Optimal Risk Sharing with Non-Monotone Monetary Functions

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Abstract

We consider the problem of sharing pooled risks among n economic agents endowed with non-necessarily monotone monetary functionals. In this framework, results of characterization and existence of optimal solutions are easily obtained as extension from the convex risk measures setting. Moreover, the introduction of the *best* monotone approximation of non-monotone functionals allows us to compare the original problem with the one which involves only *ad hoc* monotone criterions. The explicit calculation of optimal risk sharing rules is provided for particular cases, when agents are endowed with well known preference relations.

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1 Introduction

The optimal exchange of risk between economic agents is a concept that captures situations with very different characteristics and purposes: the risk sharing in insurance and reinsurance contracts; the assignment of liabilities by a company to its daughter companies; the individual hedging in an incomplete market etc. Since the early work of Arrow [2] and Borch [7] (see also Gerber [22, 23], Bühlmann [10] and Goovaerts et al. [24]) many authors have considered this problem, and the predominant method by which this has been accomplished is via expected utility. However, the introduction of notions as coherent and convex risk measures (see Artzner et al. [3], Delbaen [15, 16], Föllmer and Schied [19, 20]) has recently led several authors to study the risk sharing problem by using a new kind of approach (see Barrieu and El Karoui [4, 5, 6], Jouini et al. [27], Filipović and Kupper [18], Burgert and Rüschendorf [11, 12]). For some of the large number of references on optimal risk allocation, we refer to Dana and Scarsini [14], Burgert and Rüschendorf [12] and the references therein.

In this paper we adopt this new approach to the measures of risk/utility and weaken the requests made on choice criterions in order to consider non-monotone preferences as well. This will allow us to include, in our analysis, both the monetary utility functions and some widely used non-monotone criterions, such as the mean-variance and the standard-deviation principles. Of course in this framework we have that pathological situations may occur, where agents seem to choose in contradiction with the rational economic behaviour. Nevertheless, with regard to the risk sharing problem, fundamental results of existence and characterization of optimal solutions (as those given in [27] for the case of two monetary utility functions) are easily extended to our more general setting. In the first part of the paper we formalize these natural extensions and introduce, for any non-monotone functional, its best monotone-adjusted version, based on Maccheroni et al. [29]. This will lead us to give comparison results on the behaviour of monotone and non-monotone agents when facing the risk sharing problem, which constitute the first main contribution of the present paper. In particular, provided that at least one agent is characterized by monotone preferences, we show that there is a strict link between the solutions to the original optimal risk sharing problem and the solutions to the new one only involving "ad hoc" monotone choice criterions. We especially obtain interesting results when dealing with agents having mean-variance preferences. Indeed, in this situation we find that the Pareto optimal redistribution of the total risk is not sensitive to the lack of monotonicity by some agents.

The second contribution of the paper consists in providing the explicit solutions to some concrete risk exchange problems. We consider some widely used choice criterions and also characterize the solutions of problems where the agents have particular attitudes, such as the *strict risk-aversion conditionally on any event*. What stems from the cases studied here is that the optimal redistribution of the risk often leads to simple contracts consisting in the exchange of European options written on the total risk or in a proportional sharing of it. In this way we get typical forms of insurance contracts, such as stop-loss and quota-share rules. These examples also reveal peculiar attitudes linked to the respective preference relations: the conservative behaviour of the entropic-agent, as well as the inclination of the $AV@R$-agent towards taking extreme risks.

The paper is organized as follows. In Section 2 we define our family of choice criterions and recall some fundamental properties. In Section 3 we formulate our optimization problem. Here we first give easy generalizations of some well known results, and then compare the original problem with an associated one which only considers monotone functionals. In Section 4 we explicitly solve some risk sharing problems involving particular choice functionals, whose dual transforms and differentials are studied in Appendices A, B.

2 Choice Criterions and Related Properties

2.1 Set Up and Notations

We work in a simple model consisting of two dates: today, where everything is known, and a fixed future date (say tomorrow), where a standard probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is given. With $L^\infty := L^\infty(\Omega, \mathcal{F}, \mathbb{P})$ we mean the collection of all essentially bounded random variables, and with $(L^\infty)^* := L^\infty(\Omega, \mathcal{F}, \mathbb{P})^*$ its topological dual space, i.e., the set of all bounded finitely additive measures μ absolutely continuous w.r. to $\mathbb{P}$. We write $\mathcal{P}$ (resp. $\mathcal{P}_\sigma$) for the collection of all probability measures (resp. σ-additive measures normalized to 1) on $(\Omega, \mathcal{F})$ absolutely continuous w.r. to $\mathbb{P}$, and $\mathcal{Z}$ (resp. $\mathcal{Z}_\sigma$) for the set of their Radon-Nikodym derivatives.

Throughout the paper we will frequently make use of classical results from duality theory, for which we remind the reader to [31] and [8]. Here we just recall that, given φ and φ^* (resp. concave and convex functions) $\langle L^\infty, (L^\infty)^* \rangle$-conjugate, their gradients (resp. supergradient and subgradient) are defined as follows:

$$\partial\varphi(X) = \{\mu \in (L^\infty)^* : \varphi(Y) \leq \varphi(X) + \langle \mu, Y - X \rangle, \, \forall Y \in L^\infty\},$$

$$\partial\varphi^*(\mu) = \{X \in L^\infty : \varphi^*(\eta) \geq \varphi^*(\mu) + \langle \eta - \mu, X \rangle, \, \forall \eta \in (L^\infty)^*\}.$$

A standard result of convex analysis states, for any $(X, \mu) \in (L^\infty, (L^\infty)^*)$, the equivalences

$$\mu \in \partial\varphi(X) \iff X \in -\partial\varphi^*(\mu) \iff \varphi(X) = \varphi^*(\mu) + \langle \mu, X \rangle. \tag{1}$$

Here we consider L^∞ as space of possible financial positions occurring tomorrow, and we evaluate them by means of functionals representing agents' preferences and fulfilling suitable properties. Following the axiomatic approach of Artzner et al. [3], we first introduce the class of monetary utility functions (that, up to the sign, are exactly the convex risk measures in the sense of [20]).

Definition 2.1. *A functional* $U : L^\infty(\Omega, \mathcal{F}, \mathbb{P}) \to \mathbb{R}$*, with* $U(0) = 0$*, is called monetary utility function (m.u.f.) if it is concave, cash-invariant (*$U(X + c) = U(X) + c, \forall X \in L^\infty, c \in \mathbb{R}$*) and non-decreasing w.r. to the order of* L^∞*.*

This is exactly the family of functionals considered in [5], [27] and [12] when studying the optimal risk exchange problem, and in particular it includes the Average Value at Risk (taken with the opposite sign), as well as the entropic and the semi-deviation utilities (see Appendix A). Here we enlarge the class of functionals admitted to represent agents' preferences, by dropping the request of monotonicity. However, by doing so we lose the Lipschitz-continuity with respect to the supremum norm $\|.\|_\infty$ (ensured for m.u.f.'s), and in order to develop a theory of risk exchange based on convex analysis we need to require some regularity condition. For that reason throughout the paper we will consider choice criterions under the following assumption:

Assumption 2.2. $U : L^\infty(\Omega, \mathcal{F}, \mathbb{P}) \to \mathbb{R}$*, with* $U(0) = 0$*, is a concave and cash-invariant functional,* L^∞*-continuous (i.e., continuous w.r. to* $\|.\|_\infty$*).*

The request of L^∞-continuity guarantees the applicability of important results of functional analysis. In particular, denoting by $V : L^\infty(\Omega, \mathcal{F}, \mathbb{P})^* \to [0, \infty]$ the convex conjugate function (also called *penalty function*) of U, i.e.

$$V(\mu) := \sup_{X \in L^\infty} \{U(X) - \langle \mu, X \rangle\}, \ \forall \mu \in (L^\infty)^*, \tag{2}$$

the Fenchel-Moreau theorem ensures that U and V are $\langle L^\infty, (L^\infty)^* \rangle$-conjugate, and hence the following representation holds for U:

$$U(X) = \inf_{\mu \in (L^\infty)^*} \{V(\mu) + \langle \mu, X \rangle\}, \ \forall X \in L^\infty. \tag{3}$$

Assumption 2.2 obviously embraces the class of monetary utility functions, and it also allows to consider agents characterized by non-monotone preference criterions, as the *mean-variance* one:

$$U_\delta^{mv}(X) := \mathbf{E}[X] - \delta \mathrm{Var}(X), \quad \delta > 0, \tag{4}$$

which is widely used to shape the choice of economic agents when there is uncertainty (see, e.g., Markovitz [30] for the portfolio selection problem, as well as Bühlmann [9, §4] for the premium calculation problem).

