[r] & E^{-1} \arrow[r, "d_E"'] & E^0 \arrow[r] & \mathrm{Coker} \arrow[r] & 0
\end{tikzcd}$$

Then $K^\cdot \rightarrow E^\cdot$ is a global resolution and we have

$$\begin{tikzcd} F^{-1} \arrow[r] \arrow[d, hook] \arrow[dr, phantom, "\circlearrowright"] & F^0 \arrow[d, hook] \\ K^{-1} \arrow[r] & K^0 \end{tikzcd}$$

Thus

$$\begin{tikzcd} F_0 \arrow[r] \arrow[dr, phantom, "\circlearrowright"] & F_1 \\ K_0 \arrow[r] \arrow[u] & K_1 \arrow[u, "u"'] \end{tikzcd} \longleftrightarrow [K_1/K_0] \xrightarrow{\cong} [F_1/F_0] = \mathfrak{E}$$

Consider now

$$\begin{tikzcd}
X \arrow[r, hook, "0"] \arrow[d, equal] \arrow[dr, phantom, "\ulcorner"] & C(K) \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & C(F^\cdot) \arrow[r] \arrow[d, hook] \arrow[dr, phantom, "\mathrm{II}"] & \mathfrak{C}_X \arrow[d] \\
X \arrow[r, hook] \arrow[rr, bend right, "0_{F_1}"'] & K_1 \arrow[r, "u"'] & F_1 \arrow[r] & {[F_1/F_0] = [K_1/K_0]}
\end{tikzcd}$$

0_{K_1}

u: projection of bundles $\Rightarrow$ smooth

Therefore

$$0_{F_1}^! [C(F^\cdot)] = 0_{K_1}^! u^! [C(F^\cdot)] = 0_{K_1}^! [C(K^\cdot)]$$

■

Examples

⓪ $E = L^\bullet_X$, X l.c.i. Then

$$\begin{tikzcd}[X, L^\bullet_X] = 0^!_N [N] = [X] \\ \arrow[u] \begin{matrix} C_{X/M} \\ \shortparallel \\ N_{X/M} \\ \shortparallel \end{matrix} \\ L^\bullet_X = [\underset{\substack{\uparrow \\ \text{locally free}}}{I/I^2} \longrightarrow i^*\Omega_M]\end{tikzcd}$$

and so

$$\begin{tikzcd} N := N_{X/M} \arrow[r, equal] \arrow[d] \arrow[dr, phantom, "\square"] & N_{X/M} \arrow[d] \\ \mathfrak{C}_X \arrow[r, equal] & \eta_X \end{tikzcd}$$

① $E^\bullet \to L^\bullet_X$ perfect obstruction theory, $h^0(E^\bullet)$ locally free + $h^1(E^\bullet) = 0$

$\Rightarrow$ X is smooth, $\mathrm{vdim}(X, E) = \dim X$ and $[X, E^\bullet] = [X]$

proof

$E^\bullet = [E^{-1} \to E^0]$. $\Rightarrow h^0(E^\bullet) = \Omega_X$ is locally free $\Rightarrow X$ smooth

$\mathrm{vdim}(X, E) = \mathrm{rk}\, E^0 - \mathrm{rk}\, E^1 = \mathrm{rk}\, h^0(E^\bullet) - \mathrm{rk}\, h^{-1}(E^\bullet) = \mathrm{rk}\, h^0(E) = \dim X$

and $\underset{\substack{\shortparallel \\ 0}}{h^{-1}(E^\bullet)} \twoheadrightarrow h^{-1}(L_X) \Rightarrow h^{-1}(L^\bullet_X) = 0 \Rightarrow E^\bullet \xrightarrow{\simeq} L^\bullet_X$ q. iso.

∎

② X smooth, $E^\bullet \to L^\bullet_X$ perfect. Then

ⓘ $h^1(E^\vee)$ is locally free

ⓘⓘ $[X, E^\bullet] = c_{top}(h^1(E^{\bullet\vee}))$

Proof

We have $0 \to K \to E^{-1} \to E^0 \to \Omega_X \to 0$

E^{-1}, E^0: locally free; Ω_X: locally free + E^0 locally free $\} \Rightarrow \mathrm{Im}(E^{-1} \to E^0)$ locally free + E^{-1} locally free $\Rightarrow K$ locally free

Then

$$0 \leftarrow K^\vee \leftarrow E_1 \leftarrow E_0 \leftarrow T_X \leftarrow 0$$

$$K^\vee = \underset{\substack{\uparrow \\ \text{locally free}}}{h^{-1}(E^{\bullet\vee})}$$

and we have

$$\begin{tikzcd} C(E^\cdot) \arrow[r, "u"] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & E_1 \arrow[d] \\ \eta_X = \mathfrak{C}_X \arrow[r, hook] & \mathfrak{E} = \left[E_1/E_0\right] \end{tikzcd}$$

Claim $\Vert$ $C(E^\cdot) = \mathrm{Im}(E_0 \to E_1)$

proof of the claim

$$\begin{tikzcd} E_0 \arrow[r, two heads] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & C(E^\cdot) \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & E_1 \arrow[d] \\ X \arrow[r] & \eta_X = \mathfrak{C}_X \arrow[r] & \mathfrak{E} \end{tikzcd}$$

$\uparrow$ surjective: locally on X $\quad \eta_X\big|_V \overset{=}{} \left[N_{V/V} / T_V \right] = \left[\star / T_V \right] = \star$

$$\begin{tikzcd} U = V \arrow[d] \\ X \end{tikzcd}$$

Therefore

$$[X, E^\cdot] = 0^!_{E_1}[C(E^\cdot)] = c_{tot}\left(\overset{E_1}{\overset{\Downarrow}{N_{X/E_1}}} \Big/ N_{\underset{\underset{\mathrm{Im}(E_0 \to E_1) = C(E^\cdot)}{\|}}{X/C(E^\cdot)}} \right) \cap [X] = c_{tot}(h^1(E_0^\cdot)) \cap [X]$$

Excess intersection formula applied to:

$$\begin{tikzcd} X \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\square"] & C(E^\cdot) \arrow[d] \\ X \arrow[r, "0"] & E_1 \end{tikzcd}$$

④. Consider a cartesian diagram of schemes

$$\begin{tikzcd} X \arrow[r, hook, "j", "J"'] \arrow[d, "g"'] \arrow[dr, phantom, "\#"] & V \arrow[d, "f"] \\ Y \arrow[r, hook, "I", "i"'] & W \end{tikzcd}$$

$\leftarrow$ smooth

$\uparrow$ reg emb of cod d

$$E^{\cdot} = \left[\overset{-1}{g^* N^{\vee}_{Y/W}} \to \overset{0}{\Omega_V|_X} \right] \longrightarrow L^{\cdot}_X$$ given by

$$\begin{tikzcd} E^{\cdot} = & g^* I/_{J^2} \arrow[r] \arrow[d] \arrow[rr, bend left] & J/_{J^2} \arrow[r] \arrow[d, phantom, "\circlearrowleft"] & \Omega_V|_X \arrow[d, equal] \\ L^{\cdot}_X = & J/_{J^2} \arrow[rr] & & \Omega_V|_X \end{tikzcd}$$

Since $g^* I/_{I^2} \twoheadrightarrow J/_{J^2}$ we have $h^0(E^{\cdot}) = \Omega_X$, $h^{-1}(E) \twoheadrightarrow h^{-1}(L^{\cdot}_X)$,

$\Rightarrow E^{\cdot} \longrightarrow L^{\cdot}_X$ perfect obstruction theory and

$$\begin{tikzcd} C_{X/V} \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\square"] & g^* N_{Y/W} \arrow[d] \\ \left[C_{X/V} / T_V|_X \right] = \mathfrak{C}_X \arrow[r, hook] & \left[g^* N_{Y/W} / T_V|_X \right] \end{tikzcd}$$

$$\Rightarrow \quad [X, E^{\cdot}] = 0^{!}_{g^* N} [C_{X/V}] = i^* [V]$$

■

⑤ • $E \to \underset{\substack{\uparrow \\ \text{smooth}}}{Y}$ vector bundle, $s : Y \longrightarrow E$ section ($\Rightarrow$ reg. emb). Then we have

$$\begin{tikzcd} X := Z(s) \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & Y \arrow[d, "0"] \\ Y \arrow[r, hook, "s"'] & E \end{tikzcd}$$

and if $E^\bullet = [g^* \underbrace{N_{Y/E}}_{\substack{\| \\ E}} \longrightarrow \Omega_Y|_X]$. Then $\|$ $[X, E^\bullet] = s^![Y]$

• If $X \hookrightarrow Y$ reg. emb $\Rightarrow \|$ $s^![Y] = [Z(s)] \cdot c_{top}\left({E|_X}/{N_{X/Y}} \right)$

• Suppose $E = \underset{\substack{\uparrow \\ rk = r'}}{E'} \oplus \underset{\substack{\uparrow \\ rk = r''}}{E''}$ and $s : X \to E'$ $\overset{\text{is transverse to } 0_{E'}}{\vee}$ Then $X = Z(s) \hookrightarrow Y$ is reg emb,

$$E|_X = E'|_X \oplus E''|_X \supset N_{X/Y}$$

$\llcorner rk = r'$ and $N_{X/Y} \subset E'|_X$

$$\Rightarrow E|_X / N_{X/Y} = E''|_X \Rightarrow \| [X, E^\bullet] = c_{top}(E''|_X) \cap [X].$$

⑥ $X = V(xy, xz) \subseteq \mathbb{P}^2$. Then $X = Z(s)$ where $\underset{\substack{\| \\ (xy, xz)}}{s} : \mathbb{P}^2 \longrightarrow \mathcal{O}(2) \oplus \mathcal{O}(2) =: E$

and $\begin{cases} xy = 0 \\ xz = 0 \end{cases} \iff \underset{\substack{\| \\ \mathbb{P}^1}}{\{x = 0\}} \cup \underset{\substack{\| \\ \{\star\}}}{\{y = z = 0\}}$.

An obstruction theory on X is that obtained considering

$$\begin{tikzcd} X \arrow[r, "i"] \arrow[d, "g^*"'] \arrow[dr, phantom, "\square"] & \mathbb{P}^2 \arrow[d, "0"] \\ \mathbb{P}^2 \arrow[r, "s"'] & \mathcal{O}(2)^{\oplus 2} = E \end{tikzcd}$$

Then $E^\bullet = [E|_X \longrightarrow \Omega_{\mathbb{P}^2}|_X]$

$\rightsquigarrow i_\star [X, E^\bullet] = i_\star s^! [\mathbb{P}^2] = c_{top}(E|_X) =$ ~~$c_{top}(\mathcal{O}_{\mathbb{P}^1}(2)^{\oplus}|_E)$~~ $4[\underset{\substack{\cap \\ \mathbb{P}^1 \subset X}}{pt}] \in A_0 X$

$\mathcal{O}_{\mathbb{P}^2}(2)^{\oplus 2}|_{\mathbb{P}^1} = \mathcal{O}_{\mathbb{P}^1}(2)^{\oplus 2}$

and $c_{top}(\mathcal{O}_{\mathbb{P}^1}(2)^{\oplus 2}) = c_{top}(\mathcal{O}_{\mathbb{P}^1}(2)) \cup c_{top}(\mathcal{O}_{\mathbb{P}^1}(2))$
$= 4[pt]$

We will see in the next lecture that $i_\star[X]^{vir}$ doesn't depend on the choice of the section of $E = \mathcal{O}(2)^{\oplus 2}$ because this is somehow invariant under deformations.

Some properties

1) Products :

ⓐ $E^\bullet \to L^\bullet_X$, $F^\bullet \to L^\bullet_Y$ perfect obstruction theories

$\Rightarrow L^\bullet_{X\times Y} = L_X \boxplus L_Y$ and $E^\cdot \boxplus F^\cdot \to L^\cdot_X \boxplus L^\cdot_Y$ is a perfect obstruction theory for $X\times Y$.

ⓑ Moreover if $E^\cdot$ and $F^\cdot$ are global res. $\Rightarrow$ $E^\cdot \boxplus F^\cdot$ is a global resolution and we have

$$[X\times Y, E^\cdot \boxplus F^\cdot] = [X,E^\cdot]\otimes[Y,F^\cdot] \quad \text{in } A_{rkE+rkF}(X\times Y)$$

2) Functoriality :

Consider a cartesian diagram of DM stacks

$$\begin{tikzcd}
X' \arrow[r, "u"] \arrow[d, "g"'] \arrow[dr, phantom, "\square"] & X \arrow[d, "f"] \\
Y' \arrow[r, "v"'] \arrow[r, phantom, "\uparrow\ \text{l.c.i.}"', yshift=-14pt] & Y
\end{tikzcd}$$

Let $E^\bullet \to L_X$ and $F^\cdot \to L^\cdot_{X'}$ be perfect obstruction theories for X and X' resp

Pb : When $v^! [X,E^\cdot] = [X',F^\cdot]$?

