A note on persistent private information*

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Abstract

We study the contracting problem in the persistent private information model of Williams [16], in which an agent provides a report of a privately observed path to the principal, who in turn pays the agent in the least expensive way that induces truthful reporting. We first argue that, in the case of persistent information, the contract in [16] does not induce truthful reporting if misreporting is allowed to grow sufficiently fast. The contract becomes incentive compatible (i.e., induces truthful reporting) if one imposes additional restrictions on misreporting, as shown also in Bloedel et al [2] under different conditions. Under our restrictions, we show that the contract identified in [2] is optimal among linear contracts. On the other hand, if additional restrictions are not imposed, we show that the contract optimal in a family of generalized linear contracts is deterministic.

Keywords: Principal-Agent problem, persistent information, truthful reporting, incentive-compatible contracts

JEL classification: D86, G00, G30, G35

1 Introduction

A classical problem in economics in general, and corporate finance and executive compensation in particular, is how to compensate an economic agent (manager) to report the information the agent has access to truthfully. The present paper is inspired by Williams [16], the first paper to consider a Brownian motion continuous-time model of such a principal-agent problem, in which the principal's objective is to incentivize the agent to report the values of a process B observed by the agent truthfully; see also Zhang [17], which considered a similar problem in a continuous-time Markov chain model. We also

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complement and build on the recent paper Bloedel et al. [2] by Bloedel, Krishna, and Strulovici, with a comparison to our paper provided below.[1]

In Williams [16] and Bloedel et al. [2], as well as in the current paper, the principal minimizes the expected cost among the contracts that are incentive compatible (henceforth IC), under which the agent reports truthfully. In its main example (Section 6), Williams [16] provides an "optimal" contract, henceforth called contract W, and derives the main economic insights based on such a contract. We show that, if the agent is allowed to under-report with deviations of unbounded growth, contract W is not incentive compatible in the most interesting case in which the mean reversion parameter λ is strictly positive. Hence, it cannot be optimal, as it is outside of the set of admissible contracts. We also show that the contract W becomes incentive compatible if the difference between the misreported process and the true process is constrained to remain sufficiently bounded. However, as shown in Bloedel et al. [2], the contract is still not optimal.

In Williams [16], the agent's report y (realization of a process Y) for the realized path b of B has to be such that $y \leq b$ (i.e., $Y \leq B$). The paper justifies this by stating that, at least in some applications, one can imagine that the agent deposits the value y in an account and cannot deposit more than the actual value b. However, in addition to $Y \leq B$, Williams [16] assumes much more, namely that

$$m_t := Y_t - B_t = \int_0^t \Delta_s \, \mathrm{d}s \,, \text{ with } \Delta \leq 0.$$

This assumption not only requires that $Y - B$ is non-positive, but also that it is differentiable, with non-positive rate of change Δ. Our results do not require this assumption. In fact, we do not apply the weak formulation (in which the process Δ represents a Girsanov change of measure), used in contract theory mainly for tractability – we are able to obtain our results in the strong formulation. This is possible because as in Williams [16] and Bloedel et al. [2], we do not find the optimal contract in the general family of contracts, but in a restricted family. Studying the former, much harder problem, would require the weak formulation.

1.1 Literature Review

Comparison to Bloedel et al. [2]. The main differences between the present note and Bloedel et al. [2] are: (a) we study in more detail the case of unbounded under-reporting; (b) we consider the family of linear contracts that is more general than the family of so-called self-insurance contracts (SIC) in Bloedel et al. [2]; (c) Bloedel et al. [2] also studies contracting in the case where the agent has a hidden savings account, which we do not consider. We next provide a more detailed comparison.

Assumptions:

1. We consider the case with no assumptions on the growth of the misreporting process m separately. In the other case, in which we do impose restrictions on the growth of misreporting, our restrictions are neither implied nor imply the so-called no-Ponzi condition

[1] After we completed the first draft of this paper and made it publicly available, the existence of the paper Bloedel et al. [2] was pointed out to us by its authors. The paper, that was already under revision at that time, provides a critique and extensions of Williams [16].

of Bloedel et al. [2]. Note that Williams [16] imposes no restrictions, at least not explicitly, and this is in part leads to the failure of contract W being IC.

Results (for the case $\lambda > 0$):

2. With no restrictions on the growth of misreporting, we show that contract W and other non-deterministic linear contracts are not IC, and we identify conditions under which linear contracts are IC. Bloedel et al. [2] also shows that contract W is not IC if no additional assumptions are imposed, and that it is IC under their no-Ponzi condition. The authors also show that even under additional assumptions contract W is not optimal.

3. Bloedel et al. [2] finds the contract which is optimal among the self-insurance contracts, and shows that this contract is strictly better than contract W (which can also be implemented as a self-insurance contract). We show that the contract in Bloedel et al. [2] is also optimal in the larger family of linear contracts, which includes the self-insurance contracts. When $\lambda > 0$, none of the three papers is able to find the optimal contract in the general family of contracts.

4. Bloedel et al. [2] contains other results and interpretations. In particular, they show that contract W becomes optimal if the agent is allowed to secretly save at the interest rate equal to the discount rate. Moreover, in the so-called "permanent shocks" case, corresponding to the mean reversion parameter $\lambda = 0$, they find the optimal contract in the model with no savings account, which in this case is deterministic and, in fact, the same as in Williams [16].

Discussion:

The question arises which restrictions on the misreporting process are more natural. This is important because the optimal (linear) contract is very different under different assumptions – in one case it depends on the reported path, and in the other it is deterministic. One could argue that in some applications the restriction that m has bounded growth is natural, in others that it is not. For example, there may be cases in which it may be reasonable to assume that when the reported process becomes too different from the actual process, the principal will realize that the agent is not telling the truth; for instance, in the case of an entity reporting carbon emissions, when the reports differ significantly from the average of other similar entities. However, it may be less realistic to assume that there is an exact value of misreporting at which the principal will realize this, which corresponds to a sufficient condition we identify for a non-deterministic IC contract to be optimal to exist. In Bloedel et al. [2] an asymptotic growth condition is imposed (the no-Ponzi condition), and the principal would only know at $t = \infty$ whether the condition has been violated, which is also problematic.

Other related papers. early work that considers similar problems in continuous-time models includes Prat and Jovanovic [14] and He et al. [6]. In the model of these two papers, there is a drift component unobserved by both the principal and the agent, and they learn about it over time. These papers, as the current one, consider the problem for the agent with CARA utility, and a risk-neutral principal. Mathematically, they encounter similar difficulties: (i) there is an additional state variable that depends on the cumulative agent's action, unobserved by the principal; (ii) there are technical issues in the usual weak formulation of the problem, due to the infinite horizon. These complicate the usual first-order approach, applied, for example, in Sannikov [15], Cvitanić et al. [4], [5], and Cvitanić and Xing [3]. The first-order condition now depends additionally on the expected value of a functional of the future action. Still, Williams [16], Prat and Jovanovic [14] and He et al. [6] provide the first-order necessary conditions for the optimal contract, and also conditions which are not necessary, but sufficient together with the first-order conditions, for a contract to be optimal.

More recently, Hu et al. [7] characterize the solution for degenerate systems, which is also the case in our problem. However, they do this on a finite horizon, and the agent's objective in that paper does not allow for the objective in this paper, so their results cannot be used. Other recent papers with unobserved processes include Alvarez and Nadtochiy [1] and Huang et al. [8], and papers with (possibly) infinite horizon include Lin et al. [11] and Possamaï and Touzi [13].

Organization of the paper. Section 2 recalls the persistent private information reporting problem from Williams [16] and presents the technical assumptions needed to rigorously pose the problem. In Section 3 we present our counterexample to the contract claimed to be incentive compatible and optimal in Williams [16], and we elaborate on reasons why the argument presented there is erroneous under our assumptions. In Section 4, we show that without additional restrictions on m the optimal linear contract is deterministic. Moreover, we show that, under growth conditions on m, the optimal self-insurance contract is IC and optimal among all linear contracts, not just among self-insurance contracts. Section 5 provides a discussion and conclusions.

