Ramsey numbers of hypergraphs of a given size

Domagoj Bradač

ETH Zürich

joint work with Jacob Fox and Benny Sudakov

Ramsey numbers

Definition

The q-color Ramsey number of a k-uniform hypergraph H, denoted by $r^{(k)}(H;q)$ is the smallest integer N such that in every q-edge coloring of $K_N^{(k)}$, there is a monochromatic copy of H.

Ramsey numbers

Definition

The q-color Ramsey number of a k-uniform hypergraph H, denoted by $r^{(k)}(H;q)$ is the smallest integer N such that in every q-edge coloring of $K_N^{(k)}$, there is a monochromatic copy of H.

Theorem (Ramsey '30)

These numbers exist for every q, k, H.

Ramsey numbers

Definition

The q-color Ramsey number of a k-uniform hypergraph H, denoted by $r^{(k)}(H;q)$ is the smallest integer N such that in every q-edge coloring of $K_N^{(k)}$, there is a monochromatic copy of H.

Theorem (Ramsey '30)

These numbers exist for every q, k, H.

- $\sqrt{2}^n < r^{(2)}(K_n;2) < 3.993^n$ (Erdős '47; Campos, Griffiths, Morris, Sahasrabudhe '23+).

Ramsey numbers

Definition

The q-color Ramsey number of a k-uniform hypergraph H, denoted by $r^{(k)}(H;q)$ is the smallest integer N such that in every q-edge coloring of $K_N^{(k)}$, there is a monochromatic copy of H.

Theorem (Ramsey '30)

These numbers exist for every q, k, H.

- $\sqrt{2}^n < r^{(2)}(K_n;2) < 3.993^n$ (Erdős '47; Campos, Griffiths, Morris, Sahasrabudhe '23+).
- For $k \geq 3$, $\mathrm{tw}_{k-1}(\Omega(n^2)) \leq r^{(k)}(K_n;2) \leq \mathrm{tw}_k(O(n))$ and $r^{(k)}(K_n;4) = \mathrm{tw}_k(\Theta(n))$ (Erdős, Hajnal, Rado).

$$\mathrm{tw}_1(x) = x,\ \mathrm{tw}_2(x) = 2^x,\ \mathrm{tw}_3(x) = 2^{2^x}, \ldots$$

Ramsey numbers of graphs with m edges

Conjecture (Erdős, Graham '75)

For any graph G with $\binom{n}{2}$ edges, $r(G;2) \leq r(K_n;2)$.

Ramsey numbers of graphs with m edges

Conjecture (Erdős, Graham '75)

For any graph G with $\binom{n}{2}$ edges, $r(G;2) \leq r(K_n;2)$.

Conjecture (Erdős '84)

For any graph G with m edges, $r(G;2) = 2^{O(\sqrt{m})}$.

Ramsey numbers of graphs with m edges

Conjecture (Erdős, Graham '75)

For any graph G with $\binom{n}{2}$ edges, $r(G;2) \leq r(K_n;2)$.

Conjecture (Erdős '84)

For any graph G with m edges, $r(G;2) = 2^{O(\sqrt{m})}$.

Theorem (Sudakov '11)

For any graph G with m edges, $r(G;2) = 2^{O(\sqrt{m})}$.

Ramsey numbers of graphs with m edges

Conjecture (Erdős, Graham '75)

For any graph G with $\binom{n}{2}$ edges, $r(G;2) \leq r(K_n;2)$.

Conjecture (Erdős '84)

For any graph G with m edges, $r(G;2) = 2^{O(\sqrt{m})}$.

Theorem (Sudakov '11)

For any graph G with m edges, $r(G;2) = 2^{O(\sqrt{m})}$.

Question

Let $k \geq 3$. What is the largest value of $r^{(k)}(G;q)$ for k-graph G with m edges?

Ramsey numbers of graphs with m edges

Conjecture (Erdős, Graham '75)

For any graph G with $\binom{n}{2}$ edges, $r(G;2) \leq r(K_n;2)$.

Conjecture (Erdős '84)

For any graph G with m edges, $r(G;2) = 2^{O(\sqrt{m})}$.

Theorem (Sudakov '11)

For any graph G with m edges, $r(G;2) = 2^{O(\sqrt{m})}$.

Question

Let $k \geq 3$. What is the largest value of $r^{(k)}(G;q)$ for k-graph G with m edges? Is it $\mathrm{tw}_k(\Theta(m^{1/k}))$?