Def A compatibility datum relative to v for $E^\cdot$ and $F^\cdot$ is a triple
$$(\phi,\psi,\chi)$$
of morphisms in $D(\mathcal{O}_{X_{ét}})$ giving arise to a morphism of distinguished triangles

⌐ shift on the left

$$\begin{tikzcd}
u^*E^\bullet \arrow[r, "\phi"] \arrow[d] \arrow[dr, phantom, "\circlearrowright"] & F^\bullet \arrow[r, "\psi"] \arrow[d] \arrow[dr, phantom, "\circlearrowright"] & g^*L_{Y'/Y} \arrow[r, "\chi"] \arrow[d] \arrow[dr, phantom, "\circlearrowright"] & u^*E[-1] \arrow[d] \\
u^*L^\bullet_X \arrow[r] & L^\bullet_{X'} \arrow[r] & L_{X'/X} \arrow[r] & u^*L^\cdot_X[-1]
\end{tikzcd}$$

In this case we have a short exact sequence of vector bundle stacks over X'

$$\begin{tikzcd}
0 \arrow[r] \arrow[d, equal] & \eta_{X'/X} \arrow[r] \arrow[d] & \eta_{X'} \arrow[r] \arrow[d, hook] & u^*\eta_X \arrow[r] \arrow[d, hook] & 0 \arrow[d, equal] \\
0 \arrow[r] & g^* h^1/h^0(T_{Y'/Y}) \arrow[r] \arrow[d, equal] & C(F^\cdot) = h^1/h^0(F^{\cdot\vee}) \arrow[r] \arrow[d, equal] & u^*C(E^\cdot) = u^* h^1/h^0(E^{\cdot\vee}) \arrow[r] \arrow[d, equal] & 0 \\
 & g^*\eta_{Y'/Y} & \mathfrak{F} & u^*\mathfrak{E} &
\end{tikzcd}$$

$\uparrow$

$$\begin{tikzcd}
Y' \arrow[r, "v"] \arrow[d, hook, "\text{reg emb}"'] \arrow[dr, phantom, "\circlearrowright"] & Y \\
W \arrow[ur, "\text{sm}"'] &
\end{tikzcd}$$

l.c.i $\Rightarrow$ $L_{Y'/Y} = [I/I^2 \longrightarrow \Omega_Y|_{Y'}]$ $\Rightarrow$ $\eta_{Y'/Y}$ vector bundle stack over Y'

↖ ↗
locally free

Rmk v reg emb $\Longrightarrow \eta_{Y'/Y} = N_{Y'/Y} = N$

Prop If E and F have global resolutions, ~~[illegible]~~ are compatible and

① v is smooth

② or Y' and Y' are smooth

$\Rightarrow v^![X, E] = [X', F]$.

The error in the paper

Recall Vistoli Rational equivalence:

Consider $Y' \xrightarrow{v} Y$, $X \hookrightarrow Y$. Let $C = C_{X/Y}$ and $N = N_{Y'/Y}$. Consider the
reg. loc. emb ↖ smooth and equidimensional

Cartesian diagram

$$\begin{tikzcd}
N \times_Y C \arrow[r, "\gamma"] \arrow[d] \arrow[dr, phantom, "\square"] & u^*C \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & C \arrow[d] \\
g^*N \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & X' \arrow[r, "u"] \arrow[d, "j"] \arrow[dr, phantom, "\square"] & X \arrow[d, "i"] \\
N \arrow[r, "g"'] & Y' \arrow[r, "v"] & Y
\end{tikzcd}$$

Vistoli proved (for schemes, but the result is also true for Artin/DM stacks by 'Canonical rational equivalence of intersections of divisors' by A. Kresch) that there exists a canonical rational equivalence

$$\beta(Y, X) \in W_*(N \times_Y C)$$

s.t.

$$\partial\beta(Y',X) = \underbrace{\left[C_{u^*C/C}\right]}_{\substack{\cap\!| \\ \text{pullback of } N \\ \text{to } u^*C\text{, i.e.} \\ N\times_Y C}} - \gamma^*\underbrace{\left[C_{X'/Y'}\right]}_{\substack{\cap\!| \\ u^*C_{X/Y} = u^*C \\ \cap \;\leftarrow \text{ as the zero section} \\ N\times_Y C = N\times_{Y'} u^*C \;\leftarrow \text{ pullback of } N \text{ to } u^*C}}$$

Rmk

$$v^![C] = 0^!\left[C_{u^*C/C}\right] = 0^!\gamma^*\left[C_{X'/Y'}\right] = \qquad \left[C_{X'/Y'}\right] \in A_*(u^*C)$$

$$\begin{tikzcd} u^*C \arrow[r, "0"] & N\times_{Y'} u^*C = N\times_Y C \arrow[l, bend left, "\gamma"] \end{tikzcd}$$

■

Consider now

$$T_X \subset T_Y|_X \circlearrowright C = C_{X/Y} \subset N_{X/Y}$$

$\rightsquigarrow$ $T_X|_{X'} \circlearrowright C|_{X'} = u^*C$ and moreover we have $T_X|_{X'} \underset{\substack{\uparrow \\ \text{Trivial action}}}{\circlearrowright} N|_{X'} = \rho^*N$

$\rightsquigarrow$ $T_X|_{X'} \circlearrowright \rho^*N \times_{X'} u^*C = N\times_Y C$

Thm (A. Kresch)

The rational equivalence $\beta(Y',X)$ is invariant under the action of $T_X|_{X'}$.

Therefore we have

$$\bar\beta(Y,X) \in W_*\left(\left[{}^{N\times_Y C}/_{T_X|_{X'}}\right]\right)$$

s.t.

$$\partial\bar\beta(Y,X) = \left[{}^{C_{u^*C/C}}/_{T_X|_{X'}}\right] - \left[{}^{\gamma^* C_{X'/Y'}}/_{T_X|_{X'}}\right]$$

This fact is exploited in Lemma 5.9 of [B-F] where they invoke the uncorrect stronger claim appearing in Prop 3.5 that the rational equivalence is equivariant for the bigger group $T_Y|_{X'}$.

Partial proof of functoriality

Lemma (5.9 in [BF])

Consider

$$\begin{tikzcd}
& \mathfrak{F} \supset \mathfrak{C}_{X'} \arrow[d] & \mathfrak{E} \supset \mathfrak{C}_X \arrow[d] \\
& X' \arrow[r, "u"] \arrow[d, "g"'] \arrow[dr, phantom, "\square"] & X \arrow[d, "f"] \\
N \arrow[r] & Y' \arrow[r, "v"'] & Y \\
& \text{smooth} \arrow[u] & \text{smooth} \arrow[u]
\end{tikzcd}$$

reg. emb

. Then $\exists$ canonical rational equivalence $\beta \in W_* g^*N \times_{X'} \mathfrak{F})$

s.t.

$$\partial\beta(Y', X) = \left[\phi^* C_{u^*\mathfrak{C}_X/\mathfrak{C}_X}\right] - \left[\underbrace{g^*N \times_{X'} \mathfrak{C}_{X'}}_{\substack{\cap \\ g^*N \times_{X'} \mathfrak{F}}}\right]$$

(we we recall that we have an exact sequence of vector bundles stacks on X'

$$0 \longrightarrow g^*N \longrightarrow \mathfrak{F} \xrightarrow{\phi} u^*\mathfrak{E} \longrightarrow 0$$

)

Note that

$$\begin{tikzcd}
\mathfrak{F} \times_{X'} g^*N \arrow[d, "\phi \times \mathrm{id}"] \arrow[drrrr, phantom, "\square"] & & & & \phi^* C_{u^*\mathfrak{C}_X/\mathfrak{C}_X} \arrow[llll, hook'] \arrow[d] \\
u^*\mathfrak{E} \times_{X'} g^*N & \supset & u^*\mathfrak{C}_X \times_{X'} g^*N \arrow[d, equal] & \supset & C_{u^*\mathfrak{C}_X/\mathfrak{C}_X} \\
& & u^*\mathfrak{C}_X \times_{Y'} N & &
\end{tikzcd}$$

$\uparrow$ Pullback of N to $u^*\mathfrak{C}_X$

Proof of Lemma $\Longrightarrow$ Prop when Y, Y' smooth + v reg. emb

~~Consider~~

~~Claim We can choose global resolutions $[E^{-1} \to E^0]$ of $E^\bullet$ and $[F^{-1} \to F^0]$ of $F^\bullet$ together with epimorphisms $\phi_0 : F_0 \to u^*E_0$, $\phi_1 : F_1 \to u^*E_1$ s.t. $\phi = (\phi_0, \phi_1)$.~~

mk

Let $[E^{-1} \to E^0]$ be a global resolution of $E^{\bullet}$. Consider

$$\begin{tikzcd}
0 \arrow[r] & N \arrow[r] \arrow[d, equal] & F_1 \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & u^*E_1 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & N \arrow[r] & \mathfrak{F} \arrow[r] & u^*\mathfrak{E} \arrow[r] & 0
\end{tikzcd}$$

Then ~~[illegible]~~ Zariski-locally on X' $\mathfrak{F} = N \times_{X'} u^*\mathfrak{E}$ so $F_1 = N \times_{X'} u^*E_1$

$\Rightarrow F_1$ is a scheme. Moreover F_1 is a vector bundle stack over X'

and so it is a vector bundle on X'.

~~We~~ Then if

$$\begin{tikzcd}
F_0 \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\square"] & F_1 \arrow[d] \\
X \arrow[r, hook, "0"'] & \mathfrak{F}
\end{tikzcd}$$

$\Rightarrow F_0$ is a scheme, $F_0 \to X$ is smooth and F_0 is a cone ~~[illegible]~~ stack over X

$\Rightarrow F_0 \to X$ is a vector bundle

and $\mathfrak{F} = [F_1/F_0] = h^1/h^0([F_0 \to F_1])$

$\quad\; \| $

$h^1/h^0(F^{\bullet\vee})$

$\Rightarrow F^{\bullet\vee} = [F_0 \to F_1]$

Prop 2.6 says that $h^0(\phi)$ and $h^1(\phi)$ must be isomorphisms

Since we know that $F^{\bullet\vee} = [\tilde{F}_0 \to \tilde{F}_1]$ for some $\tilde{F}_0 \to \tilde{F}_1$

we have $F^{\bullet\vee} = [F_0 \to F_1]$.

■

Rmk ~~Lemma [illegible]~~: Call

$$\begin{tikzcd}
C_1 \arrow[r, phantom, "\subset"] \arrow[d] \arrow[dr, phantom, "\square"] & E_1 \arrow[d] \\
\mathfrak{C}_X \arrow[r, phantom, "\subset"] & \mathfrak{E}
\end{tikzcd}$$

and

$$\begin{tikzcd}
D_1 \arrow[r, phantom, "\subset"] \arrow[d] \arrow[dr, phantom, "\square"] & F_1 \arrow[d] \\
\mathfrak{C}_{X'} \arrow[r, phantom, "\subset"] & \mathfrak{F}
\end{tikzcd}$$

. Then

Lemma $\Rightarrow$ $[g^*N \times_{X'} D_1] = [\phi^* C_{u^*C_1/C_1}]$ in $A_*(g^*N \times_{X'} F_1)$

all Cartesian

$$\begin{tikzcd}
 & F_1 \times_{X'} g^*N \arrow[d] \arrow[ddl] \arrow[rr, leftrightarrow, "T"] & & \phi^*C_{u^*C_1/C_1} \arrow[dll] \arrow[ddl] \\
 & u^*E_1 \times_{X'} g^*N \arrow[ddl] & & C_{u^*C_1/C_1} \arrow[dl] \\
\mathfrak{F} \times_{X'} g^*N \arrow[d] \arrow[rr, leftrightarrow] & & \phi^*C_{u^*\mathfrak{C}_X/\mathfrak{C}_X} \arrow[d] & \\
u^*\mathfrak{E} \times_{X'} g^*N \supset u^*\mathfrak{C}_X \times_{X'} g^*N \supset & & C_{u^*\mathfrak{C}_X/\mathfrak{C}_X} &
\end{tikzcd}$$

using the fact that if

$$\begin{tikzcd}
Z' \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & X' \arrow[d, "\text{étale}"] \\
Z \arrow[r, hook] & X
\end{tikzcd}$$

$\Rightarrow C_{Z/X} \times_Z Z' = C_{Z'/X'}$

Therefore

$$[X',F^\circ] = 0^!_{F_1}[D_1] = 0^!_{g^*N\times_{X'}F_1}[g^*N\underset{X'}{\times} D_1] = 0^!_{g^*N\times_{X'}F_1}[\phi^* C_{u^*C_1/C_1}] =$$

$$= 0^!_{g^*N\times_{X'}u^*E_1}[C_{u^*C_1/C_1}] = 0^!_{u^*E_1}\, v^![C_1) \overset{\substack{\text{commutativity}\\ \downarrow}}{=} v^!\, 0^!_{E_1}[C_1] = v^![X,E^\circ]$$

$$\begin{tikzcd}
g^*N \underset{X'}{\times} u^*E_1 \arrow[r, phantom, "="] & N \underset{Y'}{\times} u^*C_1 \arrow[r] \arrow[dd] & u^*C_1 \arrow[r] \arrow[l, bend right, "0"'] \arrow[d, phantom, "\cap"] \arrow[dr, phantom, "\square"] & C_1 \arrow[d, phantom, "\cap"] \\
 & & u^*E_1 \arrow[r] \arrow[d] & E_1 \arrow[d] \\
 & g^*N \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & X' \arrow[r, hook] \arrow[d, "g"'] \arrow[uul, "0"] \arrow[u, bend right, "0"'] \arrow[ur, phantom, "\square"] \arrow[dr, phantom, "\square"] & X \arrow[d] \arrow[u, bend right, "0"'] \\
 & N \arrow[r] & Y' \arrow[r, hook] & Y
\end{tikzcd}$$

