

# Are law-invariant risk functions concave on distributions?

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October 4, 2013

## Abstract

While it is reasonable to assume that convex combinations on the level of random variables lead to a reduction of risk (diversification effect), this is no more true on the level of distributions. In the latter case, taking convex combinations corresponds to adding a risk factor. Hence, whereas asking for convexity of risk functions defined on random variables makes sense, convexity is not a good property to require on risk functions defined on distributions. In this paper we study the interplay between convexity of law-invariant risk functions on random variables and convexity/concavity of their counterparts on distributions. We show that, given a law-invariant convex risk measure, on the level of distributions, if at all, concavity holds true. In particular, this is always the case under the additional assumption of comonotonicity.

*Keywords:* convexity, law-invariant risk measure, convex order, comonotonicity

*MSC 2010 Classification:* 46N10, 60E15, 91B30

## 1 Introduction

The concept of *risk* is nowadays used to describe and capture many phenomena, different both by nature and for the context in which they arise. Therefore, when talking of risk measurement, one should first specify to which framework one refers. A *risk measure* is usually intended as a function  $f$  defined on the set  $\mathcal{X}$  of the accordingly identified risky elements, associating to each element  $x \in \mathcal{X}$  a value  $f(x)$  which expresses the riskiness of the “situation” described by  $x$ . Here we consider the case where  $\mathcal{X}$  is intended to model the set of all possible financial positions. The two most prominent approaches to describe these positions are either by random variables on some probability space, or by probability distributions, usually referred to as lotteries in decision theory. To be in line with the traditional notation, to indicate the elements in  $\mathcal{X}$ , we will use  $X, Y, \dots$  to denote random variables, and  $\mu, \nu, \dots$  for distributions. One property which is often required on risk measures defined on random variables is the so-called *law-invariance*, meaning that positions sharing the same distribution are equally risky. We have for example in mind expected losses, certainty equivalents, law-invariant coherent and convex risk measures as introduced by Artzner, Delbaen, Eber and Heath [2, 3] and by Föllmer and Schied [8] and Frittelli and Rosazza Gianin [11], deviation measures in the sense of Rockafellar, Uryasev and Zabarankin [14], and quantile-based measures. Under the paradigm of law-invariance, there is a one-to-one relation between risk functions defined on some space of random variables

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and risk functions defined on the corresponding space of distributions. Nevertheless, some care is needed when translating features from one setting to the other. In particular, we will illustrate how the properties of convexity and concavity are not transferable between the two settings. The reason is that the randomization  $\alpha\mu_X + (1 - \alpha)\mu_Y$  of the lotteries  $\mu_X$  and  $\mu_Y$  corresponding to the laws of the random variables  $X$  and  $Y$  under a probability measure generally differs from the lottery  $\mu_{\alpha X + (1 - \alpha)Y}$  corresponding to the law of the state-wise convex combination  $\alpha X + (1 - \alpha)Y$ . A prominent example is the Value at Risk, which is not quasi-convex on random variables, while the corresponding risk measure on distributions is. On this matter we recall that Frittelli, Maggis and Peri [10] prove that convexity is not compatible with translation invariance on the space of probability distributions, which is the reason why they require quasi-convexity. However, note that concavity is compatible with translation invariance.

On the level of random variables it is well understood that convex combinations corresponding to a diversification in the portfolio should reduce the overall risk. However, on the level of distributions a convex combination corresponds to an additional randomization. Indeed,  $\alpha\mu + (1 - \alpha)\nu$  can be interpreted as the sampling of a lottery between  $\mu$  and  $\nu$ , depending on the outcome of a simultaneous independent toss with probabilities  $\alpha$  and  $(1 - \alpha)$ . Hence, we have an additional factor of risk coming from the toss. Thus requiring that a risk function  $\phi$  on distributions be (quasi-) convex has not the diversification interpretation, and is not necessarily a natural property.

This paper is meant to analyze the interplay between convexity of law-invariant risk functions on random variables and convexity/concavity of their counterparts on distributions. We show that, given a law-invariant convex risk measure, on the level of distributions, if at all, concavity holds true. This is for example always the case under the additional assumption of comonotonicity; see Section 3. Under the assumption of translation invariance, Frittelli et al. [10], Drapeau and Kupper [6] study quasi-convex risk measures over lotteries, as counterparts to convex risk measures on random variables, providing robust representations. Our analysis shows that replacing quasi-convexity on distributions by concavity (these properties do not exclude each other) could probably be more appropriate. Of course this would also provide nicer robust representations. Our results are illustrated by several examples using well-know risk functions.

The remainder of the paper is organized as follows. In Section 2 we specify our setting and prove how, in general, a risk measure  $\phi$  defined on distributions is not convex (Proposition 2.1). To the contrary, under positive homogeneity we show a weak form of concavity for  $\phi$  (Proposition 2.3). Moreover, we provide a dual characterization of concavity of  $\phi$ . In Section 3 we work under the assumption of comonotonicity, and show that in this case concavity of  $\phi$  is automatically satisfied.

## 2 Setup and first results

Let  $(\Omega, \mathcal{F}, \mathbb{P})$  be a non-atomic standard probability space. For any random variables  $X, Y$  and a distribution  $\mu$ , we write  $X \stackrel{d}{=} Y$  to indicate that  $X$  and  $Y$  are equally distributed under  $\mathbb{P}$ , and  $X \sim \mu$  to indicate that the distribution of  $X$  under  $\mathbb{P}$  is  $\mu$ . We do our analysis on  $L^1 := L^1(\Omega, \mathcal{F}, \mathbb{P})$ . It will be obvious from the results and proofs that most of them carry over to any law-invariant Banach spaces of (equivalence classes of) random variables, such as the  $L^p := L^p(\Omega, \mathcal{F}, \mathbb{P})$  spaces for  $p \in [1, \infty]$  equipped with the p-norm  $\|\cdot\|_p := E[|\cdot|^p]^{1/p}$  for

