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EXISTENCE OF RADIAL SOLUTIONS FOR QUASILINEAR ELLIPTIC EQUATIONS WITH SINGULAR NONLINEARITIES

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Abstract. We prove the existence of radial solutions of the quasilinear elliptic equation $\mathrm{div}(A(|Du|)Du) + f(u) = 0$ in $\mathbb{R}^n$, $n > 1$, where f is either negative or positive for small $u > 0$, possibly singular at $u = 0$, and growths subcritically for large u. Our proofs use only elementary arguments based on a variational identity. No differentiability assumptions are made on f.

Contents

1. Introduction

Recently, for the p–Laplacian equation when $p > 1$, under general conditions for the nonlinearity f, besides other properties Tang in [18] proved existence of crossing radial solutions for f positive near at $u = 0$, while Gazzola, Serrin and Tang in [7] proved existence of radial ground states in $\mathbb{R}^n$ for f negative near at $u = 0$. Montefusco and Pucci in [9] extended the results of [7] to the general quasilinear case, considered here. The main purposes of this paper are to extend the existence results of [7] and [18] to general quasilinear elliptic problems, using a unified proof, and also to extend them and those of [9], introducing a new subcritical condition on f at infinity that, in canonical cases, interesting in applications, is more general than the subcritical condition used in [7], [18] and [9]. For instance, in the case $1 < p < n$, the two typical examples covered in our paper, but not in that of [7] and [18], are given by $f(u) = u^{p^*-1} + \log u$ and $f(u) = u^{p^*-1} + u^{q-1}$, with $1 < q < p^*$, and for $u \in \mathbb{R}^+$, where $p^* = np/(n-p)$.

In particular, we are interested in finding sufficient conditions for existence of radial ground states of the quasilinear elliptic equation

$$\mathrm{div}\{A(|Du|)Du\} + f(u) = 0 \quad \text{in } \mathbb{R}^n, \qquad n > 1, \tag{1.1}$$

when $f < 0$ near at $u = 0$. By a *ground state* we mean a non–negative non–trivial solution of (1.1) which tends to zero at infinity. Moreover, with the same technique, we are also able to prove the existence of a radial positive crossing solution of (1.1) in its maximal continuation interval where $u > 0$ and $u' < 0$, when $f > 0$ near at $u = 0$, already established in [18] for the p–Laplacian equation.

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In addition to the ground state problem, when $f < 0$ near at $u = 0$, we can also consider existence of non–trivial radial solutions of the homogeneous Dirichlet–Neumann free boundary problem

$$\begin{gathered}\operatorname{div}\{A(|Du|)Du\} + f(u) = 0 \qquad \text{in } B(0,R) \subset \mathbb{R}^n,\\ u > 0 \quad \text{in } B(0,R), \quad u = \frac{\partial u}{\partial \nu} = 0 \quad \text{on } \partial B(0,R).\end{gathered} \tag{1.2}$$

For $t > 0$ we set $\Phi(t) = tA(t)$ and assume that

($\Phi1$) Φ is of class $C^1(\mathbb{R}^+)$, $\mathbb{R}^+ := (0,\infty)$,

($\Phi2$) $\Phi'(t) > 0$ for $t > 0$, and $\Phi(t) \to 0$ as $t \to 0$,

($\Phi3$) There exists a positive number $1 < p \le n$ such that $t^{1-p}\Phi(t)$ is a non–decreasing function on $\mathbb{R}^+$.

Note that ($\Phi3$) is equivalent to

$$\Phi'(t) \ge \frac{p-1}{t}\Phi(t), \qquad t > 0. \tag{1.3}$$

As in [7], [18], [9], and more generally in natural existence settings, we are concerned with subcritical nonlinearities f since $1 < p \le n$. Specific hypotheses are given in the statements of the main Theorems 5.1–5.4 below. Throughout the paper f is assumed of type

$$f \in C(\mathbb{R}^+) \cap L^1[0,1]. \tag{$f1$}$$

Condition ($f1$) was first studied in [16] for the uniqueness problem, but without suitable attention to the difficulties attendant on this type of singularity at $u = 0$. Only recently in [6] a careful definition of solution for (1.1) under a singularity of type ($f1$) was given, in order to avoid the undefined nature of $f(u)$ at $u = 0$. We shall adopt that definition.

Moreover, putting $F(u) = \int_0^u f(v)dv$, in the main existence theorems we assume also that either

($f3$) there exists $b > 0$ such that $F(u) < 0$ for $0 < u < b$, $F(b) = 0$ and $f(b) > 0$,

as in [7] and [9], or

($f4$) there exists $c > 0$, possibly infinite, such that $f(u) > 0$ for $0 < u < c$,

as in [18].

The behavior of f near 0 is of crucial importance for the existence results given in Sections 5. We shall identify two mutually exclusive situations:

Regular case: f can be extended by continuity at $u = 0$, with $f(0) = 0$;

Singular case: f cannot be extended by continuity at $u = 0$, with $f(0) = 0$.

For a complete discussion on the wide background and literature concerning related previous results, including as well the classical scalar field equation and the regular case with $f(0) = 0$, we refer the reader to [5] and [7] and the references therein.

The paper is organized as follows: in Section 2 some preliminary qualitative properties for solutions of (1.1) and (1.2) are given, including a necessary and sufficient condition in order that a solution of (1.1) has compact support. In Section 3 we present and summarize the main properties of solutions of the corresponding initial value problem. Section 4 is devoted to prove the existence of crossing solutions and also to show the connections between the subcritical growth condition

($Q1$) The function $Q(v) = pnF(v) - (n-p)vf(v)$, $v \in \mathbb{R}^+$, is locally bounded near $v = 0$ and there exist $\mu > d$ and $\lambda \in (0,1)$ such that $Q(v) \ge 0$ for all $v \ge \mu$ and

$$\limsup_{v\to\infty} Q(\lambda_1 v)\left[\frac{v^{p-1}}{f(\lambda_2 v)}\right]^{n/p} = \infty \qquad \text{for all } \lambda_1 \text{ and } \lambda_2 \text{ in } [\lambda, 1],$$

used in [18] and [9], and the new condition

(Q) The function $Q(v) = pnF(v) - (n-p)vf(v)$, $v \in \mathbb{R}^+$, is locally bounded near $v = 0$ and there exist $\mu > d$ and $\lambda \in (0,1)$ such that $\quad Q(v) \geq 0$ for all $v \geq \mu \quad$ and

$$\limsup_{v\to\infty} Q(\lambda_1 v)[v^{p+1} f(\lambda_2 v)]^{n/p} = \infty \quad \text{for every} \quad \lambda_1,\, \lambda_2 \in [\lambda, 1],$$

introduced in this paper, where in both conditions $d = b$ under $(f3)$ and $d = 0$ under $(f4)$. Section 4 ends with some remarks and examples proving the independence of the two different growth hypotheses (Q) and $(Q1)$, and the main fact that, under $(f1)$ and $\liminf_{u\to\infty} f(u) = k_0 > 0$, condition $(Q1)$ is stronger than (Q), see Proposition 4.3. Finally, in Section 5, we give some existence results, also using the further assumption

$$k = \inf_{t>0} \frac{\Phi(t)}{t^{p-1}} > 0. \tag{$\Phi4$}$$

In particular: Theorem 5.3 is the analogue of Theorem 4.1 of [9] and Theorem 5.4 extends Theorem 5.1 of [18]. While Theorems 5.1–5.2 give the same results under the new condition (Q) and without assuming $(\Phi4)$.

2. General theory and behavior of solutions

We consider the quasilinear singular elliptic problem

$$\begin{aligned} &\operatorname{div}\{A(|Du|)Du\} + f(u) = 0 \qquad \text{in } \mathbb{R}^n, \qquad n > 1,\\ &\qquad u \geq 0, \qquad u \not\equiv 0 \qquad \text{in } \mathbb{R}^n, \end{aligned} \tag{2.1}$$

under the following main structural assumptions. For $t > 0$ we set $\Phi(t) = tA(t)$, as said in the Introduction, and assume the validity of $(\Phi1)$–$(\Phi3)$ throughout the paper. Note that Φ can be extended by continuity at $t = 0$ by $(\Phi1)$ and $(\Phi2)$, that is $\Phi \in C(\mathbb{R}_0^+) \cap C^1(\mathbb{R}^+)$, with $\Phi(0) = 0$. We introduce the function

$$H(t) = t\Phi(t) - \int_0^t \Phi(s)ds = \int_0^t s\Phi'(s)ds, \qquad t > 0. \tag{2.2}$$

Therefore, H is strictly increasing on $\mathbb{R}_0^+$, with $H(0) = 0$. Moreover, by $(\Phi3)$, that is (1.3), it is also evident that

$$\frac{p-1}{p} B(t) \leq H(t) < B(t) \qquad \text{for all } t > 0, \tag{2.3}$$

where

$$B(t) = t\Phi(t), \qquad t \geq 0, \tag{2.4}$$

is strictly increasing on $\mathbb{R}_0^+$, with $B(0) = 0$, by $(\Phi2)$, and moreover $t^{-p}B$ is non–decreasing on $\mathbb{R}^+$ by $(\Phi3)$. Clearly $\Phi(t) \to \infty$ as $t \to \infty$ by $(\Phi3)$ and in turn both $H(t) \to \infty$ and $H(t)/\Phi(t) \to \infty$ as $t \to \infty$ by (2.3).

Throughout the paper we also assume that f satisfies $(f1)$, so that it is clear that $F(u) = \int_0^u f(v)dv$ is well defined and is of class $C(\mathbb{R}_0^+) \cap C^1(\mathbb{R}^+)$, with $F(0) = 0$.

Since (2.1) is possibly singular when $u = 0$, it is necessary to carefully define the meaning to be assigned to solutions of (2.1), and in analogy with [16] and [6], we introduce the following:

Definition. *A semi–regular non–negative (weak) radial solution* u of (2.1) is a non–trivial non–negative radial function of class $C^1(\mathbb{R}^n \setminus \{0\})$, which is a classical distribution solution of (2.1) in the open (support) set $\Omega = \{x \in \mathbb{R}^n \setminus \{0\} \ : \ u(x) > 0\}$, and is bounded near $x = 0$.

Of course non–negative semi–regular radial solutions are then of class $C^1(\mathbb{R}^+)$, bounded near $r = 0$, and we shall see that satisfy

$$[r^{n-1}A(|u'|)u']' + r^{n-1}f(u) = 0 \qquad \text{in } J = \{r > 0 \ : \ u(r) > 0\}. \tag{2.5}$$

As in [6] it will first be shown that the definition of non–negative *semi–regular* radial solution is compatible with that of classical solution in the regular case, namely when f is continuous in $\mathbb{R}_0^+$ with $f(0) = 0$.