Lecture VIII (Speaker: Miguel Moreira)

Relative obs theories & virtual Pull-backs

Based on "virtual pull-backs" by Cristina Manolache

Motivation

C fixed nodal curve, X = smooth proj. variety

$$M(C,X) = \text{maps } C \longrightarrow X$$

has a perfect obstruction theory

$$(R\pi_* f^* T_X)^\vee \longrightarrow L^\bullet_{M(C,X)}$$

where

$$\begin{tikzcd} M\times C \arrow[r] \arrow[d, "\pi"'] & X \\ M \end{tikzcd}$$

$$\rightsquigarrow [M(C,X)]^{vir} \in A_*(M(C,X))$$

What if we want to do this in families? i.e. we want $[\bar{\mathcal{M}}_{gn}(X,\beta)]^{vir}$

We have

$$\begin{tikzcd} \bar{\mathcal{M}}_{g\,n+1}(X,\beta) \arrow[r, "f"] \arrow[d, "\pi"'] & X \\ \bar{\mathcal{M}}_{gn}(X,\beta) \end{tikzcd}$$

and we can still write

$$(R\pi_* f^* T_X)^\vee \longrightarrow L^\bullet_{\bar{\mathcal{M}}_{gn}(X,\beta)/\mathfrak{M}_{gn}}$$

$$\rightsquigarrow [\bar{\mathcal{M}}_{gn}(X,\beta)]^{vir} = ?$$

Manulasche : $f : F \longrightarrow G$ map of stacks with wild hypothesis.
$E_f^\bullet \longrightarrow L_f^\bullet$ relative obs. theory $\leadsto$ $f^!_{E_f^\bullet} : A_\star(G) \longrightarrow A_{\star + rk(E_f^\bullet)}(F)$

In our case we have

$$g : \overline{\mathcal{M}}_{gn}(X,\beta) \longrightarrow \mathfrak{M}_{gn}$$

and

$$\left(R\pi_\star f^\star T_X\right)^\vee \longrightarrow L_{\overline{\mathcal{M}}_{gn}(X,\beta)/\mathfrak{M}_{gn}}$$

and so $g^![\mathfrak{M}_{gn}] =: \left[\overline{\mathcal{M}}_{gn}(X,\beta)\right]^{vir}$

Detour's : Kresch's Artin Stacks (Reference: 'Cycle groups for Artin Stacks' by A. Kresch)

Chow Artin Stacks is tricky:
The naive idea would be : $A^\circ_\star(F) = \dfrac{\text{cycles}}{\text{rational equivalence}}$
$\uparrow$
has not nice properties.
For example: $\begin{array}{c} E \\ \pi \downarrow {\scriptstyle v.b \atop rk\,m} \\ X \end{array}$ $\pi^\star : A^\circ_\star X \xrightarrow{\sim} A_{\star+m} E$
is NO-longer true.

Kresch : $\hat{A}_\star(F) = \lim\limits_{\substack{E \\ \downarrow v.b \\ X}} A^\circ_\star(E)$ for F connected

Now $\hat{A}_\star(F)$ does not have proper pushforward. So

$$A_\star(F) = \lim_{Y \rightarrow F} \hat{A}_\star(Y) \Big/ \sim$$

connected $\nearrow$ Y, $\uparrow$ proper

The properties of $A_\star$

$\exists$ " functor "

$$A_\star : \begin{matrix}\text{Artin Stacks}\\ \text{finite type}/k\end{matrix} \longrightarrow \begin{matrix}\text{Graded abelian}\\ \text{groups}\end{matrix}$$

which is related to the "functors" $A^\circ_\star$ and $\hat{A}_\star$ via

$$A^\circ_\star(X) \to \hat{A}_\star(X) \longrightarrow A_\star(X)$$

Moreover we have that $A_\star$:

1) it is a contravariant functor for morphisms which are flat of constant rel dim:

$$f : X \xrightarrow[\text{flat of rel dim } n]{} Y \quad \rightsquigarrow \quad f^\star : A_\star Y \longrightarrow A_{\star+n} X$$

2) it is covariant for proper pushforward

$$f : X \xrightarrow[\text{proper}]{} Y \quad \rightsquigarrow \quad f_\star : A_\star X \longrightarrow A_\star Y$$

3) If X is smooth and satisfies (#) $\Rightarrow$ $A_\star X$ has a product

4) If X is an alg. space $\Rightarrow$ $A^\circ_\star(X) \xrightarrow{\sim} A_\star(X)$ is an iso of groups.
If in addition X is smooth it is an iso of rings.

5) If X is a DM-stack $\Rightarrow$ $A^\circ_\star(X)\otimes_{\mathbb{Z}}\mathbb{Q} \xrightarrow{\sim} A_\star(X)\otimes_{\mathbb{Z}}\mathbb{Q}$ is an iso of groups.
If in addition X is smooth $\Rightarrow$ it is an iso of rings
↳ note that DM-stacks are stratified by quotients

6) Excission holds: $Z \subset X$ closed substack, $U := X \setminus Z$ then

$$A_j Z \to A_j X \to A_j U \to 0$$

is exact

7) $A_j X = 0 \quad \forall j > \dim X$

8) $E \to X$ v.b of rk e $\Rightarrow$ $\pi^\star : A_j X \xrightarrow{\sim} A_{j+n} E$

9) $E \to X$ v.b of rk e, $p : \mathbb{P}(E) \to X$

$$\Rightarrow \theta_E : \bigoplus_{i=0}^{e-1} A_{j-(e-1)+i} X \xrightarrow{\sim} A_j\, \mathbb{P}(E)$$

$$(\dots, \alpha_i \dots) \longmapsto \sum_{i=1}^{e-1} c_1(\mathcal{O}_E(1))^i \cap p^\star \alpha_i$$

10) There are Segre and Chern classes of vector bundles and these satisfies the usual universal identities

11) There are Gysin maps for lci morphisms whenever $f: X \to Y$ and X is, which are functorial, commute with each other and are compatible with flat pullbacks and proper push forward

12) If X satisfies $(\dagger)$ and $\pi: \mathcal{E} \longrightarrow X$ is a vector bundle stack with virtual $\mathrm{rk}\, e (= \mathrm{rk}\, E_1 - \mathrm{rk}\, E_0$ if $\mathcal{E} = [E_1/E_0]$ $) \Rightarrow \pi^*: A_j X \xrightarrow{\sim} A_{j+e} \mathcal{E}$ is an iso

note that this is flat of const rel dim $= e$

Indeed we have

$$\begin{tikzcd} \bigsqcup [E_1/E_0] \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \bigsqcup U_i \arrow[d, "\text{étale covering}"] \\ \mathcal{E} \arrow[r] & X \end{tikzcd}$$

$\Rightarrow$ faithfully flat

and $[E_1/E_0] \longrightarrow U_i$ is flat because

$$\begin{tikzcd} E_1 \arrow[r, "\text{faithfully flat}"] \arrow[rr, bend right, "\text{flat}"'] & {[E_1/E_0]} \arrow[r] & U_i \end{tikzcd}.$$

$(\dagger)$ = stratified by locally closed substacks which are global quotients:

$$X = \bigsqcup \left[{}^{\text{alg stack}} / {}_{\text{Alg. group}} \right]$$

$\cap$ loc. closed

X

Example X DM stack $\Rightarrow X$ satisfies $(\dagger)$.

■

⚠ ~~Let X projective Artin stack (i.e. p: X → Spec(k) proper morphism, which~~

④

DM-type morphisms

Def A morphism $f: F \longrightarrow G$ of finite type/$\mathbb{C}$ Artin stacks is called of DM-type if one of the following equivalent conditions hold:

(i) For any
$$\begin{tikzcd} F' \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \overset{\text{scheme}}{G'} \arrow[d] \\ F \arrow[r, "f"'] & G \end{tikzcd}$$
$\Rightarrow$ F' is DM-stack over G'

(ii) $L_f^\bullet \in D^{\leq 0}(\mathcal{O}_{X_{F_{et}}})$ (i.e. $h^i(L_f)=0$ for $i>0$)

(ii) $L_f^\bullet$ satisfies ⊛

(iii) $F \xrightarrow[\Delta]{} F \times_G F$ is unramified ~~and~~ representable

Normal cones to DM-type morphisms

Lemma $f: F \longrightarrow G$ DM-type $\Rightarrow \exists$
$$\begin{tikzcd} & \text{schemes} \arrow[dl] \arrow[dr] & \\ U \arrow[rr, hook, "\text{closed emb}"'] \arrow[d, "\text{smooth + surj}"'] & & V \arrow[d, "\text{smooth + surj}"] \\ F \arrow[rr, "f"'] & \circlearrowright & G \end{tikzcd}$$

When F is a DM-stack, ~~since~~ we can choose it to be étale

reason: let
$$\begin{tikzcd} U' \arrow[r, "\text{étale}"] \arrow[d] \arrow[dr, phantom, "\square"] & U \arrow[d] \\ F' \arrow[r, "\text{étale}"'] \arrow[u, bend left, "\exists\text{ section}"] & F \end{tikzcd}$$
Then
$$\begin{tikzcd} F' \arrow[r, "a"] \arrow[d, "\text{étale}"'] \arrow[rrr, bend right, "\circlearrowright\text{ unramified}"'] & U' \arrow[r] & U \arrow[r] & V \arrow[d] \\ F \arrow[rrr] & & & G \end{tikzcd}$$
and since $F' \to V'$ is unramified
$$\begin{tikzcd} \exists\ F'' \arrow[r, hook, "\text{closed emb}"] \arrow[d, "\text{étale}"'] \arrow[dr, phantom, "\circlearrowright"] & V' \arrow[d, "\text{étale}"] \\ F' \arrow[r] & V \end{tikzcd}$$

Since $F \xrightarrow[\Delta]{} F\times_{\mathbb{C}} F$ and $G \xrightarrow[\Delta]{} G\times_k G$ are repr, these are automatically DM stacks

Now call $R := U \times_F U \longrightarrow V \times_G V = S$

↑ locally closed embedding, because
$$\begin{tikzcd} U\times_F U \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & U\times_G U \arrow[r, hook, "\text{closed emb}"] \arrow[d] & V\times_G V \\ F \arrow[r, "\Delta"'] & F\times_G F & \end{tikzcd}$$
↑ unramified

↑ In the language of stack this means that it is unramified and representable

Vistoli defined the cone $C_{R/S}$ for
$$\begin{tikzcd} R \arrow[r, "\text{unramified}"] & S \\ & \text{DM} \arrow[ul] \arrow[u] \end{tikzcd}$$

Prop $\exists\ C_{R/S} \rightrightarrows C_{U/V}$ smooth groupoid

Def $\quad \underset{\displaystyle \overset{\cap}{N_{F/G}}}{C_{F/G}}$ $:=$ stack associated to the groupoid $[C_{R/S} \rightrightarrows C_{U/V}]$

$N_{F/G} :=$ stack associated to the groupoid $[N_{R/S} \rightrightarrows N_{U/V}]$

$\uparrow$ constructed similarly

Def $\quad f : F \to G$ DM-type , $L^{\bullet}_{F/G} \in D(\mathcal{O}_{F_{et}})$

Define $\quad \mathfrak{N}_{F/G} := h^1/h^0(L^{\vee}_{F/G}) \longrightarrow F$

$\uparrow$ cone stack over F (being f of DM-type)

Rmk When in addition F is a DM-stack,

[B-F] $\Rightarrow$ $\exists \ \mathfrak{C}_{F/G} \underset{\substack{\uparrow \\ \text{closed} \\ \text{emb}}}{\subset} \mathfrak{N}_{F/G}$ s.t.

for all diagrams

$$\begin{tikzcd} U \arrow[r, hook, "\text{closed emb}", "\tilde{f}"'] \arrow[d, "\text{étale}"'] \arrow[dr, phantom, "\circlearrowright"] & V \arrow[d, "\text{smooth}"] \\ F \arrow[r, "f"'] & G \end{tikzcd}$$

$$\begin{tikzcd} \mathfrak{C}_{F/G}|_U \arrow[r, phantom, "\cong"] \arrow[d, hook] & \left[C_{U/V} / \tilde{f}^* T_V \right] \arrow[d, phantom, "\cap"] \\ \mathfrak{N}_{F/G}|_U \arrow[r, phantom, "\cong"] & \left[N_{U/V} / \tilde{f}^* T_V \right] \end{tikzcd}$$

With some work one can define $\mathfrak{C}_{F/G}$ for all $f : F \to G$ of DM-type (removing the hp that F is a DM-stack)

Prop $\quad f : F \longrightarrow G$ DM type . Then

ⓐ $N_{F/G} \cong h^1/h^0(L^{\bullet}_{F/G}) = \mathfrak{N}_{F/G}$

ⓑ If F is DM-stack $\Rightarrow$ $\mathbf{C}_{F/G} \cong \mathfrak{C}_{F/G}$

■

Relative perfect obstruction theories

Def Let $f: F \longrightarrow G$ be of DM type is

$$E_f^\bullet \xrightarrow{\phi} L_{F/G}^\bullet$$

where:

- ⓘ $E_f^\bullet$ is perfect of amplitude $[-1,0]$
- ⓘⓘ $h^0(\phi)$ is an iso
- ⓘⓘⓘ $h^{-1}(\phi)$ is surjective

Def We say that (f, ϕ) is a <u>virtually smooth map</u> if in addition F satisfies $(\dagger)$

Rmk ‖ If $E_f^\bullet$ is actually perfect of amplitude just 0 (i.e. $E_f^\bullet = [0 \to E_f^0 \to 0]$) ~~then~~ then $\Omega_{F/G} = h^0(L_f^\bullet) = h^0(E_f^\bullet) = E_f^0$ is locally free $\Longleftrightarrow$ f is smooth.