2 The model and contracting problem

We recall first the framework of Williams [16]. We fix a complete probability space $(\Omega, \mathcal{F}, \mathbf{P})$ supporting a Brownian motion W with its usual filtration $\mathbb{F} = (\mathcal{F}_t : t \geq 0)$. The agent observes the realization of the process B, whose evolution is given by

$$dB_t = (\mu_0 - \lambda B_t)\,\mathrm{d}t + \sigma\,\mathrm{d}W_t, \quad B_0 = b_0, \tag{2.1}$$

where $\mu_0 \in \mathbb{R}, \lambda \geq 0, \sigma > 0$ and $b_0 \in \mathbb{R}$.

The principal knows the dynamics of the process B, that is, she knows μ_0, λ, σ and b_0 in (2.1), but does not observe the realization of B. To elicit B, she offers a contract to the agent, to receive a report on the observed realization. More formally, the agent chooses a *reporting strategy*

$$Y : [0, \infty) \times \Omega \to \mathbb{R}$$

which is a continuous process adapted to $\mathbb{F}^B$, the filtration generated by B. One easily verifies that up to completion and closure from the right, $\mathbb{F}^B = \mathbb{F}$. We shall thus assume that Y is actually $\mathbb{F}$-adapted. Continuity is not necessary for our results, but it seems reasonable to assume that if the agent reported a discontinuous path, the principal would consider it a lie. We provide a further discussion on this issue in Section 5.

Contracts

For a fixed reporting strategy Y, we denote by $y = (y_t)_{t\in[0,\infty)}$ its realization, which is the path reported to the principal by the agent as the realization $b = (b_t)_{t\in[0,\infty)}$ of B. By assumption both y and b are elements of $\mathcal{C}([0, \infty); \mathbb{R})$, the space of continuous functions $f : [0, \infty) \to \mathbb{R}$, equipped with the topology of uniform convergence on compact sets. We use $y_{s\leq t}$ as a shorthand for $(y_s)_{s\in[0,t]}$, the reported path up to time t. In exchange for the report, the principal pays the agent a salary $s = (s_t)_{t\in[0,\infty)}$ continuously in time. Those payments depend, in an adapted way, only on the path revealed by the agent. This means that the principal chooses a jointly Borel measurable function $s : [0,\infty)\times\mathcal{C}([0,\infty);\mathbb{R}) \to \mathbb{R}$ that satisfies

$$s_t(y) = s_t\left(y_{s\leq t}\right), \quad \text{for all } t \in [0,\infty).$$

We call $s = (s_t(y))_{t\in[0,\infty)}$ a *contract*.

Decision problems

As is standard in contract theory, the principal proposes a contract to the agent. Then, the agent may either accept or reject the contract. Once accepted, the agent continuously makes the report y_t to the principal and in turn receives the contractual compensation $s_t(y)$ from the principal.

For a given contract s, the agent aims to maximize

$$J_0(s(Y), B) := \mathbf{E}\Big[\int_0^\infty e^{-\rho t} u(s_t(Y), B_t)\,\mathrm{d}t\Big] \tag{2.2}$$

with

$$u(x) = -e^{-\theta x}\,, \tag{2.3}$$

for a fixed parameter $\theta > 0$ and discount factor $\rho > 0$, over reporting strategies Y. We call any strategy Y^s that maximizes (2.2), if it exists, an *optimal report* for s.

Definition 2.1. A contract s is said to

- be *Incentive Compatible* (IC), if $Y^s = B$ is an optimal reporting strategy;
- satisfy the *Participation Constraint*, if

$$J_0\left(s(B), B\right) \geq v_0,$$

  for a given reservation utility v_0 of the agent.

As usual in contract theory, it is assumed that, if indifferent, the agent will act in the principal's best interest, which in our framework means reporting truthfully. In particular, deterministic contracts are considered incentive compatible.

Having formalized the agent's problem, we consider the principal's problem of minimizing

$$\mathbf{E}\Big[\int_0^\infty e^{-\rho t} s_t(Y^s)\,\mathrm{d}t\Big] \tag{2.4}$$

over contracts s that are incentive compatible and of the linear form as introduced in (4.1), below. Note that we impose incentive compatibility as the objective of the principal – we implicitly assume that the principal experiences utility equal to minus infinity if the truth is not reported. This is also assumed in Williams [16]. The problem of minimizing over a subset of more general contracts (possibly nonlinear and path-dependent) seems not to have a solution that can be nicely characterized, and is outside of the scope of this paper.

Denote the misreporting process by

$$m_t := Y_t - B_t. \tag{2.5}$$

Definition 2.2. We say that a misreporting strategy $m = (m_t)_{t\geq 0}$ is *admissible*, if it is a continuous $\mathbb{F}$-adapted process of locally finite variation such that

$$\mathbf{P}\Big[\int_0^t m_s \,\mathrm{d}s < \infty \text{ for all } t \geq 0\Big] = 1\,.$$

Remark 2.3. In Williams [16] the following additional conditions are imposed on reporting strategies:

$$\text{The agent can only under-report: } Y \leq B \tag{2.6}$$

and, in fact, the stronger requirement that

$$m_t = \int_0^t \Delta_s \,\mathrm{d}s\,, \text{ for some } \mathbb{F}\text{-adapted process } \Delta \leq 0. \tag{2.7}$$

We stress that neither (2.6) nor (2.7) are needed in this paper.

The justification for (2.6) provided in Williams [16] is that for some applications one can envision that the agent deposits the reported values y_t in a savings account observable by the principal, and is unable to deposit more than the actual values b_t. However, for many applications this may not be a valid restriction; see Section 5 for a discussion. The requirement that m_t satisfies (2.7), i.e., is differentiable is justified in Williams [16] by saying that "the agent's report Y must be absolutely continuous with respect to his true state B". Indeed, in theory, it could be argued that, given the reported path, the principal could keep performing, continuously in time, (uncountably) many tests for the likelihood ratio $\frac{dP^B}{0.5(dP^B+dQ)}$, where P^B is the law of the true process B, for all possible alternative laws Q. For a path that comes from a law that is singular with respect to P^B, there would be a positive probability that the tests would reject P^B in favor of some singular Q in finite time. (We thank Martin Larsson for this observation.) In practice, this would, of course, be very hard to do. $\diamond$

3 A Counterexample to Incentive Compatibility

Before studying linear contracts, we show that the contract in the main example of [16, Sec. 6] is not incentive compatible, if misreporting is not bounded from below and no other restrictions are imposed on misreporting. Note that no such restrictions are imposed in Williams [16]. In the example, the agent's expected utility is

$$\mathbf{E}\Big[\int_0^\infty e^{-\rho t} u(s_t(Y) + B_t)\,\mathrm{d}t\Big],$$

where u is as in (2.3).

We consider here the case $\lambda > 0$. In the case $\lambda = 0$, the contract in [16, Sec. 6] makes the agent indifferent, and is thus incentive compatible.

Consider the contract

$$S_t := s_t(Y) = \alpha - \frac{1}{\theta}\log(-q_t) - Y_t, \tag{3.1}$$

from [16, p. 1271]. Here α is an appropriate constant, q_t is the promised utility, i.e., the value process of the agent under truthful reporting, given by

$$dq_t = -\sigma\theta\beta q_t \, \mathrm{d}W_t^Y \ ,$$

where β is a constant and W_t^Y is given by

$$W_t^Y := \frac{1}{\sigma}\left(Y_t - b_0 - \int_0^t (\mu_0 - \lambda Y_s) \, \mathrm{d}s\right). \tag{3.2}$$

Note that the functional s in (3.1) satisfies the measurability restrictions required in the definition of a contract from Section 2. Contract (3.1) may be written as

$$S_t + Y_t = \tilde{\alpha} + \frac{1}{2}\theta\beta^2\sigma^2 t + \beta\sigma W_t^Y, \tag{3.3}$$

where $\tilde{\alpha}$ is an appropriate constant. It is claimed in Williams [16] that, with

$$\beta = \frac{\rho}{\rho + \lambda} \, , \tag{3.4}$$

this contract is optimal and thus, in particular, incentive compatible. We now show that this is not the case if misreporting is not bounded from below, and we find a sufficient condition on admissible misreporting strategies for the contract to be IC.