Example 2.3. *Consider $X \equiv 0$ and $Y \in L^\infty$ s.t. $Y = y > 0$ with probability $\alpha \in (0,1)$ and $Y = 0$ otherwise. For these financial positions, $U_\delta^{mv}(X) = 0$ and $U_\delta^{mv}(Y) = \alpha y(1 - \delta y(1-\alpha))$. Therefore, for any $y > 1/(\delta(1-\alpha))$ we have $U_\delta^{mv}(Y) < 0$, and the mean-variance agent considers X as being strictly better than Y. This means that, if someone offers her a lottery ticket with probability α of a "too-big" winning y, then the U_δ^{mv}-agent does not accept it.*

Similar anomalous situations can arise when considering the *standard-deviation* principle:

$$U_\delta^{sd}(X) := \mathbf{E}[X] - \delta \mathrm{Var}(X)^{1/2}, \quad \delta > 0, \tag{5}$$

that, however, is a "good" choice criterion, in the sense that it satisfies Assumption 2.2 as well. We just note that, thanks to positive homogeneity, concavity is equivalent to super-additivity, which follows as a straightforward application of the Cauchy-Schwarz inequality.

2.2 Representation Results and Additional Properties

Some fundamental results can be easily extended from the class of monetary utility functions to the wider family resulting from Assumption 2.2. Among them, notably, the possibility to formulate representation (3) over L^1 instead of the whole dual space $(L^\infty)^*$ (where the inclusion $L^1 \subseteq (L^\infty)^*$ holds by identification of σ-additive measures with their Radon-Nikodym derivatives), which clearly leads to more manageable situations, especially if one wants to explicitly solve particular optimization problems (as the case in Section 4). The following remark goes exactly in this direction and it can be obtained as standard result from convex duality theory (see for instance [15] and [19] for coherent and convex risk measures, as well as [1] for its statement in this framework).

Remark 2.4. *A functional U satisfying Assumption 2.2 admits the following dual representation over the set of σ-additive measures $\mathcal{P}_\sigma$:*

$$U(X) = \inf_{\mathbb{Q} \in \mathcal{P}_\sigma} \{V(\mathbb{Q}) + \boldsymbol{E}_\mathbb{Q}[X]\}, \ \forall X \in L^\infty, \tag{6}$$

if and only if it satisfies the Fatou property, i.e., for any bounded sequence $(X_n)_{n\in\mathbb{N}} \in L^\infty$ converging $\mathbb{P}$-a.s. to some X, then $U(X) \geq \limsup_n U(X_n)$.

We recall, for instance, that the Fatou property is satisfied by any U law-invariant, i.e., that satisfies $U(X) = U(Y)$ whenever X and Y have the same distribution (see [26, Theorem 1.3]). The law-invariance is a crucial property since it also ensures the existence of solutions to the optimization problem studied in this paper (Theorem 3.7), and allows a dual representation in terms of quantile functions (obtained by Kusuoka [28] and further extended in [21] and [26] as far as m.u.f.'s are concerned, see [1] for its statement in this more general setting).

Another concept stronger than the Fatou property is the following.

Definition 2.5. *A functional $U : L^\infty \to \mathbb{R}$ is said to satisfy the Lebesgue property, if for any bounded sequence $(X_n)_{n\in\mathbb{N}} \in L^\infty$ converging $\mathbb{P}$-a.s. to some X, then $\lim_{n\to\infty} U(X_n) = U(X)$.*

This concept has been introduced in [26], where the authors show that a functional U (satisfying our Assumption 2.2) has the Lebesgue property if and only if the domain of its convex conjugate is contained in L^1 (and therefore the infimum in (6) is actually a minimum). This ensures that the $AV@R$-criterion fulfils this property (see Appendix A), as well as the other choice functionals mentioned so far (by just applying dominated convergence).

Definition 2.6 ([27])**.** *A functional U defined on L^∞ is said strictly risk-averse conditionally on any event if it satisfies the following property:*

(S) $U(X) < U(X\boldsymbol{1}_{A^c} + \boldsymbol{E}[X|A]\boldsymbol{1}_A)$ for any $A \in \mathcal{F}$ and $X \in L^\infty$ s.t. $\mathbb{P}(A) > 0$ and $essinf_A X < esssup_A X$.

For example, this property is satisfied by both U_δ^{mv} and U_δ^{sd}, by applying Jensen's inequality, as well as by the entropic utility, as shown in [27].

Remark 2.7. *For any concave functional U satisfying property (S), we can easily extend a result given in [27, Lemma 5.1]: $\forall (X, Z) \in L^\infty \times L^1$ such that $Z \in \partial U(X)$ and Z is constant on some set $A \in \mathcal{F}$ (with $\mathbb{P}(A) > 0$), then X is a.s. constant on A as well.*

2.3 Monotone Adjusted Version of Non-Monotone Criterions

For each element in the family of criterions satisfying Assumption 2.2, we want to give an approximation in the smaller class of monetary utility functions, where the axiom of monotonicity is satisfied. By doing that we follow [29], where the *best monotone approximation* of non-monotone functionals is introduced in order to solve a portfolio selection problem. Let us first characterize the set of financial positions where an agent behaves monotonically.

Definition 2.8 ([29]). *Let* $U : L^\infty(\Omega, \mathcal{F}, \mathbb{P}) \to \mathbb{R}$ *be a functional satisfying Assumption 2.2, and let* $(L^\infty)^*_+ := \{\mu \in (L^\infty)^* : \langle \mu, X\rangle \geq 0$ *whenever* $X \geq 0\}$ *be the collection of all positive measures. We call domain of monotonicity of* U *the following subset of* L^∞*:*

$$M(U) := \{X \in L^\infty : \partial U(X) \cap (L^\infty)^*_+ \neq \emptyset\}. \tag{7}$$

As the definition suggests, for $X, Y \in M(U)$ s.t. $X \leq Y$, we get $U(X) \leq U(Y)$. Indeed, by hypothesis there exists $\mu_Y \in \partial U(Y) \cap (L^\infty)^*_+$, and by the equivalences in (1) and representation (3) we obtain $U(X) \leq V(\mu_Y) + \langle \mu_Y, X\rangle = U(Y) - \langle \mu_Y, Y - X\rangle \leq U(Y)$. Consider, for example, the mean-variance principle (4). In this case from Appendix B we obtain

$$M(U^{mv}_\delta) = \{X \in L^\infty : \nabla U^{mv}_\delta(X) \in \mathcal{Z}\} = \Big\{X \in L^\infty : X - \mathbf{E}[X] \leq \frac{1}{2\delta}\Big\}.$$

Let now fix any functional U satisfying Assumption 2.2 and s.t. $M(U) \neq \emptyset$ (e.g., U law-invariant, since in this case $V(1) = 0$, hence $\{c : c \in \mathbb{R}\} \subseteq M(U)$). We define the best monotone approximation of U as the most conservative m.u.f. that extends it outside $M(U)$:

$$U^m(X) = \sup\{U(Y) : Y \in L^\infty \text{ and } Y \leq X\},\ \forall X \in L^\infty. \tag{8}$$

By [29] we have that, for any $X \in L^\infty$, $X \in M(U)$ if and only if $U(X) = U^m(X)$. Moreover, denoting $V^m : (L^\infty)^* \to [0, \infty]$ as the dual conjugate of U^m, then for $\mu \in (L^\infty)^*$ we have

$$V^m(\mu) = \begin{cases} V(\mu), & \text{if } \mu \in (L^\infty)^*_+, \\ +\infty, & \text{otherwise.} \end{cases} \tag{9}$$

Remark 2.9. *From (10) and (11) we have that* U^m *defines the same concave functional as* $U \Box U_w$*, where* $U_w(X) := \inf_{\mathbb{Q} \in \mathcal{P}} \boldsymbol{E}_{\mathbb{Q}}[X]$ *is the worst-case (i.e. the most conservative) m.u.f.*

Example 2.10. *In the case of mean-variance preferences* U^{mv}_δ*, this monotone approximation corresponds to a truncation of the payoffs from above (see [29]):*

$$U^{mmv}_\delta(X) = \begin{cases} U^{mv}_\delta(X), & \textit{if } X \in M(U^{mv}_\delta), \\ U^{mv}_\delta(X \wedge k_X), & \textit{otherwise}, \end{cases}$$

for any $X \in L^\infty$*, where* $k_X = \max\{t \in \mathbb{R} : X \wedge t \in M(U^{mv}_\delta)\}$*. Now, if we consider the same payoffs as in Example 2.3, we have* $Y \in M(U^{mv}_\delta)$ *if and only if* $y \leq 1/(2\delta(1-\alpha))$*, and then* Y *is truncated at level* $k_Y = 1/(2\delta(1-\alpha))$*. Therefore, unlike the* U^{mv}_δ*-agent, the* U^{mmv}_δ*-agent behaves in a "more rational"' way, accepting tickets of any lottery.*

3 Optimal Sharing of Aggregate Risks

Consider an aggregate of n economic agents, for some $n \in \mathbb{N}$, characterized by choice functionals $(U_i)_{i=1}^n$ satisfying Assumption 2.2, and endowed with initial risky positions $(\xi_i)_{i=1}^n \in L^\infty$. The problem that arises is whether the agents may re-share the total risk $X = \sum_{i=1}^n \xi_i$ in order to make their situation better, where "better" has the meaning of "more satisfactory" in the sense of the choice criterions $(U_i)_{i=1}^n$. We study the optimal exchange of risk proceeding in two steps: first we maximize the joint level of satisfaction of the agents; next we take into consideration the individual point of view of each agent, looking for a contract that everyone agrees to sign.