$p \in [1, \infty)$  and  $\|\cdot\|_\infty := \inf\{m \in \mathbb{R} \mid \mathbb{P}(|\cdot| \geq m) = 0\}$  for  $p = \infty$ . Let  $\Phi : L^1 \rightarrow [-\infty, +\infty]$  be a law-invariant function, that is,  $\Phi(X) = \Phi(Y)$  for all  $X, Y \in L^1$  with  $Y \stackrel{d}{=} X$ . Then  $\Phi$  induces a function  $\phi : \mathcal{M}_1 \rightarrow [-\infty, +\infty]$  on the set  $\mathcal{M}_1$  of all Borel probability measures on  $\mathbb{R}$  with finite first moment by

$$(2.1) \quad \phi(\mu) = \Phi(X) \quad \text{for any } X \text{ with } X \sim \mu.$$

Vice versa, any function  $\phi : \mathcal{M}_1 \rightarrow [-\infty, +\infty]$  induces a law-invariant function on  $L^1$  by (2.1). Our main aim is to study how convexity properties of  $\Phi$  on  $L^1$  are related to convexity properties of  $\phi$  on  $\mathcal{M}_1$ . Throughout this paper we will always treat the case that  $\Phi$  is (quasi-)convex. This corresponds to the risk measurement point of view. The corresponding results for utility criteria, i.e. (quasi-)concave  $\Phi$ , are then obvious.  $\Phi$  is said to be lower semicontinuous (lsc) if the level sets  $E_k := \{X \in L^1 \mid \Phi(X) \leq k\}$ ,  $k \in \mathbb{R}$ , are all closed in  $(L^1, \|\cdot\|_1)$ . This is equivalent to  $\Phi(X) \leq \liminf_{n \rightarrow \infty} \Phi(X_n)$  whenever  $(X_n)_{n \in \mathbb{N}} \subset L^1$  is a sequence converging to  $X$  in  $(L^1, \|\cdot\|_1)$ . It is shown in [7] that, for a law-invariant convex risk function  $\Phi$  on  $L^1$ , lsc (in contrast to continuity) is a natural property. Indeed, in most examples that we have in mind, such as convex risk measures,  $\Phi|_{L^\infty}$  is lsc (even continuous) with respect to the convergence in the  $\|\cdot\|_\infty$ -norm. Then it is shown in [7] that there exists a canonical lsc extension of  $\Phi|_{L^\infty}$  to  $L^1$ . Thus requiring lsc basically means that  $\Phi$  should equal its canonical lsc extension from  $L^\infty$  to  $L^1$ . As a lsc convex function which takes the value  $-\infty$  cannot take any finite value, and as the value  $-\infty$  does not make much sense from a risk perspective, we reduce our studies to the case  $\Phi : L^1 \rightarrow (-\infty, +\infty]$ .<sup>1</sup> For the remainder of this paper  $\Phi : L^1 \rightarrow (-\infty, \infty]$  will always be a law-invariant function and the relation between the function  $\Phi$  and  $\phi : \mathcal{M}_1 \rightarrow (-\infty, +\infty]$  will be given by (2.1).

Prominent examples of law-invariant risk functions are

- expected losses  $E[l(-X)]$  or certainty equivalents  $l^{-1}(E[l(-X)])$  where the loss function  $l : \mathbb{R} \rightarrow \mathbb{R}$  is convex and strictly increasing;
- convex risk measures, that is, convex functions which also are antitone (i.e.  $X \geq Y$   $\mathbb{P}$ -a.s. implies  $\Phi(X) \leq \Phi(Y)$ ) and translation invariant:  $\Phi(0) \in \mathbb{R}$  and  $\Phi(X + x) = \Phi(X) - x$  for all  $x \in \mathbb{R}$ ;
- coherent risk measures, that is, convex risk measures which are also positively homogeneous:  $\Phi(tX) = t\Phi(X)$  for all  $t \geq 0$ ;
- quantile-based risk measures;
- deviation measures, that is, convex, positively homogeneous functions which are positive on non-constants, and constant-absorbing:  $\Phi(X + x) = \Phi(X)$  for all  $x \in \mathbb{R}$ .

## 2.1 In general $\phi$ is not convex

Let us recall the definition of the convex order  $\succeq$ : for  $X, Y \in L^1$ ,

$$(X \succeq Y) \iff (E[c(X)] \geq E[c(Y)] \text{ for every convex function } c : \mathbb{R} \rightarrow \mathbb{R}).$$

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<sup>1</sup>Note that an infinitely risky position makes sense, as it can be interpreted to be so bad that it cannot be hedged at any finite cost. Whereas to assume that there is a position which is infinitely good (risk  $-\infty$ ) is clearly not reasonable since it would imply that we could withdraw any amount of money and still face no risk.

It is proved in [4, Theorem 4.1] that if  $\Phi$  is lsc, convex and law-invariant, then it is automatically *preserving the convex order* (see also [15]), which means that  $\Phi(X) \geq \Phi(Y)$  whenever  $X \succeq Y$ . Hence, when considering convex  $\Phi$ , it is also natural to assume that  $\Phi$  is  $\succeq$ -preserving. As  $X \succeq Y$  only depends on the distributions of  $X$  and  $Y$ , we may also see  $\succeq$  as an order on  $\mathcal{M}_1$ . Clearly,  $\Phi$  is  $\succeq$ -preserving if and only if  $\phi$  is  $\succeq$ -preserving. Another natural request on a risk function (on discounted payoffs) is to be *invariant on constants*, that is,  $\Phi(x) = -x$  for every  $x \in \mathbb{R} \subset L^1$ . Indeed, when there is no risk involved, i.e., when we are facing a deterministic scenario  $X \sim \delta_x$  (where  $\delta_x$  denotes the Dirac-measure), it seems natural to assign it the value  $x$ , thus the risk  $-x$ . For instance translation invariant risk functions are, apart from the constant  $\Phi(0)$ , invariant on constants.

Having the mentioned typical properties of risk functions in mind, the message of the following proposition is that convexity is not a good property to require on  $\phi$ .

**Proposition 2.1.** *The following are equivalent:*

- (i)  $\Phi$  is lsc,  $\succeq$ -preserving, invariant on constants (resp. translation invariant), and  $\phi$  is convex;
- (ii)  $\Phi(X) = -E[X]$ , i.e.  $\phi(\mu) = -\int x d\mu$  (resp.  $\Phi(X) = \Phi(0) - E[X]$ ).

