An affine intensity model for large credit portfolios

Beatrice Acciaio* and Stefano Herzel†

July 3, 2008

Abstract

The paper proposes a reduced-form model for credit risk in a multivariate setting. The default intensities are linear combinations of three independent affine jump-diffusion processes that can be interpreted as the intensities of general, sectoral and idiosyncratic credit events. The model is rather flexible and can be efficiently calibrated to term structures of default probabilities and conditional probabilities of default given the occurrence of common credit events. We analyse the correlation of defaults and formulate an algorithm for the exact simulation of default scenarios.

key words: Credit risk, Reduced-form model, Affine jump diffusions

1 Introduction

We propose a reduced-form model for credit risk in a multivariate setting where the intensities of defaults are driven by affine jump diffusions processes. An important example of application for the model is a portfolio of bonds issued by many obligors, such as a Collateralized Debt Obligation (CDO). By modeling the intensities of defaults of each obligor, one can produce default scenarios that are necessary to simulate the cashflows of the collateral portfolio and hence to estimate the risk associated to any tranche of a CDO. We assume that defaults may occur as

*Vienna University of Technology - Dep. Financial and Actuarial Mathematics, Wiedner Hauptstrasse 8/105-1 A-1040 Vienna. *email: acciaio@fam.tuwien.ac.at*. Financial support from the European Science Foundation (ESF) "Advanced Mathematical Methods for Finance" (AMaMeF) under the exchange grant 1192 is gratefully acknowledged.

†University of Perugia - Dep. Economy Finance and Statistics, Via A.Pascoli 20 - 06123 Perugia. *email: herzel@unipg.it*.

a consequence of three independent types of credit events: the idiosyncratic one, depending only on the particular situation of the obligor, the sectoral one, affecting all the obligors belonging to a given group, the general one, affecting all the obligors. The intensities of the independent credit events are affine jump diffusion processes and the default intensity process of each obligor is a linear combination of them.

The idea of using affine processes to model the intensity of defaults has been frequently used in the financial literature. In particular, our approach is closely related to Duffie and Gârleanu [7] who focus on the analysis of the effect of correlation on the values of CDOs with different cash flow structures. Variations of the Duffie and Gârleanu model were proposed by Mortensen [18], Eckner [12] who consider a unique common factor, but allow for sensibility coefficients depending on the obligors. All these works impose some restrictions on the parameters in order to simplify computations to get affine default intensities. Chapovsky et al. [2] model all default intensities with a unique affine common factor and deterministic idiosyncratic components. In this paper we consider a generalization of such approaches, defining a a framework where single name default intensities have different sensibilities to the three types of credit events.

The model we consider belongs to the general class of doubly stochastic processes of default. The basic idea of such models is that, after conditioning for the intensities, defaults are independent, therefore, default correlation is solely determined by the correlation of the intensities. It has often been debated if the double stochastic hypothesis is able to produce sufficient default correlation. The main critique to the effectiveness of this approach is that conditional independence makes the connection between defaults too indirect. An empirical study by Das et al. [4] tests a double stochastic model on data of U.S. corporations from 1979 to 2004, showing that it does not explain all default clustering observed. In particular, it has been argued (see for instance Schönbucher [19]) that the introduction of joint jumps in the default intensities is necessary to produce significant levels of correlation. Duffie and Gârleanu [7] studied the impact of correlation on valuation of CDO's tranches. Chapovsky et al. [2] showed that a simple, one factor model can be calibrated efficiently on market prices of synthetic CDO's and reproduce the observed correlation skew. Positive results in the same direction are obtained by Mortensen [18] and Eckner [12], who calibrate to CDS, credit indices and credit tranche spreads, and prove the capability of their models to generate correlation consistent with market-implied levels. In our general setting we obtain an even more flexible correlation structure. Moreover, a nice feature of our class of models is that default correlations can be explicitly computed. We can therefore compare the impact of the diffusion and of the jump components on default correlation. We will give evidence that an appropriate choice of parameters can produce any level of default correlation.

The standard approach to simulate default scenarios is to discretize the SDEs of all the intensity processes involved. Such a straightforward methodology is affected by a discretization error, that may be controlled by decreasing the length of the time interval, with a greater computational cost. We will follow an alternative idea producing an exact simulation of the default scenarios. The simulation is "exact" in the sense that it produces the times of default and the identity of the defaulters from the exact probability distributions, without resorting to approximation or discretization. Moreover, it improves the computational efficiency because it does not require the simulation of the intensity processes of all the obligors. Such a methodology was originally proposed by Duffie and Singleton [10] for general intensity processes. Here we develop a toolbox for the specific case of affine processes and compare the performances of the method with the standard approach.

The availability of closed formulas for default probabilities is a nice feature that can also be exploited when calibrating the model to data. To show the flexibility of the model we calibrate it to a given set of marginal default probabilities and a correlation structure assigned by specifying the dependence on the sectoral and general factors. The model is compatible with different correlation of default times and provides partial positive evidence to the common concern towards reduced-form models of not being able to match empirical default correlations.

The rest of the paper is structured as follows. In Section 2 we introduce the model. Section 3 studies the problem of the correlation of defaults. In Section 4 we illustrate the exact simulation algorithm. In Section 5 we show how to calibrate the model. Section 6 is devoted to applications and Section 7 concludes. More technical details are in the Appendix.

2 The model

In this section we define a model for the times of default of N agents, which are assumed to belong to S different groups ($S \leq N$) and to a unique general environment. We start by setting a filtered probability space $(\Omega, \mathbb{F}, \mathbb{P})$, where $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ and, for any $t \geq 0$, $\mathcal{F}_t$ is the σ-algebra of all the information available at time t. We indicate by $\mathbb{E}_t$ the conditional expectation given $\mathcal{F}_t$, and assume $\mathcal{F}_0$ to be trivial so that $\mathbb{E}_0 = \mathbb{E}$.

For each agent $j = 1, \ldots, N$, we denote by τ_j the time of default and by D_j the default process:

$$D_j(t) = 1_{\{\tau_j \le t\}}.$$

We adopt the intensity-based (or reduced-form) approach (first studied by Jarrow and Turnbull [16], Duffie and Singleton [11] and Lando [17]), assuming that τ_j admits an intensity process λ_j. This means that, conditional on the realization of the intensity λ_j, D_j is an inhomogeneous Poisson process with intensity λ_j, stopped at the first jump[1] (called Cox process or doubly stochastic process). In particular, for $0 \le s < t$, the conditional probability that agent j survives until time t is given by

$$\mathbb{P}(\tau_j > t|\mathcal{F}_s) = 1_{\{\tau_j > s\}} \mathbb{E}_s \left[e^{-\int_s^t \lambda_j(u)du} \right]. \tag{1}$$

This leads to the interpretation of λ_j as the conditional expected default rate of the agent j: for $t < \tau_j$ and a small $\Delta t > 0$, the probability that the agent defaults in the time interval $(t, t + \Delta t]$ is approximated by $\lambda_j(t)\Delta t$.

In our setting defaults may occur as a consequence of three different and independent credit events. For each $j = 1, \dots, N$, the default intensity process of agent j, belonging to group $g(j)$, is given by a linear combination of three independent factors

$$\lambda_j = X_j + u_j Y_{g(j)} + v_j Z. \tag{2}$$

We interpret the processes $X_j, Y_{g(j)}$ and Z as the intensities of, respectively, the idiosyncratic component (concerning only the particular situation of agent j), the sectoral one (common to all the agents belonging to group $g(j)$) and the general one (regarding the whole economy). The coefficients u_j and v_j are constants, usually assumed to belong to the interval $(0, 1)$, and regulate the sensibility of the default intensity λ_j on the common factors $Y_{g(j)}$ and Z. More precisely, we assume the default intensity of obligor j due to sectoral (resp. general) credit events to be equal to $u_j Y_{g(j)}$ (resp. $v_j Z$). Due to the independence of the factors, the expectation in (1) can be factorized as

$$\mathbb{E}_s \left[e^{-\int_s^t \lambda_j(u)du} \right] = \mathbb{E}_s \left[e^{-\int_s^t X_j(u)du} \right] \mathbb{E}_s \left[e^{-u_j \int_s^t Y_{g(j)}(u)du} \right] \mathbb{E}_s \left[e^{-v_j \int_s^t Z(u)du} \right]. \tag{3}$$

The factors $\{X_j\}_{j=1}^N$, $\{Y_i\}_{i=1}^S$ and Z are assumed to be independent Basic Affine Processes (BAP) with parameters $\psi_j = (k_j, \theta_j, \sigma_j, \mu_j, l_j)$, $j = 1, \dots, N$, $\psi_{g_i} = (k_{g_i}, \theta_{g_i}, \sigma_{g_i}, \mu_{g_i}, l_{g_i})$, $i = 1, \dots, S$,

[1] In other words, the compensator process of D_j, which exists by the Doob-Meyer decomposition theorem, is given by $A_j(t) = \displaystyle\int_0^t \lambda_j(s) 1_{\{\tau_j > s\}} ds$.

and $\psi_z = (k_z, \theta_z, \sigma_z, \mu_z, l_z)$ respectively. A BAP with parameters $\psi := (k, \theta, \sigma, \mu, l)$ is a process X whose dynamics are given by

$$dX(t) = k(\theta - X(t))dt + \sigma\sqrt{X(t)}dW(t) + dJ(t), \tag{4}$$

where W is a standard Brownian motion,and J is a pure-jump process (independent of W) with jump sizes independent and exponentially distributed with mean μ, and the jump times of an independent Poisson process with mean jump arrival rate l (jump times and jump sizes are also independent random variables).