3.1 The Sup-Convolution Problem

Definition 3.1. *Given an aggregate of n agents and a risk $X \in L^\infty$, we define the set of attainable (resp. increasing) allocations as the following collection of n-tuples:*

$$\mathbb{A}_n(X) := \left\{ (X_i)_{i=1}^n \in L^\infty : \sum_{i=1}^n X_i = X \right\}$$

$$\left(\textit{resp.}\, \mathbb{A}_n^\uparrow(X) := \{ (X_i)_{i=1}^n \in \mathbb{A}_n(X) : X_i \uparrow X,\, \forall i \} \right),$$

where $X_i \uparrow X$ means $X_i = \phi_i(X)$ pointwise, for some non-decreasing function $\phi_i : \mathbb{R} \to \mathbb{R}$.

The first problem we consider is the sup-convolution of the involved functionals:

$$(P) \qquad U(X) := U_1 \square \cdots \square U_n(X) \equiv \sup_{(X_1,..,X_n) \in \mathbb{A}_n(X)} \sum_{i=1}^n U_i(X_i), \tag{10}$$

which provides solutions jointly optimal for the agents and define a functional U on L^∞. Since $U_i : L^\infty \to \mathbb{R}$ are concave and cash-invariant, so does U, and it also satisfies $U > -\infty$ on L^∞. In this way we get $U : L^\infty \to \mathbb{R} \cup \{+\infty\}$, with either $U \equiv +\infty$ or $\mathrm{dom}(U) = L^\infty$, by concavity. Denote by $V, V_1, \ldots, V_n : (L^\infty)^* \to [0, +\infty]$ the convex conjugate functions of $U, U_1, \ldots, U_n$ respectively. Then we obtain $V \equiv +\infty$ if $U \equiv +\infty$, and

$$V = \sum_{i=1}^n V_i, \quad \text{with} \quad \mathrm{dom}(V) = \bigcap_{i=1}^n \mathrm{dom}(V_i), \tag{11}$$

if U is proper. Now, to avoid the worthless case $U \equiv +\infty$, we make the following assumption on the dual functions V_i's, which is equivalent to have U proper with $\mathrm{dom}(U) = L^\infty$.

Assumption 3.2. *The convex conjugate functions $V_1, \dots, V_n$ are such that $\bigcap_{i=1}^{n} dom(V_i) \neq \emptyset$.*

We remark that this condition is closely related to Pareto equilibrium (see [11], [12], [25]), and it is equivalent in the case of coherent risk measures. Under Assumption 3.2 we have $\partial U(X) \neq \emptyset \ \forall X \in L^\infty$, and (U, V) are $\langle L^\infty, (L^\infty)^* \rangle$-conjugate. Moreover, when all the U_i's are law-invariant, this condition is automatically satisfied since $Z \equiv 1$ lies in the effective domain of V_i (with $V_i(1) = 0$) for all i, which also guarantees the normalization property for U by relation (11). We now study some stability properties of the preference criterions.

Lemma 3.3. *Let $(U_i)_{i=1}^n$ be choice criterions satisfying Assumption 2.2 and Assumption 3.2, and let U be the functional defined in (10). Then the following implications hold:*

(i) U_j monotone for some $j \in \{1, \dots, n\}$ $\Rightarrow$ U monotone;

(ii) $(U_i)_{i=1}^n$ law-invariant and satisfy property (S) $\Rightarrow$ U satisfies property (S);

(iii) $(U_i)_{i=1}^n$ law-invariant and U_j strictly monotone for some $j \in \{1, \dots, n\}$ $\Rightarrow$ U strictly monotone.

Proof. [Lemma 3.3-(i)] Assume U_j to be monotone for some $j \in \{1, \dots, n\}$ and fix $X, Y \in L^\infty$ such that $X \leq Y$. Let $(X_1^m, \dots, X_n^m)_{m \in \mathbb{N}}$ be a maximizing sequence in $\mathbb{A}_n(X)$ for the sup-convolution problem, and consider the allocations $(Y_1^m, \dots, Y_n^m)_{m \in \mathbb{N}} \in \mathbb{A}_n(Y)$ given by

$$Y_i^m = \begin{cases} X_j^m + (Y - X), & \text{if } i = j, \\ X_i^m, & \text{if } i \neq j. \end{cases}$$

Then $U(Y) \geq \lim_m \sum_i U_i(Y_i^m) \geq \lim_m \sum_i U_i(X_i^m) = U(X)$, by monotonicity of U_j. □

The proof of statements (iii)-(iv) will be given after Theorem 3.7. Now we recall some classical results from economic theory, well known from several references (see, e.g., the recent work [27]), that are still valid in our unusual non-monotone setting.

Definition 3.4. *An n-tuple* $(X_i)_{i=1}^n \in \mathbb{A}_n(X)$ *is said a Pareto Optimal Allocation (POA) if for any* $(\xi_i)_{i=1}^n \in \mathbb{A}_n(X)$ *such that* $U_i(\xi_i) \geq U_i(X_i)\ \forall i$, *then* $U_i(\xi_i) = U_i(X_i)\ \forall i$.

Note that, due to cash invariance, Pareto optimality is defined up to constants summing up to zero. That is, for any POA $(X_i)_{i=1}^n \in \mathbb{A}_n(X)$ and any choice of constants $(c_i)_{i=1}^n \in \mathbb{R}$ s.t. $\sum_{i=1}^n c_i = 0$, then the allocation $(X_i + c_i)_{i=1}^n$ of X is Pareto optimal as well.

A standard result that goes back to Gerber [22], is the equivalence between minimal risk and Pareto optimality. Here it can be expressed in the following form (and proved, for instance, by induction from [27, Theorem 3.1]).

Theorem 3.5. *Let* $(U_i)_{i=1}^n$ *be preference functionals satisfying Assumption 2.2 and Assumption 3.2, with associated dual convex functions* $(V_i)_{i=1}^n$. *For a given risk* $X \in L^\infty$ *and an allocation* $(\xi_i)_{i=1}^n \in \mathbb{A}_n(X)$, *the following statements are equivalent:*

(i) $(\xi_1, \ldots, \xi_n)$ *is a Pareto optimal allocation,*

(ii) $U_1 \Box \cdots \Box U_n(X) = \sum_{i=1}^n U_i(\xi_i)$,

(iii) there exists a measure $\mu \in (L^\infty)^*$ *s.t.* $U_i(\xi_i) = V_i(\mu) + \langle \mu, \xi_i \rangle\ \forall i = 1, \ldots, n$.

Remark 3.6. *By using this theorem it can easily be shown that, for any* $X \in L^\infty$,

(a) $\mu \in \partial U(X) \ \Rightarrow\ \mu \in \bigcap_{i=1}^n \partial U_i(X_i)$, *for any POA* $(X_i)_{i=1}^n \in \mathbb{A}_n(X)$;

(b) $\mu \in \bigcap_{i=1}^n \partial U_i(X_i)$, *for some* $(X_i)_{i=1}^n \in \mathbb{A}_n(X) \ \Rightarrow\ \mu \in \partial U(X)$.

Several authors (see [13], [27] and [12] among others) rely on monotonicity results in order to prove existence of optimal solutions. Here we have that the law-invariance property leads to express the optimization problem (10) in terms of increasing allocations, ensuring the existence of POAs (a fact not true in general, see [27] or [17] for a counterexample).

Theorem 3.7. *Let* $(U_i)_{i=1}^n$ *be law-invariant functionals satisfying Assumption 2.2. Then*

$$\sup_{(X_i)_{i=1}^n \in \mathbb{A}_n(X)} \sum_{i=1}^n U_i(X_i) = \sup_{(X_i)_{i=1}^n \in \mathbb{A}_n^{\uparrow}(X)} \sum_{i=1}^n U_i(X_i). \tag{12}$$

Moreover, for any $X \in L^\infty$, *the set of Pareto optimal allocations in* $\mathbb{A}_n^{\uparrow}(X)$ *is non-empty.*

Note that the existence of POAs implies, in particular, that the functional U defined in (10) is proper, with $\bigcap_{i=1}^n \mathrm{dom}(V_i) \neq \emptyset$. We can now complete the proof of Lemma 3.3.

