OPTIMAL PORTFOLIO SELECTION VIA CONDITIONAL CONVEX RISK MEASURES ON L^p

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Abstract. We consider conditional convex risk measures on L^p and show their robust representation in a standard way. Such measures are used as evaluation functionals for optimal portfolio selection in a Black&Scholes setting. We study this problem focusing on the conditional Average Value at Risk and the conditional entropic risk measure, and compare the respective optimizers.
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1. Introduction

The measurement and management of risk is a central issue in finance, and huge effort is made in order to analyze it and to understand all the related problems. Risk measures as introduced by Artzner, Delbaen, Eber, and Heath [1, 2] - and then extended by Föllmer and Schied [17] and Frittelli and Rosazza Gianin [19] to the general convex case - serve to quantify the riskiness of financial positions and to give a criterion for their acceptability. These seminal papers consider the space L^∞ of essentially bounded random variables, used to model essentially bounded financial positions. Since then the literature on convex risk measures rapidly developed also beyond such space, in order to include important risk models as those involving normal or log-normal distributions. Delbaen in [8] defines risk measures on the space L^0 of all random variables. In this case the failing of local convexity limits the use of convex analysis, hence the theory is not as rich as in the L^∞ case. One should notice, however, that most of the applications one usually has in mind are recovered by the L^p spaces of random variables with finite p-th moment, with $p \in [1, \infty)$. Since these spaces carry a natural local convex topology, classical convex analysis provides many powerful tools, see e.g. [15, 21].

Another generalization of the original concept of convex risk measure comes from the need of taking into account the additional information becoming available in time. The concept of *conditional* convex risk measures is the natural extension to this setting; see Detlefsen and Scandolo [9]. With the exception of very few works, the financial literature in conditional setting is so far mostly devoted to the study of the essentially bounded case. In this paper, instead, we consider conditional convex risk measures defined on the model space L^p, for $p \in [1, \infty)$. Filipović, Kupper and Vogelpoth in [13, 14] investigate conditional convex risk measures defined on the space L^p or in a random module generated by it, performing a careful analysis. In Section 2 we make clear the connection between our setting and the settings considered in those papers.

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One should notice that, while the axiomatic characterization of convex risk measures contributes to the immediateness of their economic interpretation, an 'explicit' representation is desirable in order to use such tools in practical decision making, that is, for the actual evaluation of financial positions. This explains the popularity of the robust representations of convex risk measures, which come as a natural result from convex duality. Also, being our setting not recovered by the previous literature, this is a motivation for us to establish a robust representation, simple generalization of the analogous result in the unconditional case. This applies in particular to the two risk measures we are especially interested in: the conditional versions of the Average Value at Risk and of the entropic risk measure.

Such measures are used to investigate the portfolio selection problem in the classical continuous-time framework pioneered by Merton [22] and nowadays mostly referred to as Black&Scholes-type market, in which the stocks dynamics are log-normal. We restrict ourselves to the case of constant proportion portfolios, where the proportion of wealth invested in each asset is constant in time. Theses policies result to be optimal for many interesting objective functions. For example, in the problem of maximizing the expected utility of terminal wealth for a logarithmic utility or a power utility. The optimality of constant policies in a utility theory setting is considered since Merton [22]. For a discussion on several cases of optimality we refer to Browne [6]. Dhaene et al. in [10] also investigate the portfolio selection problem in a Black&Scholes-type market, considering the Average Value at Risk and other quantile-based risk measures in the unconditional case. Moreover, in [10] as well the attention is restricted to the class of constant mix portfolios. Here we formulate the optimal selection problem for law-invariant conditional convex risk measures, focusing on the conditional versions of the Average Value at Risk and of the entropic risk measure. We show the existence of solutions to the optimal selection problem in these two cases, then compare their behavior in relation to the parameters that describe the stocks' dynamics and to the parameters that characterize such measures.

The rest of the paper is organized as follows. In Section 2 we introduce conditional convex risk measures on L^p-spaces and prove a robust representation result. Section 3 deals with the portfolio selection problem when choices are performed according to conditional convex risk measures. We present and compare the cases of the conditional Average Value at Risk and the conditional entropic risk measure, for which we show the existence of solutions to the optimal selection problem. Some numerical examples show the impact of risk-preferences on investment decisions and conclude the paper.

2. Conditional risk measures for unbounded risks

Throughout the paper, we fix a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ as stochastic basis. With $L^p = L^p(\Omega, \mathcal{F}, \mathbb{P})$, resp. $L^p_t = L^p(\Omega, \mathcal{F}_t, \mathbb{P})$, we mean the space of real-valued $\mathcal{F}$-measurable, resp. $\mathcal{F}_t$-measurable, random variables with finite p-th moment, for $p \in [1, \infty)$. We use the notation $\bar{\mathbb{R}} = (-\infty, +\infty]$ and $\mathbb{R}_+ = [0, \infty)$ and, for any set $A \subseteq [-\infty, +\infty]$, we denote by $L^0_t(A)$ the spaces of $\mathcal{F}_t$-measurable random variables taking values in A. Equalities and inequalities between random variables are understood in the almost sure sense.

We denote by $\mathcal{M}_1$ the sets of all probability measures on $(\Omega, \mathcal{F})$ which are absolutely continuous with respect to $\mathbb{P}$. Moreover, for $q \in [1, \infty)$, we define the set of probability measures

$$\mathcal{Q}_t^q := \left\{ \mathbb{Q} \in \mathcal{M}_1 : \frac{d\mathbb{Q}}{d\mathbb{P}} \in L^q,\ \mathbb{Q} = \mathbb{P} \text{ on } \mathcal{F}_t \right\}, \qquad t \geq 0.$$

Definition 2.1. For $p \in [1, \infty)$ and $t \geq 0$, a map $\rho_t : L^p(\Omega, \mathcal{F}, \mathbb{P}) \to L^0_t(\bar{\mathbb{R}})$ is called a *conditional convex risk measure* if it satisfies the following properties for all $X, Y \in L^p(\Omega, \mathcal{F}, \mathbb{P})$:

- Conditional cash invariance: $\rho_t(X + m_t) = \rho_t(X) - m_t,\ m_t \in L^p(\Omega, \mathcal{F}_t, \mathbb{P})$;
- Monotonicity: $\rho_t(X) \geq \rho_t(Y)$ whenever $X \leq Y$;
- Conditional convexity: for all $\lambda \in L^0_t([0,1])$,
$$\rho_t\big(\lambda X + (1-\lambda)Y\big) \leq \lambda \rho_t(X) + (1-\lambda)\rho_t(Y);$$
- Normalization: $\rho_t(0) = 0$.

In the static case $t = 0$ this definition coincides with that of a convex risk measure given in [21].

For ρ_t as in Definition 2.1, the Fenchel-Moreau theorem from classical convex analysis (see e.g. [23, 12]) does not apply, being ρ_t valued in $L^0_t(\bar{\mathbb{R}})$, and neither do the methods used by Filipović et al. in [13] to establish dual representation results for risk measures defined on L^p and taking values in L^r_t. In [14], on the other hand, risk measures are defined on the random module $L^p_{\mathcal{F}_t}(\mathcal{F}) := L^0(\mathcal{F}_t) \cdot L^p(\mathcal{F}) = \{XY | X \in L^0(\mathcal{F}_t),\ Y \in L^p(\mathcal{F})\}$. There a dual representation result à la Fenchel-Moreau is established and a rich theory is developed (see also Guo [20]). In the present paper, for the easiness of tractability, we choose to work on L^p and not in random modules as in [14], but still we do not impose regularity conditions as in [13], in order to include in our analysis one of the most known and used risk measures: the entropic one.

With Theorem 2.2 we establish a robust representation result in our framework. It is obtained as an easy generalization of the analogous result proved by Kaina and Rüschendorf [21] in the unconditional case (see Lemma A.1 for another equivalent characterization). We use the notation $Y^- = (-Y) \vee 0$ for the negative part of a random variable Y.

Theorem 2.2. *Let $\rho_t : L^p(\Omega, \mathcal{F}, \mathbb{P}) \to L^0_t(\bar{\mathbb{R}})$ be a conditional convex risk measure. Assume $\rho_t^-(X) \in L^1(\Omega, \mathcal{F}_t, \mathbb{P})$ for all $X \in L^p$. Then the following are equivalent:*

(i) *ρ_t is continuous from above: For any sequence $(X_n)_{n \in \mathbb{N}} \subset L^p$ and $X \in L^p$ with $X_n \searrow X$ $\mathbb{P}$-a. s., it follows that $\rho_t(X_n) \nearrow \rho_t(X)$ $\mathbb{P}$-a. s.*

(ii) *ρ_t has the robust representation*
$$\rho_t(X) = \operatorname*{ess\,sup}_{\mathbb{Q} \in \mathcal{Q}_t^q} \big\{ -\mathbb{E}_{\mathbb{Q}}[X \,|\, \mathcal{F}_t] - \alpha_t(\mathbb{Q}) \big\}, \quad X \in L^p,$$
where q is the conjugate index of p ($\frac{1}{p} + \frac{1}{q} = 1$) and the minimal penalty function α_t of ρ_t is given by
$$\alpha_t(\mathbb{Q}) = \operatorname*{ess\,sup}_{X \in L^p} \{ -\mathbb{E}_{\mathbb{Q}}[X \,|\, \mathcal{F}_t] - \rho_t(X) \}, \quad \mathbb{Q} \in \mathcal{Q}_t^q.$$

Note that our integrability condition on the negative part of risk measures is much weaker than the integrability condition imposed in [13], and it has the natural economical interpretation that there is no financial position that gives on average an infinite ‘utility’.