Proposition 3.1. *Any contract that satisfies* (3.3) *for an arbitrary constant* $\beta \in (0,1)$ *and an arbitrary constant* $\tilde{\alpha}$ *is not incentive compatible, if we allow all deterministic misreporting that is not bounded from below.*

The proof shows that contract (3.1) actually does not admit an optimal reporting strategy. Instead, under this contract, the higher the misreporting, the higher the agent's utility.

Proof. We provide a counterexample with misreporting processes m that satisfies properties (2.6) and (2.7) from Remark 2.3, i.e., the assumptions imposed in [16]. Just as in [16, p. 1272], equation (3.2) can be written as

$$\begin{aligned} \sigma \, \mathrm{d}W^Y &= \mathrm{d}B_t - (\mu_0 - \Delta_t - \lambda(B_t + m_t))\mathrm{d}t \\ &= \sigma \, \mathrm{d}W_t + (\Delta_t + \lambda m_t) \, \mathrm{d}t. \end{aligned}$$

Contract (3.3) may be written as

$$S_t + B_t = \tilde{\alpha} + \frac{1}{2}\theta\beta^2\sigma^2 t + \beta\sigma W_t + \lambda\beta \int_0^t m_s \, \mathrm{d}s - (1-\beta)m_t, \tag{3.5}$$

for some constant $\tilde{\alpha}$. For $k > 0$, let

$$\Delta_t = -e^{kt}. \tag{3.6}$$

Then, the sum of the last two terms in the expression for $S_t + B_t$ above is

$$\begin{aligned}\lambda\beta \int_0^t m_s \,\mathrm{d}s - (1-\beta)m_t &= \frac{1}{k}\lambda\beta \int_0^t (1-e^{ks})\,\mathrm{d}s - \frac{1}{k}(1-\beta)(1-e^{kt})\\ &= \frac{1}{k}\lambda\beta\Big(t + \frac{1}{k}(1-e^{kt})\Big) - \frac{1}{k}(1-\beta)(1-e^{kt})\\ &= \frac{1}{k}\lambda\beta t + \frac{1}{k}(1-e^{kt})\Big(\frac{1}{k}\lambda\beta - (1-\beta)\Big).\end{aligned}$$

A sufficient condition for this to be larger than zero is

$$k \geq \frac{\lambda\beta}{1-\beta}.$$

Thus, for any such k, the agent would be better off using the corresponding strictly negative Δ instead of zero. In fact, he would like to set k as large as possible. More precisely, plugging it back to the agent's expected utility $\mathbf{E}[\int_0^\infty e^{-\rho t} u(S_t + B_t)\,\mathrm{d}t]$, we see that the agent can get as close as he wants to its maximum value, equal to zero, by increasing k. □

Since $\lambda > 0$, we have $\beta < 1$ in (3.4), and it follows in particular that the contract in [16, Sec. 6] is not incentive compatible. An inspection of the direct verification of incentive compatibility in the appendix of Williams [16] reveals the following omission. On [16, p. 1272] there is an HJB equation for the agent's value given the contract (3.3), with $\beta = \rho/(\rho+\lambda)$ and with W^Y expressed in terms of q. The solution to that equation is provided as

$$V(q,m) = \frac{q\exp(\theta m)(\rho+\lambda)}{\rho + \lambda + \theta\lambda m}\,. \tag{3.7}$$

It is in fact immediately seen that this solution cannot be the value function of the agent for all values of m: since $q < 0$, this value is positive for $m < -(\rho+\lambda)/(\theta\lambda)$, hence it cannot be the value function corresponding to the negative CARA utility. An essential step neglected in Williams [16] is the verification that the solution of the HJB equation is in fact the agent's value function. For one thing, on infinite horizon the solution to the HJB equation may not be equal to the value function if it does not satisfy the following transversality condition: for all admissible strategies,

$$\lim_{T\to\infty} \mathbf{E}\Big[e^{-\rho(T-t)} V(q_T, m_T)\Big] = 0\,.$$

It can be checked that the above function does not satisfy this unless m never reaches $-(\rho+\lambda)/(\theta\lambda)$.

Williams [16] provides another argument to claim that the contract is incentive compatible, by checking that the conditions of [16, Theorem 4.1] are satisfied. However, the proof of that theorem goes through for infinite horizon only if $\mathbf{E}[e^{-\rho T} p_T m_T]$ converges to zero as T goes to infinity, where, as the paper shows, the adjoint process p satisfies $p_T = \theta q_T$, with q_T as above. Then, it is easy to verify that we do not have the desired convergence if m ever goes below the value $-\frac{\rho+\lambda}{\theta\lambda}$. This is because then $e^{-\rho T} q_T/q_t$ converges to infinity, and so does $\mathbf{E}[e^{-\rho T} p_T m_T]$.

Remark 3.2. It follows from the results of the next section that a contract of the form (3.5) with $\beta = \frac{\rho}{\rho+\lambda}$ becomes incentive compatible if we impose additional restrictions on admissible misreporting strategies. However, as shown in Bloedel et al. [2], even under conditions under which that contract is IC, it is not optimal; see Remark 4.5. $\diamond$

4 Linear Contracts

We now restrict our attention to the family of linear contracts. We follow the tradition in the contracting literature of focusing on the contracts most observed in practice, i.e., linear contracts, when considering more general contracts is not tractable. For this, let $\beta, C, k \in \mathbb{R}$ be constants, and $a : [0, \infty) \to \mathbb{R}$ be a differentiable function with $a(0) = 0$. We consider contracts s that satisfy

$$s_t(Y) + Y_t = C + a(t) + (k+1)(Y_t - Y_0) + \beta \int_0^t \lambda Y_s \, \mathrm{d}s \,. \tag{4.1}$$

Note that contracts of this form satisfy the requirements imposed in Section 2.

Lemma 4.1. *Any contract s of the linear form in* (4.1) *satisfies*

$$\begin{aligned} s_t(B) + B_t = C + a(t) + \beta\mu_0 t + (\lambda B_0 - \mu_0)\frac{1}{\lambda}(\beta - k - 1)(1 - e^{-\lambda t}) \\ + \beta\sigma W_t - (\beta - k - 1)e^{-\lambda t} \int_0^t e^{\lambda s} \sigma \, \mathrm{d}W_s. \end{aligned}$$

Proof. We have

$$\begin{aligned} A_t &:= (k+1)(B_t - B_0) + \beta \int_0^t \lambda B_s \, \mathrm{d}s \\ &= (k+1)(B_t - B_0) - \beta(B_t - B_0 - \mu_0 t - \sigma W_t). \end{aligned}$$

Using the fact that

$$B_t = e^{-\lambda t} B_0 + \mu_0 \int_0^t e^{-\lambda(t-s)} \, \mathrm{d}s + \int_0^t e^{-\lambda(t-s)} \sigma \, \mathrm{d}W_s, \tag{4.2}$$

we get

$$A_t = (k + 1 - \beta)\Big[(e^{-\lambda t} - 1)B_0 + \mu_0 \frac{1}{\lambda}(1 - e^{-\lambda t}) + \int_0^t e^{\lambda(s-t)} \sigma \, \mathrm{d}W_s\Big] + \beta\mu_0 t + \beta\sigma W_t,$$

which concludes the proof. $\square$

For convenience we introduce the notation

$$\tilde{a}(t) := a(t) + \beta\mu_0 t + (\lambda B_0 - \mu_0)\frac{1}{\lambda}(\beta - k - 1)(1 - e^{-\lambda t}). \tag{4.3}$$