*Proof.* We only prove the case when  $\Phi$  is invariant on constants. The proof in case of translation invariance follows similarly. Clearly (ii)  $\Rightarrow$  (i). In order to show (i)  $\Rightarrow$  (ii), recall that  $E[X] \preceq X$ . As  $\Phi$  is  $\succeq$ -preserving and invariant on constants, we have  $\Phi(X) \geq \Phi(E[X]) = -E[X]$ . For  $x_1, \dots, x_n \in \mathbb{R}$ ,  $\alpha_1, \dots, \alpha_n > 0$  such that  $\sum_{i=1}^n \alpha_i = 1$ , and a partition  $A_1, \dots, A_n \in \mathcal{F}$  of  $\Omega$  with  $\mathbb{P}(A_i) = \alpha_i$ , we have that  $\sum_{i=1}^n x_i 1_{A_i} \sim \sum_{i=1}^n \alpha_i \delta_{x_i}$  and

$$\begin{aligned} \phi\left(\sum_{i=1}^n \alpha_i \delta_{x_i}\right) &= \Phi\left(\sum_{i=1}^n x_i 1_{A_i}\right) \geq -E\left[\sum_{i=1}^n x_i 1_{A_i}\right] \\ &= -\sum_{i=1}^n \alpha_i x_i = \sum_{i=1}^n \alpha_i \Phi(x_i) = \sum_{i=1}^n \alpha_i \phi(\delta_{x_i}). \end{aligned}$$

As  $\phi$  is convex, the inequality in the computation must be an equality. Hence,  $\phi(X) = -E[X]$  for all simple random variables  $X$ . Now let  $X \in L^1$  be arbitrary and choose a sequence of simple random variables  $(X_n)_{n \in \mathbb{N}}$  converging to  $X$  in  $(L^1, \|\cdot\|_1)$ . Then

$$-E[X] = \lim_{n \rightarrow \infty} -E[X_n] = \liminf_{n \rightarrow \infty} \Phi(X_n) \geq \Phi(X) \geq -E[X],$$

where the first inequality follows by lsc of  $\Phi$ . □

Proposition 2.1 shows that in many typical cases, such as convex risk measures,  $\phi$  will either be concave (including the linear case) or be neither convex nor concave. Also notice that if  $\Phi$  is a deviation measure, then  $X \mapsto E[-X] + \Phi(X)$  is translation invariant. Hence, supposing that the deviation measure  $\Phi$  is lsc,  $\succeq$ -preserving and that  $\phi$  is convex, Proposition 2.1 implies that  $\Phi(X) \equiv \Phi(0)$ .

As mentioned in the introduction, many authors require quasi-convexity when they define risk measures on  $\mathcal{M}_1$ . The following Lemma 2.2 shows that starting from a quasi-convex and  $\succeq$ -preserving  $\phi$  on  $\mathcal{M}_1$  also  $\Phi$  must be quasi-convex, and thus even convex if  $\phi$  and thus also  $\Phi$  is translation invariant (see e.g. [9, p. 178]). This further justifies our approach to study the the

convexity/concavity problem through the lens of a (quasi-) convex  $\Phi$ . In order to prove this result we fix two independent sub- $\sigma$ -algebras  $\mathcal{G}_0, \mathcal{F}_0$  of  $\mathcal{F}$  such that  $(\Omega, \mathcal{G}_0, \mathbb{P})$  and  $(\Omega, \mathcal{F}_0, \mathbb{P})$  are non-atomic standard probability spaces. In particular, for any  $X, Y \in L^1(\Omega, \mathcal{F}, \mathbb{P})$  there exist  $\mathcal{G}_0$ -measurable  $\tilde{X}, \tilde{Y}$  such that  $(\tilde{X}, \tilde{Y}) \stackrel{d}{=} (X, Y)$ . Moreover, for any  $\alpha \in (0, 1)$  there is  $A \in \mathcal{F}_0$  such that  $\mathbb{P}(A) = \alpha$ , and thus if  $X$  and  $Y$  are  $\mathcal{G}_0$ -measurable with  $X \sim \mu, Y \sim \nu$ , then

$$(2.2) \quad 1_A X + 1_{A^c} Y \sim \alpha \mu + (1 - \alpha) \nu \quad \text{and} \quad \alpha X + (1 - \alpha) Y = E[1_A X + 1_{A^c} Y \mid \mathcal{G}_0].$$

Relation (2.2) will turn out to be useful throughout the paper.

**Lemma 2.2.** *If  $\phi$  is (quasi-)convex and  $\succeq$ -preserving, then  $\Phi$  is (quasi-)convex.*

*Proof.* Let  $\phi$  be quasi-convex. Fix  $X, Y \in L^1$  and  $\alpha \in [0, 1]$ , and let  $(\tilde{X}, \tilde{Y})$  be a  $\mathcal{G}_0$ -measurable random vector such that  $(\tilde{X}, \tilde{Y}) \stackrel{d}{=} (X, Y)$ , and thus also  $\alpha X + (1 - \alpha) Y \stackrel{d}{=} \alpha \tilde{X} + (1 - \alpha) \tilde{Y}$ . Moreover, let  $A \in \mathcal{F}_0$  be such that  $\mathbb{P}(A) = \alpha$ . Using that for any random variable  $X$  we have  $E[X \mid \mathcal{G}_0] \preceq X$ , and that with  $\phi$  also  $\Phi$  must be  $\succeq$ -preserving, we obtain

$$\begin{aligned} \Phi(\alpha X + (1 - \alpha) Y) &= \Phi(\alpha \tilde{X} + (1 - \alpha) \tilde{Y}) = \Phi(E[1_A \tilde{X} + 1_{A^c} \tilde{Y} \mid \mathcal{G}_0]) \\ &\leq \Phi(1_A \tilde{X} + 1_{A^c} \tilde{Y}) = \phi(\alpha \mu + (1 - \alpha) \nu) \\ &\leq \phi(\mu) \vee \phi(\nu) = \Phi(X) \vee \Phi(Y). \end{aligned}$$

The case of  $\phi$  convex follows in a similar way. □

In particular, this implies that the quasi-convex risk measures on distributions studied by Frittelli et al. in [10] either correspond to quasi-convex risk measures on random variables or they are not  $\succeq$ -preserving.