Proposition 2.1. *Let $u = u(|x|)$ be a non–negative semi–regular radial solution of* (2.1)*, where $f \in C(\mathbb{R}_0^+)$, with $f(0) = 0$. Then u is a classical C^1 solution of* (2.5) *in $\mathbb{R}^+$.*

Proof. If $J = \mathbb{R}^+$ there is nothing to prove. Otherwise let J' be any component of $J = \{r > 0 \ : \ u(r) > 0\}$, let r, $r' \in J'$ and $r_0 < r_1$ be the endpoints of J'. Then using standard distribution arguments we get

$$r^{n-1}A(|u'(r)|)u'(r) - (r')^{n-1}A(|u'(r')|)u'(r') = -\int_{r'}^{r} s^{n-1}f(u(s))ds. \tag{2.6}$$

Let $r' \to r_1$ and observe that necessarily $u(r_1) = u'(r_1) = 0$ by definition of a semi–regular non–negative solution. This gives

$$r^{n-1}A(|u'(r)|)u'(r) = -\int_{r_1}^{r} s^{n-1}f(u(s))ds. \tag{2.7}$$

Moreover, when $r_0 > 0$ we may also let $r \to r_0$ in (2.7) and thus obtain

$$\int_{r_0}^{r_1} s^{n-1}f(u(s))ds = 0. \tag{2.8}$$

Now let $\bar{r}$ be a *fixed* point of $\mathbb{R}^+ \setminus J$. Since $f(0) = 0$ it is easy to see that

$$\int_{I(r_1,\bar{r})} s^{n-1}f(u(s))ds = \sum_{J_i \subset I(r_1,\bar{r})} \int_{J_i} s^{n-1}f(u(s))ds = 0, \tag{2.9}$$

where $I(r_1, \bar{r})$ is the interval with endpoints r_1, $\bar{r}$; the sum is taken over all the components J_i of J contained in $I(r_1, \bar{r})$; and (2.8) is used at the second step.

Finally from (2.7) and (2.9) one gets

$$r^{n-1}A(|u'(r)|)u'(r) = -\int_{\bar{r}}^{r} s^{n-1}f(u(s))ds. \tag{2.10}$$

Here r can be any point in J and also by the computation of (2.9) it is clear that (2.10) is also correct if $r \in \mathbb{R}^+ \setminus J$. That is (2.10) holds for all $r \in \mathbb{R}^+$. On the other hand, both sides of (2.10) are continuously differentiable, so u satisfies (2.5) in $\mathbb{R}^+$, which was to be proved. $\square$

In analogy with [16] and [6] we give

Proposition 2.2. *Let $u = u(|x|)$ be a non–negative semi–regular radial solution of* (2.1)*, with*

$$\liminf_{r \to 0^+} u(r) = \alpha > 0. \tag{2.11}$$

Then $A(|u'|)u' \in C^1(J)$ and u is a solution in J of (2.5)*. Moreover,*

$$\lim_{r \to 0^+} u'(r) = 0 \quad \text{and} \quad \lim_{r \to 0^+} u(r) = \alpha. \tag{2.12}$$

Proof. Let u be a non–negative semi–regular radial distribution solution of (2.1), so that u is in particular of class $C^1(\mathbb{R}^n \setminus \{0\})$. It is clear that u solves (2.5) in the sense of distributions in J. By $(f1)$ we have $f \circ u \in C(J)$. Hence, as in (2.6),

$$r^{n-1}A(|u'(r)|)u'(r) - r_0^{n-1}A(|u'(r_0)|)u'(r_0) = -\int_{r_0}^{r} s^{n-1}f(u(s))ds, \tag{2.13}$$

for any interval $[r_0, r] \subset J$. Thus $A(|u'|)u'$ is actually of class $C^1(J)$, and (2.5) holds in J.

To obtain (2.12), choose $\varepsilon > 0$ so small that $f \circ u \in L^\infty[0,\varepsilon]$; this can be done since u is positive and bounded near 0 by definition and (2.11), and also by $(f1)$. Then, letting $r_0 \to 0^+$ in (2.13), we see that

$$\lim_{r_0 \to 0^+} r_0^{n-1} A(|u'(r_0)|)u'(r_0) = \ell, \tag{2.14}$$

where ℓ is a finite number. We claim that $\ell = 0$. Assume for contradiction that $\ell \neq 0$. Then by (2.14) and $(\Phi 3)$

$$|u'(r_0)| = \Phi^{-1}\left(|\ell| r_0^{-(n-1)}\right)[1+o(1)] \geq \left[|\ell|^{1/(p-1)}\Phi^{-1}(1)\right] r_0^{-(n-1)/(p-1)}[1+o(1)]$$

as $r_0 \to 0^+$ and so u cannot be bounded near 0, since $1 < p \leq n$, which is the required contradiction. Hence $\ell = 0$ in (2.14) and the claim is proved.

Now, letting $r_0 \to 0^+$ in (2.13), we obtain

$$r^{n-1}A(|u'(r)|)u'(r) = -\int_0^r s^{n-1} f(u(s))ds, \qquad 0 < r < \varepsilon,$$

that is, $\Phi(|u'(r)|) \leq Cr$, where $C > 0$ is an appropriate constant, since $f \circ u \in L^\infty[0,\varepsilon]$ by (2.11) and $(f1)$. Hence (2.12) immediately follows. $\square$

Remark. If Φ satisfies also condition $(\Phi 4)$, given in the Introduction, then it is easily seen that (2.12) holds in the stronger form

$$u'(r) = O\left(r^{1/(p-1)}\right) \quad \text{and} \quad u(r) - \alpha = O\left(r^{p/(p-1)}\right) \qquad \text{as } r \to 0^+. \tag{2.15}$$

Let u be a fixed semi–regular non–negative radial solution of either (2.1) or (1.2). In order to unify the notation we shall define $R = \infty$ for (2.1) and clearly $R < \infty$ is the positive radius given already in (1.2). By Proposition 2.2 then u is a non–negative classical solution of (2.5) in $J \subset (0,R)$ such that

$$\begin{gathered} u \in C^1[0,R), \qquad A(|u'|)u' \in C^1(J), \\ u(0) = \alpha > 0, \qquad u'(0) = 0, \qquad u \geq 0. \end{gathered} \tag{2.16}$$

It is useful to define $w(r) = A(|u'(r)|)u'(r)$.

Lemma 2.3. *The function w is of class $C^1[0,R)$ and is a solution of*

$$(r^{n-1}w)' + r^{n-1}f(u) = 0 \qquad \text{on } J. \tag{2.17}$$

Moreover, denoting by r_0 the first zero of u in $(0,R)$, if any, or otherwise $r_0 = R$, we have

$$w(r) = -\frac{1}{r^{n-1}}\int_0^r s^{n-1}f(u(s))ds, \qquad 0 < r < r_0, \tag{2.18}$$

$$w(0) = 0, \qquad w'(0) = -\frac{f(\alpha)}{n}. \tag{2.19}$$

Finally, putting $\rho(r) = |u'(r)|$, there holds

$$\lim_{r\to 0^+} \frac{B(\rho(r))}{r} = 0 \quad \text{and} \quad \frac{B(\rho)}{r} \in C[0,R). \tag{2.20}$$

Proof. Of course $w(0) = 0$, since $u'(0) = 0$ by (2.16); the function w is a solution of (2.17) by (2.5). Integrating over $[0,r]$, as long as $r < r_0$, we get

$$r^{n-1}w(r) = -\int_0^r s^{n-1}f(u(s))ds,$$

so (2.18) holds. From (2.16) we have

$$w'(r) = -f(u(r)) + \frac{n-1}{r^n}\int_0^r s^{n-1}f(u(s))ds,$$

and by L'Hôpital's rule (2.19), as well as the C^1 regularity of w at $r = 0$, follows at once.

Finally,

$$\frac{B(\rho(r))}{r} = u'(r) \cdot \frac{w(r)}{r},$$

and (2.20) follows at once by (2.19) and (2.16). $\square$

Corollary 2.4. *If $u'(r) \neq 0$ at some r, with $0 < r < r_0$, then u'' exists at this point and satisfies* (2.5) *in the form*

$$\Phi'(\rho)u'' - \frac{n-1}{r}\Phi(\rho) + f(u) = 0, \qquad \rho = |u'|. \tag{2.21}$$

Proof. By (2.18) of Lemma 2.3 and the fact that $u'(r) \neq 0$ we have by (Φ2)

$$|u'(r)| = \Phi^{-1}\left(\left|\int_0^r \left(\frac{s}{r}\right)^{n-1} f(u(s))ds\right|\right),$$

since $\Phi(\infty) = \infty$ by (Φ3). Now the integral is not zero, so that the function on the right hand side is differentiable at r by (Φ1). Hence u'' exists at r and from (2.5) we get exactly (2.21), since $|u'(r)| = \rho(r) > 0$. $\square$

A natural energy function associated to semi–regular non–negative radial solutions u of (2.1) is given by

$$E(r) = H(\rho(r)) + F(u(r)), \qquad \rho = |u'|. \tag{2.22}$$

Lemma 2.5. *The energy function E is of class $C^1(J)$, with*

$$E'(r) = -\frac{n-1}{r}B(\rho(r)) \qquad \textit{in } J. \tag{2.23}$$

Proof. Obviously, by ($f1$) and (2.16)

$$\frac{dF(u(r))}{dr} = f(u(r))u'(r),$$

this formula being valid only when $u(r) > 0$, namely in J. Moreover, by (2.2) and (Φ2),

$$H(\rho) = \int_0^\rho t d\Phi(t) = \int_0^{\Phi(\rho)} \Phi^{-1}(\omega)d\omega$$

and, since $\Phi(\rho(r)) = [\operatorname{sgn} u'(r)]w(r)$, we get

$$\frac{dH(\rho(r))}{dr} = \rho(r)[\operatorname{sgn} u'(r)]w'(r) = u'(r)w'(r).$$

Therefore, by (2.17), on J

$$E'(r) = u'(r)\Big[w'(r) + f(u(r))\Big] = -u'(r)\frac{n-1}{r}w(r)$$

and (2.23) follows at once. $\square$

Let now u be either a fixed semi–regular non–negative radial ground states of the problem (2.1), namely a semi–regular non–negative radial solution of (2.1) in the sense above such that

$$\lim_{|x|\to\infty} u(x) = 0, \tag{2.24}$$

or a fixed semi–regular non–negative radial solution of the corresponding homogeneous Dirichlet–Neumann free boundary problem (1.2).

With the respective end conditions at $R = \infty$ in (2.1), (2.24) and at $R < \infty$ in (1.2), the problems (2.1), (2.24) and (1.2) can be unified again into the single statement (2.5) and (2.16) in $J \subset (0, R)$.