Question

Let $k \geq 3$. What is the largest value of $r^{(k)}(G; q)$ for k-graph G with m edges? Is it $\mathrm{tw}_k(\Theta(m^{1/k}))$?

Ramsey numbers of graphs with m edges

Question

Let $k \geq 3$. What is the largest value of $r^{(k)}(G;q)$ for k-graph G with m edges? Is it $\mathrm{tw}_k(\Theta(m^{1/k}))$?

Theorem (Conlon, Fox, Sudakov '09)

No! There is a 3-graph H with m edges for which $r^{(3)}(H;4) > 2^{2^{c\sqrt{m}}}$.

Ramsey numbers of graphs with m edges

Question

Let $k \geq 3$. What is the largest value of $r^{(k)}(G;q)$ for k-graph G with m edges? Is it $\mathrm{tw}_k(\Theta(m^{1/k}))$?

Theorem (Conlon, Fox, Sudakov '09)

No! There is a 3-graph H with m edges for which $r^{(3)}(H;4) > 2^{2^{c\sqrt{m}}}$. On other hand, for every k-graph H with m edges:

$$r^{(k)}(H;q) = \begin{cases} \mathrm{tw}_3(O(\sqrt{m}\log m)), & \text{if } k = 3, \\ \mathrm{tw}_k(O(\sqrt{m})), & \text{if } k \geq 4. \end{cases}$$

Results

Theorem (B., Fox, Sudakov '24+)

Let $k \geq 3$ and $q \geq 2$ be fixed. For any k-uniform hypergraph H with m edges and no isolated vertices, it holds that

$$r^{(k)}(H; q) \leq \mathrm{tw}_k(c_{k,q}\sqrt{m}).$$

Results

Theorem (B., Fox, Sudakov '24+)

Let $k \geq 3$ and $q \geq 2$ be fixed. For any k-uniform hypergraph H with m edges and no isolated vertices, it holds that

$$r^{(k)}(H; q) \leq \mathrm{tw}_k(c_{k,q}\sqrt{m}).$$

Up to the constant $c_{k,q}$ this is tight for all $k \geq 3$ and $q \geq 4$.

Stepping-up: the function δ

For distinct nonnegative integers x, y, define $\delta(x, y)$ as the index counted from the right of the most significant bit at which their binary representations differ.

Stepping-up: the function δ

For distinct nonnegative integers x, y, define $\delta(x, y)$ as the index counted from the right of the most significant bit at which their binary representations differ.
E.g. $\delta(2, 3) = \delta(010_2, 011_2) = 1, \delta(5, 7) = \delta(101_2, 111_2) = 2.$

Stepping-up: the function δ

For distinct nonnegative integers x, y, define $\delta(x, y)$ as the index counted from the right of the most significant bit at which their binary representations differ.

E.g. $\delta(2, 3) = \delta(010_2, 011_2) = 1, \delta(5, 7) = \delta(101_2, 111_2) = 2.$

- If $x < y < z$, then $\delta(x, y) \neq \delta(y, z)$, and
- $\delta(x, z) = \max\{\delta(x, y), \delta(y, z)\}.$
- If $x_1 < x_2 < \cdots < x_t$, then $\delta(x_1, x_t) = \max_{1 \leq i \leq t-1} \delta(x_i, x_{i+1})$

Stepping-up construction

Suppose $k \geq 4$ and we have a coloring $\phi^{(k-1)}\binom{[N]}{k-1} \to \{A, B, C, D\}$.

Stepping-up construction

Suppose $k \geq 4$ and we have a coloring $\phi^{(k-1)}\binom{[N]}{k-1} \to \{A, B, C, D\}$. We define $\phi^{(k)}\binom{[2^N]}{k} \to \{A, B, C, D\}$. For $x_1 < x_2 < \cdots < x_k$, consider the sequence $\mathrm{s} = (\delta_1, \ldots, \delta_{k-1})$, where $\delta_i = \delta(x_i, x_{i+1})$.

Stepping-up construction

Suppose $k \geq 4$ and we have a coloring $\phi^{(k-1)} \binom{[N]}{k-1} \to \{A, B, C, D\}$. We define $\phi^{(k)} \binom{[2^N]}{k} \to \{A, B, C, D\}$. For $x_1 < x_2 < \cdots < x_k$, consider the sequence $\mathrm{s} = (\delta_1, \ldots, \delta_{k-1})$, where $\delta_i = \delta(x_i, x_{i+1})$.

