Effective bounds for induced size-Ramsey numbers of cycles

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joint work with Nemanja Draganić and Benny Sudakov

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Ramsey numbers

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The k-color Ramsey number of H, denoted by $r^k(H)$, is defined as $r^k(H) = \min\{v(G) \mid G \xrightarrow{k} H\}$.

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- a path (Beck '83),
- a bounded degree tree (Friedman, Pippenger '87),
- a cycle (Haxell, Kohayakawa, Łuczak '95),
- a bounded degree graph with bounded treewidth (Kamčev, Liebenau, Wood, Yepremyan '21; Berger, Kohayakawa, Maesaka, Martins, Mendonça, Mota, Parczyk '21),
- a logarithmic subdivision of a bounded degree graph (Draganić, Krivelevich, Nenadov '22).

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However, $\hat{r}^2(H)$ is not linear in $v(H)$ for all bounded degree graphs (Rödl, Szemerédi '00; Tikhomirov '22+).

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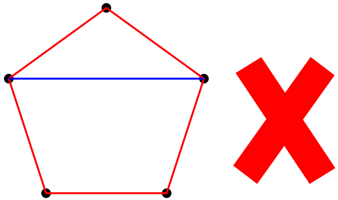

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For every k, there is $C = C(k)$ such that $\hat{r}^k_{\text{ind}}(P_n), \hat{r}^k_{\text{ind}}(C_n) \leq Cn$.

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Question

What is the best value of $C = C(k)$ for cycles in the Theorem above?

Previous results

	Lower bound		Upper bound	
$\hat{r}^k(P_n)$	$\Omega(k^2)n$	(DP '17)	$O(k^2 \log k)n$	(K '19)
$\hat{r}^k_{\text{ind}}(P_n)$	$\Omega(k^2)n$	(DP '17)	$O(k^3 \log^4 k)n$	(DGK '22)
$\hat{r}^k(C_n)$, n even	$\Omega(k^2)n$	(DP '17)	$O(k^{120} \log^2 k)n$	(JM '23)
$\hat{r}^k(C_n)$, n odd	$2^{k-1}n$	(JM '23)	$O(2^{k^2+16\log k})n$	(JM '23)
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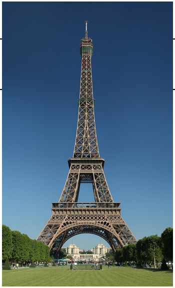

Our results

Theorem (B., Draganić, Sudakov '23+)

For any $k \geq 1$, there is n_0 such that for $n \geq n_0$, the following holds.

- $\hat{r}^k(C_n) = 2^{O(k)} n$.
- *If n is even, then* $\hat{r}^k_{\mathrm{ind}}(C_n) = O(k^{102})n$.
- *If n is odd, then* $\hat{r}^k_{\mathrm{ind}}(C_n) = 2^{O(k \log k)} n$.

Overview of results

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Our new host graph construction is designed to exploits this.

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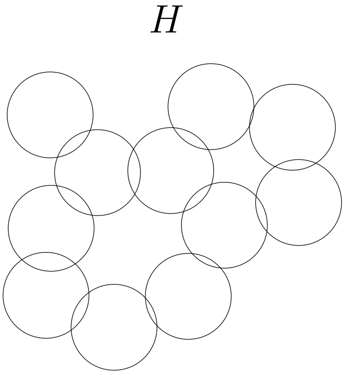
- Auxiliary graph G with $V(G) = V(\Gamma)$ and edges: for each placed copy of F, find one monochromatic induced C_5 and connect two nonadjacent vertices on this C_5.

Host graph and auxiliary graph



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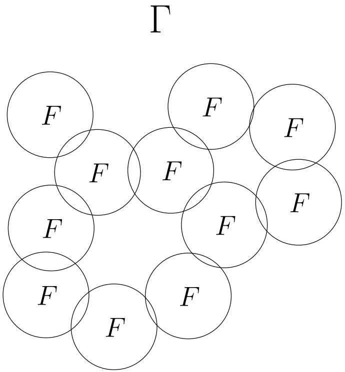

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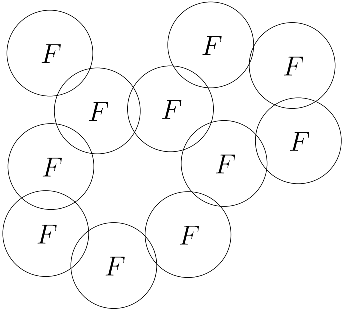