Theorem 2.6. *If $u(t_0) = 0$ for some $t_0 > 0$, then $u \equiv 0$ on $[t_0, R)$.*

Proof. Since $u \geq 0$, clearly $u'(t_0) = 0$. Hence $E(t_0) = 0$ by (2.22), (2.2) and $(f1)$. Assume for contradiction that there is t_1, with $t_0 < t_1 \leq R$ such that again $u(t_1) = 0$ and $u(r) > 0$ in (t_0, t_1). Then $(t_0, t_1) \subset J$ and $E' \leq 0$ in (t_0, t_1) by (2.23). Clearly $E(t_1) = 0$ when $t_1 < R$, and we claim that $E(r) \to 0$ as $r \to t_1 = R$. Indeed, if $R < \infty$, the claim is obvious by the boundary conditions of (1.2); while if $R = \infty$, since $u(r) \to 0$ as $r \to \infty$ by (2.24), then $F(u(r)) \to 0$ by $(f1)$, and so $E(r)$ decreases to a finite non–negative limit as $r \to \infty$ by (2.22) and (2.2). Consequently, $u'(r) \to \text{limit} = 0$ as $r \to \infty$, since $u(r) \to 0$ as $r \to \infty$, and H is strictly increasing and positive by (2.2). Therefore $E(r) \to 0$ as $r \to \infty$, as claimed. Hence $E \equiv E' \equiv 0$ in (t_0, t_1) and so $B(\rho) \equiv 0$ on $[t_0, t_1)$, by (2.23) namely $u' \equiv 0$ on $[t_0, t_1)$ by (2.4). This contradiction completes the proof. □

By Theorem 2.6 it follows that *any solution of* (2.16) *and* (2.5)*, with the given end conditions* (2.24) *and those in* (1.2)*, has as its (open) support set J exactly an initial interval* $(0, r_0)$*, with* $r_0 \leq R$. In turn, one deduces that actually $E \in C^1[0, R)$, and that (2.23) holds in the entire maximal interval $[0, R)$. Therefore for any $0 \leq s_0 < r < R$ we have

$$E(r) - E(s_0) = -\int_{s_0}^{r} \frac{n-1}{s} B(\rho(s))ds. \tag{2.25}$$

Clearly $E(0) = F(\alpha)$ by (2.22) and (2.16). Thus, letting $s_0 \to 0^+$ in (2.25), we obtain

$$E(r) = F(\alpha) - (n-1)\int_{0}^{r} \frac{B(\rho(s))}{s} ds, \quad 0 \leq r < R. \tag{2.26}$$

As in the proof of Theorem 2.6, if $R = \infty$ and $u(r) \to 0$ as $r \to \infty$, then $u'(r) \to 0$ and $E(r) \to 0$ as $r \to \infty$. Hence, by (2.24) or (1.2), the non–negative continuous function $B(\rho(s))/s$ is also integrable on $[0, R)$, $R \leq \infty$, with

$$\int_{0}^{R} \frac{n-1}{s} B(\rho(s))ds = F(\alpha). \tag{2.27}$$

In summary, *a semi–regular non–negative radial ground state of* (2.1), (2.24)*, or a semi–regular non–negative radial solution of the corresponding free boundary problem* (1.2)*, has the property that*

$$u(0) = \alpha > 0, \quad u'(0) = 0, \quad u(R) = u'(R) = 0, \tag{2.28}$$

where respectively $R = \infty$ or $R < \infty$. Furthermore, by (2.26) *and* (2.27)

$$E(r) = \int_{r}^{R} \frac{n-1}{s} B(\rho(s))ds \geq 0, \tag{2.29}$$

and clearly also

$$E(0) = F(\alpha) = \int_{0}^{R} \frac{n-1}{s} B(\rho(s))ds > 0. \tag{2.30}$$

Lemma 2.7. *If $s_0 \geq 0$ is a critical point of u, with $u(s_0) > 0$, then either $u(r) \leq u(s_0)$ for $r > s_0$ and $f(u(s_0)) \geq 0$, or $u(r) \geq u(s_0)$ for $r > s_0$ and $f(u(s_0)) \leq 0$.*

Proof. Let $s_0 \geq 0$ be a critical point of u. Assume for contradiction that there are two points t_1, $t_2 > s_0$ such that $u(t_1) > u(s_0)$ and $u(t_2) < u(s_0)$. Then, there is r in the interval, with endpoints t_1 and t_2, such that $u(r) = u(s_0)$ and u is not constant on $[s_0, r]$. Now by (2.25) and (2.22)

$$H(\rho(r)) + \int_{s_0}^{r} \frac{n-1}{s} B(\rho(s))ds = 0$$

and both terms are non–negative by $(\Phi 2)$, (2.2) and (2.4). Thus in particular $B(\rho) \equiv 0$ on $[s_0, r]$, so $u' \equiv 0$ on $[s_0, r]$ by (2.4), which is impossible. Hence we have two cases: either $u(r) \leq u(s_0)$ for $r > s_0$, or $u(r) \geq u(s_0)$ for $r > s_0$.

In the first case, since $u'(s_0) = 0$, then $w(s_0) = 0$ by ($\Phi 2$), and by (2.17) and the regularity of w established in Lemma 2.3 at $r = s_0$ we have

$$f(u(s_0)) = -w'(s_0) \geq 0.$$

Indeed, otherwise $w'(s_0) > 0$, and so there is $t_3 > s_0$ such that $w'(r) > 0$ on $[s_0, t_3]$; in turn $w(r) > w(s_0) = 0$ and $u'(r) > 0$ on $[s_0, t_3]$, which gives $u(r) > u(s_0)$ on $[s_0, t_3]$, which contradicts (2.28).

In the same way, in the second case, it follows that $f(u(s_0)) \leq 0$, completing the proof of the lemma. □

It is convenient to introduce the following further condition on f.

($f2$) There exists $\gamma > 0$, possibly infinite, such that $F(u) \leq 0$ whenever both $f(u) = 0$ and $u \in (0, \gamma)$.

Clearly if ($f2$) holds for some $\gamma > 0$, then it continues to hold for all $\gamma' \in (0, \gamma)$. Consequently there exists a maximal γ, possibly infinite, for which ($f2$) is valid. Without loss of generality we can assume that γ in ($f2$) is maximal.

One can see also from ($f2$) that if $F(u_0) \leq 0$ at some point u_0, $0 < u_0 < \gamma$, then $F(u) \leq 0$ for all $u \in [0, u_0]$. Conversely, it is evident that if $F(u) \leq 0$ on some interval $[0, u_0]$, then the maximal $\gamma \geq u_0$.

Furthermore if either ($f3$) or ($f4$) holds, where ($f3$) and ($f4$) are the main qualitative conditions of f given in the Introduction, then ($f2$) is satisfied, with

$$\gamma = \sup\{v > d \ : \ f(u) > 0 \text{ for } u \in (d, v)\}, \quad \text{where} \quad d = \begin{cases} b, & \text{if } (f3) \text{ holds}, \\ 0, & \text{if } (f4) \text{ holds}. \end{cases} \tag{2.31}$$

In this case $\gamma > d \geq 0$ and $\gamma = \infty$ if $f(u) > 0$ for all $u > d$, while $f(\gamma) = 0$ if $\gamma < \infty$.

Proposition 2.8. *Let u be a semi–regular non–negative radial ground state of* (2.1), (2.24), *or a semi–regular non–negative radial solution of the corresponding free boundary problem* (1.2), *so that* (2.28) *holds. Then $r = 0$ is a maximum of u and $u' \leq 0$ on $[0, R)$; furthermore $f(\alpha) \geq 0$ and $F(\alpha) > 0$.*

If ($f2$) *holds, then also*

(i) $f(\alpha) > 0$, *when* $0 < \alpha < \gamma$,

(ii) $u'(r) < 0$ *when* $r > 0$ *and* $0 < u(r) < \gamma$.

Proof. By Lemma 2.7 and the condition $u'(0) = 0$ one sees that $u(r) \leq u(0)$ for $r > 0$, since otherwise $u(r) \geq u(0) = \alpha > 0$ for all $r > 0$, and so u cannot approach 0 as $r \to R^-$, contradicting (2.28). The fact that $f(\alpha) \geq 0$ similarly follows from Lemma 2.7, and clearly $F(\alpha) > 0$ by (2.30).

Next assume for contradiction that $u'(s_0) > 0$ for some $s_0 > 0$. Since $u(s_0) \leq u(0)$, as shown above, there is a minimum s in $(0, s_0)$, with $u(s) < u(s_0)$, and so $u(r) \geq u(s)$ for $r > s$ by Lemma 2.7. If $u(s) > 0$, then $u(r)$ cannot approach 0 as $r \to R^-$, contradicting (2.28). Therefore $u(s) = 0$ with $s > 0$, and by Theorem 2.6 we get $u \equiv 0$ on $[s, R)$: thus $u'(s_0) = 0$, which is again a contradiction. Hence $u'(r) \leq 0$ on $[0, R)$.

To show *(i)* it is enough to observe that if $f(\alpha) = 0$ and $0 < \alpha < \gamma$, then $F(\alpha) \leq 0$ by assumption ($f2$). This is impossible by (2.30), proving *(i)*.

To obtain *(ii)*, assume now for contradiction that there is a point $s_0 > 0$ such that $u'(s_0) = 0$ and $0 < u(s_0) < \gamma$. Since $u'(r) \leq 0$ for $r \geq 0$, then both $u(r) \geq u(s_0)$ for $0 \leq r < s_0$ and $u(r) \leq u(s_0)$ for $s_0 < r < R$. Of course, $w(s_0) = 0$ by ($\Phi 2$). We claim that also $w'(s_0) = 0$. Indeed, if $w'(s_0) > 0$, then w would be strictly increasing at s_0, namely u' would change sign at s_0, which is impossible since $u'(r) \leq 0$ on $[0, R)$. Analogously, the case $w'(s_0) < 0$ also cannot occur.

Since w is a $C^1[0,R)$ solution of (2.17) by Lemma 2.3, we get $f(u(s_0)) = 0$. Also $0 < u(s_0) < \gamma$, so that $F(u(s_0)) \le 0$ by $(f2)$. Hence by (2.29) and (2.22), with $r = s_0$,

$$0 \le \int_{s_0}^{R} \frac{n-1}{s} B(\rho(s))ds = E(s_0) = F(u(s_0)) \le 0,$$

which implies $u' \equiv 0$ on $[s_0, R)$ by $(\Phi2)$ and (2.4). Thus $u(r) \equiv u(s_0) > 0$ for $s_0 \le r < R$, again contradicting (2.28). This completes the proof of *(ii)*. □

The next result gives a necessary and a sufficient condition for a semi–regular non–negative radial ground state to have compact support.

Theorem 2.9. *Let u be a semi–regular non–negative radial ground state of* (2.1), (2.24), *with $u(0) = \alpha > 0$.*

(i) If $F(u) \le 0$ for all values $0 < u < \delta$, for some $\delta > 0$, and

$$\int_{0+} \frac{du}{H^{-1}(|F(u)|)} < \infty, \tag{2.32}$$

then u has compact support.

(ii) Conversely, assume there exists $\delta > 0$ and a non–decreasing function $G : [0, \delta) \to \mathbb{R}$, with $G(0) = 0$, such that $|F(u)| \le G(u)$ for all $u \in [0, \delta)$. If u has compact support, then

$$\int_{0+} \frac{du}{H^{-1}(G(u))} < \infty. \tag{2.33}$$

Proof. Let u be a semi–regular non–negative radial ground state as in the theorem, so $R = \infty$.

(i) Suppose (2.32) holds. We denote by $r_\delta > 0$ any point such that $0 \le u(r) < \delta$ on (r_δ, ∞). Condition $(f2)$ clearly holds with $\gamma \ge \delta$, as noted above. Thus by Proposition 2.8 *(ii)* we have $u'(r) < 0$ for all $r \in (r_\delta, \infty)$ for which $u(r) > 0$. Hence by Theorem 2.6 either $u \equiv 0$ for all r sufficiently large, or $u > 0$ and $u' < 0$ on (r_δ, ∞). In the first case we are done. Otherwise, denoting by $r = r(u)$ the inverse function on (r_δ, ∞), by (2.29) and (2.22), we get on (r_δ, ∞)

$$H(\rho(r)) = -F(u(r)) + \int_r^\infty \frac{n-1}{s} B(\rho(s))ds > -F(u(r)),$$

or $u'(r) < -H^{-1}(|F(u(r))|)$ on (r_δ, ∞) by the assumption that $F(u) \le 0$ on $[0, \delta)$ and the fact that $H(\infty) = \infty$ by $(\Phi3)$. That is, writing $r = r(u)$ and putting $\varepsilon = u(r_\delta)$, we have

$$\frac{1}{r'(u)} < -H^{-1}(|F(u)|) \quad \text{for} \quad u \in u((r_\delta, \infty)) = (0, \varepsilon),$$

since $u(r) \to 0$ as $r \to \infty$. By integration over $(u(r), \varepsilon)$, $r > r_\delta$,

$$\int_{u(r)}^{\varepsilon} \frac{du}{H^{-1}(|F(u)|)} > -\int_{u(r)}^{u(r_\delta)} r'(u)du = r - r_\delta.$$

Hence, letting $r \to \infty$, there results

$$\int_0^{\varepsilon} \frac{du}{H^{-1}(|F(u)|)} = \infty.$$

This contradicts (2.32) and completes the proof of part *(i)* of the theorem.