Affine processes have been first studied by Feller [13], introduced in the financial literature by Cox et al. [3] and generalized in a path-breaking paper by Duffie and Kan [8]. An omni-comprehensive study on affine process is Duffie et al. [6]. Such processes have the appealing property that the characteristic function of $\int_0^t X(u)du$ is an affine function of $X(0)$, a fact that has been repeatedly exploited in several financial applications. In fact, if X is a BAP,

$$\mathbb{E}\left[e^{-\int_s^t mX(u)du}|\mathcal{F}_s\right] = \exp(C(X, s, t; m)), \tag{5}$$

where

$$C(X, s, t; m) = \alpha_\psi^m(t-s) + \beta_\psi^m(t-s)X(s)$$

and the functions $\alpha_{\psi,r}^m(t-s), \beta_{\psi,r}^m(t-s)$ are solutions of some ordinary differential equations (see Appendix A.2). Note that, as a linear combination of affine processes, the intensity process λ_j defined in (2) is not, in general, affine. Nevertheless, thanks to the hypothesis of independence, the affine framework guarantees an efficient computation of the survival probabilities in (3).

The survival probabilities of any obligor can be written as products of functions of the factor processes that is, by (1), (3) and (5),

$$s_j^t := \mathbb{P}(\tau_j > t) = f(X_j, t, 1)f(Y_{g(j)}, t, u_j)f(Z, t, v_j), \quad j = 1, \ldots, N, \tag{6}$$

where we use the notation

$$f(X, t, m) = e^{C(X,0,t;m,0)}.$$

The first factor in (6) can be interpreted as the probability that no idiosyncratic credit events occur to agent j before time t. For idiosyncratic credit event we mean anything affecting solely the given obligor, like for instance accounting problems. Sectoral credit events are events that specifically affect a group of obligors, an example of which could be a political distress affecting all obligors belonging to a given country. A general credit event affects at the same time all the obligors in the portfolio.

Let us indicate with $e(G(i),t)$ the occurrence of one credit event in sector i within time t, then

$$\mathbb{P}(e(G(i),t)) = 1 - f(Y_i,t,1), \quad i = 1,\dots,S.$$

The quantity $f(Y_{g(j)},t,u_j)$ can be interpreted as the probability that agent j survives up to time t to any credit event occurred in sector $g(j)$. Therefore, the intensity of credit events in sector $g(j)$ is $Y_{g(j)}$, while the joint occurrence of such events and the default of agent j is regulated by the intensity $u_j Y_{g(j)}$. Of course, the probability given by $f(Y_{g(j)},t,u_j)$ can also be obtained as the sum of the probability that no credit events occur in sector $g(j)$, and the probability that such an event occurs but agent j does not default, hence

$$\begin{aligned} f(Y_{g(j)},t,u_j) &= 1 - \mathbb{P}(e(G(g(j)),t))) + \mathbb{P}(\tau^j > t, e(G(g(j)),t)) \\ &= 1 - \mathbb{P}(e(G(g(j)),t))) + \mathbb{P}(e(G(g(j)),t))) \cdot \mathbb{P}(\tau^j > t | e(G(g(j)),t)) \\ &= 1 - \mathbb{P}(\tau^j < t, e(G(g(j)),t)). \end{aligned}$$

Therefore, the default probability of agent j conditioned to a sectoral credit event is, at time t, equal to

$$p_{j|G}(t) = \frac{1 - f(Y_{g(j)},t,u_j)}{1 - f(Y_{g(j)},t,1)}, \quad j = 1,\dots,N.$$

Likewise, indicating by $e(M,t)$ the occurrence of a general credit event, we have that

$$\mathbb{P}(e(M,t)) = 1 - f(Z,t,1)$$

and

$$f(Z,t,v_j) = 1 - \mathbb{P}(\tau^j < t, e(M,t)), \quad j = 1,\dots,N,$$

so that the probability of default of agent j conditional to a general event is, at time t, equal to

$$p_{j|M}(t) = \frac{1 - f(Z,t,v_j)}{1 - f(Z,t,1)}, \quad j = 1,\dots,N.$$

The affine setting provides us with closed formulas for the above quantities that may be used for an efficient calibration of the model.

3 Default correlation

There are different ways of modeling dependency across default times. In particular, besides the approach of conditional independence we adopt here (where dependence is introduced by assuming correlation among obligors' default intensities), a common way of describing dependency is by means of copula functions (which separate the individual default structure from the dependency structure). For a broad discussion, including pros and cons of these two methodologies, we refer the reader to Schönbucher [19] and Brigo and Mercurio [1]. Here we recall that the major concern on reduced-form models based on conditional independence regards the low level of correlation between default times produced (see e.g. Hull and White [15] and Schönbucher and Schubert [20]). On the other hand, recently (see e.g. Yu [21] and Mortensen [18]) it has been pointed out that these models can in fact reproduce realistic correlations, as long as the common factors structure is rich enough. A nice feature of the affine model we are presenting is that default correlations can be explicitly computed. We can therefore analyse the level of correlation produced separating the contributions given to it by the diffusive and the jump part.

Now we are going to discuss correlation between default times.

We study the correlation between defaults of agents i and j before time t,

$$\begin{aligned} Corr(i,j,t) &:= Corr(1_{\{\tau_i \le t\}}, 1_{\{\tau_j \le t\}}) = Corr(1_{\{\tau_i > t\}}, 1_{\{\tau_j > t\}}) \\ &= \frac{\mathbb{E}[1_{\{\tau_i > t\}} 1_{\{\tau_j > t\}}] - \mathbb{E}[1_{\{\tau_i > t\}}]\mathbb{E}[1_{\{\tau_j > t\}}]}{\sqrt{Var(1_{\{\tau_i > t\}}) Var(1_{\{\tau_j > t\}})}} \\ &= \frac{s_{ij}^t - s_i^t s_j^t}{\sqrt{s_i^t(1-s_i^t) s_j^t (1-s_j^t)}}, \end{aligned}$$

where the marginal survival probabilities are given by (6), while the joint survival probabilities are

$$s_{ij}^t = \begin{cases} f(X_i,t,1)f(X_j,t,1)f(Y_{g(i)},t,u_i)f(Y_{g(j)},t,u_j)f(Z,t,v_i+v_j), & \text{if } g(i) \neq g(j), \\ f(X_i,t,1)f(X_j,t,1)f(Y_{g(i)},t,u_i+u_j)f(Z,t,v_i+v_j), & \text{if } g(i) = g(j). \end{cases}$$

Equipped with this formula we analyse the contributions of the jump and of the diffusive part on default correlation. For a clearer exposition of such issue we consider two agents, say 1 and 2, whose default intensities are given by

$$\lambda_j = u_j Y, \;\; j = 1, 2.$$

In words, we are considering two agents whose default intensities depend on a common factor Y (being a sector or a group factor) through different coefficients u_1 and u_2 and have a negligible idiosyncratic factor. In this case, the survival probabilities are given by

$$s_j^t = f(Y, t, u_j), \quad j = 1, 2$$

and

$$s_{12}^t = f(Y, t, u_1 + u_2).$$

The process Y is basic affine, therefore

$$\mathbb{E}[e^{-\int_0^t mY(u)du}] = f(Y, t, m) = \alpha_Y^m(t) + \beta_Y^m(t)Y(0), \tag{7}$$

with $\alpha_Y^m(t)$ and $\beta_Y^m(t)$ solutions of some Riccati ODEs. In what follows, we will consider Y to be either a pure-jump process (case 1) or a diffusion process (case 2). We then compare the correlation between defaults that can be obtained in the two cases, choosing the parameters of the processes in order to have similar survival probability term structures.