To prove Theorem 2.2 we use the same arguments as in [9]. There a robust representation is proved for conditional convex risk measures on L^∞, reducing to the case of static convex risk measures on L^∞. Here we also reduce to the static case, and then use the results of Kaina and Rüschendorf [21] for static convex risk measures on L^p.

Proof. $(ii) \Rightarrow (i)$ follows in the same way as in [9, Theorem 1].

$(i) \Rightarrow (ii)$. The inequality

$$\rho_t(X) \geq \operatorname*{ess\,sup}_{\mathbb{Q}\in\mathcal{Q}_t^q} \big\{ -\mathbb{E}_{\mathbb{Q}}[X\,|\,\mathcal{F}_t] - \alpha_t(\mathbb{Q})\big\}, \quad X \in L^p, \tag{2.3}$$

easily follows from the definition of α_t.

To prove the reverse inequality we proceed as in [9, Theorem 1] and reduce to the static setting, obtaining, for all $X \in L^p$,

$$\mathbb{E}_{\mathbb{P}}[\rho_t(X)] \leq \mathbb{E}_{\mathbb{P}}\Big[\operatorname*{ess\,sup}_{\mathbb{Q}\in\mathcal{Q}_t^q} \big\{ -\mathbb{E}_{\mathbb{Q}}[X\,|\,\mathcal{F}_t] - \alpha_t(\mathbb{Q})\big\}\Big], \tag{2.4}$$

where the expectation on the left hand side is well-defined because of the integrability of $\rho_t^-(X)$. At this stage in [9] the robust representation of a conditional convex risk measure on L^∞ could be stated, while here we need to take care of integrability conditions. Suppose $X \in L^p$ is bounded from below. Then there exists $n \in \mathbb{N}$ with $X \geq -n$, and so $\rho_t(X) \leq n$ by monotonicity and cash invariance of ρ_t. Therefore the positive part of $\rho_t(X)$ is bounded and $\rho_t(X)$ is integrable. Together with (2.3), this implies

$$\mathbb{E}_{\mathbb{P}}\Big[\operatorname*{ess\,sup}_{\mathbb{Q}\in\mathcal{Q}_t^q} \big\{ -\mathbb{E}_{\mathbb{Q}}[X\,|\,\mathcal{F}_t] - \alpha_t(\mathbb{Q})\big\}\Big] < \infty.$$

Then

$$\rho_t(X) = \operatorname*{ess\,sup}_{\mathbb{Q}\in\mathcal{Q}_t^q} \big\{ -\mathbb{E}_{\mathbb{Q}}[X\,|\,\mathcal{F}_t] - \alpha_t(\mathbb{Q})\big\}$$

follows by (2.3) and (2.4), which proves the representation in (ii) for all $X \in L^p$ bounded from below.

Now consider an arbitrary $X \in L^p$ and define a sequence $(X_n)_{n\in\mathbb{N}} \subset L^p$ by $X_n := X \vee (-n)$. Then $X_n \searrow X$ and $\rho_t(X_n) \nearrow \rho_t(X)$ follows by continuity from above. For each $n \in \mathbb{N}$, the random variable X_n is bounded from below, and by the previous step we obtain

$$\begin{aligned}\rho_t(X) = \lim_{n\to\infty} \rho_t(X_n) &= \lim_{n\to\infty} \operatorname*{ess\,sup}_{\mathbb{Q}\in\mathcal{Q}_t^q} \big\{ -\mathbb{E}_{\mathbb{Q}}[X_n\,|\,\mathcal{F}_t] - \alpha_t(\mathbb{Q})\big\}\\ &= \operatorname*{ess\,sup}_{\mathbb{Q}\in\mathcal{Q}_t^q} \lim_{n\to\infty} \big\{ -\mathbb{E}_{\mathbb{Q}}[X_n\,|\,\mathcal{F}_t] - \alpha_t(\mathbb{Q})\big\}\\ &= \operatorname*{ess\,sup}_{\mathbb{Q}\in\mathcal{Q}_t^q} \big\{ -\mathbb{E}_{\mathbb{Q}}[X\,|\,\mathcal{F}_t] - \alpha_t(\mathbb{Q})\big\},\end{aligned}$$

where the exchange of limit and essential supremum follows since $(-\mathbb{E}_{\mathbb{Q}}[X_n\,|\,\mathcal{F}_t] - \alpha_t(\mathbb{Q}))$ is increasing with n, and so is $\operatorname{ess\,sup}_{\mathbb{Q}\in\mathcal{Q}_t^q}\big\{ -\mathbb{E}_{\mathbb{Q}}[X_n\,|\,\mathcal{F}_t] - \alpha_t(\mathbb{Q})\big\}$. This concludes the proof. $\square$

Example 2.5 (Conditional entropic risk measure)**.** For $p \in [1,\infty)$ and $t \geq 0$, the *conditional entropic risk measure* $\mathrm{Entr}_t^{\gamma_t} : L^p(\Omega,\mathcal{F},\mathbb{P}) \to L_t^0(\bar{\mathbb{R}})$ with risk aversion parameter $\gamma_t \in L^0_t([0,\infty])$ is defined by

$$\mathrm{Entr}_t^{\gamma_t}(X) = \frac{1}{\gamma_t} \log \mathbb{E}\left[e^{-\gamma_t X} \,\middle|\, \mathcal{F}_t\right], \quad X \in L^p. \tag{2.6}$$

In the limiting cases $A := \{\gamma_t = 0\}$ and $B := \{\gamma_t = \infty\}$, this is meant as

$$\mathrm{Entr}_t^{\gamma_t}(X 1_A) = \mathbb{E}[-X|\mathcal{F}_t] 1_A \;\text{ and }\; \mathrm{Entr}_t^{\gamma_t}(X 1_B) = \operatorname*{ess\,sup}_{\mathbb{Q}\in\mathcal{Q}_t^q} \mathbb{E}_{\mathbb{Q}}[-X\,|\,\mathcal{F}_t]\, 1_B.$$

The entropic risk measure was introduced in [18] in L^∞ in the static setting, and its conditional version appeared, among others, in [4, 5, 9, 7, 16].

Clearly (2.6) defines a conditional convex risk measure continuous from above, by monotone convergence. Moreover, $(\mathrm{Entr}_t^{\gamma_t}(X))^-$ is integrable for all $X \in L^p$ by Jensen's inequality. Therefore, by Theorem 2.2, $\mathrm{Entr}_t^{\gamma_t}$ admits a robust representation. As in [9, Proposition 4], one can show that the minimal penalty function corresponding to $\mathrm{Entr}_t^{\gamma_t}$ is given by

$$\alpha_t(\mathbb{Q}) = \frac{1}{\gamma_t} H_t(\mathbb{Q}|\mathbb{P}), \quad \text{for } \mathbb{Q} \in \mathcal{Q}_t^q,$$

where $H_t(\mathbb{Q}|\mathbb{P})$ is the conditional relative entropy

$$H_t(\mathbb{Q}|\mathbb{P}) = \mathbb{E}_{\mathbb{P}}\left[\frac{dQ}{d\mathbb{P}} \log \frac{dQ}{d\mathbb{P}} \middle| \mathcal{F}_t\right], \quad \text{for } \mathbb{Q} \in \mathcal{Q}_t^q,$$

Therefore, the functional in (2.6) has representation

$$\mathrm{Entr}_t^{\gamma_t}(X) = \operatorname*{ess\,sup}_{\mathbb{Q}\in\mathcal{Q}_t^q} \left\{ -\mathbb{E}_{\mathbb{Q}}[X\,|\,\mathcal{F}_t] - \frac{1}{\gamma_t} H_t(\mathbb{Q}|\mathbb{P}) \right\}, \quad X \in L^p. \tag{2.7}$$

Example 2.8 (Conditional Average Value at Risk)**.** For $p \in [1,\infty)$ and $t \geq 0$, the *conditional Average Value at Risk* $\mathrm{AVaR}_t^{\lambda_t} : L^p(\Omega,\mathcal{F},\mathbb{P}) \to L^0_t(\bar{\mathbb{R}})$ at level $\lambda_t \in L^0_t([0,1])$ is defined by

$$\mathrm{AVaR}_t^{\lambda_t}(X) = \operatorname{ess\,sup}\left\{ -\mathbb{E}_{\mathbb{Q}}[X\,|\,\mathcal{F}_t] \,\middle|\, \mathbb{Q} \in \mathcal{Q}_t^q, d\mathbb{Q}/d\mathbb{P} \leq \lambda_t^{-1} \right\}, \quad X \in L^p. \tag{2.9}$$

In the limiting case $A := \{\lambda_t = 0\}$, this is meant as

$$\mathrm{AVaR}_t^{\lambda_t}(X 1_A) = \operatorname*{ess\,sup}_{\mathbb{Q}\in\mathcal{Q}_t^q} \mathbb{E}_{\mathbb{Q}}[-X\,|\,\mathcal{F}_t]\, 1_A.$$

The static Average Value at Risk in L^∞ was introduced in [2], and its conditional version appeared in [3] and was also studied in [11, 25].