(ii) Let u have compact support. Then by Theorem 2.6 and the first part of Proposition 2.8 there is $r_0 > 0$ such that $u'(r) \le 0$ and $0 < u(r) \le u(0) = \alpha$ on $(0, r_0)$, while $u \equiv 0$ on $[r_0, \infty)$. Let $r_\delta \in (0, r_0)$ be some fixed point such that $0 < u(r) < \delta$ on (r_δ, r_0). By (2.25) and (2.22) for $0 < r_\delta < r < r_0$ we have $E(r_0) = 0$ and

$$H(\rho(r)) = -F(u(r)) + \int_r^{r_0} \frac{n-1}{s} B(\rho(s))ds \le G(u(r)) + c_1 \int_r^{r_0} H(\rho(s))ds,$$

by assumption, (Φ3) and (2.3), with $c_1 = p'(n-1)/r_\delta$. Applying Gronwall's inequality to $e^{-c_1(r_\delta - r)} \int_r^{r_0} H(\rho(s))ds$, we obtain

$$H(\rho(r)) \le G(u(r)) + c_1 \int_r^{r_0} G(u(s)) e^{c_1(s-r)} ds.$$

Now $G(u(r))$ is non–increasing on (r_δ, r_0), since G is non–decreasing by assumption and u is non–increasing on (r_δ, r_0). Hence

$$H(\rho(r)) \le CG(u(r)), \quad \text{with} \quad C = e^{c_1(r_0 - r_\delta)} > 1.$$

Therefore, since $u'(r) \le 0$,

$$-u'(r) \le H^{-1}(CG(u(r))) \quad \text{on} \quad (r_\delta, r_0).$$

We can now apply Lemma 1.3.3 of [5] since the main assumption *(ii)* in [5] is clearly satisfied. Indeed, here $p \int_0^t \Phi(s)ds \le B(t)$ in the entire $\mathbb{R}_0^+$ by (2.4), (2.3) and (2.2). Consequently by Lemma 1.3.3 of [5] there is a constant $D > 0$ such that

$$H^{-1}(Ct) \le DH^{-1}(t) \quad \text{for all } 0 = G(0) < t < G(u_\delta),$$

where $u_\delta = u(r_\delta) \le \delta$. Hence

$$-u'(r) \le DH^{-1}(G(u(r))) \quad \text{on} \quad (r_\delta, r_0).$$

Integrating on $[s_0, r]$, with $r_\delta < s_0 < r < r_0$, we get

$$\int_{u(r)}^{u(s_0)} \frac{du}{H^{-1}(G(u))} = -\int_{s_0}^{r} \frac{u'(s)}{H^{-1}(G(u(s)))} ds \le D(r - s_0).$$

Letting $r \to r_0^-$, this yields

$$\int_0^{u(s_0)} \frac{du}{H^{-1}(G(u))} \le D(r_0 - s_0),$$

that is (2.33) holds. This completes the proof of part *(ii)* of the theorem. □

As an immediate consequence of Theorem 2.9 we obtain

Corollary 2.10. *Let u be a semi–regular non–negative radial ground state of* (2.1), (2.24), *with $u(0) = \alpha > 0$, and assume $f \le 0$ on $(0, \delta)$, for some $\delta > 0$, and $f \not\equiv 0$.*

(i) Then $u(r) > 0$ for every $r > 0$ if and only if

$$\int_{0^+} \frac{du}{H^{-1}(|F(u)|)} = \infty. \tag{2.34}$$

(ii) In particular, $u(r) > 0$ for every $r > 0$ if

$$\int_{0^+} \frac{du}{|F(u)|^{1/p}} = \infty. \tag{2.35}$$

(iii) Furthermore, if Φ satisfies also

$$k = \inf_{t>0} \frac{\Phi(t)}{t^{p-1}} > 0, \tag{Φ4}$$

then $u(r) > 0$ for every $r > 0$ if and only if

$$\int_{0^+} \frac{du}{|F(u)|^{1/p}} = \infty.$$

Proof. Statement *(i)* is a direct consequence of Theorem 2.9, with $G = |F|$. By (Φ3), (2.4) and (2.3) it is clear that $H(t) \le B(t) \le B(1)t^p$ for all $0 \le t \le 1$, with $B(1) > 0$. Hence $t \le H^{-1}(B(1)t^p)$ since $H(\infty) = \infty$, and so $H^{-1}(s) \ge cs^{1/p}$, where $c = [B(1)]^{-1/p}$, for $0 \le s \le B(1)$. Consequently the validity of (2.35) implies (2.34), and the proof of *(ii)* is complete.

Furthermore, if (Φ4) holds, by (Φ4) and (2.3) again, we also have $H(t) \geq (p-1)kt^p/p$ for all $t \geq 0$. Consequently $H^{-1}(s) \leq Cs^{1/p}$ for every $s \geq 0$, where $C = [(p-1)k/p]^{-1/p}$, and $H^{-1}(s) \geq cs^{1/p}$ for $0 \leq s \leq B(1)$, as shown above in part *(ii)*. Hence (2.35) is now equivalent to (2.34), and this proves *(iii)* by *(i)*. □

Remarks. To see that (Φ4) is not automatic under the assumption (Φ1)–(Φ3), note that $\Phi(t) = t^{p-1}(e^t - 1)$ verifies (Φ1)–(Φ3) but not (Φ4). Of course $\Phi(t) = t^{p-1}e^t$ satisfies (Φ1)-(Φ4), with $k = 1$.

Assume also (Φ4). If f is singular at $u = 0$, with $|f(u)| \sim u^q$ as $u \to 0^+$, then by $(f1)$ we must have $q \in (-1, 0)$. Hence $|F(u)|^{1/p} \sim u^{(q+1)/p}$ as $u \to 0^+$, and so $\int_{0^+} |F(u)|^{-1/p} du < \infty$ and every semi–regular non–negative radial ground state of (2.1), with $u(0) = \alpha > 0$, is compactly supported on $\mathbb{R}^+$ by Corollary 2.10 *(iii)*. While if f is regular at $u = 0$ and $q > 0$, then $\int_{0^+} |F(u)|^{-1/p} du < \infty$ when $0 < q < p-1$ and every semi–regular non–negative radial ground state of (2.1), with $u(0) = \alpha > 0$, is compactly supported on $\mathbb{R}^+$ by Corollary 2.10 *(iii)*.

If again $|f(u)| \sim u^q$ as $u \to 0^+$, with $q > -1$ by $(f1)$, then by Corollary 2.10 *(iii)* and Proposition 2.8 it holds that $u(r) > 0$, $u'(r) < 0$ for all $r \in \mathbb{R}^+$ if and only if $q \geq p-1$. In this case, since $p > 1$, this means that f is continuous at $u = 0$, with $f(0) = 0$, namely we are in the regular case.

For a more general wider discussion on the validity of the strong maximum and compact support principles for solutions, radial or not, of quasilinear singular elliptic inequalities, as well as on applications of these principles to variational problems on manifolds and to existence of radial dead cores, we refer to [17]. See also [14].

3. Properties of solutions of the corresponding initial value problem

Semi–regular non–negative radial solutions u of (2.1), (2.24), or of (1.2), are also solutions of the initial value problem

$$\begin{cases} [r^{n-1}\Phi(\rho(r))]' - r^{n-1}f(u(r)) = 0, & r > 0, \\ u(0) = \alpha > 0, \qquad u'(0) = 0, \end{cases} \tag{3.1}$$

by (2.28) and the first part of Proposition 2.8.

From now on, besides (Φ1)–(Φ3), we assume that either $(f3)$ or $(f4)$ holds, see the Introduction, so that, as noted above, $(f2)$ is also satisfied, with γ given by (2.31). Moreover, from now on we also assume that

$$f \in \mathrm{Lip}_{\mathrm{loc}}(0, \gamma). \tag{$f5$}$$

In this section we shall consider only solution of (3.1) under the further restriction

$$d < \alpha < \gamma, \tag{3.2}$$

where d and γ are given in (2.31).

Lemma 3.1. *Problem* (3.1) *and* (3.2) *has a unique classical solution u in a neighborhood of the origin. Moreover, $u'(r) < 0$ for r small and positive. The solution is unique as long as it exists and remains in $(0, \gamma)$.*

Proof. Local existence and uniqueness of solution of the initial value problem (3.1) and (3.2) follows from Proposition *A*4 of the Appendix of [5], since (Φ3) is equivalent to (1.3) and in turn for $0 < t < p-1$

$$\Phi'(t) \geq \Phi(t) > 0, \tag{3.3}$$

which is the main condition of Proposition *A*4 of [5], in the special case $\mu = 1$. Moreover by Proposition 2.8 *(i)* and *(ii)* we have $f(\alpha) > 0$ and $u'(r)$ is negative for small r. □

By Proposition 2.8, any semi–regular non–negative radial solution u of (2.1), (2.24), or of (1.2), satisfying (3.2), remains a classical solution of (3.1) as long as it remains in $(0, \alpha]$. Moreover, by Lemma 3.1 the unique local solution of (3.1) and (3.2) has the property that $u' < 0$ and $u < \alpha$ in some interval to the right of zero. We claim that the solution can be continued either for all $r > 0$ with $u(r) > 0$ and $u'(r) < 0$, or else reaches a first point r_α where either $u(r_\alpha) = 0$ and $u'(r_\alpha) \le 0$, or $u(r_\alpha) > 0$ and $u'(r_\alpha) = 0$. To prove this note first that by (2.25), with $s_0 = 0$, and (2.22), since $F > 0$ and increasing on (d, γ) by either $(f3)$ or $(f4)$, then

$$H(|u'(r)|) \le F(\alpha) - F(u(r)) \le F(\alpha) + \max_{v \in [0,d]} |F(v)| = M_\alpha \in \mathbb{R}^+,$$

by (2.30). Therefore, since $H(\infty) = \infty$ by $(\Phi3)$, we also obtain that $|u'(r)| \le H^{-1}(M_\alpha)$ as long as u exists and $0 < u(r) \le \alpha$. This shows the claim.

In what follows we assume that the solution u of (3.1) and (3.2) is continued exactly until a first point r_α is reached where either $u(r_\alpha) = 0$ and $u'(r_\alpha) \le 0$, or $u(r_\alpha) > 0$ and $u'(r_\alpha) = 0$. If no such a point occurs, we put $r_\alpha = \infty$.

We denote by J_α the maximal domain of continuation of any semi–regular non–negative radial solution u of (2.1), (2.24), or of (1.2), satisfying (3.2), where it is positive, namely $J_\alpha = (0, r_\alpha)$, r_α finite or not, is the maximal open interval of continuation under the restriction

$$u > 0, \qquad -\infty < u' < 0 \quad \text{in } J_\alpha. \tag{3.4}$$

As mentioned earlier, the main purpose of this paper is to extend to the general problem (2.1), (2.24) and to the corresponding free boundary problem (1.2), the recent existence results given by Gazzola, Serrin and Tang in [7], under $(Q1)$ and $(f3)$, and by Tang in [18], under $(Q1)$ and $(f4)$, for the p–Laplacian case, namely when $\Phi(t) = t^{p-1}$, $p > 1$, in $(\Phi1)$–$(\Phi3)$, to the general setting of [9] on Φ, under the slightly more general growth condition (Q), when the natural assumption $(f1)$ holds, and using a unified technique. We shall also completely generalize the existence results of [9].