Using the function $\tilde{a}$ we write the objective (2.2) of the agent with exponential intertemporal utility (2.3) using an admissible misreporting strategy m as

$$\begin{aligned} J_0(s(Y), B) = -e^{-\theta C} \mathbf{E}\Big[\int_0^\infty \exp\Big(-\rho t - \theta\Big(\tilde{a}(t) + \beta\sigma W_t - (\beta - k - 1) \\ \times \int_0^t e^{\lambda(s-t)} \sigma \mathrm{d}W_s\Big) - \theta\Big(k m_t + \beta \int_0^t \lambda m_s \, \mathrm{d}s\Big)\Big) \, \mathrm{d}t\Big], \end{aligned} \tag{4.4}$$

where $Y = m + B$ from (2.5). We may consider the agent's problem from time $t \in [0, \infty)$ onward. It is a standard Markovian stochastic control problem with control being the choice of an admissible misreporting strategy m, and states given by the processes

$$\begin{aligned} \mathrm{d}W_r^w &= \mathrm{d}W_r \,, && \text{with } W_t^w = w; \\ \mathrm{d}X_r^x &= e^{\lambda r}\sigma\, \mathrm{d}W_r \,, && \text{with } X_t^x = x; \\ \mathrm{d}P_r^m &= \lambda m_r\, \mathrm{d}r \,, && \text{with } P_t^m = p; \end{aligned} \tag{4.5}$$

for $r \in [t, \infty)$ with $w, x, p \in \mathbb{R}$. We can then write the agent's objective function (4.4) from time t onward as

$$\begin{aligned} J(t,p,w,x;m) := -e^{-\theta K}\mathbf{E}_{p,w,x}\Big[\int_t^\infty \exp\Big(&-\rho v - \theta\big(\tilde{a}(v) + \beta\sigma W_v^w - (\beta - k - 1) \\ &\times e^{-\lambda v}X_v\big) - \theta(km_v + \beta P_v)\Big)\,\mathrm{d}v\Big] \,, \end{aligned}$$

where $\mathbf{E}_{p,w,x}$ denotes expectation taken with respect to the law of the unique solution of system (4.5) on $(\Omega, \mathbb{F})$. From this, we see that the HJB equation for the value function

$$V := V(t,p,w,x) := \sup\Big\{J(t,p,w,x;m) \,:\, m \text{ is an admissible control}\Big\} \tag{4.6}$$

is given by

$$\begin{aligned} 0 = \sup_m \Big\{ &- \exp\Big(-\rho t - \theta\big(C + \tilde{a}(t) + \beta\sigma w - (\beta - k - 1)e^{-\lambda t}x\big) - \theta(km + \beta p)\Big) \\ &+ \lambda m V_p + V_t + \frac{1}{2}V_{ww} + \frac{1}{2}\sigma^2 e^{2\lambda t}V_{xx} + \sigma e^{\lambda t}V_{xw}\Big\} \,, \end{aligned} \tag{4.7}$$

where subscripts denote partial derivatives. In the remainder of the section, we assume that m is allowed to take values in an interval $(-\infty, N)$, with $0 \le N \le \infty$. Thus, zero misreporting is always allowed. We first consider the case with no additional restrictions on the admissible misreporting strategies, and then the case with additional restrictions. The proofs are postponed to Section 4.3.

4.1 No additional restrictions on misreporting

In this subsection, other than $m \le N$, we impose no restrictions on the admissible processes m. When $N \in (0, \infty]$, we show that the only IC contracts are the deterministic contracts; see Proposition 4.2. In the case $N = 0$, considered in Williams [16], even if non-deterministic IC contracts may exist, we show that it is still the case that a deterministic contract is optimal; see Proposition 4.3.

Proposition 4.2. *Consider a linear contract of the form* (4.1)*.*

(i) For the contract to be IC, it is necessary that either it is deterministic, that is,

$$\beta = k = 0 \,, \tag{4.8}$$

or it satisfies

$$\beta = k + 1. \tag{4.9}$$

Moreover, we have $\beta \ge 0$ and $k \ge 0$.

(ii) If $N > 0$, then the contract is IC iff $\beta = k = 0$.

We show next that, even if non-deterministic IC contracts may exist, a deterministic contract is still optimal in a more general family of linear contracts, regardless of whether $N > 0$ or $N = 0$.

Proposition 4.3. *The deterministic contract S^C given by*

$$S_t^C = C - e^{-\lambda t} b_0 - \frac{1}{\lambda}\mu_0(1 - e^{-\lambda t}) + \frac{\theta\sigma^2}{4\lambda}(1 - e^{-2\lambda t}), \tag{4.10}$$

where constant C is chosen so that the agent's participation constraint is satisfied as equality, is optimal among all IC contracts of the form

$$S_t = S_0 + \int_0^t k(s)\,\mathrm{d}Y_s + \int_0^t \big(a(s) + \gamma(s)S_s + \beta(s)\lambda Y_s\big)\,\mathrm{d}s, \tag{4.11}$$

where λ is a constant, a, k, γ, β are deterministic, measurable functions of time, with k, β, λ and γ bounded, and such that:

- *The contract* (4.11) *is defined for all $t \geq 0$;*
- *$k \geq 0$, $\beta \geq 0$;*
- *When $m \equiv 0$ both the agent's and the principal's utility are defined and finite, i.e., $\mathbf{E}[\int_0^t e^{-\rho t} u(B_t + S_t)\,\mathrm{d}t] < \infty$ and $\mathbf{E}[\int_0^t e^{-\rho t}|S_t|\,\mathrm{d}t] < \infty$.*

Note that (4.11) is more general than the IC contracts of the linear form (4.1).

4.2 Additional restrictions on misreporting, that allow non-deterministic IC contracts

In this section, we show that additional restrictions on admissible misreporting can be added to guarantee that (specific) linear contracts of the form (4.1) with $\beta = k+1$ are IC. We also show that the contract that is optimal among the linear IC contracts is superior to the deterministic contracts.

Denote

$$r = \frac{\lambda\beta}{1-\beta}\ . \tag{4.12}$$

It follows from (4.22) in the proof of Proposition 4.2 below that if $\beta = k + 1$ (which necessarily holds unless $\beta = k = 0$), the process $\tilde{a}$ defined by (4.3) is given by

$$\tilde{a}(t) = \frac{1}{\theta}r - \frac{\rho}{\theta}t + \frac{1}{2}\theta\sigma^2\beta^2 t. \tag{4.13}$$

Using this and expression (4.18) from the proof of Proposition 4.2 below, it is straightforward to verify that the agent's indirect utility function V^0 corresponding to $m = 0$, when offered the linear contract satisfying (4.9) and (4.13), is given by

$$V^0(t, p, w) = -\frac{1}{r}e^{-\theta\left(C + \beta p + \sigma\beta w + \frac{1}{2}\theta\sigma^2\beta^2 t\right)}. \tag{4.14}$$

We then obtain the following proposition.

Proposition 4.4. *Assume that the set of admissible strategies m is reduced to a subset for which*

(a) V^0 *satisfies the HJB equation* (4.7) *with* $\beta = k+1$ *and* $\tilde{a}$ *given by* (4.13)*;*

(b) *the following transversality condition holds:*

$$\lim_{T\to\infty} e^{-\rho T}\mathbf{E}\big[V^0(T, P_T^m, W_T^w)\big] = 0; \tag{4.15}$$

(c) *the following consequence of the martingale property holds:*

$$\mathbf{E}\Big[\int_0^T e^{-\rho s}V_w^0(s, P_s^m, W_s^w)\,\mathrm{d}W_s^w\Big] = 0\,,\ \textit{for all } T > 0,. \tag{4.16}$$

Then, the linear contract of the form (4.1) *satisfying* (4.9) *and* (4.13) *is IC, and* V^0 *is the value function of the agent.*

Remark 4.5.

1. Linear contracts with $\beta = k+1$ were first introduced by Bloedel et al. [2], from a different reasoning, as the contracts in which the principal offers to the agent a savings account with interest rate r given by (4.12), called self-insurance contracts, SIC's. They provide different sufficient conditions for the contract to be IC – the associated savings account process has to satisfy a transversality condition, called no-Ponzi condition, that are neither implied by, nor imply our sufficient conditions. When the contracts are IC, Bloedel et al. [2] describe how to find optimal $r = \hat{r}$. They also observe that the contract from Williams [16] corresponds to $r = \rho$ and show that $\hat{r} < \rho$, and hence contract Williams [16] is not optimal even among SIC's, thus also not optimal among the linear contracts.