Note that (2.9) defines a conditional convex risk measure through its robust representation. An alternative formulation is given in [25] under the name of conditional Expected Shortfall. In order to obtain it, let us fix $X \in L^p(\Omega,\mathcal{F},\mathbb{P})$. Let $\kappa_{X,t} : \Omega \times \mathcal{B}(\mathbb{R}) \to [0,1]$ be the regular conditional distribution of X with respect to $\mathcal{F}_t$, so that for all $B \in \mathcal{B}(\mathbb{R})$

$$\kappa_{X,t}(\omega, B) = \mathbb{P}\left[X \in B|\mathcal{F}_t\right](\omega) \quad \text{for all } \omega \text{ } \mathbb{P}\text{-a.s.},$$

and let $F_{X,t} : \Omega \times \mathbb{R} \to [0,1]$ be the regular conditional distribution function of X given $\mathcal{F}_t$, so that for all $x \in \mathbb{R}$

$$F_{X,t}(\omega, x) = \kappa_{X,t}\big(\omega, (-\infty, x]\big) = \mathbb{P}\left[X \leq x\,|\mathcal{F}_t\right](\omega) \quad \text{for all } \omega \text{ } \mathbb{P}\text{-a.s.}$$

As reference on conditional distributions, see [24, Chapter II.7].

As in [25] we introduce the concept of conditional quantile, which will provide the desired equivalent characterization of the conditional AVaR.

Definition 2.10. For $X \in L^p(\Omega, \mathcal{F}, \mathbb{P})$ and $\lambda_t \in L^0_t((0,1])$, we call a random variable $q_{X,t} : \Omega \to \mathbb{R}$ a conditional quantile of X given $\mathcal{F}_t$ at level λ_t, if it satisfies

$$\kappa_{X,t}(\omega, (-\infty, q_{X,t}(\omega))) \leq \lambda_t(\omega) \leq \kappa_{X,t}(\omega, (-\infty, q_{X,t}(\omega)]) \quad \text{for all } \omega \ \mathbb{P}\text{-a.s.}$$

Define the random variable $I^{\lambda_t}_{X,t} : \Omega \to \mathbb{R}$ by

$$I^{\lambda_t}_{X,t} = \frac{1}{\lambda_t} \left(\mathbb{1}_{\{X < q_{X,t}\}} + \eta_{X,t} \mathbb{1}_{\{X = q_{X,t}\}} \right),$$

where

$$\eta_{X,t}(\omega) = \begin{cases} 0 & \text{if } \mathbb{P}\left[X = q_{X,t} | \mathcal{F}_t\right](\omega) = 0, \\ \frac{\lambda_t(\omega) - \mathbb{P}[X < q_{X,t}|\mathcal{F}_t](\omega)}{\mathbb{P}[X = q_{X,t}|\mathcal{F}_t](\omega)} & \text{if } \mathbb{P}\left[X = q_{X,t} | \mathcal{F}_t\right](\omega) > 0. \end{cases}$$

With this notation, we have the following equivalent formulation of (2.9):

$$\mathrm{AVaR}^{\lambda_t}_t(X) = \mathbb{E}\left[-X I^{\lambda_t}_{X,t} | \mathcal{F}_t \right], \quad X \in L^p. \tag{2.11}$$

In [25, Theorem 4.4.10] it is proven that the definitions in (2.9) and (2.11) coincide almost surely in case X is essentially bounded. The proof extends easily to all random variables in L^p.

3. Optimal portfolio selection problem

In this section we investigate the portfolio selection problem in a Black&Scholes-type market, where a riskless asset $(S_0(t))_{t \geq 0}$ and n risky assets $(S_i(t))_{t \geq 0}$, $i = 1, \dots, n$, are traded continuously. The price of the riskless asset is assumed to evolve according to the following ordinary differential equation:

$$\frac{dS_0(t)}{S_0(t)} = r dt, \quad S_0(0) = s_0 > 0,$$

where $r > 0$ is the constant interest rate. The price of each risky asset S_i evolves according to a geometric Brownian motion, represented by the following stochastic differential equation:

$$\frac{dS_i(t)}{S_i(t)} = b_i dt + \sum_{j=1}^{d} \sigma_{ij} dW_j(t), \quad S_i(0) = s_i > 0,$$

where $b = (b_1, \dots, b_n)' \in \mathbb{R}^n$ is the vector of the assets' rates of return, $\Sigma = (\sigma_{ij})_{1 \leq i \leq n, 1 \leq j \leq d} \in \mathbb{R}^{n \times d}$ is the matrix of the assets' price volatilities and $W = (W_1, \dots, W_d)$ is a d-dimensional standard Brownian motion, with $d \geq n$. We make the usual assumptions of $\Sigma\Sigma'$ positive definite and $b \neq r\mathbf{1}$, where $\mathbf{1}$ is the n-dimensional vector of ones.

Suppose that at some time $t \geq 0$ we are endowed with a wealth $V(t) > 0$ which we can invest in such market, and that we can continuously trade in a self-financing way (i.e., no money is added to or withdrawn from our portfolio). We consider the problem of how to optimally invest in the market in order to minimize the risk at some future fixed time $T > t$, when the financial positions are evaluated through conditional convex risk measures. If one does not impose any restriction on the admissible strategies, however, there is in general no hope to find a solution to such problem.

For that reason, as done in [10] for quantile-based risk measures in the unconditional case, here we restrict ourselves to a special class of investment strategies, known as constant proportion portfolio strategies or constant mix strategies. This means that we rebalance the portfolio continuously in time so that the proportions π_i's of wealth invested in the risky assets S_i's remain constant over time. The remaining proportion $\pi_0 = 1 - \sum_{i=1}^n \pi_i$ is clearly constant too, and is invested in the riskless asset S_0. Therefore, in t we decide the fraction of wealth to invest in each asset, and keep it constant until the time horizon T. On the other hand, we do not impose any condition on the signs of the π_i's, thus allowing for short selling. In this way the strategies that we consider are described by n-dimensional random variables in the set

$$\Pi = \{\pi = \{\pi_1, \dots, \pi_n\} \,:\, \pi_i \ \mathcal{F}_t\text{-measurable}, \ i = 1, \dots, n\}.$$

These policies result to be optimal for many interesting objective functions, as recalled in the Introduction.

The wealth process $(V^\pi(s))_{s\geq t}$, obtained starting in t with an amount $V(t)$ and then following the policy $\pi \in \Pi$, satisfies the stochastic differential equation

$$\begin{aligned} \frac{dV^\pi(s)}{V^\pi(s)} &= \sum_{i=0}^n \pi_i \frac{dS_i(s)}{S_i(s)} = \mu(\pi)ds + \sigma(\pi)dB(s), \quad s \geq t \\ V^\pi(t) &= V(t), \end{aligned} \tag{3.1}$$

where

$$\mu(\pi) = \pi_0 r + \pi' b, \quad \sigma(\pi)^2 = \pi'\Sigma\Sigma'\pi$$

and the process $B = (B(s))_{s\geq t}$ is defined by

$$B(s) = \frac{1}{\sqrt{\pi'\Sigma\Sigma'\pi}}\pi'\Sigma W(s), \quad s \geq t.$$

The stochastic differential equation in (3.1) was first derived in Merton [22]. It implies that

$$V^\pi(T) = V(t)e^{X^\pi(t,T)}, \tag{3.2}$$

with

$$X^\pi(t,T) = \left(\mu(\pi) - \frac{1}{2}\sigma(\pi)^2\right)(T-t) + \sigma(\pi)(B(T) - B(t)). \tag{3.3}$$

In what follows we will use the fact that $X^\pi(t,T)$ is normally distributed with mean

$$\mu = (T-t)(\mu(\pi) - \frac{1}{2}\sigma(\pi)^2) \tag{3.4}$$

and variance

$$\vartheta^2 = (T-t)\sigma(\pi)^2. \tag{3.5}$$

Our aim is to study the problem of minimizing the risk of the discounted wealth in T, when the positions are evaluated via conditional convex risk measures. The problem therefore reads as

$$\operatorname*{ess\,inf}_{\pi\in\Pi} \rho_t(V^\pi(T)e^{-r(T-t)}). \tag{3.6}$$

In the next proposition we show that this problem can be formulated in a much simpler way.