Lemma 3.2. *Let u be a solution of* (3.1) *and* (3.2)*, and let $J_\alpha = (0, r_\alpha)$ be the corresponding maximal interval of definition in the sense of* (3.4)*.*

(i) The limit

$$\ell_\alpha = \lim_{r \to r_\alpha^-} u(r) \tag{3.5}$$

belongs to $[0, b)$ if $(f3)$ holds, and $\ell_\alpha = 0$ if $(f4)$ is satisfied. Moreover if $\ell_\alpha > 0$, then $u'(r_\alpha) = 0$, with r_α possibly infinite.

(ii) Let $\lambda > d$. If $\alpha > \lambda$, then there exists a unique value $\overline{r} = \overline{r}(\alpha) \in J_\alpha$ such that $u(\overline{r}) = \lambda$.

(iii) If $r_\alpha = \infty$, then $\lim_{r \to \infty} u'(r) = 0$.

Proof. (i) Clearly the limit in (3.5) exists and is non–negative, since u is decreasing and positive in J_α by (3.4), in particular $\ell_\alpha \in [0, \gamma)$ by (3.2).

Let first $(f3)$ hold. By contradiction, suppose $\ell_\alpha \in [b, \gamma)$. Then $b \le \ell_\alpha < u(r) < \alpha$ in J_α, and this implies $[r^{n-1}\Phi(\rho(r))]' > 0$ by (3.1) and (3.2), that is $r^{n-1}\Phi(\rho(r))$ is strictly increasing on J_α.

If r_α is finite and $u(r_\alpha) = \ell_\alpha \ge b > 0$, then $u'(r_\alpha) = 0$ by (3.4), and in turn $r^{n-1}\Phi(\rho(r))$ tends to 0 as $r \to r_\alpha^-$. On the other hand, $r^{n-1}\Phi(\rho(r))$ is 0 at $r = 0$. This is impossible since $r^{n-1}\Phi(\rho(r))$ is strictly increasing on J_α.

If $r_\alpha = \infty$, then $J_\alpha = \mathbb{R}^+$ and $E \in C^1(\mathbb{R}^+)$ is non–increasing in $\mathbb{R}^+$ by Lemma 2.5. Therefore by (2.22) also $H(\rho(r))$ admits finite limit at ∞, and in turn by (2.2) also $\rho(r)$ admits limit as $r \to \infty$. Consequently $u'(r) \to 0$ as $r \to \infty$. Now, rewriting the equation in

(3.1) in the equivalent form

$$[\Phi(\rho(r))]' + \frac{n-1}{r}\Phi(\rho(r)) - f(u(r)) = 0, \quad r \in J_\alpha = \mathbb{R}^+,$$

we obtain that $\lim_{r\to\infty}[\Phi(\rho(r))]' = f(\ell_\alpha) > 0$, since $\ell_\alpha \in [b,\gamma)$. This is impossible since $\Phi(\rho(r)) > 0$ on $\mathbb{R}^+$ and $\lim_{r\to\infty}\Phi(\rho(r)) = 0$ by (Φ2).

Let now (f4) hold. If we assume by contradiction that $\ell_\alpha > 0$, then, omitting b, we can repeat the above proof word by word and reach the desired conclusion.

If $\ell_\alpha > 0$ and $r_\alpha < \infty$, then $\lim_{r\to r_\alpha^-} u'(r) = u'(r_\alpha) = 0$ again by (3.4). If $\ell_\alpha > 0$ and $r_\alpha = \infty$, then by Lemma 2.5 and (2.22), as shown above, $\lim_{r\to\infty} u'(r) = 0$. This case can occur only when (f3) holds.

(ii) The claim follows easily, by definition of J_α, since u is strictly decreasing.

(iii) Let $r_\alpha = \infty$, then the claim follows word by word as in the proof of *(i)*, since $H(\rho(r))$ approaches a finite limit as $r \to \infty$ by (2.22).

□

Lemma 3.3. *Let u be a solution of* (3.1) *with maximal interval J_α and assume* (3.2)*. Then for any $t_0 \in J_\alpha$ and $\varepsilon > 0$, there exists $\delta > 0$ such that if v is a solution of* (3.1) *with $|u(0) - v(0)| < \delta$, then $v = v(r)$ is defined on $[0, t_0]$ and*

$$\sup_{[0,t_0]}\left\{|u(r) - v(r)| + |u'(r) - v'(r)|\right\} < \varepsilon.$$

Proof. The proof of Lemma 2.3 of [9] can be repeated since it was used only the fact that $f \in \mathrm{Lip}_{\mathrm{loc}}(0,\gamma)$ together with (f1). □

Lemma 3.4. (Ni–Pucci–Serrin) *Let u be a solution of* (3.1)*. Set*

$$Q(v) = pnF(v) - (n-p)vf(v), \quad v \in \mathbb{R}^+,$$

and

$$P(r) = (n-p)r^{n-1}u(r)u'(r)A(|u'(r)|) + pr^nE(r), \quad 0 < r < r_\alpha. \tag{3.6}$$

Then

$$P(r) \ge \int_0^r Q(u(s))s^{n-1}ds. \tag{3.7}$$

This follows at once by direct calculation from (2.5), (2.3) and (2.23). See the proof of Lemma 2.3 of [9].

4. Preparatory existence results

To establish existence of semi–regular non–negative radial solutions of (2.1), (2.24) or of (1.2), we follow the main ideas used in [5], [7], [18] and [9], and shall first prove existence theorems for the corresponding initial value problem (3.1), under condition (3.2).

Using the same notation of the previous section, we introduce the set

$$I^- = \{\alpha > d \ : \ r_\alpha < \infty, \quad \ell_\alpha = 0, \quad u'_\alpha(r_\alpha) < 0\}.$$

Lemma 4.1. *Let u be a semi–regular radial solution of* (3.1)*, under the restriction* (3.2)*, in its maximal continuation interval J_α. If $\alpha \notin I^-$, then*

$$r < \frac{Cu(r)[\overline{F} + F(u(r))]}{F(u(r))B^{-1}(p'[\overline{F} + F(u(r))])} \qquad \text{for every} \quad r \in J_\alpha, \tag{4.1}$$

where p' is the Hőlder conjugate of p,

$$C = (n-1)p', \qquad \overline{F} = \max_{v\in[0,d]} F^-(v), \tag{4.2}$$

B^{-1} is the inverse function of B given in (2.4)*, and* $F^- = \max\{-F,\ 0\}$.

Proof. Assume by contradiction that there is $\overline{r}$, with $0 < \overline{r} < r_\alpha$ such that

$$\overline{r} \geq \frac{CU[\overline{F} + F(U)]}{F(U)B^{-1}(p'[\overline{F} + F(U)])}, \tag{4.3}$$

where $U = u(\overline{r})$. Put

$$M = \sup_{[\overline{r}, r_\alpha)} \rho(r) = \rho(R_1),$$

where $R_1 \in [\overline{r}, r_\alpha)$ by the continuity of u' on J_α, (3.4), Lemma 3.2 *(i)* and *(iii)*, since $\alpha \notin I^-$. By (2.2), (2.22) and (2.25) at $s_0 = \overline{r}$ and $r = r_\alpha$, we have

$$\begin{aligned} F(U) < E(\overline{r}) &= E(r_\alpha) + (n-1)\int_{\overline{r}}^{r_\alpha} \frac{B(\rho(s))}{s} ds \leq -(n-1)\frac{\Phi(M)}{\overline{r}}\int_{\overline{r}}^{r_\alpha} u'(s)ds \\ &= \frac{(n-1)\Phi(M)U}{\overline{r}}, \end{aligned}$$

by (2.29), (2.4), (3.4) and ($\Phi 2$). Hence

$$F(U) < \frac{(n-1)\Phi(M)U}{\overline{r}}. \tag{4.4}$$

Similarly, by (2.22) and (2.25) at $s_0 = R_1$ and $r = r_\alpha$, we get

$$\begin{aligned} H(M) &= F(\ell_\alpha) - F(u(R_1)) + (n-1)\int_{R_1}^{r_\alpha} \frac{B(\rho(s))}{s} ds \leq \overline{F} - (n-1)\frac{\Phi(M)}{R_1}\int_{R_1}^{r_\alpha} u'(s)ds \\ &\leq \overline{F} + \frac{(n-1)\Phi(M)U}{R_1}, \end{aligned}$$

since $u(R_1) \leq U$ being $R_1 \geq \overline{r}$, $E(r_\alpha) = F(\ell_\alpha) \leq 0$ if $(f3)$ holds and is zero if $(f4)$ is satisfied, since $u'(r_\alpha) = 0$ by Lemma 3.2 *(i)*, by assumption $\alpha \notin I^-$, and finally $u(\overline{r}) - \ell_\alpha \leq U$ since $\ell_\alpha \geq 0$. Hence

$$H(M) \leq \overline{F} + \frac{(n-1)\Phi(M)U}{\overline{r}}. \tag{4.5}$$

By (4.2)–(4.4) and (2.4) it follows at once that

$$\Phi(M) > \frac{\overline{r}}{n-1}\frac{F(U)}{U} \geq \Phi\left(\frac{n-1}{\overline{r}}\frac{U}{F(U)}p'[\overline{F} + F(U)]\right),$$

namely by ($\Phi 2$)

$$\frac{M}{p'} > \frac{n-1}{\overline{r}}\frac{U}{F(U)}[\overline{F} + F(U)]. \tag{4.6}$$

Now from (4.5), (2.3), (4.6) and (4.4) we finally have

$$\begin{aligned} \overline{F} &\geq H(M) - (n-1)\frac{\Phi(M)}{\overline{r}}U \geq \frac{M}{p'}\Phi(M) - (n-1)\frac{\Phi(M)}{\overline{r}}U = \Phi(M)\left[\frac{M}{p'} - \frac{n-1}{\overline{r}}U\right] \\ &> \frac{\overline{r}}{n-1}\frac{F(U)}{U}\frac{n-1}{\overline{r}}U\left[\frac{\overline{F} + F(U)}{F(U)} - 1\right] = \overline{F}, \end{aligned}$$

which is the required contradiction. □

Theorem 4.2 (Existence)**.** *Let $(f1)$ and either $(f3)$ or $(f4)$ hold. If $\gamma_1 = \infty$, we also assume that:*

(Q) *The function Q is locally bounded near $v = 0$ and there exist $\mu > d$ and $\lambda \in (0,1)$ such that $Q(v) \geq 0$ for all $v \geq \mu$ and*

$$\limsup_{v \to \infty} Q(\lambda_1 v)[v^{p+1} f(\lambda_2 v)]^{n/p} = \infty \quad \textit{for every} \quad \lambda_1,\ \lambda_2 \in [\lambda, 1].$$

Then $I^- \neq \emptyset$.

Proof. We discuss separately the two cases $\gamma < \infty$ and $\gamma = \infty$.