Proof. [Lemma 3.3-(ii),(iii)] (ii): Let $(U_i)_{i=1}^n$ be law-invariant functionals satisfying property (S), and let $X \in L^\infty$ and $A \in \mathcal{F}$ be s.t. $\mathbb{P}(A) > 0$ and X is not constant on A. By Theorem 3.7 there exists a POA $(X_i)_{i=1}^n \in \mathbb{A}_n(X)$, and we must have X_j not constant on A for some $j \in \{1, \dots, n\}$. Therefore we get

$$U(X) = \sum_{i=1}^n U_i(X_i) < \sum_{i=1}^n U_i(X_i \mathbf{1}_{A^c} + \mathbf{E}[X_i|A]\mathbf{1}_A) \leq U(X\mathbf{1}_{A^c} + \mathbf{E}[X|A]\mathbf{1}_A).$$

(iii): Let $(U_i)_{i=1}^n$ be law-invariant, with U_j strictly monotone for some $j \in \{1, \dots, n\}$, and let $X, Y \in L^\infty$ be such that $X < Y$ (i.e. $X \leq Y$ with $\mathbb{P}(X < Y) > 0$). Consider a POA $(X_i)_{i=1}^n \in \mathbb{A}_n(X)$, which exists by Theorem 3.7, and the allocation $(Y_i)_{i=1}^n$ of Y given by:

$$Y_i = \begin{cases} X_j + (Y - X), & \text{if } i = j, \\ X_i, & \text{if } i \neq j. \end{cases}$$

Then $U(X) = \sum_{i=1}^n U_i(X_i) < \sum_{i=1}^n U_i(Y_i) \leq U(Y)$ and the proof is complete. □

3.2 Constraints on the Sup-Convolution Problem

As pointed out before, the POAs are jointly optimal for the agents but do not pertain to the individual level of satisfaction of the agents. Therefore, by solving the optimization problem (10), we obtain the feasible designs of an optimal contract but not the (right) price. At this point we impose some constraints (a concept already introduced in [22]), which will lead us to single out a set of suitable prices for these risk exchanges.

Definition 3.8 ([27]). *Consider n agents endowed with choice criterions $(U_i)_{i=1}^n$ and initial risky positions $(\xi_i)_{i=1}^n \in L^\infty$. Let $(X_i)_{i=1}^n \in \mathbb{A}_n(X)$ be a POA of the total risk $X = \sum_{i=1}^n \xi_i$. Then we say that $(X_i)_{i=1}^n$ is an Optimal Risk Sharing (ORS) rule, if it satisfies the Individual Rationality (IR) constraints: $U_i(X_i) \geq U_i(\xi_i)$, for all $i = 1, \dots, n$.*

Theorem 3.9. *Let $(U_i)_{i=1}^n$ be choice functionals satisfying Assumption 2.2, and let $(\xi_i)_{i=1}^n \in L^\infty$. Consider a Pareto optimal allocation $(X_i)_{i=1}^n \in \mathbb{A}_n(X)$ of the total risk $X = \sum_{i=1}^n \xi_i$ and define $p_i := U_i(X_i) - U_i(\xi_i)$ $\forall i$. Then the following statements hold:*

(i) $\sum_{i=1}^n p_i \geq 0$*;*

(ii) *let* $\pi_1, \ldots, \pi_n$ *be constants s.t.* $\sum_{i=1}^n \pi_i = 0$*, then the allocation* $(X_1 - \pi_1, \ldots, X_n - \pi_n)$ *is an ORS rule if and only if* $\pi_i \leq p_i \ \forall i = 1, \ldots, n$*.*

Proof. (i) readily follows from the Pareto optimality, since $\sum_{i=1}^n p_i = U(X) - \sum_{i=1}^n U_i(\xi_i)$. (ii): For any choice of constants $(\pi_i)_{i=1}^n$ summing up to zero, $(X_i - \pi_i)_{i=1}^n$ is Pareto optimal as well. Therefore, by definition, it is an ORS rule iff $U_i(X_i - \pi_i) \geq U_i(\xi_i)$ for all i, that is $\pi_i \leq U_i(X_i) - U_i(\xi_i) = p_i$, from the cash-invariance property. □

This theorem ensures the existence of optimal risk sharing rules of a given aggregate risk, provided the existence of Pareto optimal allocations (see [27] for the case of two m.u.f.'s). We call $p_1, \ldots, p_n$ *indifference prices*, since agents are indifferent to either carrying out this transaction at these prices or not carrying it out at all.

Remark 3.10. *If the initial risk endowment is already Pareto optimal, then for any POA there is a unique vector of prices making it an ORS rule (the indifference prices). Otherwise, any POA admits an infinite set of suitable prices which form the polyhedral space*

$$\Pi := \{(\pi_i)_{i=1}^n \in \mathbb{R}^n : \sum_{i=1}^n \pi_i = 0 \text{ and } \pi_i \leq p_i \ \forall i = 1, \ldots, n\},$$

and it is the market power of the agents that determines the unique price of a contract.

3.3 Monotone Approximations in the ORS Problem

So far, in this section, no distinction is made between choice functionals which do or do not satisfy monotonicity. Now we just want to emphasize if the involved criterions have or have not this property, in order to compare the behaviour of monotone and non-monotone agents when facing the risk sharing problem.

Lemma 3.11. *Let* $(U_i)_{i=1}^n$ *be choice functionals satisfying Assumption 2.2 and Assumption 3.2, and let at least one of these be monotone. Then, for any POA* $(X_i)_{i=1}^n \in \mathbb{A}_n(X)$*,*

$$X_i \in M(U_i), \ \forall i = 1, \ldots, n, \tag{13}$$

where $M(U_i)$ *is the domain of monotonicity of* U_i*, defined in (7).*

Proof. For a POA $(X_i)_{i=1}^n \in \mathbb{A}_n(X)$, Theorem 3.5 ensures the existence of a measure $\mu \in (L^\infty)^*$ s.t. $\mu \in \partial U_i(X_i)$, for any $i = 1, \dots, n$. On the other hand, by hypothesis there is a functional U_j, for some $j \in \{1, \dots, n\}$, which is monotone. Therefore $\mu \in \partial U_j(X_j) \subseteq \mathrm{dom}(V_j) \subseteq (L^\infty)^*_+$, and (13) readily follows from (7). □

Let us now consider problem (P) given in (10) and introduce the associated problem involving the best monotone versions U_i^m's of the original criterions U_i's (see (8)):

$$(P^m) \qquad \widetilde{U}(X) := U_1^m \square \cdots \square U_n^m(X) \equiv \sup_{(X_1,\dots,X_n)\in\mathbb{A}_n(X)} \sum_{i=1}^n U_i^m(X_i). \quad (14)$$

Of course this problem is equivalent to maximize $\sum_{i=1}^n U_i(X_i)$ over the n-tuples $(X_i)_{i=1}^n$ such that $\sum_{i=1}^n X_i \leq X$.

Theorem 3.12. *Let $U_1, \dots, U_n$ be as in Lemma 3.11. Then U and $\widetilde{U}$, introduced in (10) and (14) respectively, describe the same monotone functional on $L^\infty(\Omega, \mathcal{F}, \mathbb{P})$:*

$$U(X) = \widetilde{U}(X), \ \ \forall X \in L^\infty. \quad (15)$$

Proof. We call $\widetilde{V}$ the convex conjugate function of $\widetilde{U}$, defined on $L^\infty(\Omega, \mathcal{F}, \mathbb{P})^*$. As immediate consequence of Lemma 3.3 and the arguments that precede it, we have that U and $\widetilde{U}$ are concave and cash-invariant functionals satisfying monotonicity. Moreover $(\widetilde{U}, \widetilde{V})$, the same as (U, V), are $\langle L^\infty, (L^\infty)^* \rangle$–conjugate. Now from (11) and (9) we have

$$\widetilde{V}(\mu) = \sum_{i=1}^n V_i^m(\mu) = \begin{cases} \sum_{i=1}^n V_i(\mu), & \text{on } \bigcap_{i=1}^n \mathrm{dom}(V_i) \bigcap (L^\infty)^*_+, \\ +\infty, & \text{elsewhere.} \end{cases}$$

On the other hand, $\mathrm{dom}(V) = \bigcap_{i=1}^n \mathrm{dom}(V_i) \subseteq (L^\infty)^*_+$, since $\mathrm{dom}(V_j) \subseteq (L^\infty)^*_+$ for some $j \in \{1, \dots, n\}$, so that $V = \widetilde{V}$ on $(L^\infty)^*$, and $U = \widetilde{U}$ on L^∞ as well. □

Equality (15) means that the consideration of criterions $(U_i)_{i=1}^n$ or $(U_i^m)_{i=1}^n$ leads, for any aggregate risk, to the same maximal overall level of satisfaction, although it does not say anything about which allocations realize or approximate this supremum. Let us now take into account exactly such allocations, that is, how the total risk can be optimally re-shared among the involved agents. The following result is just a first answer in this direction.