Consider the Value at Risk at level  $\lambda$ , i.e.

$$\Phi(X) = \text{VaR}_\lambda(X) = \inf\{m \in \mathbb{R} \mid \mathbb{P}(X + m \leq 0) \leq \lambda\}, \quad \text{where } \lambda \in (0, 1).$$

It is easily verified that the corresponding risk measure  $\phi$  on distributions is quasi-convex (see [10]). Since it is also well-known that  $\Phi$  is not quasi-convex, Lemma 2.2 implies that  $\text{VaR}_\lambda$  cannot preserve  $\succeq$ . So in particular there are payoff profiles  $X, Y$  such that every expected utility agent prefers  $X$  to  $Y$ , but under  $\text{VaR}_\lambda$  the profile  $Y$  is strictly less risky than  $X$ . Indeed, recall that a utility function is a concave and increasing function. Then note that  $X \succeq Y$  implies  $X \preceq_{uni} Y$ , where the uniform order  $\preceq_{uni}$  is defined as follows: for  $X, Y \in L^1$ ,

$$(X \preceq_{uni} Y) \iff (E[u(X)] \geq E[u(Y)]) \text{ for every utility function } u : \mathbb{R} \rightarrow \mathbb{R}.$$

From this we see that since  $\text{VaR}_\lambda$  does not preserve  $\succeq$ , it cannot be  $\preceq_{uni}$ -reverting ( $X \preceq_{uni} Y$  implies  $\Phi(X) \geq \Phi(Y)$ ). Hence, if the agents' behaviors in a market correspond to - or are modeled by means of - expected utility preferences, then a risk assessment via  $\text{VaR}_\lambda$  does not make sense. The same is true for any other risk measure which does not preserve  $\succeq$ .

## 2.2 A weak form of concavity for $\phi$

The next proposition shows that a weak form of concavity holds for  $\phi$  when  $\Phi$  is *coherent* in the sense that  $\Phi$  is convex and positively homogeneous. In what follows we write  $A \perp \sigma(X)$  to indicate that  $\sigma(A)$  is independent of  $\sigma(X)$ .

**Proposition 2.3.** *Let  $\Phi$  be lsc, convex and positively homogeneous, then*

$$(2.3) \quad \frac{\Phi(1_A X)}{\mathbb{P}(A)} \geq \Phi(X) \quad \text{for all } X \in L^1 \text{ and } A \perp \sigma(X) \text{ with } \mathbb{P}(A) > 0.$$

In the positively homogeneous case (where automatically  $\Phi(0) = \phi(\delta_0) = 0$ ), condition (2.3) can be seen as a weak form of concavity of  $\phi$  since it implies that  $\phi(\alpha\mu + (1-\alpha)\delta_0) \geq \alpha\phi(\mu)$  for all  $\mu \in \mathcal{M}_1$  and  $\alpha \in [0, 1]$ . However, Example 2.7 shows that there are coherent  $\Phi$  which are not ‘truly’ concave.

*Proof.* Being law-invariant, lsc, convex and positively homogeneous,  $\Phi$  may be represented as

$$\Phi(X) = \sup_{Z \in \mathcal{Q}} E[ZX] = \sup_{Z \in \mathcal{Q}|X} E[ZX],$$

where  $\mathcal{Q} \subset L^1$  is a  $\sigma(L^1, L^\infty)$ -closed convex set (see e.g. [5]), and  $\mathcal{Q} | X := \{E[Z | X] | Z \in \mathcal{Q}\}$ . Note that by law invariance it follows that  $\mathcal{Q} | X \subset \mathcal{Q}$ ; see [12, Lemma 4.2]. Thus if  $A \perp \sigma(X)$  we obtain:

$$\Phi(1_A X) \geq \sup_{Z \in \mathcal{Q}|X} E[1_A ZX] = \mathbb{P}(A) \sup_{Z \in \mathcal{Q}|X} E[ZX] = \mathbb{P}(A)\Phi(X).$$

□

From the proof of Proposition 2.3 it is also clear that  $\phi$  is actually concave in case

$$Z1_A + \tilde{Z}1_{A^c} \in \mathcal{Q} \quad \text{for all } Z \in \mathcal{Q} | X \text{ and } \tilde{Z} \in \mathcal{Q} | Y \text{ with } X, Y \in L^1.$$

Note that (2.3) is a reasonable property because conditional on  $A$  the random variable  $\hat{X} = X1_A$  has the same distribution as  $X$ . Thus the conditional risk  $\frac{\Phi(1_A X)}{\mathbb{P}(A)}$  should be at least  $\Phi(X)$ . If there is no ambiguity aversion involved, i.e.  $\Phi(\cdot) = E[\cdot]$ , then we have equality in (2.3). Otherwise  $1_A X$  may seem ‘more ambiguous’ than  $X$  since even with full knowledge of  $X$  the payoff  $1_A X$  is inaccessible given that  $A$  is independent of  $X$ .

### 2.3 Dual characterization of the concavity of $\phi$

In this section we provide an alternative way to check concavity of  $\phi$  by looking at the dual side. This approach turns out to be useful in the case when the Fenchel-Legendre transform  $\Phi^*$  of  $\Phi$  is easier to study than  $\Phi$  itself. In what follows,  $q_X(s) := \inf\{x \in \mathbb{R} | \mathbb{P}(X \leq x) \geq s\}$ ,  $s \in (0, 1)$ , denotes the (left continuous) quantile function of a random variable  $X$ .

**Proposition 2.4.** *Let  $\Phi$  be lsc and convex. Then  $\phi$  is concave if and only if  $\phi^* : \mathcal{M}_\infty \rightarrow (-\infty, \infty]$  is convex, where*

$$(2.4) \quad \phi^*(\mu) := \Phi^*(Z) = \sup_{X \in L^1} \int_0^1 q_X(t)q_Z(t)dt - \Phi(X) \quad \text{for } Z \sim \mu$$

and  $\mathcal{M}_\infty := \{\mu \in \mathcal{M}_1 | \mu \text{ has compact support}\}$ .