2. Using Proposition 4.2, the principal's value for the optimal deterministic contract ($\beta = k = 0$) can be computed as

$$v^{P,0} = -\frac{1}{2}\theta\sigma^2\frac{1}{\rho(\rho+2\lambda)}.$$

The value corresponding to the contract in Williams [16] with $r = \rho$ and $\beta = k+1$ can be computed as

$$v^{P,\rho} = -\frac{1}{2}\theta\sigma^2\frac{1}{(\rho+\lambda)^2} > v^{P,0}.$$

Thus, contract W, and hence also the optimal linear contract among those of the form (4.1) dominate deterministic contracts, under the assumptions in this section. These assumptions guarantee the IC property both of contract W and those of the form (4.1).

3. A sufficient condition for assumptions **(a)** and **(c)** from Proposition 4.4 to hold is that admissible strategies satisfy the following: there exists a constant $K > 0$ such that $\mathbf{P}$-a.s. $|m_t| \le K$, for all $t > 0$. Some sufficient conditions for **(b)** to hold are given next, which basically say that m is sufficiently bounded from below. More precisely, taking $0 = w = p = t$ without loss of generality, the transversality condition from (4.14) becomes

$$\begin{aligned} 0 &= \lim_{T\to\infty} e^{-\rho T}\mathbf{E}\big[V^0(T, P_T, W_T)\big] \\ &= -\lim_{T\to\infty}\frac{1}{r}e^{-\rho T}\mathbf{E}\Big[\exp\Big(-\theta\Big(K + \beta\int_0^T \lambda m_s\,\mathrm{d}s\Big) - \theta\sigma\beta W_T - \frac{1}{2}\theta^2\sigma^2\beta^2 T)\Big)\Big]. \end{aligned}$$

When $\beta > 0$ and m is allowed to be unbounded from below, we can find a (deterministic) process m for which this limit will not equal zero. On the other hand, a sufficient condition for the limit to be equal to zero is that, for some constant $c > 0$,

$$\rho + \theta\beta \frac{1}{T} \int_0^T \lambda m_s \, \mathrm{d}s \geq c. \tag{4.17}$$

In particular, the transversality condition holds if $m \geq -\frac{\rho}{\theta\lambda\beta} + \epsilon$ for some $\epsilon > 0$. When $0 < \beta < 1$, a sufficient condition for this is $m \geq -\frac{\rho}{\theta\lambda} + \epsilon$. This provides an alternative sufficient condition for the self-insurance contracts to be IC. $\diamond$

4.3 Proofs for Sections 4.1 and 4.2

It is possible to extend some of the proofs below to the case in which we allow β and k to be differentiable functions of time. In that case one first shows that it is necessary for the contract to be IC that β and k are constant, after which the proofs below can be applied.

Proof of Proposition 4.2. Fix an IC linear contract of the form (4.1). From (4.6), the indirect utility function corresponding to $m = 0$ is

$$\begin{aligned} V(t,p,w,x) :=& \; -e^{-\theta C} \mathbf{E}_{p,w,x}\bigg[\int_t^\infty \exp\bigg(-\rho v - \theta\beta p - \theta\tilde{a}(v) - \theta\sigma\beta(w + W_v) \\ & + \theta(\beta - k - 1)e^{-\lambda v}\Big(x + \int_t^v e^{\lambda s}\sigma \, \mathrm{d}W_s\Big)\bigg) \mathrm{d}v\bigg] \\ =& -e^{-\theta C} \int_t^\infty \exp\bigg[-\rho v - \theta\beta p - \theta\Big(\tilde{a}(v) + \sigma\beta w - (\beta - k - 1)x e^{-\lambda v}\Big) \\ & - \frac{1}{2}\theta\sigma^2 \int_t^v \Big(\beta - (\beta - k - 1)e^{-\lambda(v-s)}\Big)^2 \mathrm{d}s\bigg] \mathrm{d}v\,. \end{aligned} \tag{4.18}$$

Since $m = 0$ is optimal among all admissible strategies m, it is optimal in the subset of strategies that also satisfy $-\epsilon \leq m \leq \epsilon$ for any given $0 < \epsilon < N$ when $N > 0$, or strategies that also satisfy $-\epsilon \leq m \leq 0$, when $N = 0$. Optimizing over this domain, one gets from the usual stochastic control arguments using the dynamic programming principle when the set of control is bounded that the corresponding HJB equation is a necessary condition for the value function; see, e.g., Pham [12].

Note that the derivative with respect to m in the HJB equation is

$$D(t;m) = k\theta e^{-\theta[C + \tilde{a}(t) + \beta\sigma w - (\beta - k - 1)e^{-\lambda t}x]} e^{-\theta(km + \beta p)} + \lambda V_p(t,p,w,x). \tag{4.19}$$

Denote

$$\begin{aligned} g(v,p,w,x) := -\rho v - \theta\Big(&\beta p + \tilde{a}(v) + \sigma\beta w - (\beta - k - 1)e^{-\lambda v}x \\ & - \frac{1}{2}\theta\sigma^2 \int_0^v \Big[\beta - (\beta - k - 1)e^{-\lambda(v-s)}\Big]^2 \mathrm{d}s\Big). \end{aligned} \tag{4.20}$$

For $m=0$ to be optimal, we need to have $D(t;0)\geq 0$. That is, we must have that for all (t,p,w,x),

$$ke^{g(t,p,w,x)} \geq -\lambda\beta \int_t^\infty e^{g(v,p,w,x)}\,\mathrm{d}v\,.$$

This implies

$$k \geq -\lambda \int_t^\infty \beta e^{-\rho(v-t)} e^{-\theta\left\{\tilde a(v)-\tilde a(t)+(\beta-k-1)[e^{-\lambda t}-e^{-\lambda v}]x-\frac{1}{2}\theta\sigma^2\int_t^v\left[\beta-(\beta-k-1)e^{-\lambda(v-s)}\right]^2\mathrm{d}s\right\}}\,\mathrm{d}v.$$

Unless $\beta=0$ or $\beta=k+1$, the right-hand side can be pushed to infinity by sending x to minus or plus infinity. Notice also that when $\beta<0$ we would need to have $k>0$, which is not possible if $\beta=k+1$. Thus, $\beta\geq 0$.

We next show that necessarily $k\geq 0$. From (4.4), the agent's utility is

$$J(m) := -e^{-\theta C}\mathbf{E}\Big[\int_0^\infty e^{-\rho t} e^{-\theta\left[\tilde a(t)+\beta\sigma W_t-(\beta-k-1)e^{-\lambda t}\int_0^t e^{\lambda s}\sigma\,\mathrm{d}W_s\right]} e^{-\theta\left[km_t+\beta\int_0^t\lambda m_s\,\mathrm{d}s\right]}\,\mathrm{d}t\Big].$$

Suppose $k<0$ and introduce the (deterministic) strategy

$$m_t^A := \frac{1}{k}\int_0^t\left(1-\beta\lambda m_s^A\right)\mathrm{d}s\,,\quad m_0^A=0.$$

We then have

$$km_t^A+\beta\int_0^t\lambda m_s^A\,\mathrm{d}s = t\ .$$

Thus, strategy m^A is strictly better than $m\equiv 0$, and the contract is not IC. So, we must have $k\geq 0$.

Suppose now that $N>0$ and $\beta=k+1$. Then, for $m\equiv 0$ to be optimal we must have $D(t;0)=0$. This is equivalent to

$$(\beta-1)e^{-\rho t-\theta\tilde a(t)+\frac{1}{2}\theta^2\sigma^2\int_0^t\beta^2\,\mathrm{d}s} = -\beta\lambda\int_t^\infty e^{-\rho v-\theta\tilde a(v)+\frac{1}{2}\theta^2\sigma^2\int_0^v\beta^2\,\mathrm{d}s}\,\mathrm{d}v\,. \tag{4.21}$$

From this we conclude that β and $1-\beta$ have the same sign, so that $\beta\in(0,1)$. However, this implies $k<0$, and the contract cannot be IC. Thus, $\beta=k=0$.