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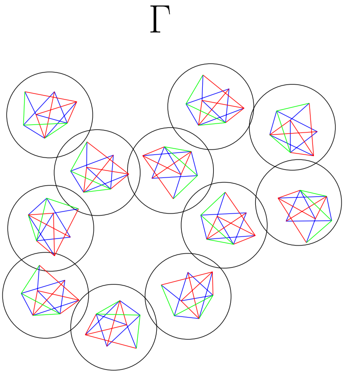

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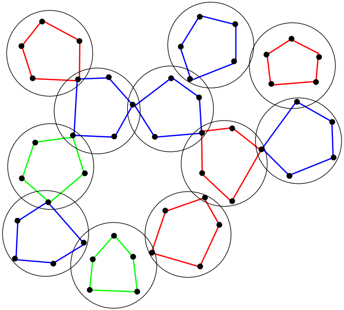

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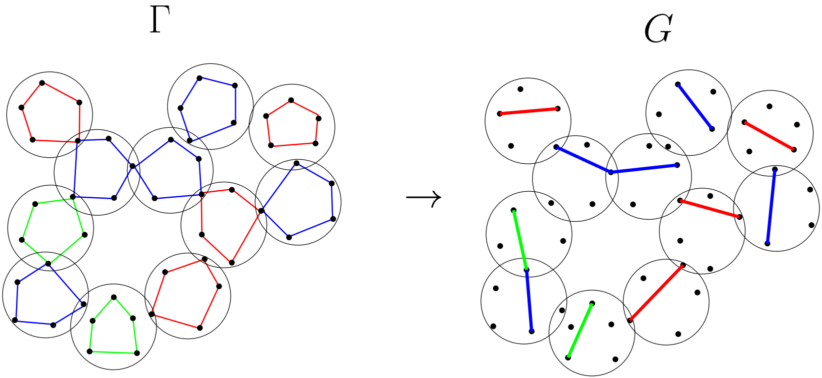

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Claim: a "good" monochromatic cycle in G of any length $\ell \in [n/3, n/2]$ gives an induced monochromatic cycle of length n in Γ.

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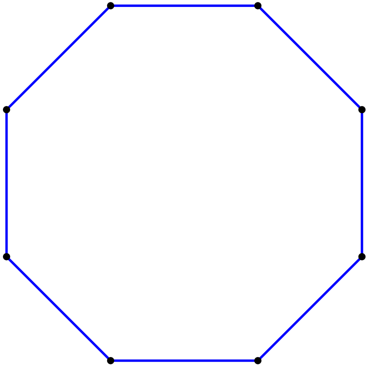

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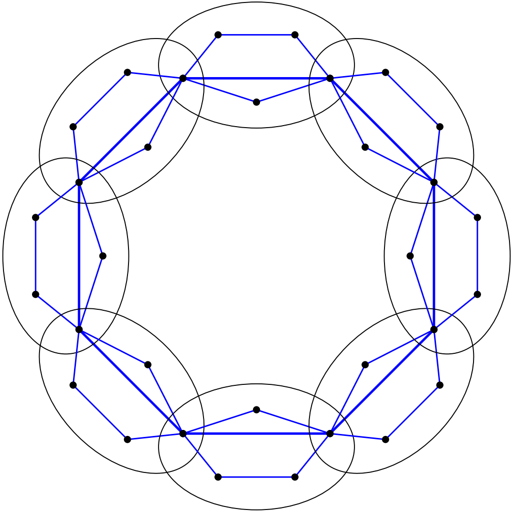

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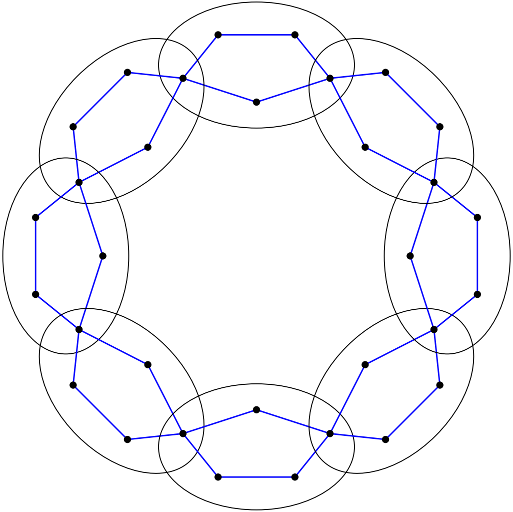