Corollary 3.13. *Let $U_1, \ldots, U_n$ be as in Lemma 3.11. Then, for any aggregate risk $X \in L^\infty$, any solution to problem (P) is also a solution to problem (P^m).*

By Theorem 3.5 this means that, for each $X \in L^\infty$, the following relation between the sets of Pareto optimal allocations holds true:

$$\{\text{POAs for } (U_1, \ldots, U_n)\} \subseteq \{\text{POAs for } (U_1^m, \ldots, U_n^m)\}.$$

Proof. If problem (P) admits no solutions, then there is nothing to prove. So, let us assume $(X_1, \ldots, X_n) \in \mathbb{A}_n(X)$ Pareto optimal w.r. to $(U_i)_{i=1}^n$. Lemma 3.11 implies $X_i \in M(U_i)$ for any i, and then $U_i(X_i) = U_i^m(X_i)$. At this point Theorem 3.12 gives us

$$\widetilde{U}(X) = U(X) = \sum_{i=1}^n U_i(X_i) = \sum_{i=1}^n U_i^m(X_i),$$

which makes $(X_i)_{i=1}^n$ Pareto optimal w.r. to the monotone functionals $(U_i^m)_{i=1}^n$ as well. □

Corollary 3.14. *Let $U_1, \ldots, U_n$ be as in Lemma 3.11, and let $(X_1, \ldots, X_n) \in \mathbb{A}_n(X)$ be a solution of both problems (P) and (P^m). Then, any vector of prices that agents characterized by U_i^m's are willing to pay for this contract, is also optimal for agents characterized by U_i's.*

Proof. Let $(\xi_i)_{i=1}^n$ be the initial endowments of the total risk X and define $p_i^m := U_i^m(X_i) - U_i^m(\xi_i)$, for $i = 1, \ldots, n$, as the indifference prices w.r. to the monotone adjusted versions $(U_i^m)_{i=1}^n$. Lemma 3.11 ensures that agents with choice criterions U_i and U_i^m give the same value to the optimal share X_i, whereas for the initial risk share ξ_i we can only say that $U_i^m(\xi_i) \geq U_i(\xi_i)$. This fact produces the inequalities $p_i^m \leq p_i$ and concludes the proof. □

Let us now focus our attention on non-monotone choice functionals of mean-variance type (4). In this case we can state more interesting results, by relying on Example 2.10.

Theorem 3.15. *Let $U_1, \ldots, U_n$ be functionals satisfying Assumption 2.2 and Assumption 3.2, such that at least one is strictly monotone, and the non-monotone ones are of type (4). Then, for any aggregate risk $X \in L^\infty$, problems (P) and (P^m) admit the same set of solutions:*

$$\{\mathit{POAs\ for}\ (U_1, \ldots, U_n)\} = \{\mathit{POAs\ for}\ (U_1^m, \ldots, U_n^m)\}.$$

Proof. The inclusion in one sense being immediate by Corollary 3.13, let us prove the other one. Assume $(X_i)_{i=1}^n \in \mathbb{A}_n(X)$ to solve problem (P^m). If $X_i \in M(U_i)$ for any i, then $(X_i)_{i=1}^n$ solves problem (P) as well. Now suppose $X_j \notin M(U_j)$ for some $j \in \{1, \dots, n\}$, which in particular implies that U_j is non-monotone, hence a mean-variance functional by assumption. By hypothesis there is an agent, say k, with strictly monotone preferences, where of course $k \in \{1, \dots, n\} \setminus \{j\}$. Since $X_j \notin M(U_j)$, then $U_j^m(X_j) > U_j(X_j)$, and Example 2.10 ensures that the supremum in (8) is actually a maximum for U_j. Therefore there exists $Y \in L^\infty$ s.t. $Y < X_j$ and $U_j^m(X_j) = U_j(Y) = U_j^m(Y)$. Let us consider the following allocation of X:

$$\zeta_i = \begin{cases} Y, & \text{if } i = j, \\ X_k + (X_j - Y), & \text{if } i = k, \\ X_i, & \forall\, i \in \{1, \dots, n\} \setminus \{j, k\}. \end{cases}$$

Strict monotonicity of U_k implies $U_k^m(\zeta_k) = U_k(\zeta_k) > U_k(X_k) = U_k^m(X_k)$, and then $\sum_i U_i^m(\zeta_i) > \sum_i U_i^m(X_i) = \widetilde{U}(X)$, which yields the desired contradiction. □

Example 3.16. *Theorem 3.15 applies, e.g., to the problem of sharing risks between a mean-variance agent and an agent with entropic utility or semi-deviation utility (for any $p \neq +\infty$).*

This fact is interesting from an economic point of view: whereas in Section 2 we have seen how the lack of monotonicity may lead to pathological situations, here we have that the optimal risk sharing does not take into account the fact that some (but not all!) choice criterions may fail this property. On the other hand, whereas the Pareto optimality is not affected by the possible lack of monotonicity by some agents, this is no longer true for the price of the contract. Indeed, by imposing the (IR) constraints and looking for ORS rules, we have only the inclusion in one sense among the sets of optimal solutions.

Corollary 3.17. *Let $U_1, \dots, U_n$ be as in Theorem 3.15. Then we have the following relation between the solutions to the ORS problem w.r. to the functionals U_i's and the solutions to the ORS problem w.r. to the functionals U_i^m's:*

$$\{\textit{ORS rules for } (U_1^m, \dots, U_n^m)\} \subseteq \{\textit{ORS rules for } (U_1, \dots, U_n)\}. \quad (16)$$

Proof. It readily follows from Theorem 3.15 and Corollary 3.14. □

Observe that, if the initial risk endowment $\xi_i \in M(U_i)$ for any agent, then the equality $p_i^m = p_i$ holds for all i, and the two sets in (16) coincide.

4 Explicit Characterization of Optimal Risk Sharing Rules

In this section we formulate and explicitly solve some concrete problems of optimal risk sharing, where the economic agents are characterized by law-invariant choice functionals. In this case Assumption 3.2 is automatically satisfied and, as far as duality is concerned, we are allowed to work in L^1 instead of the whole dual space $(L^\infty)^*$, by Remark 2.4. Moreover, the existence of Pareto optimal allocations (Theorem 3.7) gives the equality $\partial\big(\sum_{i=1}^n V_i\big)(\mu) = \sum_{i=1}^n \partial V_i(\mu)$ for all $\mu \in (L^\infty)^*$ (see [27]). Here we do not consider the initial risk endowment of the agents since, once we have a Pareto optimal allocation, only simple calculations are required to find the suitable prices, as shown in Theorem 3.9.

4.1 AV@R-Agent vs Agent with Property (S)

Proposition 4.1. *Let U_1 be the $AV@R_\lambda$-criterion given in (19) and let U_2 be a law-invariant functional satisfying Assumption 2.2 and property (S). Then, for any aggregate risk $X \in L^\infty$, there exists a unique (up to a constant) POA in $\mathbb{A}_2^\uparrow(X)$, given by*

$$(X_1, X_2) := (-(X-l)^- + (X-u)^+, (l \vee X) \wedge u), \quad \textit{for some } l, u \in \mathbb{R}. \tag{17}$$

This means that the optimal sharing consists in the exchange of at the most two European options written on X, thus producing a typical insurance contract (the so-called *limited stop-loss* contract) where the insurer's risk share X_2 has floor l and is capped at level u.

Proof. Let $(X_1, X_2) \in \mathbb{A}_2^\uparrow(X)$ be a POA of a given aggregate risk $X \in L^\infty$. Since U_1 satisfies the Lebesgue property, then Theorem 3.5 implies the existence of a density $Z \in \partial U_1(X_1) \cap \partial U_2(X_2)$, which by Appendix A is such that $Z \in [0, 1/\lambda]$ and X_1 is constant on $\big\{Z \in \big(0, \frac{1}{\lambda}\big)\big\}$. On the other hand Remark 2.7 ensures that X_2 is constant on the sets $\{Z = 0\}$ and $\{Z = 1/\lambda\}$. Now, thanks to [27, Lemma 4.1], we have that (Z, X_1), as well as (Z, X) and (Z, X_2), are anticomonotone random variables and therefore X_1, the same as X, takes its biggest values on $\{Z = 0\}$ and the smallest ones on $\{Z = 1/\lambda\}$. At this point, since X_1 and X_2 increase with X, they assume the shape declared in (17), for some thresholds l and u. Now the uniqueness stems from the fact that the set of Pareto optimal allocations is a convex space in which each element has this form. Indeed, let us fix any POA $(Y_1, Y_2) \in \mathbb{A}_2^\uparrow(X)$ and assume it is different from (X_1, X_2), in the sense that they do not differ just by a constant. From the first part of the proof, the allocation (Y_1, Y_2) is characterized by a pair $(\hat{l}, \hat{u}) \neq (l, u)$ and, by convexity, the allocation (ξ_1, ξ_2) given by $\xi_i = \alpha X_i + (1-\alpha)Y_i$, $i = 1, 2$, $\alpha \in (0, 1)$, is Pareto optimal as well. On the other hand, since $(\hat{l}, \hat{u}) \neq (l, u)$, (ξ_1, ξ_2) cannot have the desired shape, thus leading to a contradiction. □

Both the mean-variance and the standard-deviation functionals, as well as the entropic utility, satisfy the conditions required in Proposition 4.1 on agent 2. Therefore, when considering the sharing of an aggregate risk between one of such agents and an $AV@R$-agent, we have that the latter one takes on the extreme risks, thus revealing her non-conservative behaviour. However, the shape of the optimal contract in (17) may degenerate, in the sense of the following example.