$$\phi^{(k)}(\{x_1, \ldots, x_k\}) = \begin{cases} \phi^{(k-1)}(\{\delta_1, \ldots, \delta_{k-1}\}) & \text{if s is monotone, else} \\ A, & \text{if } \max \delta_i \in \{\delta_1, \delta_{k-1}\}, \\ B, & \text{if } \max \delta_i \notin \{\delta_1, \delta_{k-1}\}. \end{cases}$$

Stepping-up construction

Suppose $k \geq 4$ and we have a coloring $\phi^{(k-1)}\binom{[N]}{k-1} \to \{A, B, C, D\}$. We define $\phi^{(k)}\binom{[2^N]}{k} \to \{A, B, C, D\}$. For $x_1 < x_2 < \cdots < x_k$, consider the sequence $\mathsf{s} = (\delta_1, \ldots, \delta_{k-1})$, where $\delta_i = \delta(x_i, x_{i+1})$.

$$\phi^{(k)}(\{x_1, \ldots, x_k\}) = \begin{cases} \phi^{(k-1)}(\{\delta_1, \ldots, \delta_{k-1}\}) & \text{if s is monotone, else} \\ A, & \text{if } \max \delta_i \in \{\delta_1, \delta_{k-1}\}, \\ B, & \text{if } \max \delta_i \notin \{\delta_1, \delta_{k-1}\}. \end{cases}$$

For $k = 3$, we start with a two-coloring $\phi^{(2)}$ of $\binom{[N]}{2}$ and then $\phi^{(3)}(\{x_1, x_2, x_3\})$ records $\phi^{(2)}(\{\delta_1, \delta_2\})$ and whether $\delta_1 > \delta_2$.

Stepping-up construction lower bound

Erdős, Hajnal: $r^{(k)}(t;4) \geq 2^{r^{(k-1)}(t/2;4)}$.

Stepping-up construction lower bound

Erdős, Hajnal: $r^{(k)}(t;4) \geq 2^{r^{(k-1)}(t/2;4)}$.
Coloring $\phi^{(k-1)}\binom{[N]}{k-1} \to \{A,B,C,D\}$ with no monochromatic clique of size $t/2$. We defined $\phi^{(k)}\binom{[2^N-1]}{k} \to \{A,B,C,D\}$.

Stepping-up construction lower bound

Erdős, Hajnal: $r^{(k)}(t;4) \geq 2^{r^{(k-1)}(t/2;4)}$.

Coloring $\phi^{(k-1)}\binom{[N]}{k-1} \to \{A, B, C, D\}$ with no monochromatic clique of size $t/2$. We defined $\phi^{(k)}\binom{[2^N-1]}{k} \to \{A, B, C, D\}$.

- Assume $x_1 < x_2 < \cdots < x_t$ monochromatic t-clique in $\phi^{(k)}$.

Stepping-up construction lower bound

Erdős, Hajnal: $r^{(k)}(t;4) \geq 2^{r^{(k-1)}(t/2;4)}$.
Coloring $\phi^{(k-1)}\binom{[N]}{k-1} \to \{A,B,C,D\}$ with no monochromatic clique of size $t/2$. We defined $\phi^{(k)}\binom{[2^N-1]}{k} \to \{A,B,C,D\}$.

- Assume $x_1 < x_2 < \cdots < x_t$ monochromatic t-clique in $\phi^{(k)}$.
- Consider $(\delta_1, \ldots, \delta_{t-1})$ with $\delta_i = \delta(x_i, x_{i+1})$.

Stepping-up construction lower bound

Erdős, Hajnal: $r^{(k)}(t;4) \geq 2^{r^{(k-1)}(t/2;4)}$.
Coloring $\phi^{(k-1)}\binom{[N]}{k-1} \to \{A,B,C,D\}$ with no monochromatic clique of size $t/2$. We defined $\phi^{(k)}\binom{[2^N-1]}{k} \to \{A,B,C,D\}$.

- Assume $x_1 < x_2 < \cdots < x_t$ monochromatic t-clique in $\phi^{(k)}$.
- Consider $(\delta_1, \ldots, \delta_{t-1})$ with $\delta_i = \delta(x_i, x_{i+1})$.
- $\implies \exists$ monotone interval of length $t/2$.

Stepping-up construction lower bound

Erdős, Hajnal: $r^{(k)}(t;4) \geq 2^{r^{(k-1)}(t/2;4)}$.

Coloring $\phi^{(k-1)}\binom{[N]}{k-1} \to \{A,B,C,D\}$ with no monochromatic clique of size $t/2$. We defined $\phi^{(k)}\binom{[2^N-1]}{k} \to \{A,B,C,D\}$.