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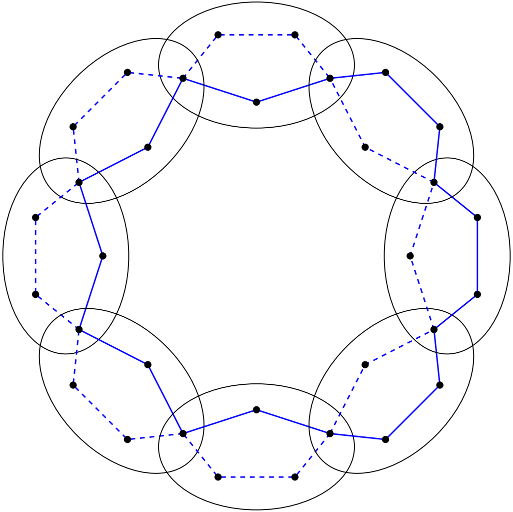

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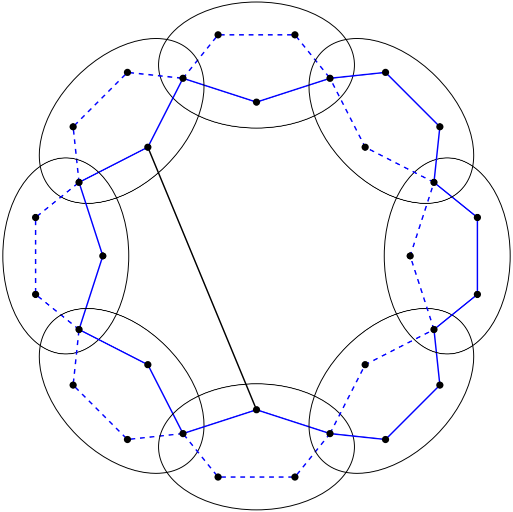

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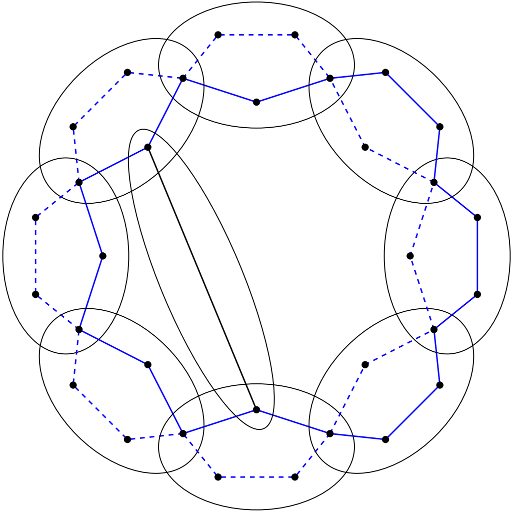

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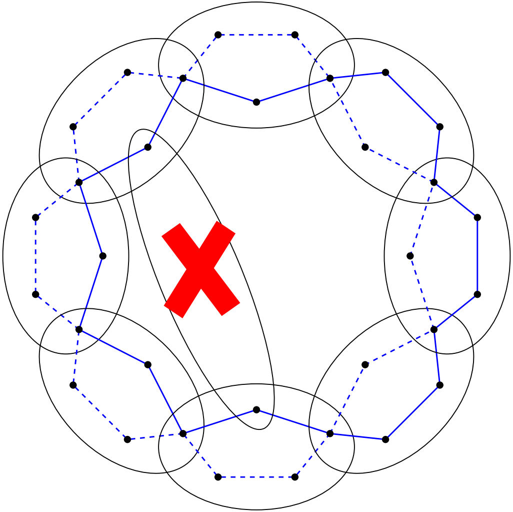

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Main tool: modification of the DFS algorithm for induced paths developed by Draganić, Glock and Krivelevich.

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- For the odd (non-induced) we want every k-edge-coloring of F to have an odd monochromatic cycle. We take $F = K_{2^k+1}$.

Concluding remarks

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