Let $\gamma < \infty$. Problem (3.1) and (3.2) admits a unique classical solution u, which depends continuously by α by Lemmas 3.1 and 3.3. Hence $u(r,\alpha) \to \gamma$ as $\alpha \to \gamma$ uniformly on every bounded interval of $\mathbb{R}_0^+$. Since $\alpha \in (d,\gamma)$, for α sufficiently close to γ, by Lemma 3.2 *(ii)*, there is a unique value $t_\alpha \in J_\alpha$ such that $u(t_\alpha,\alpha) = \tilde{\gamma}$, where $\tilde{\gamma} = (\gamma + d)/2 \in (d,\gamma)$. We claim that $\alpha \mapsto t_\alpha$ is not bounded above as $\alpha \to \gamma$. For otherwise $0 < t_\alpha \le t < \infty$ as $\alpha \to \gamma$, for an appropriate constant t, and all the corresponding solutions $u(r,\alpha)$ would be $\le \tilde{\gamma}$ as $r \ge t$, which contradicts the fact that $u(r,\alpha)$ converges uniformly on bounded sets to γ as $\alpha \to \gamma$.

Now, putting $u(t_\alpha,\alpha) = \tilde{\gamma}$ in the right–hand side of (4.1), we find a finite number which is independent of α. This is impossible by the previous argument.

Consequently there is $\alpha \in (d,\gamma)$ such that the corresponding solution $u(r,\alpha)$, with $u(t_\alpha,\alpha) = \tilde{\gamma}$, has the property that

$$t_\alpha > \frac{C\tilde{\gamma}[\overline{F} + F(\tilde{\gamma})]}{F(\tilde{\gamma})B^{-1}(p'[\overline{F} + F(\tilde{\gamma})])},$$

namely $\alpha \in I^-$ by Lemma 4.1, or u is a crossing solution by the definition of I^-, and the proof is complete when $\gamma < \infty$.

Let now $\gamma = \infty$. Assume for contradiction that $u(r,\alpha)$ is a global solution for all $\alpha \in (d,\infty)$, namely $J_\alpha = \mathbb{R}^+$ for all $\alpha \in (d,\infty)$.

We take μ and λ as stated in (Q), with $\alpha > \mu$, and, without loss of generality, we take λ sufficiently close to 1 such that:

$$\alpha > \frac{\mu}{\lambda} \qquad \text{and} \qquad 0 < \frac{\alpha(1-\lambda)}{r_\lambda} = \frac{u(0) - u(r_\lambda)}{r_\lambda} \le 1, \tag{4.7}$$

where r_λ is the unique point r such that $u(r_\lambda) = \lambda\alpha$ by Lemma 3.2 *(ii)*. This is clearly possible since by (2.16)

$$\lim_{r\to 0^+} \frac{u(r) - u(0)}{r} = u'(0) = 0 \quad \text{and} \quad \lim_{\lambda \to 1} r_\lambda = 0.$$

Integrating (3.1) on $[0,r]$, with $r \in (0,r_\lambda)$, we obtain

$$r^{n-1}\Phi(\rho(r)) = \int_0^r s^{n-1} f(u(s))ds.$$

Hence, putting

$$f(\lambda_2\alpha) = \max_{[\lambda\alpha,\alpha]} f(u) > 0, \qquad \lambda_2 \in [\lambda, 1],$$

we have

$$r^{n-1}\Phi(\rho(r)) \le f(\lambda_2\alpha)\frac{r^n}{n}.$$

Since $\Phi(\infty) = \infty$ by $(\Phi3)$, then by $(\Phi2)$

$$\rho(r) \le \Phi^{-1}\left(\frac{f(\lambda_2\alpha)r}{n}\right).$$

Integration from 0 to r_λ yields

$$\alpha(1-\lambda) \le \int_0^{r_\lambda} \Phi^{-1}\left(\frac{f(\lambda_2\alpha)r}{n}\right)dr \le r_\lambda \Phi^{-1}\left(\frac{f(\lambda_2\alpha)r_\lambda}{n}\right),$$

by $(\Phi2)$. Thus

$$\begin{aligned} \frac{f(\lambda_2\alpha)}{n} &\ge \frac{1}{r_\lambda}\Phi\left(\frac{\alpha(1-\lambda)}{r_\lambda}\right) = \frac{1}{r_\lambda}\Phi\left(\left[\frac{\alpha(1-\lambda)}{r_\lambda}\right]^{p/(p-1)}\left[\frac{r_\lambda}{\alpha(1-\lambda)}\right]^{1/(p-1)}\right) \\ &\ge \frac{1}{r_\lambda}\frac{r_\lambda}{\alpha(1-\lambda)}\Phi\left(\left[\frac{\alpha(1-\lambda)}{r_\lambda}\right]^{p/(p-1)}\right), \end{aligned}$$

by (Φ3) and (4.7). This shows that

$$\Phi^{-1}\Big(\frac{\alpha(1-\lambda)f(\lambda_2\alpha)}{n}\Big) \geq \Big[\frac{\alpha(1-\lambda)}{r_\lambda}\Big]^{p/(p-1)}, \tag{4.8}$$

namely

$$r_\lambda \geq \frac{\alpha(1-\lambda)}{\Big[\Phi^{-1}\Big(\frac{\alpha(1-\lambda)f(\lambda_2\alpha)}{n}\Big)\Big]^{(p-1)/p}}.$$

Now, we take $\lambda < 1$ sufficiently close to 1 so that

$$\frac{\alpha(1-\lambda)f(\lambda_2\alpha)}{n} \leq 1.$$

Hence

$$\Big[\frac{\alpha(1-\lambda)f(\lambda_2\alpha)}{n}\Big]^{1/(p-1)}\Phi^{-1}\Big(\frac{\alpha(1-\lambda)f(\lambda_2\alpha)}{n}\Big) \leq \Phi^{-1}(1),$$

since the function $\tau \mapsto \tau^{1/(p-1)}\Phi^{-1}(\tau)$, $\tau \in \mathbb{R}_0^+$, is non–decreasing by (Φ3). Therefore

$$\Phi^{-1}(1)\Big[\frac{n}{\alpha(1-\lambda)f(\lambda_2\alpha)}\Big]^{1/(p-1)} \geq \Phi^{-1}\Big(\frac{\alpha(1-\lambda)f(\lambda_2\alpha)}{n}\Big) \geq \Big[\frac{\alpha(1-\lambda)}{r_\lambda}\Big]^{p/(p-1)},$$

by (4.8), or

$$r_\lambda \geq \frac{(1-\lambda)^{(p+1)/p}}{[\Phi^{-1}(1)]^{(p-1)/p}n^{1/p}}\alpha^{(p+1)/p}[f(\lambda_2\alpha)]^{1/p},$$

namely

$$r_\lambda \geq C_\lambda \alpha^{(p+1)/p}[f(\lambda_2\alpha)]^{1/p}, \qquad C_\lambda = \frac{(1-\lambda)^{(p+1)/p}}{[\Phi^{-1}(1)]^{(p-1)/p}n^{1/p}}. \tag{4.9}$$

Let r_μ be the unique value of r such that

$$u(r_\mu) = \mu. \tag{4.10}$$

Hence $r_\mu > r_\lambda$, since $u(r_\lambda) = \lambda\alpha > \mu = u(r_\mu)$ by (4.7), and $u' < 0$ on $J_\alpha = \mathbb{R}^+$. Furthermore, since $\alpha \notin I^-$, by (4.1)

$$r_\mu < C_\mu := \frac{C\mu[\overline{F} + F(\mu)]}{F(\mu)B^{-1}(p'[\overline{F} + F(\mu)])}, \tag{4.11}$$

where C is given in (4.2) and B in (2.4).

Let

$$Q_\mu = \inf_{0<v\leq\mu} Q(v) > -\infty \tag{4.12}$$

by (Q). Moreover there is a suitable constant $\lambda_1 \in [\lambda, 1]$ such that

$$Q(\lambda_1\alpha) = \min_{\lambda\alpha\leq v\leq\alpha} Q(v) \geq 0 \tag{4.13}$$

by (Q) since $\lambda\alpha > \mu$. Consequently, by construction,

$$Q(u(r)) \geq \begin{cases} Q(\lambda_1\alpha), & \text{if } 0 < r < r_\lambda \\ 0, & \text{if } r_\lambda \leq r \leq r_\mu \\ -|Q_\mu|, & \text{if } r > r_\mu. \end{cases} \tag{4.14}$$

By Lemma 3.4, that is (3.6) and (3.7), with $r > r_\mu$,

$$pr^n E(r) \geq P(r) \geq \Big(\int_0^{r_\lambda} + \int_{r_\lambda}^{r_\mu} + \int_{r_\mu}^{r}\Big)s^{n-1}Q(u(s))ds. \tag{4.15}$$

Hence, using (4.14), we get

$$pr^n E(r) \geq \frac{Q(\lambda_1\alpha)}{n}C_\lambda^n[\alpha^{p+1}f(\lambda_2\alpha)]^{n/p} - \frac{|Q_\mu|r^n}{n},$$

by (4.9). By (2.22), (4.11) and the fact that F is positive and increasing on (d,∞), and so on $(d,\mu]$,

$$pnH(\rho(r)) > \left[\frac{C_\lambda}{C_\mu+1}\right]^n Q(\lambda_1\alpha)[\alpha^{p+1}f(\lambda_2\alpha)]^{n/p} - |Q_\mu| - npF(\mu),$$

for $r_\mu < r < r_\mu+1$. By (Q) we can take α sufficiently large so that $H(\rho(r)) \geq H(\mu)$, that is $\rho(r)\geq\mu$ by (2.2). Consequently

$$u(r_\mu+1) = u(r_\mu) + \int_{r_\mu}^{r_\mu+1} u'(s)ds \leq \mu-\mu = 0,$$

which contradicts the fact that $u>0$ on $J_\alpha=\mathbb{R}^+$, and completes the proof. □

We now recall the main growth condition, denoted here by $(Q1)$, for existence in the principal case $\gamma=\infty$, used in [7], [9] and [18]:

$(Q1)$ The function Q is locally bounded near $v=0$ and there exist $\mu>d$ and $\lambda\in(0,1)$ such that $Q(v)\geq 0$ for $v\geq\mu$ and

$$\limsup_{v\to\infty} Q(\lambda_1 v)\left[\frac{v^{p-1}}{f(\lambda_2 v)}\right]^{n/p} = \infty \qquad \text{for all } \lambda_1 \text{ and } \lambda_2 \text{ in } [\lambda,1].$$

Proposition 4.3. *Let $\gamma=\infty$. Assume that $(f1)$ holds and also that*

$$(F1) \qquad \liminf_{u\to\infty} f(u) = k_0 > 0,$$

with k_0 possibly infinity, is satisfied. Then

$$(Q1) \quad \Longrightarrow \quad (Q).$$

Proof. By $(F1)$ for u sufficiently large

$$f(u)>0 \qquad \text{and} \qquad uf(\lambda_2 u)\geq 1/\lambda \quad \text{for all } \lambda_2\in[\lambda,1].$$

Consequently by $(Q1)$

$$Q(\lambda_1 u)u^{(p+1)n/p}[f(\lambda_2 u)]^{n/p} \geq Q(\lambda_1 u)\left[\frac{u^{p-1}}{f(\lambda_2 u)}\right]^{n/p},$$

and the implication follows at once. □

Theorem 4.2 generalizes Lemma 3.5 of [9], since in general $(f1)$, $(F1)$, (Q) and either $(f3)$ or $(f4)$ do not imply the validity of $(Q1)$, as shown by the following examples. For brevity we define the functions f under consideration in the examples only for large u and without further mention we assume that the functions f satisfy $(f1)$ and either $(f3)$ or $(f4)$, and the corresponding functions $Q(v)=pnF(v)-(n-p)vf(v)$, $v>0$, are locally bounded near at $v=0$. Finally, in the next examples, $1<p<n$ and $p^*=\dfrac{np}{n-p}$.