- Assume $x_1 < x_2 < \cdots < x_t$ monochromatic t-clique in $\phi^{(k)}$.
- Consider $(\delta_1,\ldots,\delta_{t-1})$ with $\delta_i = \delta(x_i, x_{i+1})$.
- $\implies \exists$ monotone interval of length $t/2$.
- $\implies \exists$ monochromatic $(t/2)$-clique in $\phi^{(k-1)}$, $\Rightarrow\!\Leftarrow$.

Stepping-up construction lower bound

Erdős, Hajnal: $r^{(k)}(t;4) \geq 2^{r^{(k-1)}(t/2;4)}$.
Coloring $\phi^{(k-1)}\binom{[N]}{k-1} \to \{A,B,C,D\}$ with no monochromatic clique of size $t/2$. We defined $\phi^{(k)}\binom{[2^N-1]}{k} \to \{A,B,C,D\}$.

- Assume $x_1 < x_2 < \cdots < x_t$ monochromatic H_k on t vertices in $\phi^{(k)}$.
- Consider $(\delta_1, \ldots, \delta_{t-1})$ with $\delta_i = \delta(x_i, x_{i+1})$.
- $\implies \exists$ monotone interval of length $t/1000$.
- $\implies \exists$ monochromatic H_{k-1} on $t/1000$ vertices in $\phi^{(k-1)}$, $\Rightarrow\!\Leftarrow$.

Lower bound construction

Let G be a graph and $k \geq 2$. Define a k-uniform hypergraph $H = H(G, k)$:

- $V(H) = V(G)$.

Lower bound construction

Let G be a graph and $k \geq 2$. Define a k-uniform hypergraph $H = H(G, k)$:

- $V(H) = V(G)$.
- $E(H) = \{\{x_1, \ldots, x_{k-1}, y\} \,|\, (x_1, \ldots, x_{k-1})$ is a path in $G\}$.

Lower bound construction

Let G be a graph and $k \geq 2$. Define a k-uniform hypergraph $H = H(G,k)$:

- $V(H) = V(G)$.
- $E(H) = \{\{x_1, \ldots, x_{k-1}, y\} \,|\, (x_1, \ldots, x_{k-1})$ is a path in $G\}$.

Note: $H(G,2) = K_{v(G)}$.

Lower bound construction

Let G be a graph and $k \geq 2$. Define a k-uniform hypergraph $H = H(G, k)$:

- $V(H) = V(G)$.
- $E(H) = \{\{x_1, \ldots, x_{k-1}, y\} \,|\, (x_1, \ldots, x_{k-1})$ is a path in $G\}$.

Note: $H(G, 2) = K_{v(G)}$.

Let G be an n-vertex expander with degree $d = 10^{20k}$ and $H = H(G, k)$.

So $e(H) = O(n^2)$.

Lower bound construction

Let G be a graph and $k \geq 2$. Define a k-uniform hypergraph $H = H(G, k)$:

- $V(H) = V(G)$.
- $E(H) = \{\{x_1, \ldots, x_{k-1}, y\} \,|\, (x_1, \ldots, x_{k-1})$ is a path in $G\}$.

Note: $H(G, 2) = K_{v(G)}$.

Let G be an n-vertex expander with degree $d = 10^{20k}$ and $H = H(G, k)$. So $e(H) = O(n^2)$.

Let $\phi^{(2)}$ be a 2-coloring on $2^{c_k n}$ vertices with no monochromatic clique of size $2c_k n$. Step up $k - 2$ times
$\to$ 4-colorings $\phi^{(3)}, \phi^{(4)}, \ldots, \phi^{(k)}$, where $\phi^{(r)}$ is on $\mathrm{tw}_r(c_k n)$ vertices.

Lower bound construction

Let G be a graph and $k \geq 2$. Define a k-uniform hypergraph $H = H(G, k)$:

- $V(H) = V(G)$.
- $E(H) = \{\{x_1, \dots, x_{k-1}, y\} \mid (x_1, \dots, x_{k-1})$ is a path in $G\}$.

Note: $H(G, 2) = K_{v(G)}$.

Let G be an n-vertex expander with degree $d = 10^{20k}$ and $H = H(G, k)$. So $e(H) = O(n^2)$.

Let $\phi^{(2)}$ be a 2-coloring on $2^{c_k n}$ vertices with no monochromatic clique of size $2c_k n$. Step up $k - 2$ times
$\to$ 4-colorings $\phi^{(3)}, \phi^{(4)}, \dots, \phi^{(k)}$, where $\phi^{(r)}$ is on $\mathrm{tw}_r(c_k n)$ vertices.