This is always true if for example G' satisfies $(\dagger)$

<u>Reason</u>: $\left.\begin{array}{l} f': F' \longrightarrow G' \text{ is of DM type} \\ + G' \text{ satisfies } (\dagger) \end{array}\right\} \Longrightarrow F'$ satisfies $(\dagger)$

Rmk 3.2 in Cristina's Paper

Thm

If (f, ϕ) is virtually smooth

$\Rightarrow \exists \; f^!_{E_f^\bullet} =: f^! : A_\star G \longrightarrow A_{\star + rk(E_f^\bullet)} F$

More generally, if $G' \longrightarrow G$ is s.t. $F' := F \times_G G'$ ~~admits~~ satisfies $(\dagger)$

then we have

$$f^!_{E_f^\bullet} : A_\star G' \longrightarrow A_{\star + rk(E_f^\bullet)}(F')$$

Sketch of Proof

Step 1 ‖ $\exists \; \mathfrak{C}_{F/G} \underset{\text{closed emb}}{\hookrightarrow} \mathcal{E}_f := h^1/h^0(E_f^\bullet)$

This ~~is quite~~ follows from Prop 2.6 in [B-F]

Step 2 One construct $f^!$ as composition

$$f^! : A_\star(G) \xrightarrow{\sigma} A_\star(\mathfrak{C}_f) \xrightarrow{i_\star} A_\star(\mathfrak{E}_f) \xleftarrow{\simeq} A_{\star + rk(E_f)}(F)$$

$\uparrow$ to be defined

Definition of σ at the level of cycles:

$$\sigma^0 : A^0_\star G \longrightarrow A_\star(\mathfrak{C}_f)$$

$$\sum n_i [V_i] \longmapsto \sum n_i [\mathfrak{C}_{V_i \times_G F / V_i}]$$

where we are using the following

Proposition Consider

$$\begin{tikzcd} F' \arrow[r] \arrow[d, "p"'] & G' \arrow[d, "q"] \\ F \arrow[r, "f"'] & G \end{tikzcd}$$

Artin stacks

DM type

$\Rightarrow \exists\ \alpha : \mathfrak{C}_{F'/G'} \longrightarrow p^\star \mathfrak{C}_{F/G}$.

Moreover,

- if the diagram is cartesian $\Rightarrow \alpha$ is closed emb
- if the diagram is cartesian and q is flat $\Rightarrow \alpha$ is an iso.

so if $G' = V_i \hookrightarrow G$ we have $\mathfrak{C}_{V_i \times_G F / V_i} \subset \mathfrak{C}_{F/G}$

We want to use σ^0 to define σ.

Solution:

Thm: Let $f : F \to G$ DM-type of Artin stacks

$\Rightarrow$ $\exists\ M^0_F(G) = M \xrightarrow[\text{flat}]{p} \mathbb{P}^1$ s.t

(i) $p^{-1}(\infty) = C_{F/G} = \mathfrak{C}_{F/G}$

(ii) $M|_{\mathbb{P}^1 \setminus \{\infty\}} = (\mathbb{P}^1 \setminus \{\infty\}) \times G$

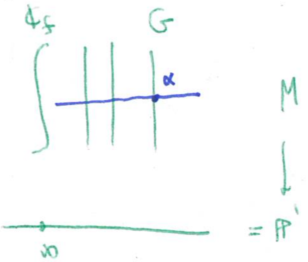
Using this then one defines

$$\begin{tikzcd}
 & & A_{\star}(G) \arrow[d, equal] & \\
 & & A_{\star+1}(G\times\mathbb{A}^1) \arrow[d, equal] & \\
A_{\star+1}(\mathbb{C}_f) \arrow[r] \arrow[dr] & A_{\star+1}(M) \arrow[r] \arrow[d, "s^!"] & A_{\star+1}(M\setminus\mathbb{C}_f) \arrow[r] \arrow[dl, dashed, "\exists\sigma"] & 0 \\
 & A_{\star}(\mathbb{C}_f) & &
\end{tikzcd}$$

$$\mathbb{C}_f \underset{\uparrow}{\sim} p^{-1}(0) \longrightarrow 0$$

rationally equivalent

← idea:



and $\sigma(\alpha) = \mathbb{C}_f \cap \overline{\alpha\times\mathbb{A}^1}$

where $0 : \underset{\underset{p^{-1}(\infty)}{\|}}{\mathbb{C}_f} \hookrightarrow M$ is the inclusion (⊨) it is a regular embedding $\overset{\text{of cod } 1}{\Gamma}$ being the zero of $\frac{1}{p}$)

<u>Simplest example</u>

Suppose $G = pt$. Then given a DM stack F and an obstruction theory $E^\circ \to L^\circ_F$ of F we get

$$\begin{tikzcd}
f^! : A_\star(pt) \arrow[r] & A_{\star+rk(E^\circ)}(F) \\
1 \arrow[u, phantom, "\in"] \arrow[r, mapsto] & f^!(1) = [F, E^\circ]
\end{tikzcd}$$

$$\uparrow$$

reason:

$$\begin{tikzcd}
\mathbb{Z} = A_\star(G) \arrow[r, "\sigma"] & A_\star(\mathbb{C}_F) \arrow[r, "i_\star"] & A_\star(\mathfrak{E}_F) \arrow[r, leftrightarrow, "\cong"] & A_{\star+rk(E^\circ)}(F) \\
1 \arrow[r, mapsto] & \sigma(1) = [\mathbb{C}_F] \arrow[r, mapsto] & {[\mathbb{C}_F]} \arrow[r, mapsto] \arrow[d, phantom, "\cap"] & 0^![\mathbb{C}_F] \\
 & & \mathfrak{E}_F &
\end{tikzcd}$$

Lecture VIII bis (Speaker: Miguel Moreira)

Last time

Def $f : F \longrightarrow G$ DM-type, F satisfies $(\dagger)$ with a relative perf obs theory

$\overset{\nearrow\ \nwarrow}{\text{Artin Stacks}}$

$$E_f^\bullet \longrightarrow L_f^\bullet$$

Main goal was: construct

$$f^!_{E_f} : A_*(G) \longrightarrow A_*(F)$$

Examples

① $f : X \to * = G$ virtually smooth $\Rightarrow [X, E^\cdot] = f^!_{E^\cdot}\,[*]$

② $f : F \longrightarrow G$ l.c.i morphism of DM type, F satisfies $(\dagger)$

$\Rightarrow L_f^\bullet \xrightarrow[\mathrm{id}]{} L_f^\bullet$ rel. perf obs. theory and $f^!_{L_f^\bullet} = f^!$ usual Gysin map for lci morphisms

proof

$$f^! : A_*(G) \to A_*(\mathfrak{C}_f = \mathfrak{N}_f) \xrightarrow{\mathrm{id}} A_*(\underset{\substack{\uparrow \\ \text{vector bundle over } F}}{\mathfrak{N}_f}) \xleftarrow{\simeq} A_*(F)$$

$$[V] \mapsto [C_{V \times_G F / V}] \longmapsto 0^![C_{V \times_G F / V}] = f^![V]$$

■

* * *

Virtual pullback compares virtual fundamental classes

X, Y DM stack with $E_X^\bullet, E_Y^\bullet$ perfect ob. theories on X and $Y \rightsquigarrow [X]^{\mathrm{vir}}, [Y]^{\mathrm{vir}}$

Consider

$$f : X \longrightarrow Y$$

Q) Can we say when $\underset{\uparrow}{f^!}[Y]^{vir} = [X]^{vir}$?

We need a rel perf obs theory on X/Y to define it.

Suppose we have $\phi : f^*E_Y \to E_X^\bullet$ as below:

$$\begin{tikzcd}
 & & \mathrm{Cone}(\phi) \arrow[d, equal] & \\
f^*E_Y^\bullet \arrow[r, "\phi"] \arrow[d] \arrow[dr, phantom, "\circlearrowright"] & E_X^\bullet \arrow[r] \arrow[d] \arrow[dr, phantom, "\circlearrowright"] & E_f^\bullet \arrow[r] \arrow[d] \arrow[dr, phantom, "\circlearrowright"] & f^*E_Y^\bullet[-1] \arrow[d] \\
f^*L_Y^\bullet \arrow[r] & L_X^\bullet \arrow[r] & L_{X/Y}^\bullet \arrow[r] & f^*L_Y[-1]
\end{tikzcd}$$

Q) Is $E_f^\bullet$ a relative perfect obstraction theory ?

A) Not in general.

Reason: We get a long exact sequence

$$\begin{tikzcd}[column sep=small]
0 \arrow[r] & h^{-2}(E_f) \arrow[r] \arrow[d] \arrow[dr, phantom, "\circlearrowright"] & h^{-1}(f^*E_Y) \arrow[r] \arrow[d, two heads] \arrow[dr, phantom, "\circlearrowright"] & h^{-1}(E_X^\bullet) \arrow[r] \arrow[d, two heads] \arrow[dr, phantom, "\circlearrowright"] & h^{-1}(E_f) \arrow[r] \arrow[d, two heads] \arrow[dr, phantom, "\circlearrowright"] & h^0(f^*E_Y) \arrow[r] \arrow[d, "\cong"] \arrow[dr, phantom, "\circlearrowright"] & h^0(E_X) \arrow[r] \arrow[d, "\cong"] \arrow[dr, phantom, "\circlearrowright"] & h^0(E_f) \arrow[r] \arrow[d, "\cong"] & 0 \\
h^{-2}(L_X^\bullet) \arrow[r] & h^{-2}(L_{X/Y}^\bullet) \arrow[r] & h^{-1}(f^*L_Y^\bullet) \arrow[r] & h^{-1}(L_X^\bullet) \arrow[r] & h^{-1}(L_{X/Y}^\bullet) \arrow[r] & h^0(f^*L_Y^\bullet) \arrow[r] & h^0(L_X^\bullet) \arrow[r] & h^0(L_{X/Y}^\bullet) \arrow[r] & {}
\end{tikzcd}$$

And also

$$\underbrace{E_f^{-i} = 0 \text{ for } i \cancel{\geq} 2,\ E_f^{-2} = f^*E_?^{-1},\ E_f^{-1} = E_X^{-1} \oplus f^*E_Y^0,\ E_f^0 = E_X^0,\ E_f^i = 0 \text{ for } i > 0}_{\text{all locally free}}$$

If we know that $h^{-2}(E_f^\bullet) = 0$ for some reason then since

$$E_f^\bullet = 0 \to E_f^{-2} \to E_f^{-1} \to E_f^0 \to 0$$

this means that $E_f^{-2} \subset E_f^{-1}$ is an inclusion. Locally on X we can write $E^{-1} = E^{-2} \oplus \tilde{E}^{-1}$

and $E_f^\bullet = \underset{\text{locally free}}{\tilde{E}^{-1} \longrightarrow E^0} \Rightarrow E_f^\bullet \to \mathbb{L}_f^\bullet$ is perfect obst. theory. ?

∎

Example If Y is smooth and $E_Y^\bullet = L_Y^\bullet \Rightarrow h^{-1}(f^*E_Y^\bullet) = 0 \Rightarrow h^{-2}(E_f^\bullet) = 0$

∎

& if $h^{-2}(f^*E_Y)=0$, then we get

$$f^!_{E_f} : A_* Y \longrightarrow A_* X$$

and

Thm $\boxed{f^!_{E_f}[\cancel{X}]^{vir} = [X]^{vir}}$

∎

More generally: given

$$F \xrightarrow{f} G \xrightarrow{g} H$$

of DM-type, F, G satisfies (†)

Q) $(g\circ f)^! \overset{?}{=} f^!\circ g^!$

Let $E_f, E_g, E_{g\circ f}$ be p.o.t. for $f, g, g\circ f$.