CASE 1. Let Y be a pure-jump process with jump sizes independent and exponentially distributed with mean μ, and the jump times of an independent Poisson process with mean jump arrival rate l. In this case, the ODEs to determine the parameters $\alpha_Y^m(t)$ and $\beta_Y^m(t)$ in (7) are

$$\begin{cases} \dfrac{\partial \beta_Y^m(t)}{\partial t} = -m \\ \beta_Y^m(0) = 0 \end{cases},$$

which gives $\beta_Y^m(t) = -mt$, and

$$\begin{cases} \dfrac{\partial \alpha_Y^m(t)}{\partial t} = -l + \dfrac{l}{\mu}\displaystyle\int_0^{+\infty} e^{z(\beta_Y^m(t)-\mu^{-1})}dz = l\left(\dfrac{1}{1+m\mu t} - 1\right) \\ \alpha_Y^m(0) = 0 \end{cases},$$

which gives $\alpha_Y^m(t) = -lt\left(1 - \dfrac{\ln(1+m\mu t)}{m\mu t}\right)$.

CASE 2. Let Y be a diffusion process

$$dY(t) = k(\theta - Y(t))dt + \sigma\sqrt{Y(t)}dW(t),$$

with W standard Brownian motion. In this case the parameters $\alpha_Y^m(t)$ and $\beta_Y^m(t)$ in (7) are given by formulas (15) and (14), with $l = 0$.

The correlation depends on the parameters chosen. As an example, following Duffie and Gârleanu [7], we set parameters of the diffusive process in case 2 to

$$k = 0.6,\ \theta = 0.0373,\ \sigma = 0.141,\ Y_0 = 0.005. \tag{8}$$

Now we estimate the parameters l and μ, for the pure jump process in case 1, so that the survival probability term structure for $t = 1, 2, \ldots, 10$ are as close as possible (in the sense of least squares). The optimization returns the values

$$l = 0.0375, \ \mu = 0.7139. \tag{9}$$

With such choice of parameters, the probability term structures of the pure jump and of the diffusion are practically equal.

Figure 1 shows the values of the default correlation for the sets of parameters in (8) and (9), when the sensibility coefficient u_1 varies in $(0, 1)$ and $u_2 = 1 - u_1$.

[Figure 1 about here]

When the driving process is pure jump, different levels of correlation (ranging from 0 to 0.664) can be obtained. In particular, we note that high levels of correlation can be obtained while keeping a good credit quality for the group factor. Indeed, in this example, the occurrence probability of a group credit event is comparable with that of a Ba rating level, accordingly to MPD Moody's default probability. On the other hand, we see that the level of default correlation produced by the pure diffusion is sensibly smaller.

We now change the level of the parameter σ, to increase the volatility of the diffusion, and estimate, with the same criterion as above, the jump parameters (l, μ), that will therefore depend on σ. Then, for u_1 and u_2 equal to 0.5, we compare the correlations produced at time $t = 10$, for σ varying between 0 and $\sqrt{2k\theta}$ (a technical condition for the existence of a strictly positive intensity process). The results are plotted in Figure 2. It is evident that the jump component is indeed necessary to produce sensible level of correlation between default times.

[Figure 2 about here]

4 Exact simulations of default scenarios

The properties of the affine structure can be exploited to produce an efficient algorithm to produce default scenarios for a set of N obligors before a given time T. Let $\tau^{(n)}$ be the time of the n-th default:

$$\tau^{(1)} = \min\{\tau_j : \tau_j \leq T,\, j = 1, \ldots, N\},$$
$$\tau^{(n)} = \min\{\tau_j :\, \tau^{(n-1)} < \tau_j \leq T, j = 1, \ldots, N\}, \quad \forall n = 2, \ldots, N_d.$$

where N_d is the number of defaults occurred within time T. Let $I^{(n)}$ be the corresponding identity of the defaulter. A *default scenario* is defined by the sequence of pairs $\{(\tau^{(n)}, I^{(n)})\}_{n=1}^{N_d}$.

The standard approach is to proceed by Euler-Marayama discretization of the SDEs of the intensity processes. This involves the simulation of the trajectories of the intensity default processes λ_j and, for each time interval Δt, the computation of the total default intensity, the simulation of the occurrence of a default and, in case there is one, the extraction of the identity of the defaulter. The main advantage of such approach is its simplicity, since it does not require any previous computation. However, it is affected by a discretization error, that may be controlled by decreasing Δt, at the expense of a greater computational effort. A problem that often arises in the discretization of square root processes, and is therefore well present in the financial literature when simulating affine processes like for Heston or Cox-Ingersoll-Ross models, is the appearance of negative values, and consequently the impossibility to take the square roots. We refer to Higham and Mao [14] for a deep analysis of the error of the Euler discretization to the case of the Heston model.

Instead, we will present an alternative approach, suggested by Duffie and Singleton [10], which produces exact simulation of the times of default, extracts the identity of the defaulter and restart the remaining intensities at appropriate values. The simulation is "exact", because it produces the times of default and the identity of the defaulters from their exact probability distributions, without resorting to approximation or discretization. Moreover, it improves the computational efficiency because it does not require the simulation of the intensity processes of the obligors. However, it is not immediately implementable, as it requires a rather elaborate construction of which we will provide the most relevant points.

The simulation algorithm of default scenarios for N obligors up to time T proceeds as follows:

(I) set $\tau^{(0)} := 0$, $C_0 := \{1, ..., N\}$ and $n := 1$;

(II) simulate the n^{th} default-time $\tau^{(n)}$, IF $\tau^{(n)} > T$ or $n = N$ STOP;

(III) from C_{n-1} extract the identity $I^{(n)}$ of the n^{th} defaulter;

(IV) re-start the intensity λ_j of any agent $j \in C_n := C_{n-1} \setminus \{I^{(n)}\}$;

(V) set $n = n+1$, GO TO (II).

We now comment on the single steps of the algorithm.

• Step (II). For $n = 1, \dots, N_d$, we define the process Λ^{n-1} on $t \in [\tau^{(n-1)}, \tau^{(n)})$, as the sum of default intensities of the obligors not defaulted by time $\tau^{(n-1)}$:

$$\Lambda^{n-1}(t) = \sum_{j \in C_{n-1}} \lambda_j(t) = \sum_{j \in C_{n-1}} \Big(X_j(t) + u_j Y_{g(j)}(t) + v_j Z(t) \Big). \tag{10}$$

Under the assumption that no more than one credit event occurs at the same time, the process Λ^{n-1} is the intensity of the time-to-next default $s_n = \tau^{(n)} - \tau^{(n-1)}$ (see Lemma A.1). We can simulate s_n by inverting its cdf, which can be computed from relation (1) and using the independence of the affine factors,

$$\begin{aligned}
&SP_n(t) := \mathbb{P}(s_n > t | \mathcal{F}_{\tau^{(n-1)}}) = \mathbb{E}_{\tau^{(n-1)}}\left[e^{-\int_{\tau^{(n-1)}}^{\tau^{(n-1)}+t} \Lambda^{n-1}(u) du} \right] \\
&= \exp\Big\{ \sum_{j \in C_{n-1}} C(X_j, \tau^{(n-1)}, \tau^{(n-1)} + t; 1, 0) + \sum_{i \in G_{n-1}} C(Y_i, \tau^{(n-1)}, \tau^{(n-1)} + t; U_i^{n-1}, 0) \\
&+ C(Z, \tau^{(n-1)}, \tau^{(n-1)} + t; V^{n-1}, 0) \Big\},
\end{aligned}$$

where

$$\begin{aligned}
V^n &= \sum_{j \in C_n} v_j, \\
U_i^n &= \sum_{j \in C_n : g(j) = i} u_j, \quad \forall i \in G_n, \\
G_n &= \{ i \in \{1, ..., S\} : \{ j \in C_n : g(j) = i \} \neq \emptyset \}
\end{aligned}$$

and $\mathbb{E}_{\tau^{(n-1)}}$ is the conditional expectation given $\mathcal{F}_{\tau^{(n-1)}}$. The numerical inversion of $SP_n(t)$ is, according to our experience, quite straightforward, as the function is usually very smooth and, obviously, monotone.