*Proof.* The equality in (2.4) for the dual function  $\Phi^*$  of  $\Phi$  holds by law-invariance, like in [9, Theorem 4.59]. In particular,  $\Phi^*$  is itself a law-invariant convex function on  $L^\infty$ .

Being a law-invariant, lsc, convex function,  $\Phi$  may be represented as

$$\begin{aligned}\Phi(X) &= \sup_{Z \in L^\infty} E[ZX] - \Phi^*(Z) = \sup_{Z \in L^\infty(\sigma(X))} E[ZX] - \Phi^*(Z) \\ &= \sup_{h \in M_b} E[h(X)X] - \Phi^*(h(X)),\end{aligned}$$

where  $M_b$  denotes the set of measurable and bounded  $h : \mathbb{R} \rightarrow \mathbb{R}$ . Here we used that  $E[Z | X] \preceq X$  and that the lsc convex law-invariant function  $\Phi^*$  is automatically  $\succeq$ -preserving. Similarly, using [7, Theorem 2.2] for the second equality, we also derive the following representation for  $\Phi^*$ :

$$\begin{aligned}\Phi^*(Z) &= \sup_{X \in L^1} E[ZX] - \Phi(X) = \sup_{X \in L^\infty} E[ZX] - \Phi(X) \\ &= \sup_{X \in L^\infty(\sigma(Z))} E[ZX] - \Phi(X) = \sup_{h \in M_b} E[Zh(Z)] - \Phi(h(Z)), \quad Z \in L^\infty.\end{aligned}$$

Now suppose that  $\phi$  is concave and let  $\mu, \nu \in \mathcal{M}_\infty$  and  $\alpha \in [0, 1]$ . Choose the random variables  $X, Y$  and  $A \in \mathcal{F}_0$  as in (2.2). Then

$$\begin{aligned}\phi^*(\alpha\mu + (1 - \alpha)\nu) &= \Phi^*(1_A X + 1_{A^c} Y) \\ &= \sup_{h \in M_b} E[(1_A X + 1_{A^c} Y)h(1_A X + 1_{A^c} Y)] - \Phi(h(1_A X + 1_{A^c} Y)) \\ &= \sup_{h \in M_b} \alpha E[Xh(X)] + (1 - \alpha)E[Yh(Y)] - \Phi(h(X)1_A + h(Y)1_{A^c}) \\ &\leq \sup_{h \in M_b} \alpha E[Xh(X)] + (1 - \alpha)E[Yh(Y)] - \alpha\Phi(h(X)) + (1 - \alpha)\Phi(h(Y)) \\ &\leq \alpha\Phi^*(X) + (1 - \alpha)\Phi^*(Y) = \alpha\phi^*(\mu) + (1 - \alpha)\phi^*(\nu),\end{aligned}$$

where we used the concavity of  $\phi$  in the first inequality since

$$h(X)1_A + h(Y)1_{A^c} \sim \alpha \text{law}(h(X)) + (1 - \alpha) \text{law}(h(Y))$$

where  $\text{law}(Z)$  denotes the distribution of a random variable  $Z$ .

Suppose that  $\phi^*$  is convex and let  $\mu, \nu \in \mathcal{M}_1$ ,  $\alpha \in [0, 1]$ ,  $X, Y \in L^1$  and  $A \in \mathcal{F}_0$  be as in (2.2). In this case we have

$$\begin{aligned}\phi(\alpha\mu + (1 - \alpha)\nu) &= \Phi(1_A X + 1_{A^c} Y) \\ &\geq \sup_{h, g \in M_b} E[(1_A X + 1_{A^c} Y)(1_A h(X) + 1_{A^c} g(Y))] - \Phi^*(1_A h(X) + 1_{A^c} g(Y)) \\ &\geq \sup_{h, g \in M_b} \alpha E[Xh(X)] + (1 - \alpha)E[Yg(Y)] - \alpha\Phi^*(h(X)) - (1 - \alpha)\Phi^*(g(Y)) \\ &= \alpha\Phi(X) + (1 - \alpha)\Phi(Y) = \alpha\phi(\mu) + (1 - \alpha)\phi(\nu),\end{aligned}$$

where we used the convexity of  $\phi^*$  in the second inequality. □

So if  $\Phi$  is convex and  $\phi$  is concave then the primal ordering is as follows:

$$\Phi(\alpha X + (1 - \alpha)Y) \leq \alpha\Phi(X) + (1 - \alpha)\Phi(Y) = \alpha\phi(\mu) + (1 - \alpha)\phi(\nu) \leq \phi(\alpha\mu + (1 - \alpha)\nu),$$

whereas for the dual we have

$$\begin{aligned}\Phi^*(\alpha X + (1 - \alpha)Y) &\leq \phi^*(\alpha\mu + (1 - \alpha)\nu) \leq \alpha\phi^*(\mu) + (1 - \alpha)\phi^*(\nu) \\ &= \alpha\Phi^*(X) + (1 - \alpha)\Phi^*(Y).\end{aligned}$$

If  $\Phi$  is lsc and coherent, then

$$\phi^*(\mu) = \delta(\mu \mid \text{dom } \phi^*) = \begin{cases} 0, & \text{if } \mu \in \text{dom } \phi^* \\ \infty, & \text{else,} \end{cases}$$

where  $\text{dom } \phi^* = \{\mu \in \mathcal{M}_\infty \mid \exists Z \in \text{dom } \Phi^* : Z \sim \mu\}$ ; see for instance [9, Corollary 4.19]. Hence, we obtain:

**Corollary 2.5.** *If  $\Phi$  is lsc and coherent, then  $\phi$  is concave if and only if  $\text{dom } \phi^*$  is convex.*

In general, not requiring coherence, convexity of  $\phi^*$  is of course equivalent to convexity of

$$\text{epi } \phi^* := \{(\mu, a) \in \mathcal{M}_\infty \times \mathbb{R} \mid \phi^*(\mu) \leq a\},$$

which corresponds to  $(1_A Z + 1_{A^c} \tilde{Z}, \alpha a + (1 - \alpha)b) \in \text{epi } \Phi^*$  whenever  $(Z, a), (\tilde{Z}, b) \in \text{epi } \Phi^*$  and  $A \perp \sigma(Z, \tilde{Z})$  with  $\mathbb{P}(A) = \alpha$ .