Proposition 3.7. *Let ρ_t be a conditional convex risk measure conditionally law-invariant, i.e. $\rho_t(X) = \rho_t(Y)$ whenever X and Y in L^p have the same conditional distribution given $\mathcal{F}_t$. Then problem* (3.6) *is equivalent to the following minimization problem:*

$$\operatorname*{ess\,inf}_{\sigma\in L_t^0(\mathbb{R}_+)} \rho_t(V^{\pi^\sigma}(T)e^{-r(T-t)}), \tag{3.8}$$

where

$$\pi^\sigma = \sigma \frac{(\Sigma\Sigma')^{-1}(b - r\mathbf{1})}{\sqrt{(b - r\mathbf{1})'(\Sigma\Sigma')^{-1}(b - r\mathbf{1})}}. \tag{3.9}$$

Proof. For $\sigma \in L_t^0(\mathbb{R}_+)$, denote by Π_σ the set of portfolios π such that $\sigma(\pi)^2 = \sigma^2$, i.e.

$$\Pi_\sigma = \{\pi \in \Pi : \pi'\Sigma\Sigma'\pi = \sigma^2\}.$$

Note that these sets are upward directed, that is,

$$\text{for } \pi_1, \pi_2 \in \Pi_\sigma \text{ there exists } \bar{\pi} \in \Pi_\sigma \text{ s.t. } \mu(\bar{\pi}) \geq \max\{\mu(\pi_1), \mu(\pi_2)\}. \tag{3.10}$$

To see this, it is sufficient to consider $A = \{\mu(\pi_1) \geq \mu(\pi_2)\} \in \mathcal{F}_t$ and $\bar{\pi} = \pi_1 1_A + \pi_2 1_{A^c}$. Moreover, for $\sigma \in L_t^0(\mathbb{R}_+)$ fixed and π running through Π_σ, the conditional distribution function of $V^\pi(T)$ given $\mathcal{F}_t$ is non-increasing in $\mu(\pi)$ by (3.2) and (3.3), that is, for $\pi_1, \pi_2 \in \Pi_\sigma$ with $\mu(\pi_1) \geq \mu(\pi_2)$,

$$F_{V^{\pi_1}(T),t} \leq F_{V^{\pi_2}(T),t}.$$

By conditional law-invariance and monotonicity, ρ_t preserves the first order stochastic dominance in the conditional sense, so that

$$\rho_t(V^{\pi_1}(T)) \leq \rho_t(V^{\pi_2}(T)).$$

This implies that if problem (3.6) admits some solution π^*, then by (3.10) π^* solves

$$\max_{\pi \in \Pi} \mu(\pi) \quad \text{subject to} \quad \sigma(\pi) = \sigma, \tag{3.11}$$

where $\sigma = \sigma(\pi^*)$, and the maximum holds ω-wise. Note that (3.11) is a conditional version of the well-known Markovitz mean-variance problem, which admits a unique solution since $\Sigma\Sigma'$ is positive definite, $b \neq r\mathbf{1}$ and short selling is allowed. By Lagrange optimization we then get that the unique optimizer of problem (3.11) is π^σ given in (3.9). Therefore problem (3.6) is reduced to problem (3.8), as claimed. □

In what follows we will consider the optimization problem (3.6) (equiv. (3.8)) both for the conditional AVaR given in (2.9) and for the conditional entropic risk measure given in (2.6).

3.1. **Optimal portfolio selection minimizing the conditional AVaR.** Here we consider problem (3.6) for $\rho_t = \mathrm{AVaR}_t^{\lambda_t}$ defined in (2.9), with parameter $\lambda_t \in L_t^0((0,1])$. One version of the conditional distribution function $F_{V^\pi(T),t}$ of $V^\pi(T)$ given $\mathcal{F}_t$ is given by

$$\begin{aligned} F_{V^\pi(T),t}(\omega, x) &= \mathbb{P}\big[V^\pi(T) \leq x \big| \mathcal{F}_t\big](\omega) = \mathbb{P}\Big[X^\pi(t,T) \leq \log \frac{x}{V(t)} \Big| \mathcal{F}_t\Big](\omega) \\ &= \Phi\left(\frac{\log \frac{x}{V(t)(\omega)} - \mu(\omega)}{\vartheta(\omega)}\right), \qquad \omega \in \Omega,\ x \in (0,\infty), \end{aligned}$$

where Φ is the standard normal cumulative distribution function and $X^\pi(t,T)$ is given in (3.3). Therefore the conditional quantile $q_{V^\pi(T),t}$ of $V^\pi(T)$ at level λ_t is given by

$$q_{V^\pi(T),t} = V(t) \exp\left(\vartheta\, \Phi^{-1}(\lambda_t) + \mu\right).$$

By (2.11), this implies that the conditional AVaR of the discounted wealth in T, obtained following the strategy $\pi \in \Pi$ from t to T, is given by

$$\begin{aligned}
\mathrm{AVaR}_t^{\lambda_t}\left(V^\pi(T)e^{-r(T-t)}\right) &= \frac{1}{\lambda_t}\,\mathbb{E}\big[-V^\pi(T)e^{-r(T-t)}\mathbb{1}_{\{V^\pi(T)<q_{V^\pi(T),t}\}}\,\big|\,\mathcal{F}_t\big]\\
&= -\frac{1}{\lambda_t}V(t)e^{-r(T-t)}\,\mathbb{E}\Big[e^{X^\pi(t,T)}\mathbb{1}_{\{X^\pi(t,T)<\vartheta\Phi^{-1}(\lambda_t)+\mu\}}\,\Big|\,\mathcal{F}_t\Big]\\
&= -\frac{1}{\lambda_t}V(t)e^{-r(T-t)+\mu+\frac{\vartheta^2}{2}}\,\Phi\big(\Phi^{-1}(\lambda_t)-\vartheta\big)\\
&= -\frac{1}{\lambda_t}V(t)e^{(T-t)\pi'(b-r\mathbf{1})}\,\Phi\left(\Phi^{-1}(\lambda_t)-\sqrt{(T-t)\pi'\Sigma\Sigma'\pi}\right).
\end{aligned}$$

By Proposition 3.7, problem (3.6) reduces to problem (3.8), which in this case reads as

$$\operatorname*{ess\,inf}_{\sigma\in L_t^0(\mathbb{R}_+)} -\frac{1}{\lambda_t}V(t)e^{(T-t)\pi^{\sigma\prime}(b-r\mathbf{1})}\Phi\left(\Phi^{-1}(\lambda_t)-\sqrt{(T-t)}\sigma\right).$$

From (3.9), this means that we want to find $\sigma^*\in L_t^0(\mathbb{R}_+)$ such that

$$f(\sigma^*) = \operatorname*{ess\,sup}_{\sigma\in L_t^0(\mathbb{R}_+)} f(\sigma), \tag{3.12}$$

where

$$f(\sigma) = e^{(T-t)\sigma\sqrt{(b-r\mathbf{1})'(\Sigma\Sigma')^{-1}(b-r\mathbf{1})}}\,\Phi\left(\Phi^{-1}(\lambda_t)-\sqrt{(T-t)}\sigma\right).$$

Here, besides σ, the only dependence on ω is through λ_t, so we start assuming $\lambda_t\in(0,1]$ deterministic and looking for $\sigma\geq 0$ that maximizes f in $[0,\infty)$ (with abuse of notation we still write f for $f\big|_{[0,\infty)}$). Moreover, we will use the notation

$$C_1=\sqrt{(b-r\mathbf{1})'(\Sigma\Sigma')^{-1}(b-r\mathbf{1})}\quad\text{and}\quad C_2(\sigma)=\Phi^{-1}(\lambda_t)-\sqrt{(T-t)}\sigma.$$