• Step (III). As suggested in [10], the conditional probability that the j-th obligor is the n-th to default, given $\tau^{(n)}$, is given by

$$p_j^n = \mathbb{P}\big(I^{(n)} = j | \mathcal{F}_{\tau^{(n-1)}} \vee \tau^{(n)}\big) = \frac{\mathbb{E}_{\tau^{(n-1)}}\left[\lambda_j(\tau^{(n)}) e^{-\int_{\tau^{(n-1)}}^{\tau^{(n)}} \Lambda^{n-1}(s) ds} \right]}{\mathbb{E}_{\tau^{(n-1)}}\left[\Lambda^{n-1}(\tau^{(n)}) e^{-\int_{\tau^{(n-1)}}^{\tau^{(n)}} \Lambda^{n-1}(s) ds} \right]}, \quad \forall j \in C_{n-1}.$$

Note that, due to the standard assumption of no-jumps at the default times, the evaluation of the intensities in $\tau^{(n)}$ is intended at time $\tau^{(n)}$-. Let us denote

$$q_j^n = \mathbb{E}_{\tau^{(n-1)}}\left[\lambda_j(\tau^{(n)})e^{-\int_{\tau^{(n-1)}}^{\tau^{(n)}} \Lambda^{n-1}(s)ds}\right]$$

and observe that

$$p_j^n = \frac{q_j^n}{\sum_{k\in C_{n-1}} q_k^n}.$$

Therefore, to extract the the identity $I^{(n)}$ of the n^{th}-defaulter given $\tau^{(n)} = t_n$ is sufficient to compute q_j^n for any $j \in C_{n-1}$. By using the independence of the intensity processes (similarly to what is done in detail in Appendix B), we obtain, for $\tau^{(n-1)} = t_{n-1}$,

$$\begin{aligned}\mathbb{E}_{t_{n-1}}\left[\lambda_j(t_n)e^{-\int_{t_{n-1}}^{t_n} \Lambda^{n-1}(s)ds}\right] = c\cdot \Big[&D(X_j, t_{n-1}, t_n; 1, 0) + u_j D(Y_{g(j)}, t_{n-1}, t_n; U_{g(j)}^{n-1}, 0) \\ &+ v_j D(Z, t_{n-1}, t_n; V^{n-1}, 0)\Big],\end{aligned}$$

for some constant c independent of j, where

$$D(X, t, t+s; m, r) = A_{\psi,r}^m(s) + B_{\psi,r}^m(s)X(t)$$

and the coefficients $A_{\psi,r}^m(s)$ and $B_{\psi,r}^m(s)$ are solutions of some ODEs (see Appendix A.2).

Instead of q_j^n we can compute $\bar{q}_j^n = q_j^n/c$, and get

$$p_j^n = \frac{\bar{q}_j^n}{\sum_{k\in C_{n-1}} \bar{q}_k^n}.$$

• Step (IV). Let us suppose, to fix the ideas, that the first default time and the first defaulter are, respectively, $\tau^{(1)} = t$ and $I^{(1)} = k$. The conditional expected intensity of the surviving obligor $j \neq k$ is given by

$$\mathbb{E}[\lambda_j(\tau^{(1)})|\tau^{(1)} = t, I^{(1)} = k] = \mathbb{E}[X_j(\tau^{(1)}) + u_j Y_{g(j)}(\tau^{(1)}) + v_j Z(\tau^{(1)})|\tau^{(1)} = t, I^{(1)} = k].$$

We compute separately the conditional expectations of each factor from the first derivative of the respective moment generating functions:

$$\begin{aligned}\mathbb{E}[X_j(\tau^{(1)})|\tau^{(1)} = t, I^{(1)} = k] &= \frac{\partial}{\partial r}\Big|_{r=0} H^j(r; t, k),\ \ j = 1, ..., N \\ \mathbb{E}[Y_i(\tau^{(1)})|\tau^{(1)} = t, I^{(1)} = k] &= \frac{\partial}{\partial r}\Big|_{r=0} M^i(r; t, k),\ \ i = 1, ..., S \\ \mathbb{E}[Z(\tau^{(1)})|\tau^{(1)} = t, I^{(1)} = k] &= \frac{\partial}{\partial r}\Big|_{r=0} N(r; t, k).\end{aligned}$$

To compute the moment generating functions of the factor processes, we use a result (see e.g. Duffie, [5] Section 10) which relates them to the total intensity Λ, which is the Λ^0 defined in (10). For the sake of a shorter exposition, we study here only the idiosyncratic factor, and refer to Appendix B for the other factors. The conditional moment generating function for $X_j(\tau^{(1)})$ is

$$H^j(r;t,k) := \mathbb{E}[e^{rX_j(\tau^{(1)})}|\tau^{(1)} = t, I^{(1)} = k] = \frac{\mathbb{E}[e^{-\int_0^t \Lambda(s)ds}\lambda_k(t)e^{rX_j(t)}]}{\mathbb{E}[e^{-\int_0^t \Lambda(s)ds}\lambda_k(t)]}. \tag{11}$$

The right hand side of equation (11) can be efficiently computed in closed form. In fact, thanks to the properties of the affine processes, we get

$$H^j(r;t,k) = \exp\{C(X_j, 0, t; 1, r) - C(X_j, 0, t; 1, 0)\}$$

(see Appendix B.1 for the details).

Therefore the conditional expectation is obtained as

$$\mathbb{E}[X_j(\tau^{(1)})|\tau^{(1)} = t, I^{(1)} = k] = \frac{\partial}{\partial r}\Big|_{r=0} H^j(r;t,k) = \frac{\partial}{\partial r}\Big|_{r=0} C(X_j, 0, t; 1, r).$$

It is worth noticing that, for any $j \neq k$ in $\{1, \dots, N\}$, and independently of the fact that the intensities are affine, the moment generating function $H^j(r;t,k)$ (and therefore the conditional value of X_j) is independent of the defaulter's identity $I^{(1)} = k$. In fact,

$$\begin{aligned}\mathbb{E}[e^{rX_j(\tau)}|\tau^{(1)} = t, I^{(1)} = k] &= \frac{\mathbb{E}[e^{-\int_0^t X_j(s)ds + rX_j(t)}]}{\mathbb{E}[e^{-\int_0^t X_j(s)ds}]} \\ &= \mathbb{E}[e^{rX_j(\tau^{(1)})}|\tau^{(1)} = t].\end{aligned}$$

We note that $H^j(r;t,k)$ is also independent of the number of obligors N. On the other hand, the moment generating functions of the sectoral factors and of the common factor are indeed dependent on the identity of the defaulter. To summarize this point: the identity of the defaulter does not affect the idiosyncratic factors while it has an influence on the other factors.

We proceed in an analogous way to compute the conditional moment generating functions of the common and the sectoral factors. We refer to the Appendix B for the specific results. The new, conditional values of the factors can be immediately computed from the explicit formulas for the conditional expectations.

This concludes the exposition of the exact simulation algorithm for default scenarios. An application of the algorithm is presented in Section 6.

5 Model Calibration

In this section we develop a procedure to fit the parameters of the model. The data that will be used for the calibration are:

1. The probabilities for any obligor j to survive up to a time t, for a finite set of times $\mathcal{T} := \{t_1, \dots, t_s\}$:
$$s_j^t = \mathbb{P}(\tau_j > t), \quad j = 1, \dots, N, \quad t \in \mathcal{T}.$$
2. The probabilities of general and sectoral credit events:
$$p^M(t) := \mathbb{P}(e(M,t))), \quad p^i(t) := \mathbb{P}(e(G(i),t))), \quad i = 1, \dots, S, \quad t \in \mathcal{T}.$$
3. For each obligor j, the probability of default conditioned on a general or sectoral credit event:
$$p_{j|G}(t) := \mathbb{P}(\tau^j < t | e(G(g(j)),t)), \quad p_{j|M}(t) := \mathbb{P}(\tau^j < t | e(M,t)), \quad j = 1, \dots, N, \quad t \in \mathcal{T}.$$

We observe that the data needed for calibration should be readily available, at least as to what regards the survival probabilities for each obligor. As for the probabilities of general and sectoral credit events, they may be provided by the user of the model from her own perception of the riskiness of a sector, perhaps by associating it to a given credit ranking class. The conditional probabilities of default should also be provided by the user on the basis of the sensibility of the particular agent to the general market or to the sector to which she belongs. Of course, another possibility is to calibrate the model from the prices of contracts related to the default risk of the obligors, like CDS or similar products.

To calibrate the model we can now proceed as follows:

1. Estimate the parameters sets $\hat{\psi}_c$ and $\{\hat{\psi}_{g_i},\, i = 1, ..., S\}$ that better fit the input data $\{p^M(t),\, t \in \mathcal{T}\}$ and $\{p^i(t),\, i = 1, ..., S, t \in \mathcal{T}\}$.
2. For each $j = 1, \dots, N$, from $p_{j|G}(t)$ and $p^{g(j)}(t)$, extract $\mathbb{P}(\tau^j < t, e(G(g(j)),t))$ and compute u_j that better fits such quantity
$$u_j := \text{argmin}_u \|\mathbb{P}(\tau^j < t, e(G(g(j)),t)) - (1 - f(Y_{g(j)}, t, u)))\|.$$
3. For each $j = 1, \dots, N$, from $p_{j|M}(t)$ and $p^M(t)$ extract $\mathbb{P}(\tau^j < t, e(M,t))$ and compute v_j that better fit such quantity
$$v_j := \text{argmin}_v \|\mathbb{P}(\tau^j < t, e(M,t)) - (1 - f(Z, t, v)))\|.$$

4. For each $j = 1, \dots, N$, compute

$$\hat{f}(X_j, t, 1) = \frac{s_j^t}{\hat{f}(Y_{g(j)}, t, u_j)\hat{f}(Z, t, v_j)}.$$

5. For each $j = 1, \dots, N$, from $\hat{f}(X_j, t)$ determine the parameters $\hat{\psi}_j$ of the idiosyncratic factor X_j that give the best fitting.