Examples. *1.* Let

$$f(u) = u^{p^*-1}+\frac{1}{u}, \quad \text{for} \quad u\geq u_0, \quad \text{with} \quad u_0>0.$$

Clearly $(F1)$ holds with $k_0=\infty$. Here

$$Q(u) = c_1 + np\log u, \qquad u\geq u_0,$$

where c_1 is an appropriate constant. Hence for $u\geq\mu$ and $\mu>d$ sufficiently large, certainly $Q(u)>0$. Now, as $u\to\infty$

$$Q(\lambda_1 u)\left[\frac{u^{p-1}}{f(\lambda_2 u)}\right]^{n/p} = [c_1+np\log(\lambda_1 u)]\frac{u^{n(p-1)/p}}{\left[(\lambda_2 u)^{p^*-1}+\dfrac{1}{\lambda_2 u}\right]^{n/p}} \longrightarrow 0,$$

so that clearly $(Q1)$ fails, while

$$Q(\lambda_1 u)[u^{p+1}f(\lambda_2 u)]^{n/p} = [c_1 + np\log(\lambda_1 u)]u^{n(p+1)/p}\Big[(\lambda_2 u)^{p^*-1} + \frac{1}{\lambda_2 u}\Big]^{n/p} \longrightarrow \infty,$$

namely (Q) holds.

2. Define

$$f(u) = u^{p^*-1} \pm \frac{1}{u^q} \quad \text{for} \quad u \ge u_0, \quad \text{with} \quad q > 1, \tag{4.16}$$

where $u_0 > 1$ when the minus sign is considered and $(f4)$ holds, since $\gamma = \infty$ in the examples, while $u_0 > 0$ in all the other cases. Again $(F1)$ holds with $k_0 = \infty$. Hence, for $c_0 = n - p + np/(q-1)=$ const. > 0 and $u \ge u_0$,

$$Q(u) = \frac{\mp c_0}{u^{q-1}} + c_1,$$

where the appropriate constant c_1 is now assumed to be positive.[1] As $u \to \infty$

$$Q(\lambda_1 u)\left[\frac{u^{p-1}}{f(\lambda_2 u)}\right]^{n/p} = \Big[\frac{\mp c_0}{(\lambda_1 u)^{q-1}} + c_1\Big]\frac{u^{n(p-1)/p}}{\Big[(\lambda_2 u)^{p^*-1} \pm \dfrac{1}{(\lambda_2 u)^q}\Big]^{n/p}} \longrightarrow 0,$$

that is $(Q1)$ fails, while

$$Q(\lambda_1 u)[u^{p+1}f(\lambda_2 u)]^{n/p} = \Big[\frac{\mp c_0}{(\lambda_1 u)^{q-1}} + c_1\Big]u^{n(p+1)/p}\Big[(\lambda_2 u)^{p^*-1} \pm \frac{1}{(\lambda_2 u)^q}\Big]^{n/p} \longrightarrow \infty,$$

that is property (Q) holds.

3. Take $u_0 > 0$ if $(f3)$ holds and $u_0 \ge 0$ if $(f4)$ is satisfied, then define

$$f(u) = u^{p^*-1} + u^{q-1}, \qquad 1 < q < p^*.$$

Again $(F1)$ holds with $k_0 = \infty$ and, for $u \ge u_0$,

$$Q(u) = c_0 u^q + c_1, \qquad \text{with } c_0 = \frac{np}{q} - (n-p) = \text{ const.} > 0,$$

and c_1 is an appropriate constant. As $u \to \infty$,

$$Q(\lambda_1 u)\left[\frac{u^{p-1}}{f(\lambda_2 u)}\right]^{n/p} = [c_0(\lambda_1 u)^q + c_1]\frac{u^{n(p-1)/p}}{[(\lambda_2 u)^{p^*-1} + (\lambda_2 u)^{q-1}]^{n/p}} \longrightarrow 0,$$

namely $(Q1)$ fails, while

$$Q(\lambda_1 u)[u^{p+1}f(\lambda_2 u)]^{n/p} = [c_0(\lambda_1 u)^q + c_1]u^{n(p+1)/p}[(\lambda_2 u)^{p^*-1} + (\lambda_2 u)^{q-1}]^{n/p} \longrightarrow \infty,$$

that is (Q) holds.

4. Now let $p = n$ and

$$f(u) = pu^{p-1}e^{u^p} \qquad \text{for} \quad u \ge u_0,$$

with $u_0 > 0$ if $(f3)$ holds and with $u_0 \ge 0$ when $(f4)$ is satisfied. Clearly $(F1)$ is satisfied with $k_0 = \infty$ and $Q(u) = p^2\left(e^{u^p} + c_1\right)$, where c_1 is an appropriate constant. Hence, as $u \to \infty$, for $\lambda_1 = \lambda_2$,

$$Q(\lambda_1 u)\frac{u^{p-1}}{f(\lambda_2 u)} = \frac{p[e^{(\lambda_1 u)^p} + c_1]}{\lambda_2^{p-1}e^{(\lambda_2 u)^p}} \longrightarrow p\lambda_1^{1-p},$$

[1] To have $c_1 > 0$, it is enough to define $f(u) = \log u + 2$ for $u \in (0,1]$, when in (4.16) the plus sign is considered, $u_0 = 1$ and $(f3)$ holds; while $f(u) = u^s + 1$ for $u \in (0,1]$, $s > -1$, when in (4.16) the plus sign is considered, $u_0 = 1$ but now $(f4)$ holds. Analogously, when in (4.16) the minus sign is taken, to get $c_1 > 0$, we can define $f(u) = u^{p^*-1} - u^{-s} + c(1-u)$ for $u \in (0,1]$, with $s \in (0,1)$ and the positive constant $c > 2/(1-s) + 2/(q-1)$, when $(f3)$ holds; while $f(u) = u_0^{p^*-1} - u_0^{-q} + 2q(u_0 - u)/(q-1)$ for $u \in [0, u_0]$, $u_0 > 1$, when $(f4)$ holds.

namely $(Q1)$ fails, while for all λ_1, $\lambda_2 \in [\lambda, 1]$, with $\lambda \in (0,1)$,

$$Q(\lambda_1 u)[u^{p+1} f(\lambda_2 u)] = \lambda_2^{p-1} p^3 [e^{(\lambda_1 u)^p} + c_1] u^{2p} e^{(\lambda_2 u)^p} \longrightarrow \infty,$$

that is (Q) holds.

Proposition 4.4. *If $\gamma = \infty$, and f satisfies $(f1)$ and*

$$(F2) \qquad \lim_{u\to\infty} uf(u) = k_1 \geq 0, \qquad \text{with} \quad k_1 \quad \text{finite},$$

then the reverse implication of Proposition 4.3 holds, namely

$$(Q) \qquad \Longrightarrow \qquad (Q1).$$

Proof. Indeed, for u sufficiently large

$$\frac{Q(\lambda_1 u) u^n}{[uf(\lambda_2 u)]^{n/p}} \geq c_0 Q(\lambda_1 u) u^n [uf(\lambda_2 u)]^{n/p},$$

where c_0 is a positive constant, and the implication follows at once. $\square$

Clearly $(F2)$ implies that $f(u) \to 0$ as $u \to \infty$, a situation which is not so interesting in applications and in any case the existence problem could be treated for such nonlinearities using much simpler techniques.

In general $(f1)$, $(F2)$, $(Q1)$ and either $(f3)$ or $(f4)$, do not imply (Q), as shown by the following examples. As before, for brevity the functions f under consideration in the next examples are defined only for large u and assumed to satisfy $(f1)$ and either $(f3)$ or $(f4)$, without further mention. Again the corresponding functions $Q(v) = pnF(v) - (n-p)vf(v)$, $v > 0$, are supposed to be locally bounded near at $v = 0$. Finally, in the next examples $1 < p \leq n$, with $p^* = \infty$ if $p = n$ and $p^* = \dfrac{np}{n-p}$ if $1 < p < n$.

Examples. *5.* Let

$$f(u) = u^{q-1} \quad \text{for } u \geq u_0, \quad u_0 > 0, \qquad q \leq -p.$$

Clearly $(F2)$ holds with $k_1 = 0$ and $Q(u) = c_1 - c_2 u^q$ for $u \geq u_0$, where the appropriate constant c_1 is now assumed to be positive[2] and $c_2 = n - p - np/q =$ const.> 0. Hence

$$Q(\lambda_1 u)\left[\frac{u^{p-1}}{f(\lambda_2 u)}\right]^{n/p} = {\lambda_2}^{n(1-q)/p}\,[c_1 - c_2(\lambda_1 u)^q]\, u^{n(p-q)/p} \longrightarrow \infty,$$

that is $(Q1)$ holds, while

$$Q(\lambda_1 u)[u^{p+1} f(\lambda_2 u)]^{n/p} = {\lambda_2}^{n(q-1)/p}\,[c_1 - c_2(\lambda_1 u)^q]\, u^{n(p+q)/p} \longrightarrow \ell,$$

where $\ell = c_1 {\lambda_2}^{-n(p+1)/p}$ when $q = -p$ and $\ell = 0$ when $q < -p$, so that (Q) fails.

6. Indeed, if

$$f(u) = u^{q-1} \quad \text{for } u \text{ large, with} \quad q > -p,$$

then (Q) and $(Q1)$ are valid if and only if $q < p^*$.

7. Let

$$f(u) = e^{-u} \quad \text{for } u \geq u_0,$$

with $u_0 > 0$ under $(f3)$ and with $u_0 \geq 0$ under $(f4)$. Condition $(F2)$ is satisfied with $k_1 = 0$, and for $u \geq u_0$ $Q(u) = c_1 - [np + (n-p)u]e^{-u}$, where the appropriate constant $c_1 > 0$ is again assumed to be positive. [3]

$$Q(\lambda_1 u)\left[\frac{u^{p-1}}{f(\lambda_2 u)}\right]^{n/p} = \{c_1 - [np + (n-p)\lambda_1 u]e^{-\lambda_1 u}\} u^{n(p-1)/p} e^{n\lambda_2 u/p} \longrightarrow \infty,$$

[2] To have $c_1 > 0$, it is enough to define $f(u) = \log u + 1$ for $u \in (0,1]$, when $u_0 = 1$ and $(f3)$ holds; while $f(u) = u^s$ for $u \in (0,1]$, $s > -1$, when $u_0 = 1$ but now $(f4)$ holds.

[3] To get $c_1 > 0$, it is enough to define $f(u) = \log u + \log 2 + e^{-1/2}$ for $u \in (0, 1/2]$, when $u_0 = 1/2$ and $(f3)$ holds; while it is enough to take $u_0 = 0$ when $(f4)$ holds.

that is $(Q1)$ holds, while

$$Q(\lambda_1 u)[u^{p+1}f(\lambda_2 u)]^{n/p} = \{c_1 - [np + (n-p)\lambda_1 u]e^{-\lambda_1 u}\}u^{n(p+1)/p}e^{-n\lambda_2 u/p} \longrightarrow 0,$$

namely (Q) fails.

Therefore Theorem 4.2 *extends the result of Theorem* 5.1 *of* [18] *also to the nonlinearity satisfying* (Q).

8. Define

$$f(u) = u^{r-1} \pm u^{q-1}, \quad 1 < q < r < p^*,$$

for $u \geq u_0$, where $u_0 > 0$ when the plus sign is considered and $(f3)$ holds, and $u_0 > 1$ when the minus sign is considered and $(f4)$ is satisfied, while in the other cases $u_0 \geq 0$. Clearly $(F1)$ holds with $k_0 = \infty$ and $Q(u) = c_1 + \sigma_r u^r \pm \sigma_q u^q$ for $u \geq u_0$, where $c_1 > 0$ is an appropriate constant and

$$\sigma_r = \frac{pn}{r} - (n-p) > 0 \quad \text{and} \quad \sigma_q = \frac{pn}{q} - (n-p) > 0.$$

Then $(Q1)$ is satisfied, and so also the weaker condition (Q), since $(F1)$ holds.