Def We say that $(E_f, E_g, E_{g\circ f})$ is a <u>compatible triple</u> if

$\exists$

$$\begin{tikzcd}
f^*E^\bullet_g \arrow[r, "\alpha"] \arrow[d] \arrow[dr, phantom, "\circlearrowright"] & E^\bullet_{g\circ f} \arrow[r] \arrow[d] \arrow[dr, phantom, "\circlearrowright"] & E^\bullet_f \arrow[r] \arrow[d] \arrow[dr, phantom, "\circlearrowright"] & f^*E_g[1] \arrow[d] \\
f^*L^\bullet_g \arrow[r] & L^\bullet_{g\circ f} \arrow[r] & L^\bullet_f \arrow[r] & f^*L_g[1]
\end{tikzcd}$$

Thm $\boxed{\text{In this case } (f\circ g)^!_{E_{gf}} = f^!_{E_f}\circ g^!_{E_g}}$

∎

<u>Examples</u>

① Take $H=pt$, then $F=X$, $G=Y$ are DM stacks, $E_g = E_Y$, $E_{g\circ f}=E_X$

Then $$\underset{[X]^{vir}}{\underset{\|}{(g\circ f)^!(1)}} = f^!\,\underset{[Y]^{vir}}{\underset{\|}{\underbrace{g^![1]}}}$$

② in [BF] they ~~take~~ consider in Section 7

$$f : \overset{\text{DM stack}}{F} \longrightarrow \underset{\text{smooth}}{G}$$

$E_f^\bullet \to L_f^\bullet$ p.o.t. Then they define a class in $A_*(F)$ that for us is just $f^!_{E_f^\bullet}[G]$.

③ In section 5 of [BF] they prove that given $X \xrightarrow{f} Y$, $E_X^\bullet$, $E_Y^\bullet$ and

$$\begin{tikzcd}
f^*E_Y^\bullet \arrow[r] \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & E_X^\bullet \arrow[r] \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & L_{X/Y}^\bullet \arrow[r] \arrow[d, equal] \arrow[dr, phantom, "\circlearrowleft"] & f^*E_Y[-1] \arrow[d] \\
f^*L_Y^\bullet \arrow[r] & L_X^\bullet \arrow[r] & L_{X/Y}^\bullet \arrow[r] & f^*L_Y^\bullet[-1]
\end{tikzcd}$$

then if f is smooth or X, Y are smooth ($\Leftarrow$ f l.c.i)

$$\Rightarrow \underbrace{f^!}_{f^!_{L_{X/Y}}} [Y]^{vir} = [X]^{vir}$$

Thm $f : F \to G$ virtually smooth, then $f^!_{E_f}$ defines a class in $A^*(F \to G)$

i.e. consider a fiber square

$$\begin{tikzcd}
F'' \arrow[r] \arrow[d, "q"'] \arrow[dr, phantom, "\square"] & G'' \arrow[d, "p"] \\
F' \arrow[r, "f'"'] \arrow[d, "g"'] \arrow[dr, phantom, "\square"] & G' \arrow[d, "h"] \\
F \arrow[r, "f"'] & G
\end{tikzcd}$$

where F', F'' satisfies (†)

1) p proper and $\alpha \in A_*(G'')$ $\Rightarrow$ $f^! p_*(\alpha) = q_* f^!(\alpha)$ in $A_*(F')$

2) p flat and $\alpha \in A_*(G')$ $\Rightarrow$ $f^! p^*\alpha = q^* f^!\alpha$ in $A_*(F'')$

3) $\begin{matrix} E_f^\bullet \\ \downarrow \\ L_f^\bullet \end{matrix}$ p.o.t $\Rightarrow$ $f^!_{\mathfrak{E}_f}(\alpha) = f'^!_{g^*\mathfrak{E}_f}(\alpha)$

A result we didn't see

Explanation:

$g^* \mathfrak{E}_f \supset g^* \mathfrak{C}_{F/G} \supset \mathfrak{C}_{F'/G'}$

$\Rightarrow$ $\exists$ (unique up to q.iso) p.o.t on F'/G', sall. it $E'' \cancel{=E}$

s.t. $g^*\mathcal{E} = h^0/h^1(E'^{\vee}) \hookleftarrow \mathfrak{C}_{F'/G'}$

Example (Park)

X any scheme, $K^\bullet$ a 2-term perfect complex in $[-1,0]$

$$\mathbb{P}_X(K^\bullet) := \mathrm{Proj}_X(\mathrm{Sym}^\circ(K^{\bullet\vee})) \longrightarrow X$$

(if $K^\bullet = \mathcal{E}$ is a vector bundle $\Rightarrow \mathbb{P}_X(K^\bullet) = \mathbb{P}(\mathcal{E}) \xrightarrow[\text{smooth}]{} X$)

Claim This map is virtually smooth with natural p.o.t.

$$\mathrm{Cone}\left(\left(\mathcal{O}_{\mathbb{P}(K^\bullet)} \longrightarrow p^*K(1)\right)^\vee\right) \longrightarrow L^\bullet_{\mathbb{P}(K^\bullet)/X}$$

When K is v.b

$$\mathcal{O}_{\mathbb{P}(X)}(-1) \hookrightarrow p^*K$$

$$0 \to \mathcal{O}_{\mathbb{P}(V)} \to p^*V(1) \to \overset{\shortparallel}{T}_{\mathbb{P}(V)/X} \to 0$$

$$0 \longrightarrow \mathcal{O} \longrightarrow p^*K(1) \longrightarrow \Omega^{(\vee)}_{\mathbb{P}(X)/K} \longrightarrow 0$$

Applications (Manolache)

X, Y smooth proj varieties, $f: X \to Y$ gives $^{\text{most of the times}}$ a map

$$\bar{f}: \overline{\mathcal{M}}_{g,m}(X,\beta) \longrightarrow \overline{\mathcal{M}}_{g,n}(Y, f_*\beta)$$

One get a can get a candidate for $E^\bullet_f$. In certain conditions, $\bar{f}$ is virtually smooth.

Cases treated in the paper

- $g=0$, $X \overset{f}{\hookrightarrow} Y = \mathbb{P}^N$
- $g=0$, $X \xrightarrow{f} Y$ blow-up
- $g=0$, $X = \mathbb{P}_Y(\mathcal{E}) \xrightarrow{f} Y$

So one can compare $g=0$ GW inv on X and on Y

Lecture IX (Speaker: Alessio Cela)

Construction of $\left[\overline{\mathcal{M}}_{gn}(X,\beta)\right]^{vir}$

Notation Call $\mathcal{M} := \overline{\mathcal{M}}_{gn}(X,\beta)$, $\mathcal{C} = \overline{\mathcal{M}}_{gn+1}(X,\beta)$ and let

$$p: \mathcal{C} \longrightarrow \mathcal{M}$$

be the universal curve

Also let $\omega := \omega_p$ be the relative dualizing sheaf of p, i.e:

$$\omega = \omega_p = \det(\Omega_{\mathcal{C}/\mathcal{M}})$$

Finally let $\mathfrak{M} = \mathfrak{M}_{gn}$. We have

$$\begin{tikzcd}
\mathcal{C} \arrow[r, "f = ev_{n+1}"] \arrow[d, "p"'] & X \\
\mathcal{M} \arrow[d, "\text{recalling the domain curve}"] & \\
\mathfrak{M} &
\end{tikzcd}$$

Rmk $\omega_p = \omega_{\mathcal{C}} \otimes p^* \omega_{\mathcal{M}}^{\vee}$ [proof Take the determinants of $0 \to p^*\Omega_{\mathcal{M}} \to \Omega_{\mathcal{C}} \to \Omega_{\mathcal{C}/\mathcal{M}} \to 0$]

Some duality results

① Grothendieck-Verdier duality

$f: X \longrightarrow Y$, $d = \dim X - \dim Y$. Let $\mathcal{F}^\bullet \in D^b(X)$, $\mathcal{E}^\bullet \in D^b(Y)$. Then

↖ ↗
smooth

∃ functorial isomorphism

$$Rf_* R\mathcal{H}om(\mathcal{F}^\bullet, Lf^*(\mathcal{E}^\bullet) \otimes \underbrace{\omega_f[-d]}_{\uparrow}) \cong R\mathcal{H}om(Rf_*\mathcal{F}^\bullet, \mathcal{E}^\bullet)$$

$[\to 0 \to \underset{\uparrow}{\omega_f} \to 0 \to \ldots]$

position $-d$

Reference: Thm 3.34 in Huybrechts 'Fourier-Mukai transforms in algebraic geometry'.

∎

② Upgraded Serre duality

$f: X \longrightarrow Y$ flat, of rel dimn n, rel Gorenstein proj morphism

↑ i.e. the relative dualizing sheaf $\omega_f^\bullet = \omega_f[n]$ is a line bundle in degree $-n$

For $G^\bullet \in D^b(Y)$, $F^\bullet \in D^-(X)$ and $k \in \mathbb{Z}$

∃ canonical isomorphisms

$$\mathrm{Ext}^k_{\mathcal{O}_X}(F^\bullet, \underbrace{f^! G^\bullet}_{\overset{\shortparallel}{Lf^*G^\bullet \overset{L}{\otimes} \omega_f^\bullet}}) = \mathrm{Ext}^k_{\mathcal{O}_Y}(Rf_* F^\bullet, G^\bullet)$$

Recall ‖ instead that (Lf^*, Rf_*) ~~is~~ (for every f) an adjoint pair

$$R\mathcal{H}om_{\mathcal{O}_X}(Lf^* G^\bullet, F^\bullet) = R\mathcal{H}om_{\mathcal{O}_Y}(G^\bullet, Rf_* F^\bullet)$$

$$(\Rightarrow \mathrm{Ext}^k_{\mathcal{O}_X}(Lf^* G^\bullet, F^\bullet) = \mathrm{Ext}^k_{\mathcal{O}_Y}(G^\bullet, Rf_* F^\bullet))$$

proof $F^\bullet \cong \underset{\text{of injective}}{I^\bullet}$ in $D(X)$ ⇒ $\mathrm{Ext}^k(Lf^* G^\bullet, F^\bullet) = H^k(R\mathcal{H}om(Lf^* G^\bullet, I^\bullet)) =$

$$= H^k(R\mathcal{H}om(G^\bullet, \underbrace{Rf_* I_\bullet}_{\overset{\shortparallel}{f_* I_\bullet}})) = \mathrm{Ext}^k(G^\bullet, Rf_* F^\bullet)$$

■

Example (Upgraded SD ⇒ SD)

Take $Y = pt = \mathrm{Spec}(\mathbb{C})$, $F \in \mathrm{QCoh}(X)$, $G = \mathcal{O}_{pt}$

$$\mathrm{xt}^k_{\mathcal{O}_X}(F, \omega_X) = \mathrm{Ext}^k_{\mathcal{O}_X}(F[-n], \omega_X[-n]) = \mathrm{Ext}^k_{\mathcal{O}_{pt}}(Rf_* F[-n], \mathcal{O}_{pt}) = \mathrm{Hom}_{D(\mathcal{O}_{pt})}(Rf_* F[-n], \mathbb{C}[-k])$$

$$= \mathrm{Hom}_{D(\mathcal{O}_{pt})}(Rf_* F, \mathbb{C}[n-k]) = H^{n-k}(X, F)^\vee$$

■

Lemma

For any cartesian diagram

$$\begin{tikzcd} N \arrow[r, "g"] \arrow[d, "q"'] \arrow[dr, phantom, "\square"] & \mathcal{C} \arrow[d, "p"] & F^\circ \in D^-(\mathcal{O}_{\mathcal{C}}) \arrow[l] \\ T \arrow[r, "f"'] & \mathcal{M} & \\ G^\circ \in D^b(\mathcal{O}_T) \arrow[u] & & \end{tikzcd}$$

$\Rightarrow$ $\exists$ canonical isomorphism

$$\mathrm{Ext}^k_{\mathcal{O}_N}(Lg^* F^\circ, Lq^* G^\circ) \cong \mathrm{Ext}^k_{\mathcal{O}_T}\left(Lf^*\left(Rp_*(F^\circ \otimes \omega[-1])\right), G^\circ\right)$$

proof

$$\mathrm{Ext}^k(Lg^* F^\cdot, Lq^* G^\cdot) \underset{\text{adjointness}}{=} \mathrm{Ext}^k(F^\cdot, Rg_*(Lg^* G^\circ)) \underset{\text{p is flat}}{=} \mathrm{Ext}^k(F^\cdot, Lp^*(Rf_* G^\circ))$$

$$\underset{\text{tensoring with } \omega[-1]}{=} \mathrm{Ext}^k(F^\cdot \overset{L}{\otimes} \omega[-1], \underbrace{Lp^*(Rf_* G^\circ) \overset{L}{\otimes} \omega[-1]}_{p^!(Rf_* G^\circ)}) \underset{\text{Serre Duality}}{=}$$

$$= \mathrm{Ext}^k(Rp_*(F^\circ \overset{L}{\otimes} \omega[-1]), Rf_* G^\circ) =$$

$$\underset{\text{adjointness}}{=} \mathrm{Ext}^k(Lf^*(Rp_*(F^\circ \overset{L}{\otimes} \omega[-1])), G^\circ)$$

Definition of the obstruction theory $E^\circ \longrightarrow L^\circ_{\mathcal{M}/\mathfrak{M}}$