**Examples 2.6.** 1. The entropic risk measure:  $\Phi(X) = \gamma \ln E[\exp(-X/\gamma)]$ ,  $\gamma > 0$ . In this case it is known that  $\phi^*(\mu) = \gamma \int (-x) \log(-x) \mu(dx)$  whenever  $\mu \in \mathcal{M}_\infty$  has support on  $\mathbb{R}_-$  and  $\int x \mu(dx) = -1$  (i.e.  $\mu$  is apart from the sign the distribution of a probability density), and  $\phi^*(\mu) = \infty$  otherwise. Being linear on its convex support,  $\phi^*$  is convex on  $\mathcal{M}_\infty$ , and thus  $\phi$  is concave which is of course also easily verified directly.

2. The Average Value at Risk:  $\Phi(X) = \text{AVaR}_\lambda(X) := -\frac{1}{\lambda} \int_0^\lambda q_X(t) dt$ ,  $\lambda \in (0, 1]$ . In this case  $\text{dom } \phi^*$  is the set of all  $\mu \in \mathcal{M}_\infty$  which have support on  $[-1/\lambda, 0]$  and satisfy  $\int x \mu(dx) = -1$ . Clearly,  $\text{dom } \phi^*$  is convex, so  $\phi$  is concave.

3. The mean-variance evaluation principle:  $\Phi(X) = -E[X] + \delta \text{Var}(X)$ ,  $\delta > 0$ . In this case  $\phi^*(\mu) = \frac{1}{4\delta} (\int x^2 \mu(dx) - 1)$  whenever  $\mu$  has support on  $\mathbb{R}_-$ , finite second moment and  $\int x \mu(dx) = -1$ , and  $\phi^*(\mu) = \infty$  otherwise. Hence  $\phi^*$  is convex, and so  $\phi$  is concave.  $\diamond$

In the following example we construct coherent risk measures  $\Phi$  for which the corresponding functions  $\phi$  are not concave.

**Example 2.7.** Let  $\mu, \nu \in \mathcal{M}_\infty$  be nondegenerate such that  $\int x \mu(dx) = -1$  and  $\int x \nu(dx) = 0$ . In particular, neither  $\mu \succeq \nu$  nor  $\nu \succeq \mu$ , because either of the relations would imply equal expectation. Let  $C(\mu) := \{Z \in L^\infty \mid \text{law}(Z) \preceq \mu\}$  and define  $C(\nu)$  analogously. Recalling that the convex order is indeed an order on the distributions clarifies the definition of  $C(\mu)$  and  $C(\nu)$ . Note that the convex sets  $C(\mu)$  and  $C(\nu)$  seen as subsets of  $L^1$  are weakly compact (see e.g. [15]) and thus the convex hull  $C := \text{co}(C(\mu) \cup C(\nu))$  is weakly closed (even weakly compact) in  $L^1$ ; see [1, Lemma 5.29]. As  $C$  is a convex set it must also be closed in the norm topology on  $L^1$ . This also implies that  $C$  is closed in  $(L^\infty, \|\cdot\|_\infty)$  and thus, as a convex set, also in  $\sigma(L^\infty, L^1)$ . Now

$$\Phi(X) := \sup_{Z \in C} E[ZX], \quad X \in L^1$$

is lsc, law-invariant, and coherent. The law invariance follows from the law invariance of the set  $C$ ; see [9, Theorem 4.59]. Moreover, we have that  $\Phi^* = \delta(\cdot \mid C)$ , thus  $\text{dom } \Phi^* = C$ . In the



following we will show that  $\text{dom } \phi^* = \{\eta \in \mathcal{M}_\infty \mid \exists Z \in C : Z \sim \eta\}$  is not convex. Suppose it were, then there would exist a convex combination  $\lambda Z + (1 - \lambda)\tilde{Z} \in C$ , with  $Z$  and  $\tilde{Z}$  being elements of  $C(\mu)$  or  $C(\nu)$ , such that  $\hat{Z} := \lambda Z + (1 - \lambda)\tilde{Z} \sim \frac{1}{2}(\mu + \nu)$ . Apparently it cannot happen that  $Z$  and  $\tilde{Z}$  are both in  $C(\mu)$  or both in  $C(\nu)$ , as  $\frac{1}{2}(\mu + \nu) \not\preceq \mu$  and  $\frac{1}{2}(\mu + \nu) \not\preceq \nu$ . Therefore, we may assume that  $Z \in C(\mu)$  and  $\tilde{Z} \in C(\nu)$ . Computing the expectation of  $\hat{Z}$  and noting that  $E[Z] = -1$  and  $E[\tilde{Z}] = 0$ , we deduce that  $\lambda = 1/2$ . Without loss of generality we may assume that  $Z, \tilde{Z}$  are  $\mathcal{G}_0$ -measurable. Otherwise we find a two dimensional  $\mathcal{G}_0$ -measurable random vector with the same distribution as  $(Z, \tilde{Z})$  and such that the corresponding convex combination has the same distribution as  $\hat{Z}$ . Then, for  $A \in \mathcal{F}_0$  with  $\mathbb{P}(A) = 1/2$ , we have that  $\hat{Z} = E[1_A Z + 1_{A^c} \tilde{Z} \mid \mathcal{G}_0]$ , and Jensen's inequality for strict convex functions shows that  $\hat{Z} \preceq 1_A Z + 1_{A^c} \tilde{Z}$  but not  $\hat{Z} \succeq 1_A Z + 1_{A^c} \tilde{Z}$ . This fact contradicts  $\hat{Z} \sim \frac{1}{2}(\mu + \nu)$ , since  $1_A Z + 1_{A^c} \tilde{Z} \sim \frac{1}{2}(\mu + \nu)$ . Hence  $\text{dom } \phi^*$  is not convex, which in turn implies that  $\phi$  is not concave by Corollary 2.5.

