Powers of Hamilton cycles of high discrepancy are unavoidable

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- A K_r-tiling of a graph is a partition of its vertices into disjoint r-cliques.

Theorem (Hajnal, Szemerédi, 1972)

If r divides n then any graph G with $\delta(G) \geq (1 - 1/r)n$ contains a K_r-tiling.

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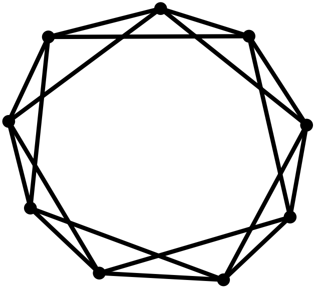

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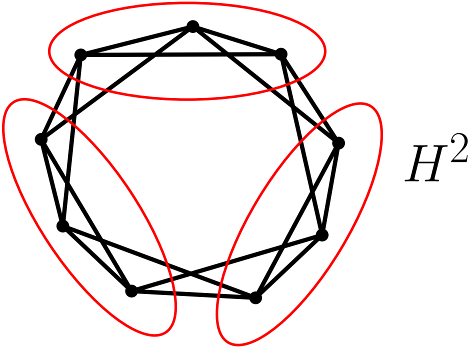

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For any $r \in \mathbb{N}$ *and* $\varepsilon > 0$ *there exists* $n_0 \in \mathbb{N}$ *so that any graph* G *on* $n \geq n_0$ *vertices with* $\delta(G) \geq \left(1 - \frac{1}{r+1} + \varepsilon\right) n$ *has a copy of* H^r.

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We are given a graph G with $\delta(G) \geq \alpha n$. Does G contain, for every coloring $f \colon E(G) \to \{-1, 1\}$, a copy of H with high discrepancy, i.e. a subgraph F isomorphic to H such that $|f(F)|$ is large?

Previous results

Theorem (Balogh, Csaba, Jing and Pluhár, 2020)

Let G be a graph with $\delta(G) \geq (3/4 + \eta)n$. Given any edge coloring $f\colon E(G) \to \{-1, 1\}$, there exists a Hamilton cycle of absolute discrepancy at least $\eta n/32$ with respect to f.

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$$\left(1 - \frac{1}{r+2}\right) n?$$

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For any $\eta > 0$, *there exist* $n_0 \in \mathbb{N}$ *and* $\gamma > 0$ *such that the following holds. Suppose a graph* G *on* $n \geq n_0$ *vertices with minimum degree* $\delta(G) \geq (3/4 + \eta)n$ *and an edge coloring* $f : E(G) \to \{-1, 1\}$ *are given. Then in* G *there exists the square of a Hamilton cycle* H^2 *satisfying* $|f(H^2)| > \gamma n$.

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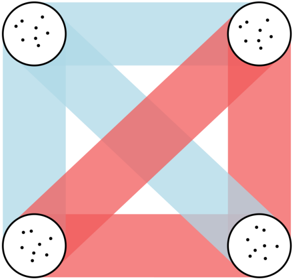
Theorem

For any integer $r \geq 3$ and $\eta > 0$, there exist $n_0 \in \mathbb{N}$ and $\gamma > 0$ such that the following holds. Suppose a graph G on $n \geq n_0$ vertices with minimum degree $\delta(G) \geq (1 - 1/(r+1) + \eta)n$ and an edge coloring $f\colon E(G) \to \{-1, 1\}$ are given. Then in G there exists the r^{th} power of a Hamilton cycle H^r satisfying $|f(H^r)| > \gamma n$.

Threshold comparison

	Threshold		Discrepancy threshold	
K_r-tiling	$(1 - \frac{1}{r})n$	[HS, '70]	$(1 - \frac{1}{r+1})n$	[BCPT, '20]
H	$\frac{1}{2}n$	[D, '52]	$\frac{3}{4}n$	[BCJP, '20]
H^2	$\frac{2}{3}n$	[KSS, '98]	$\frac{3}{4}n$	[**B**, '20]
$H^r,\ r \geq 3$	$(1 - \frac{1}{r+1})n$	[KSS, '98]	$(1 - \frac{1}{r+1})n$	[**B**, '20]

Lower bound for $r = 1, 2$



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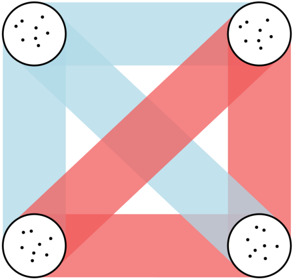