In this way one gets an almost perfect fitting of the marginal survival probabilities, while retaining enough flexibility for the fitting of conditional events. Moreover, by separating the calibration in three steps, the optimization procedure is greatly simplified, which is necessary because of the number of parameters involved. Of course, for practical implementation of the model, one may also consider to impose some restrictions on the parameters, to reduce their number at the expense of model flexibility.

6 An example

As a test and example of the calibration procedure, we considered a set of ninety obligors belonging to three different sectors. We took the default probabilities from Moody's transition matrix from years 1980-1999, as in Yu [21], assuming that within each sector, obligors 1 to 10 are rated Ba, obligors 11 to 20 are rated B and obligors 21 to 30 are rated Caa. We also assumed that the general credit events are rare and may be associated to ranking Aa, while the sectoral credit events are more common, respectively rated as Baa, Ba, B. This means that the third sector has higher probability of a credit event than the other two sectors. From the transition matrix we computed the respective survival probabilities for times $t = 1, 2, \dots, 10$. We considered three possible levels of dependence on general or sectoral credit events E at time $t = 1$, that is we assumed

$$p_{j|E}(t) = L, M, H$$

where $L = 0.25, M = 0.5, H = 0.75$. Within each group, obligors 1 to 10 have dependence (with their group and with the general factor) L, obligors 11 to 20 have dependence M and obligors 21-30 have dependence H.

The calibration routine was implemented in Matlab, adopting, for the optimization part, the routine "lsqnonlin", that implements standard techniques of non linear least squares. The results of the calibration of the marginal probabilities are shown in Figure 3. The top panel represent the complementary probability of a general event, the mid panel those of the three sectoral events and the bottom panel those of the ninety obligors. The input data are represented by circles, the output of the calibration procedure by crosses.

[Figure 3 about here]

From the plots it is rather evident that the fitting capabilities to the marginal probabilities are very satisfactory. The calibration error to the conditional one year probabilities are of the order of 10^{-9}, that is, they are perfectly fitted by the procedure.

Once the model is calibrated, we can use it to check the effectiveness of our exact simulation algorithm. To this purpose we computed by simulation the expectations of easily computable stopping times, so that we can compare the values obtained through simulations to the exact ones. More precisely, we considered the time of default τ_j of each obligor j and computed the expectation of $\tau_j \wedge T$, with time horizon $T = 5$. We computed the exact values of such expectations by numerical integration of the appropriate densities that can be easily recovered from the survival probabilities. The approximated values were computed with the exact simulation algorithm and with the standard discretization algorithm. Figure 4 presents the result of such computations on the ninety obligors. Both methodologies produced 1000 simulations. The discrete method used a time step of 1/500. The average cpu time to produce one simulation was 1.93 seconds for the discrete method and 1.76 seconds for the exact one. It is apparent that the exact simulation algorithm produces more precise results.

[Figure 4 about here]

According to our experience, a finer discretization grid does not improve the results produced by the standard algorithm, while considerably increasing the simulation time. For instance, for $N = 2000$ time steps per year the standard algorithm produces a mean squared error of the order of $2 \cdot 10^{-2}$, that is almost ten times greater than that of the exact algorithm, which is also more than four times faster.

7 Conclusions

We presented a model for the default processes of a large set of obligors. The model is based on the double stochastic hypothesis and assumes the the intensity of default of any obligor is a linear combination of three basic affine processes. The theory of affine processes makes many important quantities, like survival probabilities or default correlations, available in closed form. We developed a toolbox for such computations and used it for calibration and for simulation of default scenarios. An important feature of any model for large credit portfolio is the ability to produce sensible default correlations. We showed by numerical examples that the jump component of the common factors is necessary for a high enough correlation. We believe that the model has many interesting and promising features that make it a good candidate for further applications.

A Appendix: Some Useful Results

In the first part of the Appendix we collect few known results frequently used through the paper.

A.1 Intensity of the First Default Event

The following lemma characterizes the intensity of the first-to-default among a group of agents having given default intensities.

Lemma A.1 (Duffie [5], Lemma 1). *Let $\{\lambda_j, j = 1, ..., n\}$ be intensities processes associated to the event times $\{\tau_j, j = 1, ..., n\}$, and suppose $\mathbb{P}(\tau_i = \tau_j) = 0$ for any $i \neq j$. Then $\sum_{j=1}^{n} \lambda_j$ is an intensity process for $\tau := \min\{\tau_1, ..., \tau_n\}$.*

For the sake of completeness we provide a proof.

Proof. Let use the notations introduced in Section 2 and define the indicator of 'at least one default occurred before t':

$$D_t = 1_{\{\tau \leq t\}}.$$

We claim that the process A defined as

$$A_t = \int_0^t \sum_{j=1}^{n} \lambda_j(s) 1_{\{\tau > s\}} ds$$

is the compensator process of D, that is, $(D_t - A_t)_t$ is a martingale. Indeed, we have

$$D_t - A_t = 0 - \int_0^t \sum_{j=1}^n \lambda_j(s)ds = \sum_{j=1}^n (D_t^j - A_t^j), \quad \forall\, t \le \tau,$$

and

$$D_t - A_t = 1 - \int_0^\tau \sum_{j=1}^n \lambda_j(s)ds = D_\tau - A_\tau, \quad \forall\, t \ge \tau,$$

from which the claimed result follows. □

A.2 Basic Affine Processes

For the affine process in (4) we have the following useful results (see Duffie et al. [9]):

$$\mathbb{E}[e^{-\int_t^{t+s} mX(u)du + rX(t+s)}|\mathcal{F}_t] = \exp(C(X,t,t+s;m,r)), \tag{12}$$

$$\mathbb{E}[X(t+s)e^{-\int_t^{t+s} mX(u)du + rX(t+s)}|\mathcal{F}_t] = D(X,t,t+s;m,r)\exp(C(X,t,t+s;m,r)), \tag{13}$$

where

$$C(X,t,t+s;m,r) = \alpha^m_{\psi,r}(s) + \beta^m_{\psi,r}(s)X(t),$$

$$D(X,t,t+s;m,r) = A^m_{\psi,r}(s) + B^m_{\psi,r}(s)X(t)$$

and the coefficients $\alpha^m_{\psi,r}(s), \beta^m_{\psi,r}(s), A^m_{\psi,r}(s)$ and $B^m_{\psi,r}(s)$ are solutions of the following Riccati ODEs (with $\xi^m_{\psi,r}(s) = \xi(t,t+s,r,m)$ for any $t \ge 0$, and for $\xi = \alpha, \beta, A, B$):

$$\begin{aligned}
\frac{\partial \beta(t,T,r,m)}{\partial t} &= k\beta(t,T,r,m) - \frac{1}{2}\beta(t,T,r,m)^2\sigma^2 + m, \quad \beta(T,T,r,m) = r \\
\frac{\partial \alpha(t,T,r,m)}{\partial t} &= -k\theta\beta(t,T,r,m) - l\int_{\mathbb{R}} e^{\beta(t,T,r,m)z} d\nu(z) + l \\
&= -k\theta\beta(t,T,r,m) - \frac{l}{\mu}\int_0^\infty e^{z(\beta(t,T,r,m)-1/\mu)}dz + l, \quad \alpha(T,T,r,m) = 0 \\
\frac{\partial B(t,T,r,m)}{\partial t} &= kB(t,T,r,m) - \beta(t,T,r,m)B(t,T,r,m)\sigma^2, \quad B(T,T,r,m) = 1 \\
\frac{\partial A(t,T,r,m)}{\partial t} &= -k\theta B(t,T,r,m) - lB(t,T,r,m)\int_{\mathbb{R}} z e^{\beta(t,T,r,m)z} d\nu(z) \\
&= -k\theta B(t,T,r,m) + l\mu\frac{B(t,T,r,m)}{(\mu\beta(t,T,r,m)-1)^2}, \quad A(T,T,r,m) = 0.
\end{aligned}$$

For example, for $m > 0$, if we define

$$\begin{aligned}
c_1 &= \frac{k + \sqrt{k^2 + 2m\sigma^2}}{-2m}, \;\; d_1 = (1 - c_1 r)\frac{-k + r\sigma^2 + \sqrt{k^2 + 2m\sigma^2}}{-2kr + \sigma^2 r^2 - 2m}, \\
a_1 &= (d_1 + c_1)r - 1, \;\; b_1 = \frac{-d_1(+k + 2mc_1) + a_1(-kc_1 + \sigma^2)}{a_1 c_1 - d_1}, \\
a_2 &= \frac{d_1}{c_1}, \;\; b_2 = b_1, \;\; c_2 = 1 - \frac{\mu}{c_1}, \;\; d_2 = \frac{d_1 - \mu a_1}{c_1},
\end{aligned}$$

then from [7] we have

$$\beta^m_{\psi,r}(t) = \frac{1 + a_1 e^{b_1 t}}{c_1 + d_1 e^{b_1 t}}, \tag{14}$$

$$\begin{aligned}
\alpha^m_{\psi,r}(t) &= \frac{k\theta(a_1 c_1 - d_1)}{b_1 c_1 d_1} \ln \frac{c_1 + d_1 e^{b_1 t}}{c_1 + d_1} + \frac{k\theta t}{c_1} + \frac{l(a_2 c_2 - d_2)}{b_2 c_2 d_2} \ln \frac{c_2 + d_2 e^{b_2 t}}{c_2 + d_2} + \\
&\quad + tl\Big(\frac{1}{c_2} - 1\Big).
\end{aligned} \tag{15}$$

B Appendix: Conditional Moment Generating Functions

In this section we provide the closed formulas required in step (IV) of our simulation algorithm (in Section 4) in order to evaluate the value at which to re-start the default intensities after the occurrence of a default.