Similarly, we can also construct a lsc, translation invariant, and coherent  $\Phi$ , i.e. a coherent risk measure, such that  $\phi$  is not concave. To this end, choose nondegenerate  $\mu, \nu \in \mathcal{M}_\infty$  such that the support of  $\mu$  and  $\nu$  is contained in  $\mathbb{R}_-$  and such that  $\int x\mu(dx) = \int x\nu(dx) = -1$ , but neither  $\mu \succeq \nu$  nor  $\nu \succeq \mu$ . Construct  $\Phi$  as above, and suppose that there are elements  $Z$  and  $\tilde{Z}$  of  $C(\mu)$  or  $C(\nu)$ , such that  $\hat{Z} := \lambda Z + (1 - \lambda)\tilde{Z} \sim \frac{1}{2}(\mu + \nu)$ . Again we may assume that  $Z \in C(\mu)$  and  $\tilde{Z} \in C(\nu)$ . For any convex function  $c : \mathbb{R} \rightarrow \mathbb{R}$  we obtain that

$$\begin{aligned} \frac{1}{2} \left( \int c(x)\mu(dx) + \int c(x)\nu(dx) \right) &= E[c(\hat{Z})] \leq \lambda E[c(Z)] + (1 - \lambda)E[c(\tilde{Z})] \\ &\leq \lambda \int c(x)\mu(dx) + (1 - \lambda) \int c(x)\nu(dx). \end{aligned}$$

As there must exist some convex functions  $c_1, c_2 : \mathbb{R} \rightarrow \mathbb{R}$  such that

$$\int c_1(x)\mu(dx) < \int c_1(x)\nu(dx) \quad \text{and} \quad \int c_2(x)\mu(dx) > \int c_2(x)\nu(dx),$$

we conclude that  $\lambda = 1/2$ . Finally, the same arguments as above show that  $\text{dom } \phi^*$  is not convex.  $\diamond$

### 3 The comonotonic case

In many situations the risk of a combined position  $X + Y$  turns out to be lower than the sum of the risks given by the individual positions. This is due to the fact that one position may serve as a hedge against unfavorable outcomes of the other. However, if this hedge is not possible because the two random variables are perfectly positively correlated, then the situation looks very different. This concept is what is captured by the so-called comonotonicity property.  $\Phi$  is said to be comonotonic if it is linear on comonotone elements, that is  $\Phi(X + Y) = \Phi(X) + \Phi(Y)$  whenever  $X, Y$  satisfy  $(X(\omega) - X(\omega'))(Y(\omega) - Y(\omega')) \geq 0$  for all  $(\omega, \omega') \mathbb{P} \times \mathbb{P}$ -a.s.

The main message of this section is that under comonotonicity, concavity of  $\phi$  is completely determined by the preservation of the convex order (Proposition 3.5).

**Lemma 3.1.** *If  $\Phi$  is comonotonic and  $\succeq$ -preserving, then  $\phi$  is concave.*

*Proof.* Suppose that  $\Phi$  is comonotonic and  $\succeq$ -preserving. Let  $\mu, \nu \in \mathcal{M}_1$  and  $\alpha \in [0, 1]$ . Moreover, let  $U$  be a  $\mathcal{G}_0$ -measurable  $(0, 1)$ -uniform random variable, and set  $X := q_\mu(U) \sim \mu$

and  $Y := q_\nu(U) \sim \nu$ , so that  $X, Y$  are  $\mathcal{G}_0$ -measurable and comonotone. Here  $q_\eta(s) := \inf\{x \in \mathbb{R} \mid \eta(-\infty, x] \geq s\}$ ,  $s \in (0, 1)$ , is the quantile function of the distribution  $\eta$ . Moreover, take  $A \in \mathcal{F}_0$  such that  $\mathbb{P}(A) = \alpha$ . Then we obtain

$$\begin{aligned} \phi(\alpha\mu + (1 - \alpha)\nu) &= \Phi(1_A X + 1_{A^c} Y) \geq \Phi(E[1_A X + 1_{A^c} Y \mid \mathcal{G}_0]) \\ &= \Phi(\alpha X + (1 - \alpha)Y) = \alpha\Phi(X) + (1 - \alpha)\Phi(Y) \\ &= \alpha\phi(\mu) + (1 - \alpha)\phi(\nu) \end{aligned}$$

where the comonotonicity of  $\Phi$  enters in the second but last equality.  $\square$

**Remark 3.2.** It follows from Lemma 3.1 that for every law-invariant comonotonic convex risk measure  $\Phi$  the corresponding risk measure  $\phi$  on  $\mathcal{M}_1$  is concave. However, Kusuoka's representation of law-invariant convex risk measures shows what goes wrong in the general non-comonotone case:

$$(3.1) \quad \Phi(X) = \sup_{\mu \in \mathcal{M}(0,1]} \int_{(0,1]} \text{AVaR}_\lambda(X) \mu(d\lambda) - \alpha(\mu),$$

where  $\mathcal{M}(0, 1]$  is the set of all probability measures on  $(0, 1]$ , and  $\alpha$  is the Fenchel-Legendre transform of  $\Phi$  seen as a function on  $\mathcal{M}(0, 1]$ . On the level of distributions we see that the building blocks  $\int_{(0,1]} \text{AVaR}_\lambda(X) \mu(d\lambda)$  in (3.1) are all comonotonic and  $\succeq$ -preserving, so the corresponding risk measures on distributions are concave according to Lemma 3.1. However, when taking the supremum, the concavity of the function  $\phi$  corresponding to  $\Phi$  does not necessarily follow.  $\diamond$

The following lemma translates convexity of  $\Phi$  to a property (3.2) of  $\phi$ . It will prove to be useful later on.