In what follows we prove the existence of a unique maximizer for f in $[0,\infty)$. For $\sigma\geq 0$, the derivative of f with respect to σ is continuous and given by

$$\begin{aligned}
f'(\sigma) &= e^{(T-t)C_1\sigma}\left[(T-t)C_1\Phi\left(C_2(\sigma)\right)-\frac{\sqrt{T-t}}{\sqrt{2\pi}}e^{-C_2(\sigma)^2/2}\right]\\
&= \exp\big((T-t)C_1\sigma\big)\frac{\sqrt{T-t}}{\sqrt{2\pi}}\int_{-\infty}^{C_2(\sigma)} e^{-x^2/2}(\sqrt{T-t}\,C_1+x)\,dx,
\end{aligned}$$

where by $f'(0)$ we mean the right derivative at zero, and where we use the fact that

$$e^{-C_2(\sigma)^2/2} = -\int_{-\infty}^{C_2(\sigma)} xe^{-x^2/2}\,dx.$$

In particular, for $\sigma\geq C_1+\Phi^{-1}(\lambda_t)/\sqrt{T-t}$ we have that $C_2(\sigma)\leq -C_1\sqrt{T-t}$, which implies $f'(\sigma)<0$, that is, f is monotone decreasing in the interval $\big[C_1+\Phi^{-1}(\lambda_t)/\sqrt{T-t},\infty\big)$. Therefore, being f continuous with $f(0)<\infty$, f attains its maximum in the interval $I=\big[0,C_1+\Phi^{-1}(\lambda_t)/\sqrt{T-t}\big)$. Here $f'(\sigma)=0$ if and only if

$$-\int_{-\infty}^{-C_1\sqrt{T-t}} e^{-x^2/2}(\sqrt{T-t}\,C_1+x)\,dx = \int_{-C_1\sqrt{T-t}}^{C_2(\sigma)} e^{-x^2/2}(\sqrt{T-t}\,C_1+x)\,dx.$$

The left-hand side is greater than zero and independent of σ, while the right-hand side is strictly monotone decreasing in σ. Therefore, there is at most one root of f'

in I. In particular, if $f'(0) \le 0$, then f reaches its maximum in 0. Else f reaches is maximum at some interior point of I. In other words, if

$$\sqrt{T-t}\sqrt{(b-r\mathbf{1})'(\Sigma\Sigma')^{-1}(b-r\mathbf{1})}\lambda_t > \frac{1}{\sqrt{2\pi}}\exp\Big(-\frac{(\Phi^{-1}(\lambda_t))^2}{2}\Big),$$

then it is optimal to invest also in the risky assets, otherwise it is optimal to invest only in the riskless asset. From the previous arguments it follows that, for a generic $\lambda_t \in L_t^0((0,1])$, there is still a unique $\sigma^* \in L_t^0(\mathbb{R}_+)$ satisfying (3.12), which is given by

$$\sigma^* = \frac{\Phi^{-1}(\lambda_t) - c^*}{\sqrt{T-t}} 1_{\left\{\sqrt{T-t}\sqrt{(b-r\mathbf{1})'(\Sigma\Sigma')^{-1}(b-r\mathbf{1})}\lambda_t > \frac{1}{\sqrt{2\pi}}\exp\left(-\frac{(\Phi^{-1}(\lambda_t))^2}{2}\right)\right\}}, \tag{3.13}$$

where $c^* \in (-C_1\sqrt{T-t}, \infty)$ is the unique constant such that

$$-\int_{-\infty}^{-C_1\sqrt{T-t}} e^{-x^2/2}(\sqrt{T-t}\,C_1 + x)\,dx = \int_{-C_1\sqrt{T-t}}^{c^*} e^{-x^2/2}(\sqrt{T-t}\,C_1 + x)\,dx.$$

Note that (3.13) implies that the higher the λ_t, the greater is the optimizer σ^*, and the bigger are the amounts traded in the risky assets, by (3.9).

3.2. **Optimal portfolio selection minimizing the conditional entropic risk measure.** Here we consider the optimization problem (3.6) for $\rho_t = \mathrm{Entr}_t^{\gamma_t}$ defined in (2.6), with risk aversion parameter $\gamma_t \in L_t^0((0,\infty))$. The conditional entropic risk of the discounted wealth obtained in T, following the strategy $\pi \in \Pi$ from t to T, is given by

$$\mathrm{Entr}_t^{\gamma_t}(V^\pi(T)e^{-r(T-t)}) = \frac{1}{\gamma_t}\log\mathbb{E}\left[e^{-\gamma_t V(t)\exp(X^\pi(t,T)-r(T-t))}\,\middle|\,\mathcal{F}_t\right].$$

Being $X^\pi(t,T)$ normally distributed with parameters μ and ϑ given in (3.4) and (3.5), we obtain

$$\begin{aligned}\mathrm{Entr}_t^{\gamma_t}(V^\pi(T)e^{-r(T-t)}) &= \frac{1}{\gamma_t}\log\mathbb{E}\left[e^{-\gamma_t V(t)\exp(-r(T-t)+\mu+\vartheta Z)}\,\middle|\,\mathcal{F}_t\right]\\ &= \frac{1}{\gamma_t}\log\mathbb{E}\left[e^{-\gamma_t V(t)\exp((T-t)(\pi'(b-r\mathbf{1})-\frac{1}{2}\pi'\Sigma\Sigma'\pi)+\sqrt{T-t}\sqrt{\pi'\Sigma\Sigma'\pi}Z)}\,\middle|\,\mathcal{F}_t\right],\end{aligned}$$

where Z is standard-normal distributed and independent on $\mathcal{F}_t$.

Again by Proposition 3.7, problem (3.6) reduces to problem (3.8), that is, to find $\sigma^* \in L_t^0(\mathbb{R}_+)$ such that

$$g(\sigma^*) = \operatorname*{ess\,inf}_{\sigma\in L_t^0(\mathbb{R}_+)} g(\sigma), \tag{3.14}$$

where

$$g(\sigma) = \mathbb{E}\left[e^{-\gamma_t V(t)\exp\left((T-t)(\sigma C_1 - \frac{1}{2}\sigma^2)+\sqrt{T-t}\sigma Z\right)}\,\middle|\,\mathcal{F}_t\right].$$

Here the conditioning on $\mathcal{F}_t$ simply means "given γ_t and $V(t)$", being Z independent on $\mathcal{F}_t$. So we consider γ_t and $V(t)$ as given values, reducing to the problem of minimizing g on $[0,\infty)$ (with abuse of notation we still write g for $g\big|_{[0,\infty)}$). For $\sigma \ge 0$, the derivative of g with respect to σ is continuous and given by

$$\begin{aligned}g'(\sigma) = \gamma_t V(t)\sqrt{T-t}\,\mathbb{E}\Big[&(\sqrt{T-t}(\sigma - C_1) - Z)e^{(T-t)(\sigma C_1 - \frac{1}{2}\sigma^2)+\sqrt{T-t}\sigma Z}\\ &e^{-\gamma_t V(t)\exp\left((T-t)(\sigma C_1-\frac{1}{2}\sigma^2)+\sqrt{T-t}\sigma Z\right)}\Big],\end{aligned}$$

where by $g'(0)$ we mean the right derivative at zero. Note that $g'(0) < 0$ and that, for $\sigma \to \infty$, $g'(\sigma) \to 0$ with $g(\sigma) \to 1 = \sup_{\sigma \geq 0} g(\sigma)$. This implies that there exists a minimizer for g in $[0, \infty)$, and therefore there exists $\sigma^* \in L_t^0(\mathbb{R}_+)$, function of γ_t and $V(t)$, which satisfies (3.14).

3.3. **Comparison of AVaR and entropic risk measure.** Before showing some numerical results, we briefly comment on the two choice functionals (2.9) and (2.6). In both cases we showed the existence of an optimizer for the portfolio selection problem (3.6), thought without obtaining an explicit expression for it. In the comparison of the optimal choice made according to one rather than the other risk measure, the parameters λ_t and γ_t play an important role. From (2.9), indeed, it is clear that the greater the parameter λ_t, the smaller the risk measured by $\mathrm{AVaR}_t^{\lambda_t}$, i.e., the less prudent is the agent that chooses according to $\mathrm{AVaR}_t^{\lambda_t}$. In particular, for the discounted terminal value of a portfolio $\pi \in \Pi$, (3.2) gives

$$\mathrm{AVaR}_t^{\lambda_t}(V^\pi(T)e^{-r(T-t)}) \in \left[-V(t)e^{(T-t)\pi'(b-r\mathbf{1})}, 0\right], \quad \lambda_t \in L_t^0([0,1]),$$

where the lowest value is obtained for $\lambda_t \equiv 1$ and the highest one for $\lambda_t \equiv 0$. On the other hand, (2.7) implies that the greater the risk aversion parameter γ_t, the greater the risk measured by $\mathrm{Entr}_t^{\gamma_t}$, i.e., the more prudent is the agent that chooses according to $\mathrm{Entr}_t^{\gamma_t}$. Therefore, for the discounted terminal value of a portfolio $\pi \in \Pi$, (3.2) gives

$$\mathrm{Entr}_t^{\gamma_t}(V^\pi(T)e^{-r(T-t)}) \in \left[-V(t)e^{(T-t)\pi'(b-r\mathbf{1})}, 0\right], \quad \gamma_t \in L_t^0([0,\infty]),$$

where the lowest value is obtained for $\gamma_t \equiv 0$ and the highest one for $\gamma_t \equiv \infty$.