Example 4.2. *Let agent 2 in Proposition 4.1 have mean-variance functional (4) with parameter $\delta > 0$, and let the aggregate risk X have essential oscillations bounded as follows:*

$$(esssup X - essinf X) < \frac{1}{2\delta} \wedge \frac{1}{2\delta}\Big(\frac{1}{\lambda} - 1\Big). \tag{18}$$

Then the optimal sharing corresponds to totally charge the aggregate risk to agent 2.

Indeed, the bound in (18) in particular produces

$$X - \mathbf{E}[X] \leq \operatorname{ess\,sup} X - \operatorname{ess\,inf} X < \frac{1}{2\delta} \qquad \text{and}$$

$$X - \mathbf{E}[X] \geq -(\operatorname{ess\,sup} X - \operatorname{ess\,inf} X) > -\frac{1}{2\delta} \vee -\frac{1}{2\delta}\big(\frac{1}{\lambda} - 1\big) \geq -\frac{1}{2\delta}\big(\frac{1}{\lambda} - 1\big),$$

so that $\overline{Z} := 1 - 2\delta(X - \mathbf{E}[X]) \in (0, 1/\lambda)$. Note that $\overline{Z} \in \partial U_1(0) \cap \partial U_2(X)$ (see Appendices A, B), which implies that $(0, X)$ is a POA, by Theorem 3.5, and in fact the unique one (up to a constant), by the previous proposition. This shape of the optimal re-sharing is not surprising if we consider the fact that the mean-variance principle only penalizes the variance of financial positions. Therefore, when a payoff has a sufficiently small variability, a mean-variance agent associates a high level of satisfaction to it, thus making it favourable for her to take on the entire prospect. In line with the reasoning that follows (17), we can consider the $AV@R$-agent as an insurant and the U^{mv}-agent as an insurer. From this point of view, what we obtain as optimal risk sharing under condition (18) is a *full-insurance* contract, where the insurer takes on the whole risk.

4.2 Entropic vs Mean-variance vs Standard-Deviation

Proposition 4.3. *Let U_1 be the entropic utility (20) with parameter $\gamma > 0$, U_2 the mean-variance principle (4) with $\delta_1 > 0$, and U_3 the standard-deviation principle (5) with $\delta_2 > 0$. Then for any aggregate risk $X \in L^\infty$ there exists a unique (up to constants summing up to zero) POA $(X_1, X_2, X_3) \in \mathbb{A}_3^\uparrow(X)$, such that X_1 is a convex function of X, whereas X_2 and X_3 are concave functions of X, with X_2 proportional to X_3.*

This means that, when optimally sharing an aggregate risk X among such agents, the entropic one especially takes the lowest risks. An analogous situation occurs in the optimal exchange of risk between an entropic and a semi-deviation agent, which confirms that an agent endowed with entropic utility is prudent towards extreme risks. On this subject we recall that, in the problem of sharing a risk between an entropic-agent and an $AV@R$-agent (explicitly solved in [27]), the resulting optimal contract consists of a call option written on the total risk and offered to the entropic-agent (*stop-loss* contract).

Proof. Fix $X \in L^\infty$ and consider any POA $(X_1, X_2, X_3) \in \mathbb{A}_3^\uparrow(X)$. Here we use the notation introduced in the preceding section, denoting U as the result of the sup-convolution: $U(X) = \Box_{i=1}^3 U_i(X)$, and $V_1, V_2, V_3, V = \sum_{i=1}^3 V_i$ as the convex conjugate functions of U_1, U_2, U_3, U respectively. Once again, Theorem 3.5 and the Lebesgue property imply that there exists a density $Z \in \cap_{i=1}^3 \partial U_i(X_i)$, and therefore $Z \in \partial U(X)$ by Remark 3.6. On the other hand V inherits strict convexity on its effective domain from V_1, V_2 (see Appendices A, B), thus leading to a unique supergradient of U at X, by duality theory: $\partial U(X) = \{Z_X\}$ for some $Z_X \in \mathcal{Z}$. Now, by (1) we have $X \in -\partial V(Z_X) = -\sum_{i=1}^3 \partial V_i(Z_X)$, so that

$$X = -\gamma \ln Z_X - c_X Z_X + d_X$$

for some constants $c_X \in \mathbb{R}^+$ and $d_X \in \mathbb{R}$ univocally determined by $\mathbf{E}[Z_X] = 1$ (see Appendices A, B). Note that the pointwise relation between X and Z_X can be written as $X = f(Z_X)$ (meaning that $X(\omega) = f(Z_X(\omega))$ for all $\omega \in \Omega$), for some convex and decreasing function $f : \mathbb{R}^+ \to \mathbb{R}$. Now, since it is a one-to-one function, we can also write $Z_X = g(X)$, pointwise, with $g : \mathbb{R} \to \mathbb{R}^+$ convex and decreasing. Therefore a generic Pareto optimal allocation (X_1, X_2, X_3) takes the form:

$$X_1 = -\gamma \ln Z_X + c_1 = -\gamma \ln(g(X)) + c_1,$$

$$X_2 = -\frac{Z_X}{2\delta} + c_2 = -\frac{g(X)}{2\delta} + c_2,$$

$$X_3 = -c_4 Z_X + c_3 = -c_4 g(X) + c_3,$$

for some $c_4 \in \mathbb{R}_0^+$ univocally determined, and for any $(c_i)_{i=1}^3 \in \mathbb{R}$ s.t. $\sum_{i=1}^3 X_i = X$. □

Besides that, we can also calculate the exact proportion between the shares X_2 and X_3 of the mean-variance and the standard-deviation agents, as shown in [1, Proposition 5.6].

Remark 4.4. *In addition to the agents considered in Proposition 4.3, we can introduce a further agent characterized by the semi-deviation utility (21), with parameters $p = 2$ and $\delta_3 \in (0, 1]$. By Theorem A.2, one of the following situations occur: either the semi-deviation agent does not take any risk (i.e., $X_4 =$const and the other agents share the risk as described in Proposition 4.3), or she takes on a non-trivial share of the total risk. In the second case the set $\{\omega : Z_X(\omega) = minZ_X\}$ (where the total risk X takes its biggest values) characterizes a subinterval $[\beta, ess\ supX]$ of $[ess\ infX, ess\ supX]$, where the risk is totally charged to the semi-deviation agent. On the other hand, when considering the interval $[ess\ infX, \beta)$, we find that the risk is shared among all the agents and, in particular, the entropic-agent's share of risk is, pointwise, a convex function of X, whereas the other agents proportionally share the rest of the risk (each one taking on a quota of risk which is a concave function of X).*

Another consideration can be made with regard to the parameters characterizing the involved choice functionals. Indeed, if in a problem of optimal risk sharing there are a standard-deviation agent with parameter $\delta_1 > 0$ and a semi-deviation agent with parameters $p = 2$ and $\delta_2 \in (0, 1]$ s.t. $\delta_2 \leq \delta_1$, then the former one does not take any risk. In order to see that, denote U_1 and U_2 the respective functionals, and fix any $X_1, X_2 \in L^\infty$. Then, by the positive homogeneity of U_2, we clearly have $U_1(X_1) + U_2(X_2) \leq U_2(X_1) + U_2(X_2) = U_2(X_1 + X_2)$, the inequality being strict whenever X_1 is non-constant.

4.3 AV@R vs Entropic vs Mean-variance vs Standard-Deviation

Proposition 4.5. *Let U_0 be the $AV@R_\lambda$-criterion (19), and let U_1, U_2, U_3 be as in Proposition 4.3. Then, for any $X \in L^\infty$, there exists a unique (up to constants summing up to zero) POA $(X_i)_{i=0}^3 \in \mathbb{A}_4^\uparrow(X)$, such that*

$X_0 = -(X-k)^-$ for some $k \in \mathbb{R}$, whereas X_1 (resp. X_2, X_3) is a convex (resp. concave) function of $(X \vee k)$, with X_2 proportional to X_3.

Proof. By the associative property of the sup-convolution operator, we can consider the problem $\Box_{i=0}^3 U_i(X)$ as $U_0 \Box U(X)$, where the functional $U = \Box_{i=1}^3 U_i$ results to be law-invariant, strictly monotone and strictly risk-averse conditionally on any event by Lemma 3.3. Now Proposition 3.2 in [27] provides the unique (up to a constant) POA of X w.r. to (U_0, U):

$$(\xi_0, \xi) = (-(X-k)^-, X \vee k), \quad \text{for some } k \in \mathbb{R}.$$

This means that the interval of essential oscillations of X can be shared in two subintervals ($[\operatorname{ess\,inf} X, k]$ and $(k, \operatorname{ess\,sup} X]$) such that ξ is constant on the first one whereas ξ_0 is constant on the second one. Therefore, the $AV@R$-agent takes the worst risks (i.e. the whole risk in $[\operatorname{ess\,inf} X, k]$), whereas the others agents optimally share the rest of the risk as described in Proposition 4.3. □

We can proceed in the same way to solve the optimal risk exchange problem when considering a further agent endowed with the semi-deviation functional. Once again we first apply Proposition 3.2 of [27], and then we use the results given in Remark 4.4.