**Lemma 3.3.** *Let  $\Phi$  be  $\succeq$ -preserving. Then the following are equivalent:*

- (i)  $\Phi$  is convex;
- (ii) For all  $\mu \in \mathcal{M}_1$ ,  $f, g : \mathbb{R} \rightarrow \mathbb{R}$  increasing such that  $\mu \circ f^{-1}, \mu \circ g^{-1} \in \mathcal{M}_1$ , and  $\alpha \in [0, 1]$ , the following holds:

$$(3.2) \quad \phi(\mu \circ (\alpha f + (1 - \alpha)g)^{-1}) \leq \alpha\phi(\mu \circ f^{-1}) + (1 - \alpha)\phi(\mu \circ g^{-1}).$$

*Proof.* (i)  $\Rightarrow$  (ii): Let the random variable  $X \in L^1$  have distribution  $\mu$ . Moreover, let  $f, g : \mathbb{R} \rightarrow \mathbb{R}$  be measurable functions such that  $f(X), g(X) \in L^1$  and let  $\alpha \in [0, 1]$ . Then  $f(X) \sim \mu \circ f^{-1}$ ,  $g(X) \sim \mu \circ g^{-1}$ , and  $\alpha f(X) + (1 - \alpha)g(X) \sim (\mu \circ (\alpha f + (1 - \alpha)g)^{-1})$ . Now it is clear how (3.2) follows from the convexity of  $\Phi$ .

(ii)  $\Rightarrow$  (i): Let  $X, Y \in L^1$  and  $\alpha \in [0, 1]$ . Let  $U$  be a  $\mathcal{G}_0$ -measurable  $(0, 1)$ -uniformly distributed random variable and call  $\mu = \text{law}(U)$ . Then we have  $X \stackrel{d}{=} q_X(U)$  and  $Y \stackrel{d}{=} q_Y(U)$ , thus

$$\begin{aligned} \alpha\Phi(X) + (1 - \alpha)\Phi(Y) &= \alpha\phi(\mu \circ q_X^{-1}) + (1 - \alpha)\phi(\mu \circ q_Y^{-1}) \geq \phi(\mu \circ (\alpha q_X + (1 - \alpha)q_Y)^{-1}) \\ &= \Phi(\alpha q_X(U) + (1 - \alpha)q_Y(U)) \geq \Phi(\alpha X + (1 - \alpha)Y), \end{aligned}$$

where the last inequality follows from the fact that

$$\alpha X + (1 - \alpha)Y \preceq \alpha X^c + (1 - \alpha)Y^c$$

for all  $X^c, Y^c$  comonotone such that  $X^c \stackrel{d}{=} X, Y^c \stackrel{d}{=} Y$ ; see Kaas et al. [13, Theorem 6] and references therein.  $\square$

From the proof of Lemma 3.3 it is clear that if one replaces convexity with quasi-convexity, then (3.2) needs to be replaced by

$$\phi(\mu \circ (\alpha f + (1 - \alpha)g)^{-1}) \leq \phi(\mu \circ f^{-1}) \vee \phi(\mu \circ g^{-1}).$$

The Value at Risk is an example which shows that the request of being  $\succeq$ -preserving in Lemma 3.3 cannot be removed. Indeed,  $\text{Var}_\lambda$  is comonotone and thus satisfies (3.2) according to the following Lemma 3.4. However  $\text{Var}_\lambda$  is not  $\succeq$ -preserving and not convex.

**Lemma 3.4.** *Let  $\Phi$  be positively homogeneous. Then,  $\Phi$  is comonotonic if and only if  $\phi$  satisfies (3.2) with equality.*

*Proof.* The ‘only if’ implication is clear from the first part of the proof of Lemma 3.3 since for increasing  $f$  and  $g$  the random variables  $f(X)$  and  $g(X)$  are comonotone. Now, let  $X, Y \in L^1$  be comonotone and  $U$  be a random variable which is uniformly distributed on  $(0, 1)$  such that  $q_X(U) = X$  and  $q_Y(U) = Y$ . Then

$$\begin{aligned} \Phi(\alpha X + (1 - \alpha)Y) &= \Phi(\alpha q_X(U) + (1 - \alpha)q_Y(U)) = \phi(\mu \circ (\alpha q_X + (1 - \alpha)q_Y)^{-1}) \\ &= \alpha \phi(\mu \circ q_X^{-1}) + (1 - \alpha) \phi(\mu \circ q_Y^{-1}) = \alpha \Phi(X) + (1 - \alpha) \Phi(Y). \end{aligned}$$

□

Note that if  $\Phi$  is comonotonic, then automatically  $\Phi(rX) = r\Phi(X)$  for all rational numbers  $r \geq 0$ . Thus under some continuity condition on  $\Phi$  positive homogeneity automatically follows.

**Proposition 3.5.** *Let  $\Phi$  be lsc and comonotonic. Then the following are equivalent:*

- (i)  $\Phi$  is positively homogeneous and  $\succeq$ -preserving;
- (ii)  $\Phi$  is coherent and  $\phi$  is concave.

*Proof.* (i)  $\Rightarrow$  (ii):  $\Phi$  is convex by Lemmas 3.3 and 3.4 and thus coherent as it is also positively homogeneous.  $\phi$  is concave by Lemma 3.1.

(ii)  $\Rightarrow$  (i): This is again the fact that law-invariant lsc convex functions are automatically  $\succeq$ -preserving; see e.g. [4, Theorem 4.1] or [15]. □

**Remark 3.6.** Proving that the Average Value at Risk is convex is typically not an easy task; see e.g. [9, Theorem 4.52]. However, by applying Proposition 3.5 we can deduce the convexity of the AVaR quickly. Indeed, AVaR is  $\succeq_{uni}$ -reverting by [9, Theorem 2.57: (a) $\Leftrightarrow$ (e)], hence it preserves the convex order. Moreover, it is continuous, comonotonic, and positively homogenous. Thus AVaR is coherent by Proposition 3.5. Therefore, it admits dual representation, and the definition of the Fenchel-Legendre transform easily excludes measures with Radon-Nikodym derivatives greater than  $1/\lambda$ . In this way the well-known dual representation  $\text{AVaR}_\lambda(X) = \sup\{E_{\mathbb{Q}}[-X] : d\mathbb{Q}/d\mathbb{P} \leq 1/\lambda\}$  readily follows (cf. [9, Theorem 4.52]). ◇

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