This means that, for any fixed portfolio $\pi \in \Pi$, by varying the parameters γ_t and λ_t, the Average Value at Risk measures and the entropic risk measures span the same set of values. Therefore the choice of λ_t and γ_t is crucial when we compare such risk measures, and for that reason in our examples we will calibrate those parameters to some benchmarks in the market.

As for the optimal portfolios and the value functions, besides the respective parameters γ_t and λ_t, the results also depend on the parameters characterizing the dynamics of the stocks: rates of return and volatilities. In Section 3.4 we illustrate the different behavior of the two risk measures by some numerical examples, where we consider different sets of parameters. From those results it is clear how the optimal strategies obtained under these risk measures highly depend on such parameters.

The impact of the value $V(t)$ available at time t on the risk evaluation is also different. Being the AVaR a *coherent* risk measure, that is, proportional on linear payoffs:

$$\mathrm{AVaR}_t^{\lambda_t}(hX) = h\mathrm{AVaR}_t^{\lambda_t}(X), \quad h \in L_t^0(\mathbb{R}_+),$$

it is clear that the optimizer π^* does not depend on the value $V(t)$, and that the value function is proportional to $V(t)$. A completely different situation occurs for the entropic risk measure where, for $h \in L_t^0(\mathbb{R}_+)$, one has

$$\begin{aligned} \mathrm{Entr}_t^{\gamma_t}(hX) &\leq h\mathrm{Entr}_t^{\gamma_t}(X), \quad \text{on } \{h \in [0,1)\}, \\ \mathrm{Entr}_t^{\gamma_t}(hX) &\geq h\mathrm{Entr}_t^{\gamma_t}(X), \quad \text{on } \{h \geq 1\}. \end{aligned}$$

In this case the value $V(t)$ plays a different role, since the optimizer π^* depends on it and the value function is no more proportional to it.

3.4. **Numerical examples.** In this section we illustrate the different behavior of the functionals (2.9) and (2.6) in the optimal portfolio selection problem, by presenting some numerical examples in a market with a riskless asset with interest rate $r = 0$ and two risky assets. To consider dependence between the risky assets, we assume that we have three driving Brownian motions and that the matrix of the assets price volatility is

$$\Sigma = \begin{pmatrix} \sigma_1\sqrt{\zeta} & \sigma_1\sqrt{1-\zeta} & 0 \\ \sigma_2\sqrt{\zeta} & 0 & \sigma_2\sqrt{1-\zeta} \end{pmatrix}, \quad \text{with } \zeta \in [0,1], \sigma_1, \sigma_2 > 0.$$

The parameter ζ clearly measures the dependence between the price of risky assets. If $\zeta = 0$ they evolve independently of each other, if $\zeta = 1$ they are driven by the same Brownian motion and basically represent the same asset. For simplification we assume that we are in the unconditional case, i.e. $t = 0$, and our initial portfolio value is $V(0) = 1$.

To compare the optimal portfolio selection when using the risk measures (2.9) and (2.6) respectively, we fix the parameter $\lambda = \lambda_0 = 0.05$ for the Average Value at Risk and calibrate the risk aversion parameter $\gamma = \gamma_0$ of the entropic risk measure with respect to it. To this end, we consider three benchmark portfolios. In the first one $\pi_1 = 1$, i. e. we only invest in the first risky asset. In the second portfolio $\pi_2 = 1$, i. e. we only invest in the second risky asset. In the third one $\pi_1 = 1/2 = \pi_2$, i. e. we invest half of the wealth in the first risky asset and the other half in the second risky asset. For each different choice of drift and volatility parameters, we minimize with respect to γ the quadratic difference of the AVaR and the entropic risk of these three portfolios for the independent case $\zeta = 0$. The optimal value $\hat{\gamma}$ so found, is then used in the simulations for different values of ζ.

In our example the Average Value at Risk simplifies to

$$\mathrm{AVaR}_0^\lambda\big(V^{\pi,\zeta}(T)\big) = -\frac{1}{\lambda}e^{T\pi' b}\Phi\Big(\Phi^{-1}(\lambda) - \sqrt{T(\pi_1^2\sigma_1^2 + 2\pi_1\pi_2\sigma_1\sigma_2\zeta + \pi_2^2\sigma_2^2)}\Big).$$

Therefore, for a fixed portfolio $\pi = (\pi_1, \pi_2)'$, the Average Value at Risk is monotone increasing with respect to ζ if $\pi_1\pi_2 \geq 0$, and monotone decreasing if $\pi_1\pi_2 \leq 0$. This implies that, if for a $\zeta \in [0,1]$ the minimizing portfolio $\pi^*(\zeta) = (\pi_1^*(\zeta), \pi_1^*(\zeta))'$ satisfies $\pi_1^*(\zeta)\pi_2^*(\zeta) \geq 0$, then also $\pi_1^*(\zeta')\pi_2^*(\zeta') \geq 0$ for all $\zeta' \leq \zeta$. To see this, assume that some $\pi = (\pi_1, \pi_2)' \in \Pi$ with $\pi_1\pi_2 < 0$ minimizes the Average Value of Risk for some $\zeta' \in [0, \zeta]$. Using the uniqueness of the minimum of the Average Value at Risk, we obtain

$$\begin{aligned}\mathrm{AVaR}_0^\lambda\big(V^{\pi,\zeta}(T)\big) &\leq \mathrm{AVaR}_0^\lambda\big(V^{\pi,\zeta'}(T)\big) \\ &< \mathrm{AVaR}_0^\lambda\big(V^{\pi^*(\zeta),\zeta'}(T)\big) \leq \mathrm{AVaR}_0^\lambda\big(V^{\pi^*(\zeta),\zeta}(T)\big),\end{aligned}$$

which is a contradiction. Analogously, if for a $\zeta \in [0,1]$ the minimizing portfolio $\pi^*(\zeta)$ satisfies $\pi_1^*(\zeta)\pi_2^*(\zeta) \leq 0$, then for any $\zeta' \geq \zeta$ the minimizing portfolio $\pi^*(\zeta')$ also satisfies $\pi_1^*(\zeta')\pi_2^*(\zeta') \leq 0$. Furthermore, for $\zeta = 0$ the minimizing portfolio $\pi^*(0)$ satisfies $\pi_i^*(0) \geq 0$ for $i = 1, 2$. This means that in case of no correlation we are long in the risky assets (or do not invest in them at all). Then, by increasing the dependence ζ, the minimal risk increases as well, up to a critical point where in the optimal portfolio we go short in one of the risky assets, and from that point on the minimal risk decreases. This analytical result can also be seen in the simulations. All simulations are performed with the help of the software MATLAB. In the case

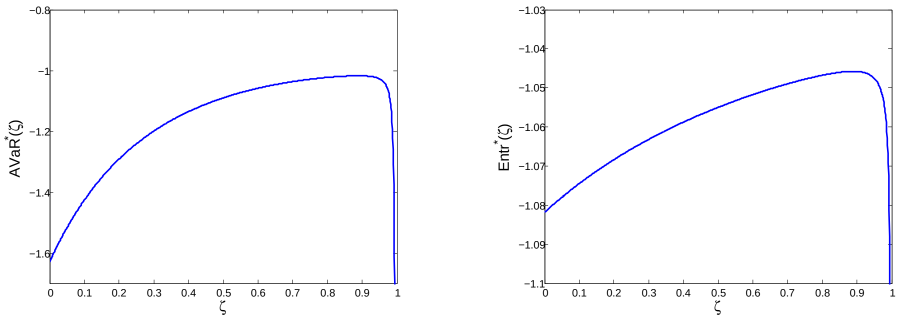

FIGURE 1. Minimal value of AVaR with $\lambda = 0.05$ and Entr with $\hat{\gamma} = 54.55$ for $\zeta \in [0, 1]$. The drift is $b = (0.03, 0.04)'$, the volatility parameters are $\sigma_1 = 0.03$ and $\sigma_2 = 0.045$, and the time horizon is $T = 5$.

of the entropic risk measure the numerical simulation results suggest that the same argument is true, see Figure 1.

Considering a market with a riskless asset and n risky assets correlated by an analogous volatility structure, we obtain similar results. With analogous volatility structure we mean that each asset is driven by an idiosyncratic Brownian motion and by a Brownian motion common to all assets, and that the correlation to the common Brownian motion is given by ζ for all risky assets. Also in this case, the minimal risk initially increases with ζ. Then, for ζ bigger than a critical point, we go short in one of the risky assets and the risk starts to decrease.