Lemma

If in $\phi^{(k)}$ there is a monochromatic copy of $H(G, k)$, then in $\phi^{(k-1)}$ there is a monochromatic copy of $H(G[U], k-1)$ with $U \subseteq V(G)$, $|U| \geq n/1000$.

Lower bound construction

Let G be a graph and $k \geq 2$. Define a k-uniform hypergraph $H = H(G, k)$:

- $V(H) = V(G)$.
- $E(H) = \{\{x_1, \ldots, x_{k-1}, y\} \mid (x_1, \ldots, x_{k-1})$ is a path in $G\}$.

Note: $H(G, 2) = K_{v(G)}$.

Let G be an n-vertex expander with degree $d = 10^{20k}$ and $H = H(G, k)$.
So $e(H) = O(n^2)$.

Let $\phi^{(2)}$ be a 2-coloring on $2^{c_k n}$ vertices with no monochromatic clique of size $2c_k n$. Step up $k - 2$ times
$\rightarrow$ 4-colorings $\phi^{(3)}, \phi^{(4)}, \ldots, \phi^{(k)}$, where $\phi^{(r)}$ is on $\mathrm{tw}_r(c_k n)$ vertices.

Lemma

If in $\phi^{(k)}$ there is a monochromatic copy of $H(G, k)$, then in $\phi^{(k-1)}$ there is a monochromatic copy of $H(G[U], k-1)$ with $U \subseteq V(G)$, $|U| \geq n/1000$.

$\implies \exists U \subseteq V(G), |U| \geq n/1000^{k-2}$ s.t. there is a monochromatic copy of $H(G[U], 2) = K_{|U|}$ in $\phi^{(2)}$, $\Rightarrow\!\Leftarrow$.

Upper bound

Ingredients:

Upper bound

Ingredients:

Lemma

Let H be a k-uniform hypergraph with m edges and no isolated vertices. Then, H is a subgraph of $K^{(k)}_{s_1,\ldots,s_t}$ for some $t, s_1, \ldots, s_t$ satisfying

- $t = O(\sqrt{m})$,
- $\prod_{i=1}^{t} s_i = 2^{O(\sqrt{m})}$.

Upper bound

Ingredients:

Lemma

Let H be a k-uniform hypergraph with m edges and no isolated vertices. Then, H is a subgraph of $K^{(k)}_{s_1,\ldots,s_t}$ for some $t, s_1, \ldots, s_t$ satisfying

- $t = O(\sqrt{m})$,
- $\prod_{i=1}^{t} s_i = 2^{O(\sqrt{m})}$.

$$r^{(k)}(K_t; q) \leq \mathrm{tw}_k(O(t)).$$

Upper bound

Ingredients:

Lemma

Let H be a k-uniform hypergraph with m edges and no isolated vertices. Then, H is a subgraph of $K^{(k)}_{s_1,\dots,s_t}$ for some $t, s_1, \dots, s_t$ satisfying

- $t = O(\sqrt{m})$,
- $\prod_{i=1}^{t} s_i = 2^{O(\sqrt{m})}$.

$$r^{(k)}(K_t; q) \le \mathrm{tw}_k(O(t)).$$

Supersaturation argument: many monochromatic t-cliques in one color.

Upper bound

Ingredients:

Lemma

Let H be a k-uniform hypergraph with m edges and no isolated vertices. Then, H is a subgraph of $K^{(k)}_{s_1,\ldots,s_t}$ for some $t, s_1, \ldots, s_t$ satisfying

- $t = O(\sqrt{m})$,
- $\prod_{i=1}^{t} s_i = 2^{O(\sqrt{m})}$.

$$r^{(k)}(K_t; q) \leq \mathrm{tw}_k(O(t)).$$

Supersaturation argument: many monochromatic t-cliques in one color.

Theorem (Erdős '64)

Let $s_1, \ldots, s_t \geq 1$ and denote $P = \prod_{i=1}^{t} s_i$. Then for all $n \geq 1$,

$$\mathit{ex}(n, K^{(k)}_{s_1,\ldots,s_t}) \leq P n^{t-1/P}.$$

Theorem (B., Fox, Sudakov '24+)

Let $k \geq 3$ and $q \geq 2$ be fixed. For any k-uniform hypergraph H with m edges and no isolated vertices, it holds that

$$r^{(k)}(H; q) \leq \mathrm{tw}_k(c_{k,q}\sqrt{m}).$$

Up to the constant $c_{k,q}$ this is tight for all $k \geq 3$ and $q \geq 4$.

Thank you!