Remark 4.6. *Consider the case of risk sharing when more agents of the same type intervene. For example, if there are more agents of standard-deviation (resp. $AV@R$) type, with parameters δ_i (resp. λ_i), then the only one among them that can take on a non-trivial (i.e. non-constant) share of risk is the one with parameter $\delta_{i^*} := min_i \delta_i$ (resp. $\lambda_{i^*} := max_i \lambda_i$) (see [1]). A completely different situation occurs if we consider more agents of mean-variance (resp. entropic) type, with parameters δ_i (resp. γ_i). These are particular cases of dilated utility measures (see [5, 1]), where the optimal risk sharing turns out to be proportional to the risk-tolerance coefficients δ_i^{-1} (resp. γ_i).*

A APPENDIX: Dual Characterization of Monotone Criterions

- **The Average Value at Risk**

The Average Value at Risk -taken with the opposite sign- is the most representative coherent utility measure:

$$U_\lambda(X) := -AV@R_\lambda(X) = -\frac{1}{\lambda}\int_0^\lambda V@R_t(X)dt, \ \lambda \in (0,1], \tag{19}$$

where the Value at Risk is defined as $V@R_t(X) = -q_X^+(t) = -\inf\{x \in \mathbb{R} : F_X(x) > t\}$, for any $t \in [0,1)$, and F_X is the cdf associated to X. It is known that its convex conjugate V_λ is the indicator function (in the sense of the convex analysis) of the convex set $\mathcal{P}_\lambda := \left\{\mathbb{Q} \in \mathcal{P} : 0 \le \frac{d\mathbb{Q}}{d\mathbb{P}} \le \frac{1}{\lambda}\right\}$, and this leads to the representation $U_\lambda(X) = \inf_{\mathbb{Q}\in\mathcal{P}_\lambda} \mathbf{E}_\mathbb{Q}[X]$. Moreover, for any $X \in L^\infty$, a generic element Z in $\partial U_\lambda(X)$ can be written as

$$Z = \begin{cases} 1/\lambda, & \text{on } \{X < q_X(\lambda)\}, \\ \in [0, 1/\lambda], & \text{on } \{X = q_X(\lambda)\}, \\ 0, & \text{on } \{X > q_X(\lambda)\}, \end{cases}$$

such that $\mathbf{E}[Z] = 1$ (see, e.g., [20]), where the lower-quantile function is defined as $q_X(t) = \inf\{x \in \mathbb{R} : F_X(x) \ge t\}$ $\forall t \in (0,1]$. At this point the relations in (1) give us the recipe to characterize the gradients of the dual function. Roughly speaking, for any $Z \in$dom(V_λ), a random variable X in $\partial V_\lambda(Z)$ takes its biggest values where $Z = 1/\lambda$, the smallest ones where $Z = 0$, and it is constant on $\{Z \in (0, 1/\lambda)\}$.

- **The Entropic Utility**

For any probability measure $\mathbb{Q}$ on $(\Omega, \mathcal{F})$, the relative entropy w.r. to $\mathbb{P}$ is defined as $H(\mathbb{Q};\mathbb{P}) = \mathbf{E}\left[\frac{d\mathbb{Q}}{d\mathbb{P}} \ln\left(\frac{d\mathbb{Q}}{d\mathbb{P}}\right)\right]$ if $\mathbb{Q} \ll \mathbb{P}$, and $H(\mathbb{Q};\mathbb{P}) = +\infty$ otherwise. In the following theorem we state a strict link between this function and the entropic utility function:

$$U_\gamma^{en}(X) := -\gamma \ln \mathbf{E}\left[\exp\left(-X/\gamma\right)\right], \quad \gamma > 0, \tag{20}$$

which gives a justification for the name of the latter.

Theorem A.1. *Let* $U_\gamma^{en} : L^\infty(\Omega, \mathcal{F}, \mathbb{P}) \to \mathbb{R}$ *be the entropic utility defined in (20) and* $V_\gamma^{en} : L^\infty(\Omega, \mathcal{F}, \mathbb{P})^* \to [0, \infty]$ *its convex conjugate function. Then*

(i) for any $Z \in \mathcal{Z}$, $V_\gamma^{en}(Z) = \gamma H(\mathbb{Q};\mathbb{P})$, *with* $\dfrac{d\mathbb{Q}}{d\mathbb{P}} = Z$;

(ii) for any $X \in L^\infty$, $\partial U_\gamma^{en}(X) = \left\{\dfrac{\exp(-X/\gamma)}{\|\exp(-X/\gamma)\|}\right\}$;

(iii) for any $Z \in$ *dom*(V^{en}), $\partial V_\gamma^{en}(Z) = \{\gamma \ln Z + c, \forall c \in \mathbb{R}\}$.

From this fact stems the dual representation:
$U_\gamma^{en}(X) = \inf\limits_{\mathbb{Q}\in\mathcal{P}} \left\{\gamma \mathbf{E}_\mathbb{Q}\left[\ln\left(\dfrac{d\mathbb{Q}}{d\mathbb{P}}\right)\right] + \mathbf{E}_\mathbb{Q}[X]\right\}$.

Proof. The Lebesgue property ensures that the effective domain of V_γ^{en} is contained in $\mathcal{Z}$. Now, since U_γ^{en} is strictly concave on L_0^∞, by duality theory we know that the differential of V_γ^{en} consists, for any Z in its domain, of a unique element in L_0^∞, with

$$\partial V_\gamma^{en}(Z) = -\arg\max_{X \in L^\infty}\{U_\gamma^{en}(X) - \mathbf{E}[ZX]\}.$$

For $Z \in \mathcal{Z}$, the functional $f(X) := U_\gamma^{en}(X) - \mathbf{E}[ZX]$ is well defined on L^∞, it is concave and Gateaux differentiable, with differential $\nabla f(X) = -Z + \dfrac{\exp(-X/\gamma)}{\|\exp(-X/\gamma)\|}$. Now we have $X_Z := -\gamma \ln Z$ satisfying $\nabla f(X_Z) = 0$, with f that attains its maximum at X_Z, and then (iii) holds. At this point (ii) readily follows by (1) and $\mathbf{E}[Z] = 1$, and we can compute

$$V_\gamma^{en}(Z) = U_\gamma^{en}(X_Z) - \mathbf{E}[ZX_Z] = \gamma \mathbf{E}[Z \ln Z], \ \forall Z \in \mathrm{dom}(V_\gamma^{en}),$$

which shows (i) and ends the proof. □

- **The Semi-Deviation Utility**

Consider now the semi-deviation utility (classical one-sided measure):

$$U_\delta^p(X) := \mathbf{E}[X] - \delta\|(X - \mathbf{E}[X])^-\|_{L^p}\,, \ 1 \le p \le \infty, \ 0 < \delta \le 1, \tag{21}$$

Once again the positive homogeneity ensures that the convex conjugate $V_\delta^p(Z)$ is the indicator function of some convex set $\mathcal{C}^p \subseteq \mathcal{Z}$, leading to the representation $U_\delta^p(X) = \inf_{Z \in \mathcal{C}^p} \mathbf{E}[ZX]$. Indeed, as shown in [16], U_δ^p can be obtained by the set of probability measures with density in $\{1 + \delta(g - \mathbf{E}[g]) : g \ge 0, \|g\|_{L^q} \le 1\}$, where $q = p/(p-1)$ is the conjugate of p. Here we consider the particular case $p = 2$, and characterize the gradients of U_δ^2 and V_δ^2 as follows:

Theorem A.2. *Let $U_\delta^2 : L^\infty(\Omega, \mathcal{F}, \mathbb{P}) \to \mathbb{R}$ be the semi-deviation utility (21) with $p = 2$ and $\delta \in (0,1]$, and let $V_\delta^2 : L^\infty(\Omega, \mathcal{F}, \mathbb{P})^* \to [0, \infty]$ be its convex conjugate. On $L^\infty \setminus \{c : c \in \mathbb{R}\}$ we define the following function: $h(X) = \dfrac{(X - \boldsymbol{E}[X])^- - \|(X - \boldsymbol{E}[X])^-\|_{L^1}}{\|(X - \boldsymbol{E}[X])^-\|_{L^2}}$. Then*

(i) for any $X \in L^\infty$, $\partial U_\delta^2(X) = \begin{cases} \mathit{dom}(V_\delta^2), & \textit{if } X = \textit{const}, \\ \{1 + \delta h(X)\}, & \textit{otherwise}, \end{cases}$

(ii) for any $Z \in \mathit{dom}(V_\delta^2)$, $\partial V_\delta^2(Z) = \{c : c \in \mathbb{R}\} \cup \{X \in L^\infty : Z = 1 + \delta h(-X)\}$.