From the diagram

$$\begin{tikzcd} \text{Universal curve over } \mathfrak{M}_{g,n} \overset{\text{open subs}}{\supset} \mathfrak{M}_{g,n+1} \arrow[d] \arrow[dr, phantom, "\square"] & \mathcal{C} \arrow[l] \arrow[r, "f"] \arrow[d, "p"] & X \\ \mathfrak{M}_{g,n} = \mathfrak{M} & \mathcal{M} \arrow[l] & \end{tikzcd}$$

we get $\quad f^* L^\circ_X \to L^\circ_{\mathcal{C}} \longrightarrow L^\circ_{\mathcal{C}/\mathfrak{M}_{g,n+1}} \overset{\text{p flat}}{=} p^* L^\circ_{\mathcal{M}/\mathfrak{M}}$

Tensoring with $\omega[-1]$:

$$f^*L_X^\bullet \overset{L}{\otimes} \omega[-1] \longrightarrow p^*L_{\mathcal{M}/\mathfrak{M}} \overset{L}{\otimes} \omega[-1] = p^! L^\bullet_{\mathcal{M}/\mathfrak{M}}$$

Since $\quad \mathrm{Ext}^0\left(f^*L_X^\bullet \otimes \omega[-1],\ p^! L^\bullet_{\mathcal{M}/\mathfrak{M}}\right) = \mathrm{Ext}^0\left(Rp_*\left(f^*L_X^\bullet \otimes \omega[-1]\right),\ L^\bullet_{\mathcal{M}/\mathfrak{M}}\right)$

we obtain

$$E^\bullet := Rp_*\left(f^*L_X^\bullet \overset{L}{\otimes} \omega[-1]\right) \longrightarrow L^\bullet_{\mathcal{M}/\mathfrak{M}}$$

Obs $\quad Rp_*\left(f^*L_X^\bullet \overset{L}{\otimes} \omega[-1]\right) = Rp_*\left(R\mathcal{H}om\left(\mathcal{O},\ f^*L_X^\bullet \overset{L}{\otimes} \omega[-1]\right)\right) = Rp_*\left(R\mathcal{H}om\left(f^*T_X^\bullet,\ \overset{\overset{p^!\mathcal{O}}{\|}}{\omega[-1]}\right)\right)$

$$\underset{\text{GV Duality}}{=} R\mathcal{H}om\left(Rp_* f^*T_X^\bullet,\ \mathcal{O}\right) = \left(Rp_*\left(f^*T_X^\bullet\right)\right)^\vee$$

Thm $\quad$ $E^\bullet = \left(Rp_*\left(f^*T_X^\bullet\right)\right)^\vee \longrightarrow L^\bullet_{\mathcal{M}/\mathfrak{M}}$ is a perfect relative obstruction theory.

Proof of the Main Theorem

① ‖ ~~Pf~~ Since $\mathcal{M} = \overline{\mathcal{M}}_{g,n}(X,\beta)$ is a DM-stack $\longrightarrow$ $\mathcal{M} \longrightarrow \mathfrak{M}$ is of DM-type

proof

Suppose given

This must be an alg. space

~~[illegible]~~ scheme

and $S'' \to S' \to X'$ is an étale cover of X'

$$\begin{tikzcd} S'' \arrow[r, "\exists\text{étale cover}"'] & S' \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & X' \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & X \arrow[d] \\ & S \arrow[r, "\exists\text{étale cover}"'] & \mathcal{M} \arrow[r] & \mathfrak{M} \end{tikzcd}$$

scheme

scheme

② || $E^{\cdot}$ is perfect of amplitude $[-1,0]$

proof

This follows from the following

Prop (Prop 5 in 'Gromov Witten invariants in Algebraic Geometry' by K. Behrend)

~~Let~~ Consider

$$\begin{tikzcd} C \arrow[r, "f"] \arrow[d, "p"] & X \\ T \arrow[u, bend left, "s_i"] & \end{tikzcd}$$

$i=1,..,m$

a stable map over T where T is any algebraic stack. Let $E \to C$ be a vector bundle on C

Then $Rp_* E = [E^0 \to E^1]$ in $D(\mathcal{O}_T)$

vector bundles

So ② ~~from~~ follows from this prop applied with

$$\begin{tikzcd} \mathcal{C} \arrow[r, "f"] \arrow[d, "p"] & X \\ \mathcal{M} & \end{tikzcd}$$

and $E = f^* T_X \to \mathcal{C}$.

Proof of the prop

Fact || Let M be an ample invertible sheaf on X. Then

$$L := \omega_p \otimes f^* M^{\otimes 3}$$

is ample on $C|_t$ $\forall\, t \in T$.

proof See Prop 3.9 in 'Stacks of stable maps and GW invariants' by K. Behrend and Y. Manin. ■

and Fact $\Rightarrow$ $\exists N > 0$ s.t

1. $p_*(E \otimes L^N)$ is a vector bundle
2. $p^* p_*(E \otimes L^N) \twoheadrightarrow E \otimes L^N$ is surjective
3. $R^1 p_*(E \otimes L^N) = 0$
4. $\forall\, t \in T$ $H^0(C_t, L^N) = 0$

Consider now

$$\begin{tikzcd} 0 \arrow[r] & H \arrow[r] & F \arrow[r] \arrow[d, equal] & E \arrow[r] & 0 & ⊛ \\ & & p^*(p_*(E\otimes L^N))\otimes L^{-N} & & & \end{tikzcd}$$

Then H is a v. bundle ~~invertible sheaf~~ on C and $\forall t \in T$ we have

$$\underset{\underset{H^0(C_t, H)}{\cup}}{H^0(C_t, F)} = H^0(C_t, L^{-N}|_{C_t} \otimes p^* p_*(E\otimes L^N)|_{C_t}) = \underset{\underset{0}{\parallel}}{H^0(C_t, L^{-N}|_{C_t})} \otimes H^0(C_t, E\otimes L^N) = 0$$

$\Rightarrow$ $p_* F = p_* H = 0$ and $R^1_{p_*} F$, $R^1 p_* H$ are locally free.

Applying $Rp_*(-)$ to ⊛ we obtain an exact triangle

$R^i p_*(-) = 0$ for $i \geq 1$ because $C \to T$ has fibers C_t that are curves

$$\begin{tikzcd} Rp_* H \arrow[r] \arrow[d, equal] & Rp_* F \arrow[r] \arrow[d, equal] & Rp_* E \arrow[r] \arrow[d, "\cong"] & Rp_* H[-1] \\ R^1 p_* H[+1] \arrow[r, "u"] & Rp_* F[+1] \arrow[r] & \mathrm{Cone}(u) \arrow[r] & R^1_{p_*} H[-2] \end{tikzcd}$$

where $\mathrm{cone}(u)^i = Rp_* F[+1]^i \oplus Rp_* H[+1]^{i+1}$

$$\Rightarrow \mathrm{Cone}(u) = [\underset{\underset{\deg 0}{\uparrow}}{Rp_* H} \longrightarrow \underset{\underset{\deg 1}{\uparrow}}{Rp_* F}]$$

∎

③ ‖ $E^\bullet = (R p_{\mathfrak{A}*} f^* T_X^\bullet)^\vee \longrightarrow L^\bullet_{\mathcal{M}/\mathfrak{M}}$ is an obstruction theory

proof

Fact (Upgraded version of thm 4.5 in [BF])

Consider

$$\begin{tikzcd} T \arrow[r, hook, "j"] \arrow[d, "g"'] & \bar{T} \arrow[d] \arrow[dl, dashed, "\exists ? \bar{g} \ \circlearrowright"'] \\ F \arrow[r, "f"'] & G \end{tikzcd} \qquad J^2 = 0$$

f ↳ DM type

We have

$$\omega(g) : Lg^* L^\bullet_{F/G} \longrightarrow L^\bullet_{T/G} \longrightarrow L^\bullet_{T/\bar{T}} = J[-1]$$

$$\cap$$

$$Ext^1(Lg^* L^\bullet_{F/G}, J)$$

Then ‖ $[\exists \bar{g}] \Longleftrightarrow [\omega(g) = 0]$

and in this case $\{\bar{g} \text{ extensions}\}$ is a torsor under $Ext^0(Lg^* L^\bullet_{F/G}, J)$

proof of ($\Longleftarrow$)

Factors $\omega(g)$ as

$$\begin{tikzcd} Lj^* L\bar{g}^* L^\bullet_{F/G} \arrow[r, phantom, "="] \arrow[d] \arrow[dr, phantom, "\circlearrowright"] & Lg^* L^\bullet_{F/G} \arrow[d] \arrow[dr, "0"] & \\ Lj^* L^\bullet_{\bar{T}/G} \arrow[r] \arrow[rr, bend right, "0"'] & L^\bullet_{T/G} \arrow[r] & L^\bullet_{T/\bar{T}} \end{tikzcd}$$

■

Now suppose given $\phi : E^\bullet \longrightarrow L^\bullet_{F/G}$. Then we have

$$\begin{array}{ccc} Hom(Lg^* L^\bullet_{F/G}, J[-1]) & \longrightarrow & Hom(Lg^* E^\bullet, J[-1]) \\ \cup & & \cup \\ \omega(g) & \longmapsto & \phi^* \omega(g) \end{array}$$

Thm

$f : F \to G$ DM type, $E^\bullet \xrightarrow{\Phi} L^\bullet_{F/G}$. Then

$$\left[\begin{array}{c} E^\bullet \text{ is a rel. ~~perf~~} \\ \text{obstruction theory} \end{array}\right] \iff \left[\begin{array}{c} \phi^*\omega(g) = 0 \iff \exists\, \bar g \\ \text{and if } \exists\, \bar g \text{ then } \{\bar g\} \text{ form} \\ \text{a torsor under } \mathrm{Hom}(Lg^*E, J) \end{array}\right]$$

Proof of ③

We will use the theorem above. Consider

$$\begin{tikzcd} T \arrow[r, hook, "J"] \arrow[d, "g"'] & \bar T \arrow[d] \arrow[dl, dashed, "\exists?\,\bar g"'] \\ \mathcal{M} \arrow[r] & \mathfrak{M} \end{tikzcd} \quad \overset{1:1}{\longleftrightarrow} \quad \begin{tikzcd} \mathcal{C}_{\bar T} \arrow[r, phantom, "\supset"] \arrow[dd, "q_{\bar T}"'] \arrow[rrrr, dashed, bend right=15, "\exists\,\bar h?"] & \mathcal{C}_T \arrow[rr, "g_T"] \arrow[d, "q_T"] \arrow[rrr, bend left, "h"] & & \overline{\mathcal{M}}_{g,n+1}(X,\beta) = \mathcal{C} \arrow[r, "f"] \arrow[d, "p"] & X \\ & T \arrow[rr, "g"] & & \mathcal{M} \arrow[dll] & \\ \bar T \arrow[r, phantom, "\supset"] \arrow[urrr, dashed, "\exists\,\bar g?"'] \arrow[dr] & & & & \\ & \mathfrak{M} & & & \end{tikzcd}$$

We know that

$$\exists\,\bar h \iff \overset{0}{\overset{\|}{\omega(h)}} : Lh^*L^\bullet_X \longrightarrow L^\bullet_{\mathcal{C}_T} \longrightarrow L^\bullet_{\mathcal{C}_T/\mathcal{C}_{\bar T}} = q_T^*\, L^\bullet_{T/\bar T} = q_T^*\, J[-1]$$

$$\rotatebox{90}{$\in$}$$

$$\omega(g) \longmapsto \phi^*\omega(g)$$

$$\mathrm{Ext}^1(\underbrace{Lh^*}_{\overset{\|}{Lg_T^* \circ Lf^*}} L^\bullet_X, q_T^*J) \underset{\overset{\uparrow}{\text{Lemma on page ③}}}{=} \mathrm{Ext}^1(Lg^*(\underbrace{R p_*(Lf^*L^\bullet_X \otimes \omega_p[-1])}_{\overset{\|}{E^\bullet}}), J)$$

and if $\exists\,\bar h$ then $\{\bar h\}$ is a torsor under $\mathrm{Ext}^0(Lh^*L^\bullet_X, q_T^*J) = \mathrm{Ext}^0(Lg^*E^\bullet, J)$.