In the first simulation we set the drift parameter $b = (0.03, 0.04)'$ and the volatility parameters $\sigma_1 = 0.15$ and $\sigma_2 = 0.2$, and measure the risk after $T = 5$ years, see Figure 2. This choice of parameters seems reasonable according to empirical studies. For this set of parameters $\hat{\gamma} = 30.28$ is the optimal value calibrated to $\lambda = 0.05$ in the sense described above. In the portfolio selection problem with respect to the Average Value at Risk, it is never optimal to invest in the risky assets, for any value of ζ, see Table 1. The amount $V(0)$ is invested in the riskless asset and kept there until the time horizon T. On the other hand, in the portfolio selection problem with respect to the entropic risk measure, the optimal strategy always counts a part invested in the risky assets, hence the minimal risk is lower. In this case, the agent whose preferences are represented by the entropic risk measure is less prudent than the agent whose preferences are represented by the Average Value at Risk.

In case of lower volatility, also the optimal portfolio of the AVaR agent includes a portion invested in the risky assets. For $b = (0.03, 0.04)'$, $\sigma_1 = 0.03$ and $\sigma_2 = 0.045$, the calibrated value of γ is $\hat{\gamma} = 54.55$ and the results of the simulations are illustrated in Figure 3 and Table 2. Here the size of the optimal investments in the risky assets is bigger in the AVaR case than in the entropic one, and the AVaR is more sensible to changes in correlation between the risky stocks, see Figure 1.

With respect to the initial 'standard' case, instead of a lower volatility we can consider a higher drift vector. With $b = (0.14, 0.15)'$, $\sigma_1 = 0.15$ and $\sigma_2 = 0.2$, the

	$\mathrm{AVaR}_0^{0.05}(V^{\pi^*}(T))$			$\mathrm{Entr}_0^{\hat{\gamma}}(V^{\pi^*}(T))$		
ζ	π_1^*	π_2^*	AVaR^*	π_1^*	π_2^*	Entr^*
0	0	0	-1	0.0442	0.0331	-1.0066
0.3	0	0	-1	0.0339	0.0254	-1.0051
0.5	0	0	-1	0.0293	0.0220	-1.0044
0.7	0	0	-1	0.0258	0.0193	-1.0039
0.9	0	0	-1	0.0230	0.0173	-1.0035

Table 1. $b = (0.03, 0.04)'$, $\sigma_1 = 0.15$, $\sigma_2 = 0.2$, $\hat{\gamma} = 30.28$, $T = 5$

	$\mathrm{AVaR}_0^{0.05}(V^{\pi^*}(T))$			$\mathrm{Entr}_0^{\hat{\gamma}}(V^{\pi^*}(T))$		
ζ	π_1^*	π_2^*	AVaR^*	π_1^*	π_2^*	Entr^*
0	11.5864	6.8660	-1.6292	0.6082	0.3604	-1.0819
0.3	6.5429	3.5028	-1.1998	0.4911	0.2629	-1.0632
0.5	4.4613	2.0819	-1.0903	0.4509	0.2104	-1.0550
0.7	3.0629	1.0210	-1.0367	0.4522	0.1507	-1.0490
0.9	3.1415	-0.1164	-1.0176	0.6406	-0.0237	-1.0458

Table 2. $b = (0.03, 0.04)'$, $\sigma_1 = 0.03$, $\sigma_2 = 0.045$, $\hat{\gamma} = 54.55$, $T = 5$

	$\mathrm{AVaR}_0^{0.05}(V^{\pi^*}(T))$			$\mathrm{Entr}_0^{\hat{\gamma}}(V^{\pi^*}(T))$		
ζ	π_1^*	π_2^*	AVaR^*	π_1^*	π_2^*	Entr^*
0	1.6149	0.9733	-1.2401	0.3492	0.2104	-1.2033
0.3	0.7480	0.3722	-1.0521	0.2924	0.1455	-1.1582
0.5	0.3845	0.1463	-1.0129	0.2803	0.1067	-1.1390
0.7	0.1351	0.0240	-1.0012	0.3018	0.0536	-1.1265
0.9	0.3663	-0.0957	-1.0032	0.5121	-0.1338	-1.1299

Table 3. $b = (0.14, 0.15)'$, $\sigma_1 = 0.15$, $\sigma_2 = 0.2$, $\hat{\gamma} = 17.47$, $T = 5$

calibrated value of γ is $\hat{\gamma} = 17.47$. Here qualitatively the results are like in the previous case, see Table 3.

Appendix A. The Fatou property

In what follows we will use the notions of essential limit inferior and essential limit superior of a sequence of random variables $(Y_n)_{n\in\mathbb{N}}$, which are respectively given by

$$\operatorname*{ess\,lim\,inf}_{n\to\infty} Y_n = \sup_{n\to\infty}\left(\operatorname*{ess\,inf}_{m\geq n} Y_m\right)$$

and

$$\operatorname*{ess\,lim\,sup}_{n\to\infty} Y_n = \inf_{n\to\infty}\left(\operatorname*{ess\,sup}_{m\geq n} Y_m\right).$$

Lemma A.1. *Let $\rho_t : L^p(\Omega,\mathcal{F},\mathbb{P}) \to L_t^0(\bar{\mathbb{R}})$ be a conditional convex risk measure and consider the following properties:*

1. *ρ_t is continuous from above (see Theorem 2.2)*
2. *ρ_t has the Fatou-property: For any sequence $(X_n)_{n\in\mathbb{N}} \subset L^p$ with $|X_n| \leq Y$ $\forall n$ for some $Y \in L^p$, and s. t. X_n converges $\mathbb{P}$-almost surely to some $X \in$*

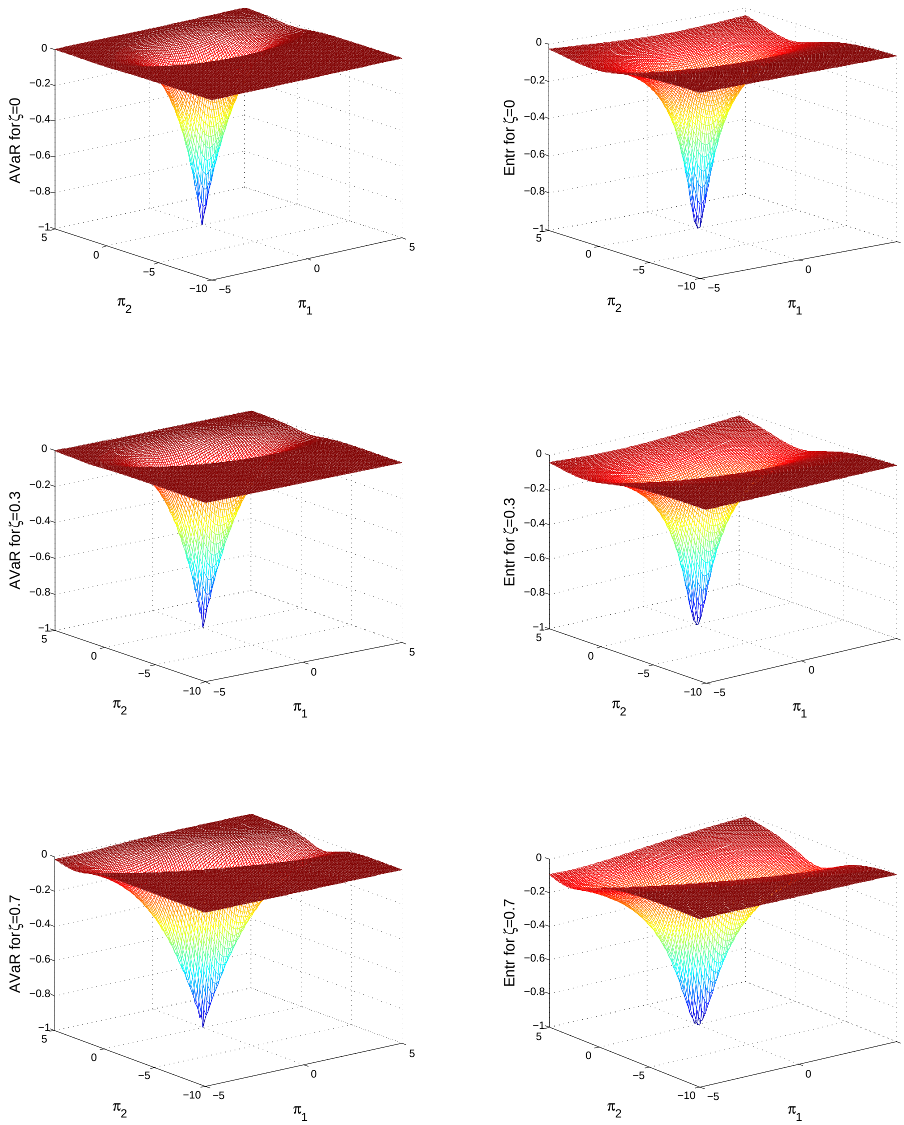

FIGURE 2. $\mathrm{AVaR}_0^{0.05}(V^\pi(T))$ and $\mathrm{Entr}_0^{30.28}(V^\pi(T))$ for $\zeta = 0, 0.3$ and 0.7, drift $b = (0.03, 0.04)'$, volatility $\sigma_1 = 0.15$ and $\sigma_2 = 0.2$, $T = 5$.