B.1 The Idiosyncratic Part

By using the independence of $\{X_1, \dots, X_N, Y_1, \dots, Y_S, Z\}$, the numerator in (11) can be rewritten as follows

$$\begin{aligned}
&\mathbb{E}[e^{-\int_0^t X_j(s)ds + rX_j(t)}]\Big\{\mathbb{E}[e^{-V\int_0^t Z(s)ds}]\mathbb{E}[X_k(t)e^{-\int_0^t X_k(s)ds}] \prod_{\substack{p=1,..,N \\ p\neq j,k}} \mathbb{E}[e^{-\int_0^t X_p(s)ds}] \prod_{i=1,..,S} \mathbb{E}[e^{-U_i\int_0^t Y_i(s)ds}] \\
&\qquad + \mathbb{E}[v_k Z(t) e^{-V\int_0^t Z(s)ds}] \prod_{\substack{p=1,..,N \\ p\neq j}} \mathbb{E}[e^{-\int_0^t X_p(s)ds}] \prod_{i=1,..,S} \mathbb{E}[e^{-U_i\int_0^t Y_i(s)ds}] \\
&\qquad + \mathbb{E}[e^{-V\int_0^t Z(s)ds}] \prod_{\substack{p=1,..,N \\ p\neq j}} \mathbb{E}[e^{-\int_0^t X_p(s)ds}] \prod_{\substack{i=1,..,S \\ i\neq g(k)}} \mathbb{E}[e^{-U_i\int_0^t Y_i(s)ds}]\mathbb{E}[u_k Y_{g(k)}(t) e^{-U_{g(k)}\int_0^t Y_{g(k)}(s)ds}]\Big\},
\end{aligned}$$

where we set

$$V=\sum_{j=1}^{N} v_j \quad \text{and} \quad U_i=\sum_{j:g(j)=i} u_j.$$

On the same way, the denominator of (11) is given by

$$\begin{aligned}\mathbb{E}[e^{-V\int_0^t Z(s)ds}]\mathbb{E}[X_k(t)e^{-\int_0^t X_k(s)ds}]\prod_{\substack{p=1,..,N\\ p\neq k}}\mathbb{E}[e^{-\int_0^t X_p(s)ds}]\prod_{i=1,..,S}\mathbb{E}[e^{-U_i\int_0^t Y_i(s)ds}]\\
+\mathbb{E}[v_kZ(t)e^{-V\int_0^t Z(s)ds}]\prod_{p=1,..,N}\mathbb{E}[e^{-\int_0^t X_p(s)ds}]\prod_{i=1,..,S}\mathbb{E}[e^{-U_i\int_0^t Y_i(s)ds}]\\
+\mathbb{E}[e^{-V\int_0^t Z(s)ds}]\prod_{p=1,..,N}\mathbb{E}[e^{-\int_0^t X_p(s)ds}]\prod_{\substack{i=1,..,S\\ i\neq g(k)}}\mathbb{E}[e^{-U_i\int_0^t Y_i(s)ds}]\mathbb{E}[u_kY_{g(k)}(t)e^{-U_{g(k)}\int_0^t Y_{g(k)}(s)ds}].\end{aligned}$$

After some obvious simplifications, we get

$$H^j(r;t,k)=\frac{\mathbb{E}[e^{-\int_0^t X_j(s)ds+rX_j(t)}]}{\mathbb{E}[e^{-\int_0^t X_j(s)ds}]}=\exp(C(X_j,0,t;1,r)-C(X_j,0,t;1,0)),$$

where the parameter α's and β's are solution of some Riccati ODEs (see Appendix A.2).

B.2 The Common Factor

The conditional moment generating function of the common factor Z is

$$N(r;t,k):=\mathbb{E}[e^{rZ(\tau^{(1)})}|\tau^{(1)}=t,I^{(1)}=k]=\frac{\mathbb{E}[e^{-\int_0^t \Lambda(s)ds}\lambda_k(t)e^{rZ(t)}]}{\mathbb{E}[e^{-\int_0^t \Lambda(s)ds}\lambda_k(t)]}.$$

In this case, the numerator can be obtained as

$$\begin{aligned}\mathbb{E}[e^{-V\int_0^t Z(s)ds+rZ(t)}]\mathbb{E}[X_k(t)e^{-\int_0^t X_k(s)ds}]\prod_{\substack{p=1,..,N\\ p\neq k}}\mathbb{E}[e^{-\int_0^t X_p(s)ds}]\prod_{i=1,..,S}\mathbb{E}[e^{-U_i\int_0^t Y_i(s)ds}]\\
+\mathbb{E}[v_kZ(t)e^{-V\int_0^t Z(s)ds+rZ(t)}]\prod_{p=1,..,N}\mathbb{E}[e^{-\int_0^t X_p(s)ds}]\prod_{i=1,..,S}\mathbb{E}[e^{-U_i\int_0^t Y_i(s)ds}]\\
+\mathbb{E}[e^{-V\int_0^t Z(s)ds+rZ(t)}]\prod_{p=1,..,N}\mathbb{E}[e^{-\int_0^t X_p(s)ds}]\prod_{\substack{i=1,..,S\\ i\neq g(k)}}\mathbb{E}[e^{-U_i\int_0^t Y_i(s)ds}]\mathbb{E}[u_kY_{g(k)}(t)e^{-U_{g(k)}\int_0^t Y_{g(k)}(s)ds}],\end{aligned}$$

and the denominator as

$$\mathbb{E}[e^{-V\int_0^t Z(s)ds}]\mathbb{E}[X_k(t)e^{-\int_0^t X_k(s)ds}]\prod_{\substack{p=1,..,N\\ p\neq k}}\mathbb{E}[e^{-\int_0^t X_p(s)ds}]\prod_{i=1,..,S}\mathbb{E}[e^{-U_i\int_0^t Y_i(s)ds}]$$

$$+\mathbb{E}[v_k Z(t)e^{-V\int_0^t Z(s)ds}]\prod_{p=1,..,N}\mathbb{E}[e^{-\int_0^t X_p(s)ds}]\prod_{i=1,..,S}\mathbb{E}[e^{-U_i\int_0^t Y_i(s)ds}]$$

$$+\mathbb{E}[e^{-V\int_0^t Z(s)ds}]\prod_{p=1,..,N}\mathbb{E}[e^{-\int_0^t X_p(s)ds}]\prod_{\substack{i=1,..,S\\ i\neq g(k)}}\mathbb{E}[e^{-U_i\int_0^t Y_i(s)ds}]\mathbb{E}[u_k Y_{g(k)}(t)e^{-U_{g(k)}\int_0^t Y_{g(k)}(s)ds}].$$

In this way, by (13) we obtain

$$N(r;t,k)=\frac{e^{C(Z,0,t;V,r)}[D(X_k,0,t;1,0)+v_k D(Z,0,t;V,r)+u_k D(Y_{g(k)},0,t;U_{g(k)},0)]}{e^{C(Z,0,t;V,0)}[D(X_k,0,t;1,0)+v_k D(Z,0,t;V,0)+u_k D(Y_{g(k)},0,t;U_{g(k)},0)]}$$

where again the C's and D's are specified in Appendix A.2.