We are now also in a position to prove that (4.13) necessarily holds. Taking the derivative with respect to t in (4.21), we get

$$(\beta-1)\Big(-\rho-\theta\tilde a'(t)+\frac{1}{2}\theta^2\sigma^2\beta^2\Big)=\beta\lambda\,.$$

From this, since $\tilde a(0)=0$,

$$\tilde a(t)=\frac{1}{\theta}\int_0^t\Big(-\frac{\lambda\beta}{\beta-1}-\rho+\frac{1}{2}\theta^2\sigma^2\beta^2\Big)\,\mathrm{d}s\,. \tag{4.22}$$

$\square$

Proof of Proposition 4.3. We begin with some preliminary considerations. The contract (4.11) is specified by an ordinary differential equation with random coefficients. Since γ is bounded, in order for S_t to be defined for all $t \geq 0$, we must have that $\mathbf{P}[\int_0^t k(s)\,\mathrm{d}Y_s + \int_0^t (a(s) + \beta(s)\lambda Y_s)\,\mathrm{d}s < \infty] = 1$ for all $t \geq 0$. Since k is bounded and Y is a continuous semimartingale, it follows that $(\int_0^t k(s)\,\mathrm{d}Y_s)_{t\geq 0}$ is a continuous process so that $\mathbf{P}[\int_0^t k_s\,\mathrm{d}Y_s < \infty] = 1$ for all $t \geq 0$. Therefore, we must have that $\mathbf{P}[\int_0^t a(s) + \beta(s)\lambda Y_s\,\mathrm{d}s < \infty] = 1$ for all $t \geq 0$. Since β and λ are bounded and Y is a continuous semimartingale, we conclude that $\mathbf{P}[\int_0^t \beta(s)\lambda Y_s\,\mathrm{d}s < \infty] = 1$ for all $t \geq 0$. But from this it follows that we must have $\int_0^t a(s)\,\mathrm{d}s < \infty$ for all $t \geq 0$. These observations allow us to write (4.11) as

$$e^{-\int_0^t \gamma(s)\,\mathrm{d}s} S_t = S_0 + \int_0^t e^{-\int_0^s \gamma(u)\,\mathrm{d}u} k(s)\,\mathrm{d}Y_s + \int_0^t e^{-\int_0^s \gamma(u)\,\mathrm{d}u}[a(s) + \lambda\beta(s)Y_s]\,\mathrm{d}s\,,$$

from which we obtain

$$\begin{aligned} S_t = e^{\int_0^t \gamma(s)\,\mathrm{d}s} S_0 &+ \int_0^t e^{\int_s^t \gamma(u)\,\mathrm{d}u} k(s)\,\mathrm{d}B_s + \int_0^t e^{\int_s^t \gamma(u)\,\mathrm{d}u} k(s)\,\mathrm{d}m_s \\ &+ \int_0^t e^{\int_s^t \gamma(u)\,\mathrm{d}u} \big(a(s) + \lambda\beta(s)(B_s + m_s)\big)\,\mathrm{d}s\,. \end{aligned} \tag{4.23}$$

Let us introduce the shorthand notation

$$\Gamma_{s,t} := e^{\int_s^t \gamma(u)\,\mathrm{d}u} \quad \text{for } 0 \leq s \leq t < \infty.$$

For each $t \geq 0$, $\mathbf{P}$-a.s. each summand in (4.23) is finite and well defined. Indeed, for the first summand this is clear since we take γ to be bounded. The stochastic integral $(\int_0^t \Gamma_{s,t} k(s)\,\mathrm{d}B_s)_{t\geq 0}$ in the second summand is a continuous semimartingale since we assume that k and γ are bounded. Thus $\mathbf{P}[\int_0^t \Gamma_{s,t} k(s)\,\mathrm{d}B_s < \infty] = 1$ for $t \geq 0$. For the third summand we note that Lebesgue-Stiltjes integral against $\mathrm{d}m_s$ is well defined in view of Definition 2.2. Moreover, for each $t \geq 0$, $\mathbf{P}$-a.s., it is finite. The final term is bounded by our our initial observation that we must have $\int_0^t a(s)\,\mathrm{d}s < \infty$, the fact that the paths of Brownian motion are continuous and Definition 2.2, respectively. Using our assumption that $\mathbf{E}[\int_0^\infty e^{-\rho t}|S_t|\,\mathrm{d}t] < \infty$, we get

$$\begin{aligned} \mathbf{E}[e^{-\rho t} S_t] = \Gamma_{0,t} e^{-\rho t} S_0 + \mathbf{E}\Big[&\int_0^t \Gamma_{s,t} e^{-\rho t}\big(a(s) + \lambda\beta(s)(B_s + m_s)\big)\,\mathrm{d}s \\ &+ \int_0^t e^{\int_s^t \gamma(u)\,\mathrm{d}u} k(s)\,\mathrm{d}B_s + \int_0^t \Gamma_{s,t} e^{-\rho t} k(s)\,\mathrm{d}m_s\Big] < \infty\,, \end{aligned}$$

for almost all $t \geq 0$. Assume now that the contract is IC. Then, $m = 0$ is optimal for the agent. The fact that k, β, λ and γ are bounded together with the previous display implies that we must have

$$-\infty < \int_0^\infty e^{-\rho t} \int_0^t a(s)\Gamma_{s,t}\,\mathrm{d}s\,\mathrm{d}t < \infty\,. \tag{4.24}$$

With these preliminary observations, we consider the principal's problem which is to minimize,

$$\mathbf{E}\Big[\int_0^\infty e^{-\rho t}\big(-Lu(S_t + B_t) + S_t\big)\,\mathrm{d}t\Big] \tag{4.25}$$

over contracts of the form (4.23), with L being the Lagrange multiplier for the participation constraint of the agent. Note that here we have not included the reservation utility v_0 from the participation constraint since we do not need to calculate the Lagrange multiplier. The reservation utility only influences the constant C in (4.11). Note that the Lagrange multiplier L must be non-zero at any optimum. Indeed, if not, then the optimal contract would need to minimize the objective $\mathbf{E}[\int_0^\infty e^{-\rho t} S_t \,\mathrm{d}t]$ which is linear in the contract S. However, the minimal value for the principal would then be $-\infty$, in contradiction with our assumption that $\mathbf{E}[\int_0^\infty e^{-\rho t} |S_t| \,\mathrm{d}t] < \infty$. We may thus take $L \neq 0$, and, in fact, $L > 0$, since ours is a minimization problem with a lower bound on agent's utility.

Instead of minimizing (4.25) over contracts of the form (4.23), it is more convenient to study a potentially relaxed problem. To this end, let us write (4.25) as

$$\begin{aligned}
&\mathbf{E}\Big[\int_0^\infty e^{-\rho t}\big(-Lu(S_t+B_t)+S_t\big)\,\mathrm{d}t\Big] \\
&\quad = \int_0^\infty e^{-\rho t}\mathbf{E}\Big[f_{a,\gamma}(t)+g_{\beta,k,\gamma}(t)-B_t+L\exp\Big(-\theta\big(f_{a,\gamma}(t)+g_{\beta,k,\gamma}(t)\big)\Big)\Big]\,\mathrm{d}t\,, \qquad (4.26)
\end{aligned}$$

where

$$\begin{aligned}
f_{a,\gamma}(t) &:= S_0\Gamma_{0,t} + \int_0^t \Gamma_{s,t}a(s)\,\mathrm{d}s\,, \\
g_{\beta,k,\gamma}(t) &:= B_t + \int_0^t \Gamma_{s,t}k(s)\,\mathrm{d}B_s + \int_0^t \Gamma_{s,t}\lambda\beta(s)B_s\,\mathrm{d}s\,. \qquad (4.27)
\end{aligned}$$