L^p, *then*

$$\rho_t(X) \leq \operatorname*{ess\,lim\,inf}_{n\to\infty} \rho_t(X_n) \quad \mathbb{P}\text{-}a.\ s.$$

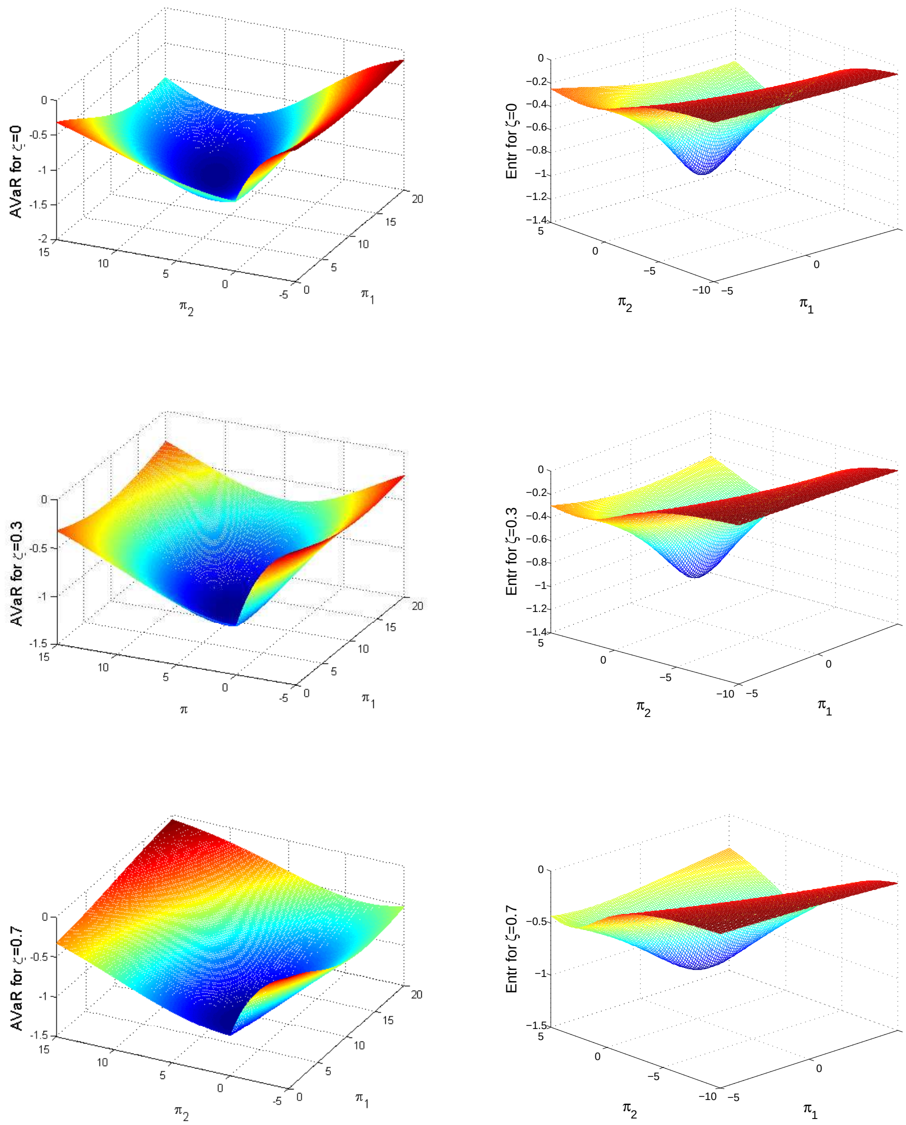

FIGURE 3. $\mathrm{AVaR}_0^{0.05}(V^\pi(T))$ and $\mathrm{Entr}_0^{54.55}(V^\pi(T))$ for $\zeta = 0, 0.3$ and 0.7, drift $b = (0.03, 0.04)'$, volatility $\sigma_1 = 0.03$ and $\sigma_2 = 0.045$, $T = 5$.

(3) *ρ_t is $\|\cdot\|_p$-lower semi continuous: For any sequence $(X_n)_{n\in\mathbb{N}}$ in L^p with $X_n \to X$ in L^p, then*

$$\rho_t(X) \le \operatorname{ess\,lim\,inf}_{n\to\infty} \rho_t(X_n) \quad \mathbb{P}\text{-a. s.}$$

(4) *The set $\{X \in L^p : \rho_t(X) \le Y\}$ is $\|\cdot\|_p$-closed for each $Y \in L_t^0$.*

Then continuity from above is equivalent to the Fatou-property, i. e. (1) $\Leftrightarrow$ (2). Moreover, (3) $\Rightarrow$ (4) $\Rightarrow$ (1). On the other hand, if ρ_t has the Fatou-property, ρ_t is in general not $\|\cdot\|_p$-lower semi continuous, i. e., (2) $\not\Rightarrow$ (3).

Proof. (1) $\Rightarrow$ (2): The proof is analogous to the proof of Lemma 4.20 in [18].

(2) $\Rightarrow$ (1): The proof is analogous to the proof of Lemma 3.2 in [21].

(3) $\Rightarrow$ (4) is obvious.

(4) $\Rightarrow$ (1): Let $(X_n)_{n\in\mathbb{N}} \subset L^p$ and $X \in L^p$ such that $X_n \searrow X$ $\mathbb{P}$-a. s. By monotonicity of ρ_t, we obtain $\rho_t(X_m) \leq \rho_t(X_n)$ a. s. for all $m \leq n$, and $\rho_t(X_n) \leq \rho_t(X)$ a. s. for all $n \in \mathbb{N}$. Since $\big(\rho_t(X_n)\big)_{n\in\mathbb{N}}$ is monotone and bounded, $\big(\rho_t(X_n)\big)_{n\in\mathbb{N}}$ converges almost surely and

$$\lim_{n\to\infty} \rho_t(X_n) \leq \rho_t(X) \quad \mathbb{P}\text{-a. s.}$$

Define the set $\mathcal{S} := \big\{\tilde{X} \in L^p : \rho_t(\tilde{X}) \leq \lim_{n\to\infty} \rho_t(X_n)\big\}$. Then $X_n \in \mathcal{S}$ for all $n \in \mathbb{N}$. By dominated convergence, $X_n \to X$ in L^p. Therefore $X \in \mathcal{S}$ by (4) and

$$\rho_t(X) \leq \lim_{n\to\infty} \rho_t(X_n) \quad \mathbb{P}\text{-a. s.}$$

Altogether, $\rho_t(X_n) \nearrow \rho_t(X)$ a. s. for $n \to \infty$.

(2) $\not\Rightarrow$ (3): To show the last statement of the lemma, we provide the following counterexample. Let Ω be the interval $[0,1]$, let the σ-algebras $\mathcal{F}_t = \mathcal{F} = \mathcal{B}([0,1])$ coincide for all $t \geq 0$, and let $\mathbb{P}$ be the Lebesgue measure λ on $[0,1]$. Define the map $\rho_t : L^p(\Omega,\mathcal{F},\mathbb{P}) \to L^p(\Omega,\mathcal{F},\mathbb{P})$ by $\rho_t(X) = -X$. Then ρ_t is a conditional convex risk measure. Furthermore, for any sequence $(X_n)_{n\in\mathbb{N}} \in L^p$ with $X_n \to X$ λ-a. s. it follows

$$\operatorname{ess}\liminf_{n\to\infty} \rho_t(X_n) = -\operatorname{ess}\limsup_{n\to\infty} X_n = -X = \rho_t(X) \quad \lambda\text{-a. s.}$$

Therefore ρ_t has the Fatou-property. Now define the sequence $(X_n)_{n\in\mathbb{N}}$ by

$$X_{2^m+k} = 1_{\left(\frac{k}{2^m},\frac{k+1}{2^m}\right]}, \quad \text{for } m \in \mathbb{N}_0 \text{ and } k \in \{0,\dots,2^m-1\}.$$

Then for each $m \in \mathbb{N}_0$ and $k \in \{0,\dots,2^m-1\}$ we obtain

$$\|X_{2^m+k}\|_p^p = \int_0^1 1_{2^m+k}(\omega)\,\lambda(d\omega) = \frac{1}{2^m}.$$

Therefore X_n converges to 0 in L^p. On the other hand,

$$\operatorname{ess}\liminf_{n\to\infty} \rho_t(X_n) = -\operatorname{ess}\limsup_{n\to\infty} X_n = -1 < 0 = \rho_t(0) \quad \lambda\text{-a. s.},$$

hence ρ_t is not $\|\cdot\|_p$-lower semi continuous. $\square$

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