Therefore, the conditional expectation is equal to

$$\mathbb{E}[Z(\tau^{(1)})|\tau^{(1)}=t,I^{(1)}=k]=\frac{\partial}{\partial r}\Big|_{r=0}N(r;t,k)=\frac{\partial}{\partial r}\Big|_{r=0}C(Z,0,t;V,r)+$$

$$\frac{v_k\frac{\partial}{\partial r}\Big|_{r=0}D(Z,0,t;V,r)}{D(X_k,0,t;1,0)+v_k D(Z,0,t;V,0)+u_k D(Y_{g(k)},0,t;U_{g(k)},0)}.$$

B.3 The Sectoral Factor

The conditional moment generating function of the sectoral factors Y_i is

$$M^i(r;t,k):=\mathbb{E}[e^{rY_i(\tau^{(1)})}|\tau^{(1)}=t,I^{(1)}=k]=\frac{\mathbb{E}[e^{-\int_0^t \Lambda(s)ds}\lambda_k(t)e^{rY_i(t)}]}{\mathbb{E}[e^{-\int_0^t \Lambda(s)ds}\lambda_k(t)]},$$

and to compute them we proceed as before by using the independence of the affine processes. So we skip the details and just emphasize the fact that here a distinction must be made whether the defaulter agent k does or does not belong to group i.

In the case $i\in\{1,..,S\}\setminus\{g(k)\}$ we have

$$\begin{aligned}M^i(r;t,k) &= \frac{\mathbb{E}[e^{-U_i\int_0^t Y_i(s)ds+rY_i(t)}]}{\mathbb{E}[e^{-U_i\int_0^t Y_i(s)ds}]}\\ &= \exp(C(Y_i,0,t;U_i,r)-C(Y_i,0,t;U_i,0)),\end{aligned}$$

and then

$$\mathbb{E}[Y_i(\tau^{(1)})|\tau^{(1)}=t,I^{(1)}=k]=\frac{\partial}{\partial r}\Big|_{r=0}M^i(r;t,k)=\frac{\partial}{\partial r}\Big|_{r=0}C(Y_i,0,t;U_i,r).$$

On the other hand, for $i = g(k)$ we obtain

$$M^i(r;t,k) = \frac{e^{C(Y_{g(k)},0,t;U_{g(k)},r)}[D(X_k,0,t;1,0) + v_k D(Z,0,t;V,0) + u_k D(Y_{g(k)},0,t;U_{g(k)},r)]}{e^{C(Y_{g(k)},0,t;U_{g(k)},0)}[D(X_k,0,t;1,0) + v_k D(Z,0,t;V,0) + u_k D(Y_{g(k)},0,t;U_{g(k)},0)]},$$

and then

$$\mathbb{E}[Y_i(\tau^{(1)})|\tau^{(1)} = t, I^{(1)} = k] = \frac{\partial}{\partial r}\Big|_{r=0} M^i(r;t,k) = \frac{\partial}{\partial r}\Big|_{r=0} C(Y_{g(k)},0,t;U_{g(k)},r) + \frac{u_k \frac{\partial}{\partial r}\Big|_{r=0} D(Y_{g(k)},0,t;U_{g(k)},r)}{D(X_k,0,t;1,0) + v_k D(Z,0,t;V,0) + u_k D(Y_{g(k)},0,t;U_{g(k)},0)}.$$

References

[1] Brigo D. and Mercurio F. (2006). Interest Rate Models - Theory and Practice: With Smile, Inflation and Credit. 2nd Edition, Springer Finance.

[2] Chapovsky A., Rennie A. and Tavares P. A. C. (2007). “Stochastic intensity modelling for structured credit exotics”, International Journal of Theoretical and Applied Finance 10 (4), 633-652.

[3] Cox J., Ingersoll J. and Ross S. (1985). “A theory of the term structure of interest rates”, Econometrica 53, 385-408.

[4] Das R. Duffie D., Kapadia N. and Saita L. (2007). “Common Failings: How Corporate Defaults Are Correlated”, Journal of Finance 62, 93-117.

[5] Duffie D. (1998). “First to Default Valuation”, Working Paper, Graduate School of Business, Stanford University.

[6] Duffie D., Filipović D. and Schachermayer W. (2003). “Affine processes and applications in finance”, Annals of Applied Probability 13, 984-1053.

[7] Duffie D. and Gârleanu N. (2001). “Risk and Valuation of Collateralized Debt Obligation”, Financial Analysts Journal 57, (1) January-February, 41-62.

[8] Duffie D. and Kan R. (1996). “A yield-factor model of interest rates”, Mathematical Finance 6, 379-406.

[9] Duffie D., Pan J. and Singleton K. (2000). “Transform Analysis and Asset Pricing for Affine Jump Diffusions”, Econometrica 68, 1343-1376.

[10] Duffie D. and Singleton K. (1998). “Simulating Correlated Defaults”, Working Paper, GSB, Stanford University.

[11] Duffie D. and Singleton K. (1999). “Modeling term structures of defaultable bonds”, Review of Financial Studies 12, 687-720.

[12] Eckner A. (2007). “Computational Techniques for basic Affine Models of Portfolio Credit Risk”, Working Paper, Stanford University.

[13] Feller W. (1951). “Two singular diffusion problems”, Annals of Mathematics 54 , 173-182.

[14] Higham D.J. and Mao X. (2005). “Convergence of Monte Carlo simulations involving the mean- reverting square root process”, Journal of Computational Finance 8 , 35-62.

[15] Hull J. and White A. (2001). “Valuing credit default swaps II: Modeling default correlations”, Journal of Derivatives 8, 12-22.

[16] Jarrow R. and Turnbull S. (1995). “Pricing derivatives on financial securities subject to credit risk”, Journal of Finance 50(1), 53-86.

[17] Lando D. (1998). “On Cox processes and Credit Risk Securities”, Review of Derivatives Research 2(2-3), 99-120.

[18] Mortensen A. (2006). “Semi-Analytical Valuation of Basket Credit Derivatives in Intensity-Based Models”, Journal of Derivatives 13, 8-26.

[19] Schönbucher P. (2003). Credit Derivatives Pricing Models: Models, Pricing and Implementation. John Wiley and Sons Ltd.

[20] Schönbucher P. and Schubert D. (2001). “Copula-dependent default risk in intensity models”, Working paper, Bonn University

[21] Yu F. (2005). “Default correlation in reduced-form models”, Journal of Investment Management 3(4), 33-42.

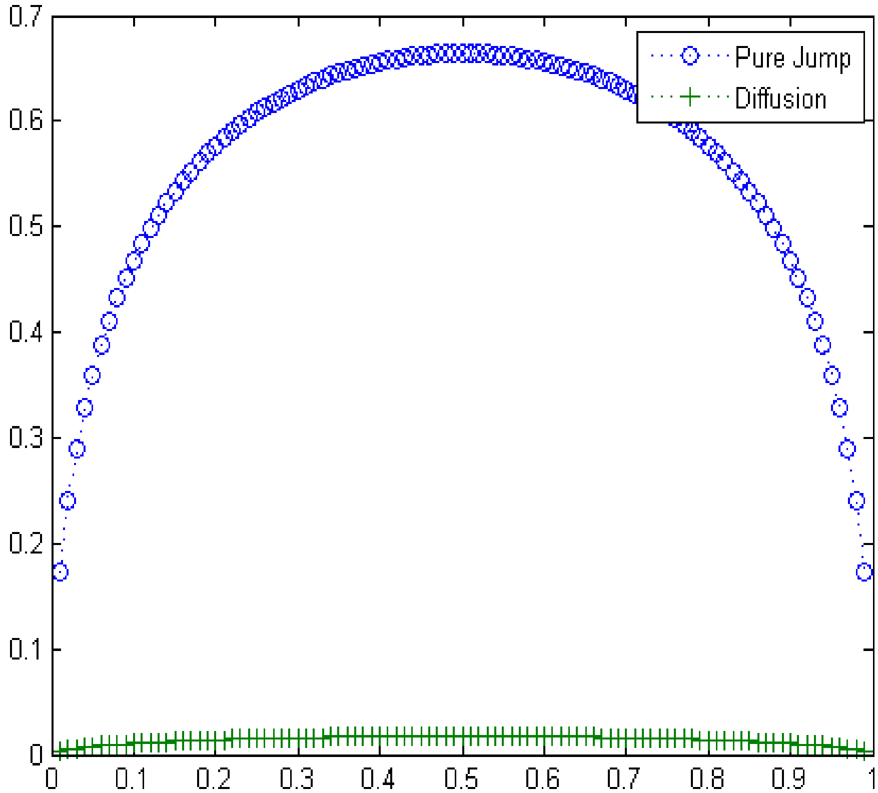


Figure 1: Comparison of the correlation between defaults, obtained in a toy example where the common factor of the default intensities is either a pure-jump process (Case 1) or a diffusion process (Case 2).