The application of Fubini's theorem leading to (4.26) is justified since we assume that $\mathbf{E}[\int_0^\infty e^{-\rho t} |S_t| \,\mathrm{d}t] < \infty$ and that the agent's utility is finite. In addition, by (4.24), we have $f_{a,\gamma}(t) < \infty$ for all $t \geq 0$. We now consider the expectation under the time-integral in (4.26), specifically the term

$$\mathbf{E}\Big[f_{a,\gamma}(t)+g_{\beta,k,\gamma}(t)-B_t+L\exp\Big(-\theta\big(f_{a,\gamma}(t)+g_{\beta,k,\gamma}(t)\big)\Big)\Big] \qquad (4.28)$$

for $t \geq 0$. Again, since $\mathbf{E}[\int_0^\infty e^{-\rho t} |S_t| \,\mathrm{d}t] < \infty$ and the agent's utility is finite, we may differentiate with respect to $f_{a,\gamma}(t)$ under the integral in (4.26), and thus also under the expectation in (4.28), for almost every $t \geq 0$. Setting the derivative of (4.28) with respect to $f_{a,\gamma}(t)$ equal to zero gives the first order condition, abbreviating the notation to $f(t) = f_{a,\gamma}(t)$,

$$-\theta\Big[f(t)-\frac{1}{\theta}\log\Big(\mathbf{E}\big[\exp\big(-\theta g_{\beta,k,\gamma}(t)\big)\big]\Big)\Big] = -\log(L\theta) \quad \text{for almost every } t \geq 0\,.$$

It is easily checked that the first order condition is also a sufficient condition for minimizing the function $x \to x + ae^{-bx+c}$, as we have inside the integral in (4.26) and (4.28). We rewrite the first order condition as

$$f(t) = \frac{1}{\theta}\log(L\theta) - G_{\beta,k,\gamma}(t) \qquad (4.29)$$

where

$$G_{\beta,k,\gamma}(t) := -\frac{1}{\theta}\log\Big(\mathbf{E}\big[\exp\big(-\theta g_{\beta,k,\gamma}(t)\big)\big]\Big)\,. \qquad (4.30)$$

Next, we represent (4.30) more explicitly, as follows. Starting from (4.30), we find $e^{-\theta G_t(\beta,k,\gamma)} = \mathbf{E}[\exp(-\theta g_{\beta,k,\gamma}(t))]$. Moreover, by (2.1) and (4.27), $g_{\beta,k,\gamma}(t)$ may be written as

$$\begin{aligned} g_{\beta,k,\gamma}(t) &= b_0 + \int_0^t \Gamma_{s,t}\lambda\beta(s)B_s \,\mathrm{d}s + \int_0^t K_{k,\gamma}(s,t)\,\mathrm{d}B_s \\ &= b_0 + \lambda\int_0^t (\Gamma_{s,t}\beta(s) - K_{k,\gamma}(s,t))B_s + K_{k,\gamma}(s,t)\mu_0\,\mathrm{d}s \\ &\quad + \int_0^t K_{k,\gamma}(s,t)\sigma\,\mathrm{d}W_s\,, \end{aligned}$$

with shorthand notation

$$K_{k,\gamma}(s,t) := 1 + \Gamma_{s,t}k(s)$$

and the explicit representation of B_t from (4.2),

$$B_t = e^{-\lambda t}b_0 + \int_0^t \mu_0 e^{-\lambda(t-u)}\,\mathrm{d}u + \int_0^t e^{-\lambda(t-u)}\sigma\,\mathrm{d}W_u\,. \tag{4.31}$$

From this it is clear that $g_{\beta,k,\gamma}(t)$ is normally distributed. The mean and variance are easily computed to be

$$\begin{aligned} M_{\beta,k,\gamma}(t) &:= b_0 + \mu_0\int_0^t K_{k,\gamma}(s,t)\,\mathrm{d}s \\ &\quad + \lambda\int_0^t \Big(\Gamma_{s,t}\beta(s) - K_{k,\gamma}(s,t)\Big)\Big(e^{-\lambda s}b_0 + \int_0^s \mu_0 e^{-\lambda(s-u)}\mathrm{d}u\Big)\,\mathrm{d}s \end{aligned}$$

and

$$\Sigma^2_{\beta,k,\gamma}(t) := \sigma^2\int_0^t \left(K_{k,\gamma}(s,t) + \lambda e^{\lambda s}\int_s^t e^{-\lambda u}\Big(\Gamma_{s,t}\beta(u) - K_{k,\gamma}(s,t)\Big)\,\mathrm{d}u\right)^2\mathrm{d}s\,,$$

respectively. From (4.30) we thus get that

$$G_{\beta,k,\gamma}(t) = M_{\beta,k,\gamma}(t) - \frac{1}{2}\theta\Sigma^2_{\beta,k,\gamma}(t). \tag{4.32}$$

From (4.32) we get following explicit version of the first order condition (4.29),

$$f(t) = \frac{1}{\theta}\log(L\theta) - M_{\beta,k,\gamma}(t) + \frac{1}{2}\theta\Sigma^2_{\beta,k,\gamma}(t)\,, \tag{4.33}$$

for almost every $t \geq 0$. Returning to (4.28) and using (4.33) we see that

$$\begin{aligned} &\mathbf{E}\Big[f(t) + g_{\beta,k,\gamma}(t) - B_t + L\exp\Big(-\theta\big(f(t) + g_{\beta,k,\gamma}(t)\big)\Big)\Big] \\ &= \frac{1}{\theta}(1 + \log(L\theta)) - G_t(\beta,k,\gamma) + \mu_0\int_0^t \Gamma_{s,t}k(s)\,\mathrm{d}s \\ &\quad + \int_0^t \Gamma_{s,t}\Big(\lambda(\beta(s) - k(s))\mathbf{E}[B_s]\Big)\,\mathrm{d}s \\ &= \frac{1}{\theta}(1 + \log(L\theta)) - M_{\beta,k,\gamma}(t) + \frac{1}{2}\theta\Sigma^2_{\beta,k,\gamma}(t), \\ &\quad + \mu_0\Gamma_{s,t}k(s)\,\mathrm{d}s + \int_0^t \Gamma_{s,t}\Big(\lambda(\beta(s) - k(s))\mathbf{E}[B_s]\Big)\,\mathrm{d}s \\ &= \frac{1}{\theta}(1 + \log(L\theta)) + \lambda\int_0^t \mathbf{E}[B_s]\,\mathrm{d}s + \frac{1}{2}\theta\Sigma^2_{\beta,k,\gamma}(t)\,, \end{aligned} \tag{4.34}$$

where the last line uses $\mathbf{E}[B_t] = e^{-\lambda s} b_0 + \int_0^s \mu_0 e^{-\lambda(s-u)} \mathrm{d}u$, see (4.31), and the definition of $M_{\beta,k,\gamma}$. With this we have arrived at our reduced problem. Instead of minimizing (4.25), we consider the relaxed problem of minimizing (4.34) with respect to (β, k, γ), point wise for each $t \geq 0$. Note that we may neglect the first two summands in this minimization as well as multiplicative constants. Thus, the relaxed problem is equivalent to minimizing the function $(\beta, k, \gamma) \mapsto \Sigma^2_{\beta,k,\gamma}(t)/\sigma^2$ for each $t \geq 0$. For this function a simple calculation shows that

$$\begin{aligned}\Sigma^2_{\beta,k,\gamma}(t)/\sigma^2 &= \int_0^t \Big(e^{-\lambda(t-s)} + \Gamma_{s,t} k(s) + \lambda \int_s^t e^{-\lambda(u-s)} \Gamma_{u,t} [\beta(u) - k(u)] \,\mathrm{d}u\Big)^2 \mathrm{d}s \\ &= \int_0^t e^{2\lambda s} \Big(e^{-\lambda t} + e^{-\lambda s} \Gamma_{s,t} k(s) + \lambda \int_s^t e^{-\lambda u} \Gamma_{u,t} [\beta(u) - k(u)] \,\mathrm{d}u\Big)^2 \mathrm{d}s\,.\end{aligned}$$