9. Let

$$f(u) = u^{q-1}\log u \quad \text{for } u \text{ large}, \quad \text{with} \quad 0 \leq q < p^*.$$

Then for $u > 0$ sufficiently large

$$Q(u) = \begin{cases} u^q\Big\{\log u\Big[\dfrac{np}{q} - (n-p)\Big] - \dfrac{np}{q^2}\Big\} + c_0, & 0 < q < p^*, \\ \dfrac{np}{2}\log^2 u - (n-p)\log u + c_1, & q = 0, \end{cases}$$

Clearly $Q(u) \geq 0$ for $u \geq \mu$ and an appropriate $\mu > d$, since here $q < p^*$, as requested in the main assumptions. Moreover, after some calculation, one sees that both (Q) and $(Q1)$ hold.

Indeed, it is worth noting that the existence Theorem 4.2, as well as those in [7] and [18] for the p–Laplacian case, and in [9] for general divergence equations, can be applied in particular when

- $f(u) = u^{q-1}$ for u large, $-p < q < p^*$;
- $f(u) = u^{q-1}\log u$ for u large, $0 \leq q < p^*$,

since both (Q) and $(Q1)$ hold.

Now, assuming the further condition $(\Phi4)$, given in the Introduction, we obtain another criteria, which extends recent results of [7] and [18], established for the p–Laplacian case, and also Theorem 4.1 of [9].

Theorem 4.5. *Suppose that* $(\Phi4)$*,* $(Q1)$*,* $(f1)$ *and either* $(f3)$ *or* $(f4)$ *hold. Then* $I^- \neq \emptyset$*.*

Proof. Obviously the only new case to be proved is when $\gamma = \infty$. Indeed, we can proceed essentially as in the proof of Theorem 4.2. By $(Q1)$ there is $\mu > d$ such that $Q(u) \geq 0$ for $u \geq \mu$. Choose $\alpha > \mu/\lambda$, where λ is the constant specified in $(Q1)$, and let u_α be the corresponding solution of (3.1), with maximal domain $J_\alpha = (0, r_\alpha)$ in the sense of (3.4). We denote by r_μ and r_λ respectively the unique values of r in which u_α reaches μ and $\lambda\alpha$, by Lemma 3.2 *(ii)*. Clearly $r_\lambda < r_\mu$.

Hence, putting

$$f(\lambda_2\alpha) = \max_{[\lambda\alpha,\alpha]} f(u) > 0, \qquad \lambda_2 \in [\lambda, 1],$$

and integrating (3.1) on $[0, r]$, for any $r \in (0, r_\lambda)$, we have

$$kr^{n-1}\rho^{p-1}(r) \leq r^{n-1}\Phi(\rho(r)) = \int_0^r s^{n-1}f(u(s))s \leq \frac{f(\lambda_2\alpha)}{n}r^n,$$

by (Φ4). Thus

$$\rho(r) \le \left[\frac{f(\lambda_2\alpha)}{kn}\right]^{1/(p-1)} r^{1/(p-1)}.$$

Integration from 0 to r_λ yields

$$\alpha(1-\lambda) \le \frac{p-1}{p}\left[\frac{f(\lambda_2\alpha)}{kn}\right]^{1/(p-1)} r_\lambda^{p/(p-1)},$$

and therefore

$$r_\lambda \ge C_\lambda \left[\frac{\alpha^{p-1}}{f(\lambda_2\alpha)}\right]^{1/p}, \quad \text{where } C_\lambda = \left\{kn\left[\frac{(1-\lambda)p}{p-1}\right]^{p-1}\right\}^{1/p}, \tag{4.17}$$

which is the exact analogue of (4.9). Continuing the proof as that of Theorem 4.2, we obtain (4.10)–(4.15). Now, by (4.15), (2.22) and (4.17),

$$pnH(\rho(r)) > \left[\frac{C_\lambda}{C_\mu+1}\right]^n Q(\lambda_1\alpha)\left[\frac{\alpha^{p-1}}{f(\lambda_2\alpha)}\right]^{n/p} - |Q_\mu| - npF(\mu),$$

for $r_\mu < r < r_\mu + 1$, where C_μ, Q_μ and $Q(\lambda_1\alpha)$ are given in (4.11), (4.12) and (4.13), respectively. Thus by $(Q1)$ we can take α sufficiently large so that $H(\rho(r)) \ge H(\mu)$ and conclude the proof exactly as in Theorem 4.2. □

5. Main existence results

Here we turn to the main problems (2.1), (2.24) and (1.2) under the main conditions (Φ1)–(Φ3).

Theorem 5.1. *Assume $(f1)$, $(f3)$ and (Q). Then problem* (2.1) *admits a positive semi–regular radial ground state if* (2.34) *holds, while a compactly supported semi–regular non–negative non–trivial ground state if* (2.34) *fails. In this second case the ground state u is also a solution of the free boundary problem* (1.2).

Proof. Put

$$I^+ = \{\alpha \ge b \;:\; \ell_\alpha > 0\}.$$

Following the proofs of Lemmas 3.2 and 3.3 of [9], word by word, we prove that I^+ is not empty and open. Similarly, using the proof of Lemma 3.6 of [9], we can also show that I^- is also open, and by the above Theorem 4.2 is not empty. Therefore there should be a number $\alpha \notin I^+ \sqcup I^-$, with $b < \alpha < \gamma$, since $b \in I^+$ by Lemma 3.2 of [9], such that (3.1) admits a positive solution u_α, with $u_\alpha(0) = \alpha$, in its maximal continuation open interval $J_\alpha = (0, r_\alpha)$. Since $\alpha \notin I^+$, then $\ell_\alpha = 0$, while since $\alpha \notin I^-$, then either $r_\alpha = \infty$ or $r_\alpha < \infty$, and in both cases $u'_\alpha(r_\alpha) = 0$ by Lemma 3.2 *(i)* and *(iii)*. In the first case u_α is a semi–regular positive radial ground state of (2.1), and the latter u_α is a semi–regular positive radial solution of (1.2), with $R = r_\alpha$, or a compactly supported radial ground state of (2.1), accordingly to Corollary 2.10 *(i)*. □

Theorem 5.2. *Assume $(f1)$, $(f4)$ and (Q). Then problem*

$$\begin{gathered} \operatorname{div}(A(|Du|)Du) + f(u) = 0 \qquad \text{in } \mathbb{R}^n, \qquad n > 1, \\ u \not\equiv 0 \qquad \text{in } \mathbb{R}^n, \end{gathered} \tag{5.1}$$

admits a semi–regular crossing radial solution u in its maximal continuation open interval $J_\alpha = (0, r_\alpha)$, with $u(0) = \alpha \in (0,\gamma)$, $u'(0) = 0$, $r_\alpha < \infty$, $u(r_\alpha) = 0$ and $u'(r_\alpha) < 0$.

Proof. By Theorem 4.2 there is $\alpha \in I^-$ and the conclusion follows at once. □

Theorem 5.3. *Assume* (Φ4), *$(f1)$, $(f3)$ and $(Q1)$. Then problem* (2.1) *admits a positive semi–regular radial ground state if* (2.35) *holds, while a compactly supported semi–regular non–negative non–trivial ground state if* (2.35) *fails. In this second case the ground state u is also a solution of the free boundary problem* (1.2).

Proof. We proceed, word for word, as in the proof of Theorem 5.1, where now $I^- \neq \emptyset$ by Theorem 4.5 and by Corollary 2.10 *(iii)*. □

Theorem 5.4. *Assume* $(\Phi 4)$, $(f1)$, $(f4)$ *and* $(Q1)$. *Then problem* (5.1) *admits a semi–regular crossing radial solution u in its maximal continuation open interval $J_\alpha = (0, r_\alpha)$, with $u(0) = \alpha \in (0, \gamma)$, $u'(0) = 0$, $r_\alpha < \infty$, $u(r_\alpha) = 0$ and $u'(r_\alpha) < 0$.*

Proof. This is an immediate consequence of Theorem 4.5. □

References

[1] F.V. Atkinson and L.A. Peletier, Ground states of $-\Delta u = f(u)$ and the Emden-Fowler equation, *Arch. Rational Mech. Anal.* **93** (1986), 103-127.

[2] H. Berestycki and P.L. Lions, Non-linear scalar field equations, L Existence of a ground state, *Arch. Rational Mech. Anal.* **82** (1983), 313-345.

[3] S. Coleman, V. Glazer and A. Martin, Actions minima among solutions to a class of Euclidean scalar field equations, *Comm. Math. Phys.* **58** (1978), 211-221.

[4] L. Erbe and M. Tang, Uniqueness theorems for positive radial solutions of quasilinear elliptic equations in a ball, *J. Differential Equations* **138** (1997), 351-379.

[5] B. Franchi, E. Lanconelli and J. Serrin, Existence and uniqueness of non–negative solutions of quasilinear equations in $\mathbb{R}^n$, *Advances in Math.* **118** (1996), 177-243.

[6] M. García-Huidobro, R. Manásevich, P. Pucci and J. Serrin, Qualitative properties of ground states for singular elliptic equations with weights, submitted for publication, pages 27.

[7] F. Gazzola, J. Serrin and M. Tang, Existence of ground states and free boundary problems for quasilinear elliptic operators, *Adv. Differential Equations* **5** (2000), 1-30.

[8] N. Kawano, E. Yanagida and S. Yotsutani, Structure theorems for positive radial solutions to $\Delta u + K(|x|)u^p = 0$ in $\mathbb{R}^n$, *Funkcial. Ekvac.* **36** (1993), 557-579.

[9] E. Montefusco and P. Pucci, Existence of radial ground states for quasilinear elliptic equations, *Adv. Differential Equations* **6** (2001), 959-986.

[10] W.–M. Ni and J. Serrin, Non–existence theorems for quasilinear partial differential equations, *Rend. Circ. Mat. Palermo, Suppl.,* **8** (1985), 171-185.

[11] W.–M. Ni and J. Serrin, Existence and non–existence theorems for ground states of quasilinear partial differential equations. The anomalous case, *Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur.* **77** (1986), 231-257.

[12] L.A. Peletier and J. Serrin, Uniqueness of positive solutions of semilinear equations in $\mathbb{R}^n$, *Arch. Rational Mech. Anal.* **81** (1983), 181-197.

[13] L.A. Peletier and J. Serrin, Uniqueness of non–negative solutions of semilinear equations in $\mathbb{R}^n$, *J. Differential Equations* **61** (1986), 380-397.

[14] P. Pucci, M. Rigoli and J. Serrin, The compact support principle for singular elliptic inequalities on complete manifolds, in preparation.

[15] P. Pucci and J. Serrin, A general variation identity, *Indiana Univ. Math. J.* **35**(1986), 681-703.

[16] P. Pucci and J. Serrin, Uniqueness of ground states for quasilinear elliptic operators, *Indiana Univ. Math. J.* **47** (1998), 501-528.

[17] P. Pucci and J. Serrin, The strong maximum principle revisited, to appear in *J. Differential Equations*, pages 48.

[18] M. Tang, Existence and uniqueness of fast decay entire solutions of quasilinear elliptic equations, *J. Differential Equations* **164** (1999), 155-179.

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