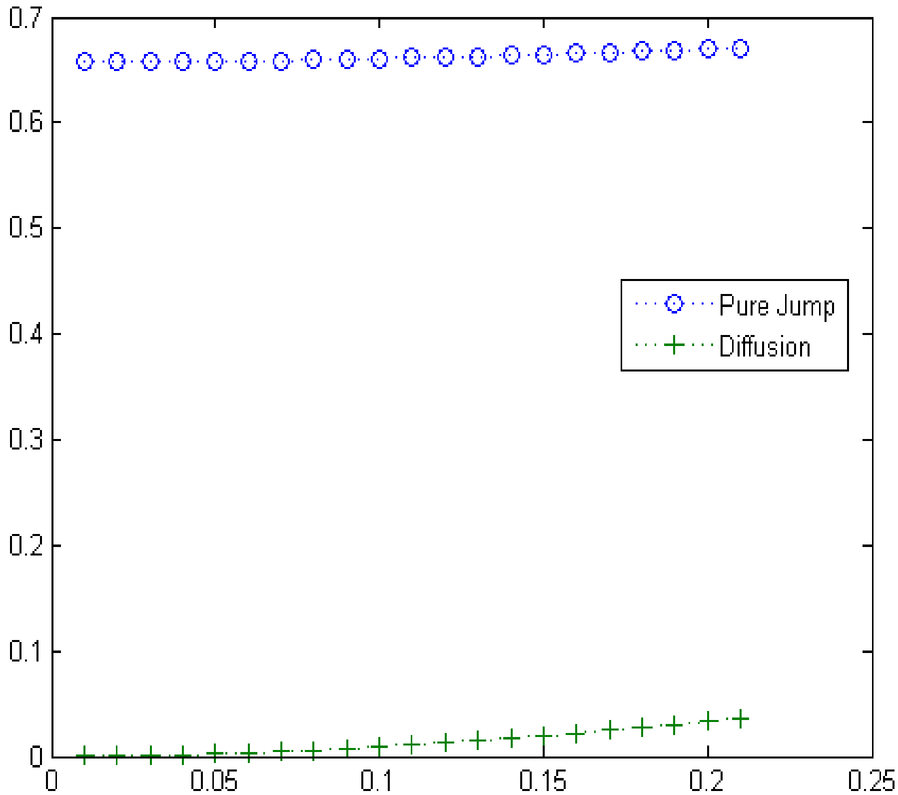
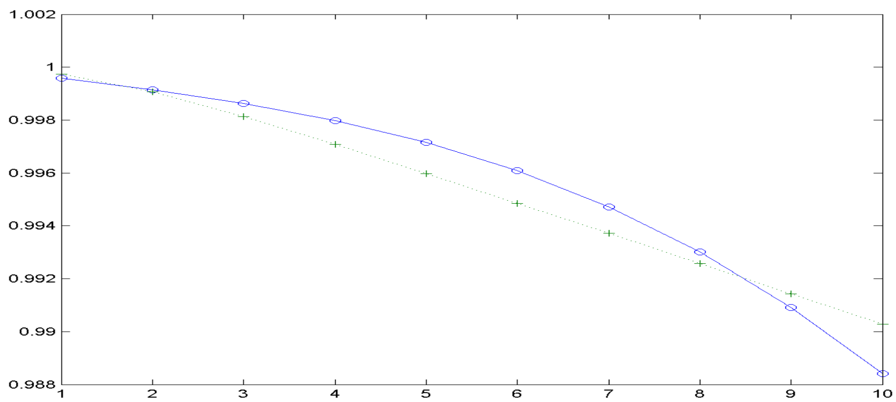
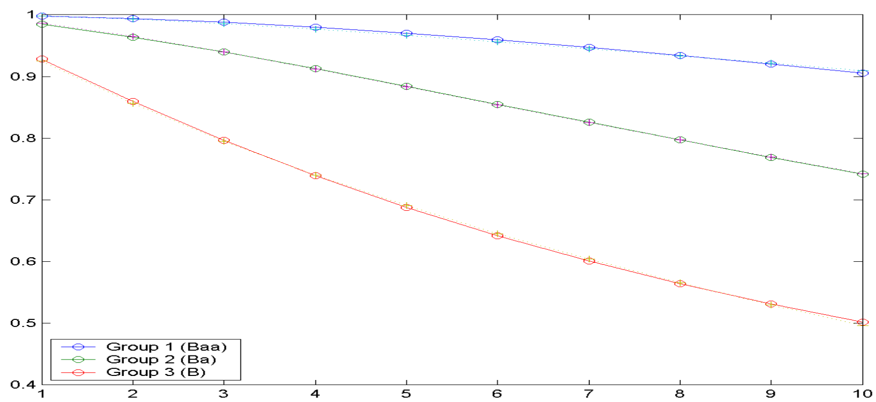


Figure 2: Comparison of the correlation between defaults, obtained in a toy example where the common factor of the default intensities is either a pure-jump process (Case 1) or a diffusion process (Case 2).





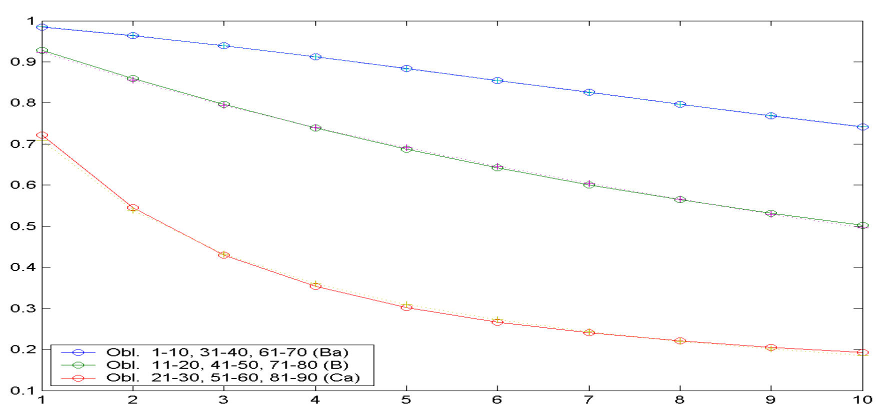


Figure 3: Comparison of the calibrated data to the input data. The three figures report the marginal survival probabilities for the general (top), sectoral (middle) and individual (bottom) credit events. The input data are represented with circles, the output from the calibration procedure are crosses. Note that scales on the three plots are different.

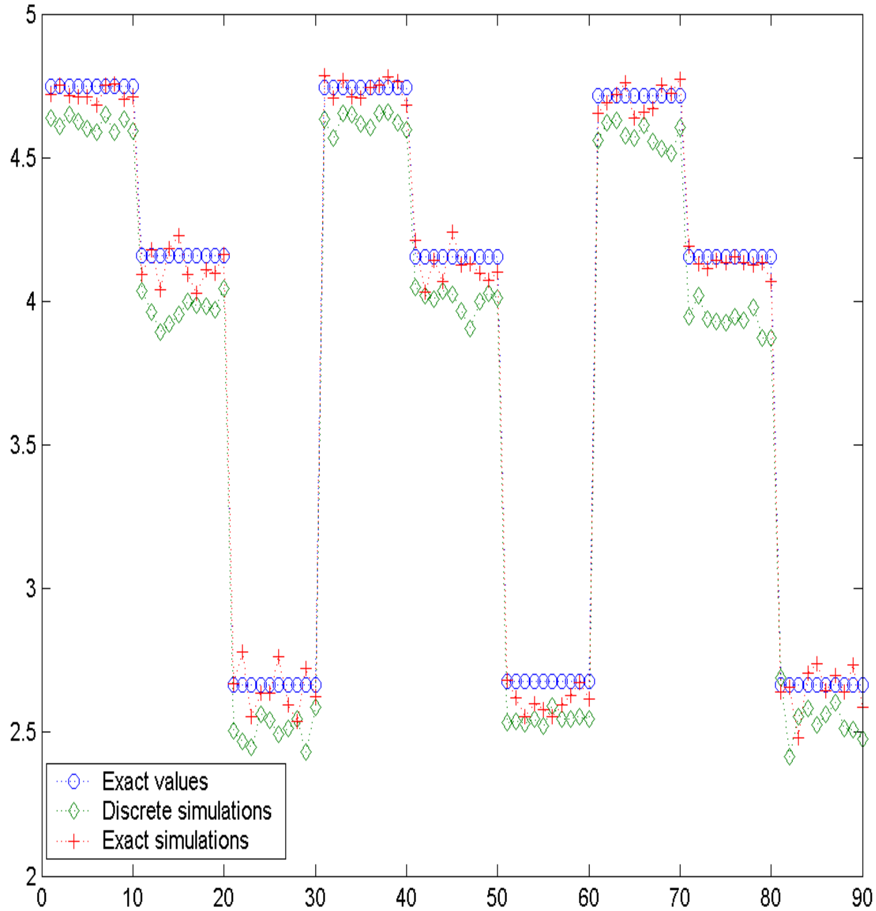


Figure 4: A comparison of the simulation methods. The figure shows the expected values of the minimum between the time of defaults and $T = 5$ for each obligor. The exact values (circles) are computed by numerical quadrature. The approximation were obtained by 1000 simulated scenarios with the exact and the discrete method. The discrete method adopted a time step equal to 1/500. The exact algorithm was 9% faster than the discrete algorithm.