As it turns out, a further relaxation is convenient. For it, let us define

$$F_t(x, y, z) := \Big(e^{-\lambda t} + \lambda(y - x) + z\Big)^2 \quad \text{for } x, y, z \in \mathbb{R} \text{ and } t \geq 0\,,$$

and set

$$g_t(s) := -\int_s^t e^{-\lambda u} \Gamma_{u,t} k(u) \,\mathrm{d}u \tag{4.35}$$

$$h_t(s) := -\int_s^t e^{-\lambda u} \Gamma_{u,t} \beta(u) \,\mathrm{d}u\,. \tag{4.36}$$

With this notation we observe that

$$\Sigma^2_{\beta,k,\gamma}(t)/\sigma^2 = \int_0^t e^{2\lambda s} F_t\big(h_t(s), g_t(s), g_t'(s)\big) \,\mathrm{d}s\,, \quad \text{for all } t \geq 0\,.$$

The new relaxed problem is thus to minimize for each $t \geq 0$

$$F_t(h(s), g(s), g'(s)) = \big(e^{-\lambda t} + g'(s) + \lambda[g(s) - h(s)]\big)^2,$$

over functions g and h, subject to the constraints

$$g'(s) \geq 0,\ h'(s) \geq 0,\ g(t) = h(t) = 0.$$

Note that the choice $(g, h) = (0, 0)$ satisfies the constraints and can be achieved via (4.35) and (4.36) by setting $\beta, k = 0$ and letting γ be arbitrary. Thus this choice is also admissible for our original relaxed problem of minimizing $(\beta, k, \gamma) \mapsto \Sigma^2_{\beta,k,\gamma}(t)/\sigma^2$. Let us compare this choice to any other admissible choice. Since $(x, y, z) \mapsto F_t(x, y, z)$ is convex for all $t \geq 0$, we have that

$$\begin{aligned}&\int_0^t e^{2\lambda s} (F_t(0, 0, 0) - F_t(h(s), g(s), g'(s))) \,\mathrm{d}s \\ &\leq -\int_0^t e^{2\lambda s} (h(s) \partial_x F_t(0, 0, 0) + g(s) \partial_y F_t(0, 0, 0) + g'(s) \partial_z F_t(0, 0, 0)) \,\mathrm{d}s \\ &= -2 \int_0^t e^{2\lambda s} e^{-\lambda t} \Big(\lambda\big(g(s) - h(s)\big) + g'(s)\Big) \,\mathrm{d}s\end{aligned}$$

using integration by parts and $g(t) = 0$ in the last equality, we have

$$= -\int_0^t e^{2\lambda s} 2e^{-\lambda t}\Big(\lambda(g(s) - h(s)) - 2\lambda g(s)\Big)\,\mathrm{d}s + 2e^{-\lambda t}(g(0) - g(t)e^{2\lambda t})$$
$$\leq 0\,.$$

Therefore $(g, h) = (0, 0)$ is optimal. Via (4.27) we see that $g_{0,0,\gamma}(t) = B_t$ for any γ and $t \geq 0$. From (4.30) and (4.29) we then get $f(t)$ by setting $\gamma = 0$ and choosing $a(t)$ accordingly. This all gives rise to the contract of the form S^C. $\square$

Proof of Proposition 4.4: To show that V^0 is the value function, we perform the usual verification argument from stochastic control theory to verify that $m = 0$ is optimal. Take any admissible strategy m. Since we assume that the set of admissible strategies m is such that the corresponding value function V of (4.14) satisfies the HJB equation, we have, by Ito's rule,

$$\begin{aligned} e^{-\rho T} V^0(T, P_T^m, W_T^w) \leq\ & e^{-\rho t} V^0(0, p, w) \\ & + \int_0^T e^{-\rho s - \theta[C + \tilde{a}(s) + \beta\sigma W_s^w]} e^{-\theta(k m_s + \beta P_s^m)}\,\mathrm{d}s \\ & + \int_0^T e^{-\rho s} V_w^0(s, P_s^m, W_s^w)\,\mathrm{d}W_s^w\,. \end{aligned}$$

Taking expectation, and using assumption **(c)**, we get

$$\begin{aligned} & -\mathbf{E}\Big[\int_0^T e^{-\rho s - \theta[C + \tilde{a}(s) + \beta\sigma W_s^w]} e^{-\theta(k m_s + \beta P_s^m)}\,\mathrm{d}s\Big] \\ & \quad \leq V^0(0, p, w) - e^{-\rho T}\mathbf{E}[V^0(T, P_T^m, W_T^w)]\,. \end{aligned}$$

Taking the limit as $T \to \infty$ and using assumption **(b)** concludes the proof. $\square$

5 Discussion and Conclusions

Discussion: ways the agent can misreport. We now discuss, somewhat informally, what kind of reports by the agent could be considered credible by the principal. In Williams [16] it is assumed that the only way the agent can lie is with a reporting process Y that equals the true state process plus a non-positive differentiable drift. In general, we can ask the following question:

For a reported path $y = (y_t)_{t\in[0,\infty)}$ to be accepted by the principal as credible, what are the properties it has to satisfy?

The answer is not obvious. For example, since the diffusion process (2.1) has continuous paths almost surely, we may want to impose a restriction that the reported path y be continuous, almost surely. However, the qualifier 'almost sure' is relevant in this context, for – depending on the space on which the diffusion is defined – a discontinuous path could actually happen, and some principals might be willing to accept even such reports. Nevertheless, it is reasonable that the principal insists on the reported path y having properties that are known to hold for almost all paths of B. This can be rephrased as saying that the principal fixes a set of paths deemed as unlikely to happen and thus deemed as lies if reported by the agent:

The principal fixes a Borel measurable set $N \subset \mathcal{C}([0,\infty),\mathbb{R})$, with the property that $\mathbf{P}[B \in N] = 0$. Then, if the agent reports $y \in N$, the principal considers it a lie.

The set N could, for instance, include all differentiable paths, for Brownian motion is almost surely nowhere differentiable. It could also include paths that do not have a quadratic variation process equal to $\sigma^2 t$, because quadratic variation can be calculated pathwise; see e.g. Karandikar [9].

On the other hand, no continuous $\mathbb{L}^2$ drift can be recognbiblioized as a lie in the above sense. Indeed, for every $t < \infty$, the law of B in (2.1) is equivalent to the law of $\sigma W.$ on $\mathcal{C}([0,t];\mathbb{R})$. This follows from Cameron-Martin-Girsanov theory, see e.g. [10, Ch. 3.5]: if $A \subset \mathcal{C}([0,t],\mathbb{R})$ is measurable and such that $\mathbf{P}[\sigma W. \in A] > 0$, then also $\mathbf{P}[\sigma W. \in A_\mu] > 0$ for any absolutely continuous $\mu \in \mathbb{L}^2([0,t])$, where $A_\mu := \{s \mapsto w_s + \mu_s \ : \ w \in A\}$ is the set of path translated by μ. Thus, if $N_t \subset \mathcal{C}([0,t],\mathbb{R})$ is a null set, then set $A = N^c$ has a full measure, and so does A_μ, so that reporting an additional drift μ to any report that is not considered a sure lie, is not a sure lie. Evidently same conclusion holds also for B.

We leave for further research these issues, and, in particular, the following question: What is the minimal set of restrictions that are reasonable to assume on the reported process, while still having interesting examples, that is, examples in which the optimal contract is not such that the agent is indifferent with respect to how much to misreport?

Conclusions. In this paper, we show that the optimal contract provided in Williams [16] is not incentive compatible in the case of mean-reverting shocks, if the growth on under-reporting is not restricted. It becomes incentive compatible if the difference between the reported process and the true process is sufficiently bounded, but, as shown in Bloedel et al. [2], it is not optimal. In the case when there is no limit on how much the agent can under-report, we show that deterministic contracts are the only IC contracts in an extended family of linear contracts. If additional restrictions on admissible contracts are imposed, we show that the contract that Bloedel et al. [2] identifies as optimal among self-insurance contracts is also optimal among the more general class of linear contracts. It is still an open question what the optimal contract is if we allow general contracts, and/or under additional restrictions on misreporting, for example, if it is restricted to take values in a bounded interval. We also leave for future research the question of what types of misreporting would be credible in continuous-time models driven by Brownian motion, while still resulting in non-trivial optimal contracts.

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