In particular, $\{1 + \delta h(X) : X \in L^\infty \setminus \{c : c \in \mathbb{R}\}\}$ is the minimal set $C \subseteq \mathcal{Z}$ which allows us to represent U^2_δ in the form $U^2_\delta(X) = \min_{Z \in C} \mathbf{E}[ZX]$.

Proof. For any $Z \in \mathcal{Z}$ we know that

$$V^2_\delta(Z) = \sup_{X \in L^\infty} \{\mathbf{E}[X(1-Z)] - \delta \|(X - \mathbf{E}[X])^-\|_{L^2}\} \tag{22}$$

$$= 0 \vee \sup_{X \in L^\infty, X \neq \text{const}} \{\mathbf{E}[X(1-Z)] - \delta \|(X - \mathbf{E}[X])^-\|_{L^2}\}. \tag{23}$$

In order to solve the last optimization problem, we construct the Lagrangian function L and impose the optimality condition $\nabla L = 0$, thus obtaining $Z = 1 + \delta h(X)$. Now, for $Z \in \mathcal{Z}$ admitting a payoff $X \in L^\infty$ s.t. $Z = 1 + \delta h(X)$, the maximization over non-constant prospects yields zero as result, so that X solves the problem in (22) as well (U^2_δ coherent implies V^2_δ equal to zero on its domain). Therefore, such a Z lies in $\text{dom}(V^2_\delta)$ and such a X belongs to $\arg\max_{\xi \in L^\infty} \{U^2_\delta(\xi) - \mathbf{E}[Z\xi]\}$, that is $X \in -\partial V^2_\delta(Z)$. In this case we have $Var(Z) < \delta^2$ and the differential of V^2_δ can be rewritten as follows:

$$\partial V^2_\delta(Z) = \left\{ \frac{\mathbf{E}[Y]}{1-z} Z - Y + c : c \in \mathbb{R},\ Y \in L^\infty_+ \text{ and } Y \mathbf{1}_{\{Z \neq z\}} \equiv 0 \right\}, \tag{24}$$

where $L^\infty_+ := \{M \in L^\infty : M \geq 0\}$ and $z := \min_\omega Z(\omega) = 1 - \sqrt{\delta^2 - \text{Var}(Z)}$, with $\mathbb{P}(Z = z) > 0$. On the other hand, if a density Z in $\text{dom}(V^2_\delta)$ cannot be written as $Z = 1 + \delta h(X)$, then $\partial V^2_\delta(Z)$ just contains the constant payoffs. This concludes the proof of (ii) and, by the equivalences in (1), statement (i) holds as well. □

B APPENDIX: Dual Characterization of Non-Monotone Criterions

- **The Mean-Variance Principle**

Theorem B.1. *Let $U^{mv}_\delta : L^\infty(\Omega, \mathcal{F}, \mathbb{P}) \to \mathbb{R}$ be the mean-variance principle (4), and $V^{mv}_\delta : (L^\infty(\Omega, \mathcal{F}, \mathbb{P}))^* \to \mathbb{R}$ its dual transform. Then*

(i) $V^{mv}_\delta(Z) = \max_{X \in L^\infty} \{U^{mv}_\delta(X) - \boldsymbol{E}[ZX]\} = \frac{VarZ}{4\delta}$, $\forall Z \in$ *dom* $(V^{mv}_\delta) = \mathcal{Z}_\sigma \cap L^2$;

(ii) $\partial U^{mv}_\delta(X) = \{1 - 2\delta(X - \boldsymbol{E}[X])\}$, $\forall X \in L^\infty$;

(iii) $\partial V^{mv}_\delta(Z) = \left\{ \frac{Z}{2\delta} + c, \forall c \in \mathbb{R} \right\}$, $\forall Z \in$ *dom* (V^{mv}_δ).

In particular the following representation holds:
$U^{mv}_\delta(X) = \min_{Z\in\mathcal{Z}_\sigma}\left\{\frac{\mathbf{E}[Z^2]}{4\delta} + \mathbf{E}[ZX]\right\} - \frac{1}{4\delta}.$

Proof. Given X and Y in L^∞, the Gateaux differential of U^{mv}_δ at X with perturbation Y is

$$J_\delta(X;Y) := \lim_{\epsilon\to 0}\frac{U^{mv}_\delta(X+\epsilon Y) - U^{mv}_\delta(X)}{\epsilon} = \mathbf{E}[Y(1-2\delta(X-\mathbf{E}[X]))].$$

By differential theory we know that linearity of the functional $J_\delta(X;.) = \nabla U^{mv}_\delta(X)$ means differentiability of U^{mv}_δ, with $\partial U^{mv}_\delta(X) = \{\nabla U^{mv}_\delta(X)\} = \{Z_X\}$, where $Z_X = 1-2\delta(X-\mathbf{E}[X]) \in \mathcal{Z}_\sigma$. At this point, strict concavity of U^{mv}_δ on L^∞_0 implies the existence of a unique gradient of V^{mv}_δ on L^∞_0 for any element in the domain of V^{mv}_δ, and relation (1) gives us (iii). Moreover, for any $X_Z \in -\partial V^{mv}_\delta(Z)$, we may calculate

$$V^{mv}_\delta(Z) = U^{mv}_\delta(X_Z) - \mathbf{E}[ZX_Z] = \mathbf{E}\Big[-\frac{Z}{2\delta}(1-Z)\Big] - \delta Var\Big(-\frac{Z}{2\delta}\Big) = \frac{\mathrm{Var}(Z)}{4\delta}.$$

□

- **The Standard-Deviation Principle**

Theorem B.2. *Let $U^{sd}_\delta : L^\infty(\Omega,\mathcal{F},\mathbb{P}) \to \mathbb{R}$ be the standard-deviation principle defined in (5) and $V^{sd}_\delta : L^\infty(\Omega,\mathcal{F},\mathbb{P})^* \to [0,\infty]$ its convex conjugate function. Then*

(i) for any $X \in L^\infty$, $\partial U^{sd}_\delta(X) = \begin{cases} dom(V^{sd}_\delta), & \text{if } X = const, \\ \left\{1-\delta\frac{X-\boldsymbol{E}[X]}{\sqrt{Var(X)}}\right\}, & \text{otherwise,}\end{cases}$

(ii) for any $Z \in dom(V^{sd}_\delta)$, $\partial V^{sd}_\delta(Z) = \{c : c\in\mathbb{R}\} \cup \left\{X \in L^\infty : Z = 1+\delta\frac{X-\boldsymbol{E}[X]}{\sqrt{Var(X)}}\right\}.$

In particular, $\left\{1-\delta\frac{X-\mathbf{E}[X]}{\sqrt{\mathrm{Var}(X)}} : X\in L^\infty\setminus\{c: c\in\mathbb{R}\}\right\}$ is the minimal set C which allows us to represent U^{sd}_δ in the form $U^{sd}_\delta(X) = \min_{Z\in C}\mathbf{E}[ZX]$.

Proof. Since U^{sd}_δ is positively homogeneous, we know that V^{sd}_δ is equal to zero on its domain and, in particular, for any $Z \in \mathcal{Z}_\sigma$ we have

$$V^{sd}_\delta(Z) = \sup_{X\in L^\infty}\{\mathbf{E}[X(1-Z)] - \delta\|X-\mathbf{E}[X]\|_{L^2}\} \quad (25)$$
$$= 0 \vee \sup_{X\in L^\infty, X\neq \mathrm{const}}\{\mathbf{E}[X(1-Z)] - \delta\|X-\mathbf{E}[X]\|_{L^2}\}. \quad (26)$$

As in the case of the semi-deviation utility, we solve the last optimization problem by constructing the Lagrangian function L and imposing the optimality condition $\nabla L = 0$. In this way we obtain $Z = 1 - \delta(X - \mathbf{E}[X])/\sqrt{\mathrm{Var}(X)}$. Therefore, if $Z \in \mathcal{Z}_\sigma$ admits a payoff X s.t. Z can be written in this way, then $Z \in L^\infty$ with $\mathrm{Var}(Z) = \delta^2$, and the maximization over non-constant prospects yields zero as result. From this fact it follows that X solves the problem in (25) as well, that is, it belongs to $\arg\max_{\xi \in L^\infty}\{U_\delta^{sd}(\xi) - \mathbf{E}[Z\xi]\}$, which means $X \in -\partial V_\delta^{sd}(Z)$. In this case we have

$$\partial V_\delta^{sd}(Z) = \{d_1 Z + d_2 : d_1 \geq 0, d_2 \in \mathbb{R}\}.$$

On the other hand, for each $Z \in$dom(V_δ^{sd}) that cannot be written as $Z = 1 - \delta(X - \mathbf{E}[X])/\sqrt{\mathrm{Var}(X)}$, then $\partial V_\delta^{sd}(Z)$ just contains the constant payoffs. This shows (ii) and, by (1), statement (i) is also proved. $\square$

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