Def $\mathcal{M} = \overline{\mathcal{M}}_{gn}(X,\beta) \xrightarrow[\varphi]{} \mathfrak{M} = \mathfrak{M}_{gn}$ is virtually smooth with

$E^\bullet_\varphi = (R p_* (f^* T_X))^\vee \longrightarrow L^\bullet_\varphi$, so we have

$$\varphi^!_{E_\varphi} : A_*(\mathfrak{M}) \longrightarrow A_*(\mathcal{M})$$

$$\rotatebox{90}{$\in$}$$

$$[\mathfrak{M}] \longmapsto \varphi^!_{E^\bullet_\varphi}[\mathfrak{M}] =: [\overline{\mathcal{M}}_{gn}(X,\beta)]^{\mathrm{vir}}$$

Obs $\cdot$ $\mathrm{vdim}\left(\overline{\mathcal{M}}_{gn}(X,\beta)\right) = \underbrace{\dim \mathcal{M}_{gn}}_{\substack{\| \\ 3g+n-3}} + \mathrm{rk}(E^\circ)$

So we want to compute $\mathrm{rk}(E^\circ)$

Claim $\Big\|$ $\mathrm{rk}(E^\circ) = r(1-g) + \int_\beta c_1(T_X)$ $\qquad$ $\underline{\left(\Rightarrow \mathrm{vdim} = (r-3)(1-g) + n + \int_\beta c_1(T_X)\right)}$

proof

Consider

$$\begin{tikzcd}
C \arrow[r] \arrow[d, "p_C"'] \arrow[rr, bend left, "f_c"] \arrow[dr, phantom, "\square"] & \mathcal{C} \arrow[r, "f"] \arrow[d, "p"] & X \\
(C, x_i, f_c) \arrow[r, phantom, "\in"] & \mathcal{M} &
\end{tikzcd}$$

Then $\quad Rp_\star(f^\star T_X)\Big|_{(C,x_1,f)} \underset{\substack{| \\ p \text{ flat}}}{=} Rp_{C\star}(f_c^\star T_X) = \left[\ldots \to H^i(C, f_c^\star T_X) \overset{0}{\to} H^{i-1}(C, f_c^\star T_X) \to \right.$

$$= \Big[0 \to \underset{\substack{\uparrow \\ \deg 0}}{H^0(C, f_c^\star T_X)} \overset{0}{\to} \underset{\substack{\uparrow \\ \deg 1}}{H^1(C, f_c^\star T_X)} \to 0 \Big]$$

$\Rightarrow \forall\ x = (C, x_i, f) \in \mathcal{M} \quad \mathrm{rk}_x E_0 - \mathrm{rk}_x E_1 = \chi(f_c^\star T_X) \underset{RR}{=} \underbrace{\mathrm{rk}(f_c^\star T_X)}_{\substack{\| \\ r}}(1-g) + \deg(f_c^\star T$

$$= r(1-g) + \underbrace{\int_C c_1(f_c^\star T_X)}_{\substack{\| \\ c_1(T_X) \cap \underbrace{f_{c\star}[C]}_{\substack{\| \\ \beta}}}} = r(1-g)\int_\beta c_1(T_X$$

Simplest examples

① $g=0,\ X=\mathbb{P}^r \Rightarrow [\mathcal{M}_{0n}(\mathbb{P}^r,d)]^{vir} = [\mathcal{M}_{0n}(\mathbb{P}^r,d)]$.

Indeed, $E = \left(Rp_*(f^*T_{\mathbb{P}^r})\right)^\vee \longrightarrow L^\bullet_{\mathcal{M}/\mathfrak{M}}$

<u>Obs</u> $h^1(E^{\bullet\vee}) = h^1(E_\bullet) = h^1(Rp_*(f^*T_{\mathbb{P}^r})) = R^1p_*(f^*T_X) = 0$

<u>Reason</u>: It is enough to check that for all $(C, x_i, f_C) \in \overline{\mathcal{M}}_{0n}(\mathbb{P}^r,d)$

We have $H^1(C, f_C^*T_{\mathbb{P}^r}) = 0$

But $C = \mathbb{P}^1_1 \cup \dots \cup \mathbb{P}^1_k$ ~~[illegible]~~ has genus 0, so

$$\underbrace{f_C^*\underbrace{T_{\mathbb{P}^r}}_{\substack{\uparrow \\ \text{globally generated}}}\Big|_{\mathbb{P}^1_i}}_{\text{globally generated}} \cong \bigoplus_j \mathcal{O}(d_{ij}) \Longrightarrow d_{ij} \geq 0 \ \forall i,j$$

$$\Rightarrow H^1(\mathbb{P}^1_i, f_C^*T_{\mathbb{P}^r}|_{\mathbb{P}^1_i}) = \bigoplus_j H^1(\mathbb{P}^1_i, \mathcal{O}(d_{ij})) = 0$$

$$\| \qquad H^0(\mathbb{P}^1_i, \mathcal{O}(\underbrace{-d_{ij}-2}_{<0})) = 0$$

$$\Rightarrow H^1(\mathbb{P}^1, f_C^*T_{\mathbb{P}^r}) = H^1(\mathbb{P}^1_1 \sqcup \dots \sqcup \mathbb{P}^1_k, \nu^* f_C^* T_{\mathbb{P}^r}) = 0.$$

$\nu : \bigsqcup_i \mathbb{P}^1_i \longrightarrow C$ normalization $\Rightarrow$ $0 \to E \to \nu_*\nu^*E \to \bigoplus_{P \text{ nodes}} \mathbb{C}_p^{rk(E)} \to 0$

$\Rightarrow 0 \to H^0(E) \to H^0(\nu^*E) \to \bigoplus \mathbb{C}_p^{\oplus rkE} \to H^1(E) \to H^1(\nu^*E) = 0$

onto being $T_{\mathbb{P}^n}$ globally gen.

$v \mapsto (\dots, v(p_1) - v(p_2), \dots)$

$$\Rightarrow E_\bullet = [0 \to h^0(E_\bullet) \to 0] \quad \text{and} \quad E^\bullet = [0 \to h^0(E_\bullet)^\vee \to 0] = [0 \to h^0(E^\bullet) = \Omega_{\mathcal{M}/\mathfrak{M}} \to 0]$$

$\uparrow$ locally free being the kernel of $E_0 \to E_1 \to 0$

$\Rightarrow \mathcal{M} \xrightarrow{t} \mathfrak{M}$ is a smooth map and

$$\begin{aligned} A_*(\mathfrak{M}) &\longrightarrow A_*(\mathfrak{C}_t) \xrightarrow{id} A_*(\overset{\|}{\mathfrak{E}_t}) = A_{*+rk(\Omega_{\mathcal{M}/\mathfrak{M}})}(\mathcal{M}) \\ [\mathfrak{M}] &\longmapsto [\mathfrak{C}_t] = [\mathfrak{C}_t] = [\mathcal{M}] = 0^![\mathfrak{C}_t] \end{aligned}$$

② Now take $\beta = 0$, ~~$g \geq 1$~~ $\underset{g\geq 1}{}$ $\Rightarrow \overline{\mathcal{M}}_{gn}(X,0) = \overline{\mathcal{M}}_{gn} \times X$ is smooth of pure dim

└ whenever it makes sense

Obs ‖ $\dim = 3g-3+m+r > 3g-3+m+r(1-g) + \int \cancel{c_1}(T_X) = \text{vdim}$.

_{0}

Claim ‖ $[\overline{\mathcal{M}}_{gn} \times X]^{vir} = c_{eg}(\mathbb{E}^\vee \boxtimes T_X)$ where $\mathbb{E} \to \overline{\mathcal{M}}_{gn}$ is the Hodge bundle (with fibers over $(C, x_i) = H^0(C, \omega_C)$) ~~and~~

proof of the claim

Consider

$$0 \to h^0(E_\cdot) \to E_0 \xrightarrow{d_E} E_1 \to h^1(E_\cdot) \to 0$$

$h^0(E_\cdot) = R^0 p_*(f^* T_X)$

$h^1(E_\cdot) = h^1(R p_*(f^* T_X)) = R^1 p_*(f^* T_X)$

and $\forall (C, x_i, f_C) \in \mathcal{M}$

$h^0(C, f_C^* T_X) = h^0(C, \mathcal{O}) = 1$

$\Rightarrow R^0 p_*(f^* T_X)$ is a v.b on $\overline{\mathcal{M}}_{gn} \times X$ with fibers $H^0(C, f_C^* T_X)$ over (C, x_i, f).

Now ~~for~~ all $(C, x_i, f_C) \in \mathcal{M}$

$h^1(C, \underbrace{f_C^* T_X}_{\overset{\shortparallel}{\mathcal{O}}}) = h^1(C, \mathcal{O}) = g$ is constant

$\Rightarrow h^1(E_\cdot)$ is a vector bundle on $\mathcal{M} = \mathcal{M}_{gn} \times X$ with fibers over $(C, x_i, f_C) = H^1(C, f_C^* T_X) = H^0(C, f_C^* \Omega_X \otimes \omega_C)$

Consider now

$$A_*(\mathfrak{M}_{gn}) \xrightarrow{\sigma} A_*(\mathfrak{C}_t) \longrightarrow A_*([E_0/E_1]) = A_{*+rk(E^\cdot)}(\overbrace{\overline{\mathcal{M}}_{gn} \times X}^{\overset{\mathcal{M}}{\shortparallel}})$$

$$[\mathfrak{M}] \longmapsto [\mathfrak{C}_t] \longmapsto \; ?$$

To understand ? consider

$$\begin{tikzcd}
E_0 \arrow[r] \arrow[d] \arrow[rr, bend left, "d_E"] \arrow[dr, phantom, "\square"] & C \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\square"] & E_1 \arrow[d] \\
\mathcal{M} \arrow[r] & \mathfrak{C}_t \arrow[r, hook] & {[E_1/E_0]}
\end{tikzcd}$$

$$\Rightarrow 0^!_{[E_0/E_1]}(\mathfrak{C}_t) = 0^!_{E_1}[C] = e\left({}^{N_{X/E_1}}\big/_{N_{X/\mathrm{Im}(d_E)}}\right) = e(h^1(E_\cdot)) = c_{eg}(\mathbb{E}^\vee \boxtimes T_X)$$

$$\begin{tikzcd}
0 \arrow[r] & C = \mathrm{Im}(d_E) \arrow[r, "\text{subbundle}"] & E_1 \arrow[r] & h^1(E_\cdot) \arrow[r] & 0 \\
& X \arrow[u, "0"] \arrow[r, equal] \arrow[ur, phantom, "\square"] & X \arrow[u, "0"'] & &
\end{tikzcd}$$

Exercise Let $\pi : \overline{\mathcal{M}}_{g\,n+1}(X,\beta) \longrightarrow \overline{\mathcal{M}}_{g\,n}(X,\beta)$ be the universal curve. Show that $\pi^* \left[\overline{\mathcal{M}}_{gn}(X,\beta)\right]^{vir} = \left[\overline{\mathcal{M}}_{g\,n+1}(X,\beta)\right]^{vir}$

<u>proof</u>

Consider

$$\begin{tikzcd}
\mathcal{C}_{m+1} \arrow[r, "p_{n+1}"] \arrow[d, "\pi_{m+1}"'] \arrow[dd, bend right=60, "f_{m+1}"'] \arrow[dr, phantom, "\square"] & \overline{\mathcal{M}}_{g\,n+1}(X,\beta) \arrow[r, "t_{n+1}"] \arrow[d, "\pi_n"] \arrow[dl] \arrow[dr, phantom, "\square"] & \mathfrak{M}_{g\,n+1} \arrow[d, "s"] \\
\mathcal{C}_m \arrow[r, "\mathcal{E}_n\, p_{n+1}"'] \arrow[d, "f_n"'] & \overline{\mathcal{M}}_{gn}(X,\beta) \arrow[r, "t_m"'] \arrow[d, bend left, dash] & \mathfrak{M}_{gn} \\
X & \mathcal{E} \arrow[u] &
\end{tikzcd}$$

$\Rightarrow \exists\ \alpha :$

$$\begin{tikzcd}
\mathcal{C}_{\mathcal{E}_{n+1}} \arrow[r, "\sim"] \arrow[d] \arrow[dr, phantom, "\square"] & \pi^* \mathcal{C}_{t_m} \arrow[d] \\
\mathcal{E}_{n+1} \arrow[r, "\sim"] & \pi^* \mathcal{E}_m
\end{tikzcd}$$

$\Leftarrow$ commutativity here can be checked by hand.

$\uparrow$

given by :

$$L\pi_n^* R p_{n*} Lf_m^* T_X \cong R p_{m+1*} \underbrace{L\pi_{m+1}^* Lf_m^* T_X}_{\substack{\shortparallel \\ Lf_{m+1}^* T_X}}$$

So $\pi_m^* \left[\overline{\mathcal{M}}_{gn}(X,\beta)\right]^{vir} = \pi_m^*\, 0^!_{\mathcal{E}_m} \left[\mathcal{C}_{t_m}\right] = 0^!_{\mathcal{E}_{n+1}}\, \pi^* \left[\mathcal{C}_{t_n}\right] = 0^!_{\mathcal{E}_{n+1}} \left[\mathcal{C}_{t_{m+1}}\right] = \left[\overline{\mathcal{M}}_{g\,n+1}(X,\beta)\right]^{vir}$

$$\begin{tikzcd}
\mathcal{E}_{n+1} \arrow[r] \arrow[d, "\pi"'] \arrow[dr, phantom, "\square"] & \overline{\mathcal{M}}_{g\,n+1}(X,\beta) \arrow[d, "\pi_m"] \arrow[l, bend right, "0"'] \\
\mathcal{E}_m \arrow[r] & \overline{\mathcal{M}}_{gn}(X,\beta) \arrow[l, bend left, "0"]
\end{tikzcd}$$

$\blacksquare$

Lecture X (Speaker: Miguel Moreira)

Localization of virtual cycles by cosection (Kiem-Li)

· Tools to compute virtual fundamental class:

- virtual pullback
- cosection
- Torus localization
- Wall-crossing.

Recall $E^{\bullet} \to L^{\bullet}_M$ p.o.t. on DM stack $M \longmapsto [M]^{vir}$

On M we have $Ob_M := h^1(E_{\bullet}) \leftarrow$ sheaf on M

Example ‖ M smooth $\Rightarrow Ob_M$ is a bundle and $[M]^{vir} = c_{top}(Ob_M) \cap [M]$

Obs if $\exists \, Ob_M \twoheadrightarrow \mathcal{O} \Rightarrow c_{top}(Ob_M) = 0$

if $\exists \, U \subseteq M$ open s.t. $\exists \, Ob_M|_U \twoheadrightarrow \mathcal{O}_U \Rightarrow c_{top}(Ob_M)|_U = 0$

This suggest that : $[M]^{vir}$ lives in $M \setminus U$

Def Given $E^{\bullet} \xrightarrow{\phi} L^{\bullet}_{M/S}$ relative obstruction theory one can define $Ob_M = h^1(\hat{E}_{\bullet})$ where $\hat{E}^{\bullet} \xrightarrow{\hat{\phi}} L^{\bullet}_M$ is an obstruction thery on M obtained from $\hat{\phi}$. When $S = \{pt\} \Rightarrow \hat{\phi} = \phi$.

Thm 1

Suppose $\exists \, \underset{open}{U \subseteq M}$ and a surjective cosection

$$Ob_M|_U \overset{\sigma}{\twoheadrightarrow} \mathcal{O}$$

Let $M(\sigma) := M \setminus U$. Then there is a class $[M]^{vir}_{loc,\sigma} \in A_* M(\sigma)$ s.t. $[M]^{vir} = j_* [M]^{vir}_{loc}$.

$$\begin{tikzcd} M(\sigma) \arrow[d, hook, "j"] \\ M \end{tikzcd}$$

■

Corollary

$$\boxed{\text{If } \exists \, Ob_M \overset{\sigma}{\twoheadrightarrow} \mathcal{O}_M \Rightarrow [M]^{vir} = 0}$$

■

Example

X proj + smooth$/\mathbb{C}$; $\Theta \in H^{0,2}(X) = H^0(X, \Lambda^2\Omega_X)$ holomorphic 2-form

$\longmapsto$ cosection on $\overline{\mathcal{M}}_{gn}(X,\beta)$ for every β.

Recall: r.p.o.t.

$$(R\pi_* f^*T_X)^\vee \longrightarrow L^{\circ}_{\mathcal{M}/\mathfrak{m}} \qquad \text{where} \qquad \begin{tikzcd} \mathcal{C} \arrow[r, "f"] \arrow[d, "\pi"] & X \\ \mathcal{M} = \overline{\mathcal{M}}_{gn}(X,\beta) \arrow[d] & \\ \mathfrak{m} & \end{tikzcd}$$

Then $Ob_{\mathcal{M}} = R^1\pi_* f^*T_X \leftarrow$ sheaf whose fiber over $(C, x_1, .., x_n, f)$ is $H^1(C, f^*T_X)$

Define $\hat{\Theta}: T_X \xrightarrow{id\otimes\Theta} T_X \otimes \Lambda^2\Omega_X \longrightarrow \Omega_X$

$$\underset{\alpha\wedge\beta}{\underset{\parallel}{v\otimes w}} \longmapsto \alpha(v)\beta - \beta(v)\alpha = w(v,-)$$

and then the cosection is given by

$$H^1(C, f^*T_X) \xrightarrow{f^*\hat{\Theta}} H^1(C, f^*\Omega_X) \longrightarrow H^1(C, \Omega_C) \underset{\substack{\uparrow \\ \text{iso when} \\ C \text{ is smooth}}}{\to} H^1(C,\omega_C) \cong H^0(C,\mathcal{O})^\vee = \mathbb{C}$$

so we get a morphism

we didn't really define this sheaf in the relative case

$$R^1\pi_* f^*T_X \longrightarrow \mathcal{O}_M$$

Which induces $Ob_{\mathcal{M}} \xrightarrow{\sigma} \mathcal{O}_M$

Q) where is σ surjective ? σ not surjective is ~~not surjective~~ related to $f(C) \subset$ vanishing locus of Θ

Corollary | if X admits a nowhere vanishing $\Theta \in H^{0,2}(X) \Longrightarrow [\overline{\mathcal{M}}_{gn}(X,\beta)]^{vir} = 0$.

■

Example ‖ $X = K3$ surface $\Rightarrow K_X = 0 \Rightarrow \exists$ nowhere vanishing $\Theta \in H^{0,2}(X) \Rightarrow [\overline{\mathcal{M}}_{gn}(K3,\beta)]^{vir} = 0$

■

That's very sad... $\rightsquigarrow$ reduced virtual fundamental class

Thm 2 Suppose given

$$\sigma : Ob_{\mathcal{M}} \twoheadrightarrow \mathcal{O}$$

globally surjective.

$$\Rightarrow \exists\ [\mathcal{M}]^{vir}_{red,\sigma} \in A_{vdim+1}(\mathcal{M})$$

Example 1 $[\overline{\mathcal{M}}_g(K3,\beta)]^{vir}_{red} \in A_{g+n}(\mathcal{M})$

Example 2 I ideal sheaf considered in $D(\mathcal{O}_X)$ X being ~~[illegible]~~ proj smooth variety of dim 3

$$\begin{tikzcd} \mathrm{Ext}^2(I,I)_0 \arrow[r, "-\cup At(I)"] & \mathrm{Ext}^3(I, I\otimes\Omega_X) \arrow[r, "tr"] & H^3(\Omega_X)=H^{3,1}(X) \arrow[r, "\int_X -\cup\theta"] & \mathbb{C} \\ \mathrm{Ext}^2(I,I) \arrow[u, phantom, "\cap"] & & & \end{tikzcd}$$

Atiyah class

where $At(I) \in \mathrm{Ext}^1(I, I\otimes\Omega_X)$ is

We always have $\mathrm{Ext}^i(I,J)\otimes \mathrm{Ext}^j(J,K) \longrightarrow \mathrm{Ext}^{i+j}(I,K)$
given by $\mathrm{Hom}(I, J[-i]) \times \mathrm{Hom}(J[i], K[-i-j]) \to \mathrm{Hom}(I, K[-i-j])$

We always have
$tr : \mathrm{Ext}^i(A, A\otimes B) \to H^i(B)$

If $X = K3 \times$ Elliptic curve, then this map is surjective

$$\rightsquigarrow [PT_n(K3\times E,\beta)]^{vir}_{red} \in A_1(PT_n(K3\times E,\beta))$$

Example 3 S surface

$$\underset{\uparrow}{\mathrm{Quot}_S}(\mathcal{O}_S, v=(\overset{H^0}{\overset{\downarrow}{0}},\overset{H^2}{\overset{\downarrow}{0}},\overset{H^4}{\overset{\downarrow}{m}})) = \{\mathcal{O}_S \twoheadrightarrow F \mid ch(F)=v\} = \underset{\substack{\uparrow \\ \dim = 2n \\ \text{smooth}}}{\mathrm{Hilb}^n(S)}$$

$$\Rightarrow [\mathrm{Quot}]^{vir} = c_m(((K_S^{[m]})^\vee)) \cap [\mathrm{Hilb}^n(S)]$$

If $C \subseteq S$ is a smooth canonical curve, i.e. $C = Z(s)$ where $s \in H^0(K_S)$

Then you can find a cosection and you can prove that

$$[\mathrm{Quot}]^{vir} = j_*[C^{[m]}]$$

$$\|$$

$$C^{\times m}/S_m$$

Two ingredients for the proofs:

① Localized Gysin map (for v.b stacks)

② Reduction of the intrinsic normal cone.

Construction of $[M]^{vir}_{loc,\sigma}$ in Thm 1 (~~wh~~ in the non-relative case)

$$E^\bullet \longrightarrow L^\bullet_M \quad \rightsquigarrow \quad \mathfrak{E} = h^1/h^0(E_\bullet) \hookleftarrow \mathfrak{C}_X$$

The cosection

$$\sigma : Ob_M|_U \twoheadrightarrow \mathcal{O}_U$$

gives

$$E_\bullet|_U \to h^1(E_\bullet)[1]\Big|_U = Ob_M|_U[1] \xrightarrow{\sigma} \mathcal{O}_U[1]$$

and so a map of cones

$$h^1/h^0(E_\bullet)|_U = \mathfrak{E}|_U \xrightarrow{\tilde{\sigma}} h^1/h^0(\mathcal{O}_U[1]) = U \times \mathbb{C}$$

if
$$\begin{tikzcd} E_\bullet \arrow[d] & = & E_0 \arrow[r] \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & E_1 \arrow[d, "\sigma"] \\ \mathcal{O}_U[1] & = & 0 \arrow[r] & \mathcal{O}_U \end{tikzcd}$$
Then $Ker(\sigma)$ is a vb on M and $Ker(\tilde{\sigma}) = \left[{Ker(\sigma)}/{E_0}\right]$

The kernel cone ~~co~~ stack is

$$\mathfrak{E}(\sigma) := \mathfrak{E}|_{M(\sigma)} \cup Ker(\tilde{\sigma})$$

$\cap$ closed substack considered with

$\mathfrak{E}$ the reduced structure

where $M(\sigma) = M \setminus U$

↑ considered with the reduced structure

<u>Example</u> Suppose
$$\begin{tikzcd} E_\bullet \arrow[d] & = & E_0 \arrow[r, hook, "\text{subbundle}"] \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & E_1 \arrow[d, "\sigma"] \\ \mathcal{O}_M[1] & = & 0 \arrow[r] & \mathcal{O}_M \end{tikzcd}$$
($\rightsquigarrow U = M$) Then $\mathfrak{E}(\sigma) = ~~\mathit{K}~~ \left[{Ker(\sigma)}/{E_0}\right] = h^1(E_\bullet)$

② means that:

The cycle $[\mathfrak{C}_X] \in Z_*(\mathfrak{E})$ is contained in $Z_*(\mathfrak{E}(\sigma))$ ← ⚠ This doesn't mean that $\mathfrak{C}_X \subset \mathfrak{E}(\sigma) \subset \mathfrak{E}$

<u>Example</u> if $E = Spec(\mathbb{C}) = \{0\} \in \mathbb{A}^1$ and $C = Spec\ \mathbb{C}[t]/(t) \in \mathbb{A}^1$
$\Rightarrow C \not\subset E$ but $[C] \in Z_*(E)$

While ③ means that:

$\exists$ localized Gysin map:

$$0^!_{\mathfrak{E},\sigma} : A_*(\mathfrak{E}(\sigma)) \longrightarrow A_{*+rkE^\bullet}(M(\sigma))$$

One define:

$$s^!_{\mathfrak{E},\sigma}[\mathfrak{C}_X] = [M]^{vir}_{loc,\sigma} \in A_{vdim}(M(\sigma))$$

$\uparrow$ $\mathfrak{C}_X$ has pure dim 0 so $s^!_{\mathfrak{E},\sigma}[\mathfrak{C}_*] \in A_{\underset{\underset{vdim(M)}{\shortparallel}}{rk(E')}}(M(\sigma))$

Finally, a property of $s^!_{\mathfrak{E},\sigma}$ is the following:

$$\begin{tikzcd}
\mathfrak{E}(\sigma) \arrow[r, hook, "j"] & \mathfrak{E} \arrow[d] \arrow[d, bend left, leftarrow, "0"] \\
M(\sigma) \arrow[u, bend left, "0"] \arrow[r, hook, "i"'] & M
\end{tikzcd}
\quad \text{gives} \quad
\begin{tikzcd}
A_*(\mathfrak{E}(\sigma)) \arrow[r, "j_*"] \arrow[d, "s^!_{\mathfrak{E},\sigma}"'] \arrow[dr, phantom, "\circlearrowright"] & A_*(\mathfrak{E}) \arrow[d, "s^!_{\mathfrak{E}}"] \\
A_{*+rkE'}(M(\sigma)) \arrow[r, "i"'] & A_{*+rk(E)}(M)
\end{tikzcd}$$

$$\Rightarrow i_*[M]^{vir}_{loc,\sigma} = i_* s^!_{\mathfrak{E},\sigma}[\mathfrak{C}_X] = s^!_{\mathfrak{E}}[\mathfrak{C}_X]$$

■

Definition of $[M]^{vir}_{red,\sigma}$ in the non-relative case

Given $Ob_M \xrightarrow{\sigma} \mathcal{O}$ then we have

$$0 \to \ker(\tilde{\sigma}) \to \mathfrak{E} \xrightarrow{\tilde{\sigma}} M \times \mathbb{C} \to 0$$

$$\underset{\underset{\text{v.b. stack over } M \text{ of } rk = -(rk(E_0)+1) = rk(\mathfrak{E})-1}{\uparrow}}{\underset{\mathfrak{E}^{red}}{\shortparallel}}$$

(the rank of $\mathfrak{E}[E_1/E_0]$ is $rk(\mathfrak{E}) = rk E_1 - rk E_0 = -rk$

and one defines $[M]^{vir}_{red} := s^!_{\mathfrak{E}^{red}}[\mathfrak{C}_X] \in A_{-rk(\mathfrak{E}^{red}) = \underbrace{rk(E_0)}_{=vdim}+1}(M)$

■

⑤