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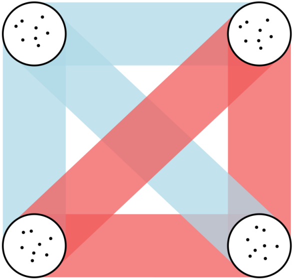

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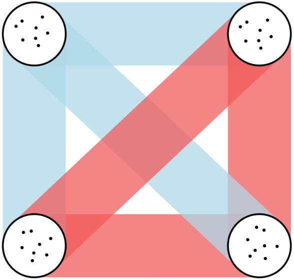

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- In a copy of H^r, we have $\frac{n}{4} \cdot 2r = \frac{nr}{2}$ blue edges.
- H^r has nr edges, so $f(H^r) = 0$.

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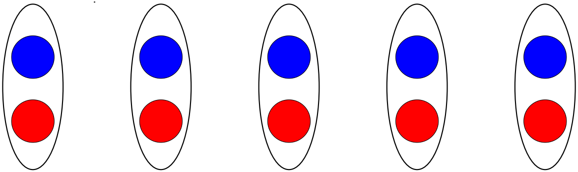
$r + 1$ clusters



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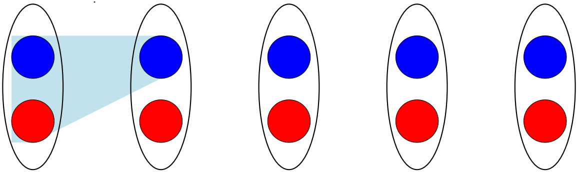

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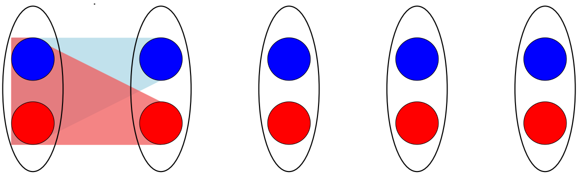

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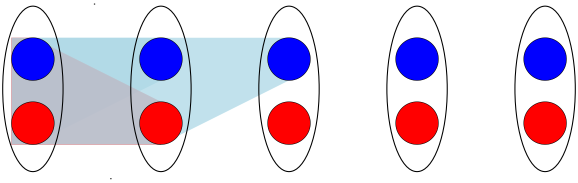

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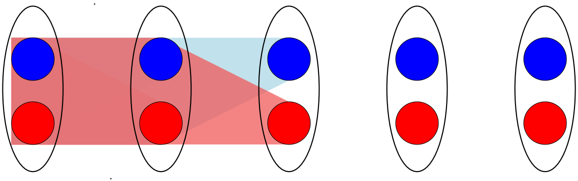

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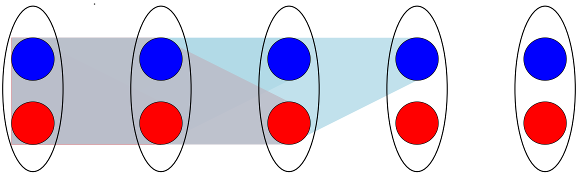

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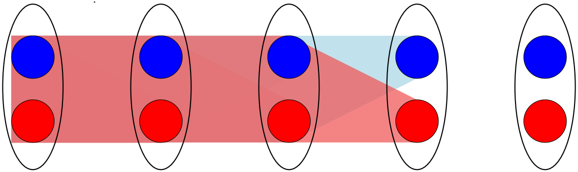

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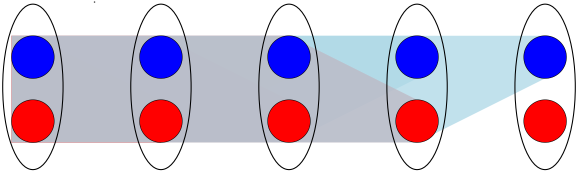

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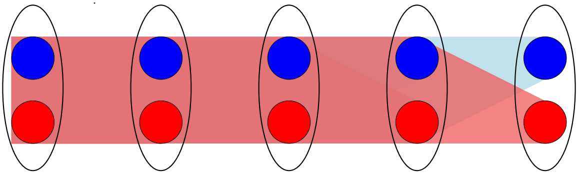

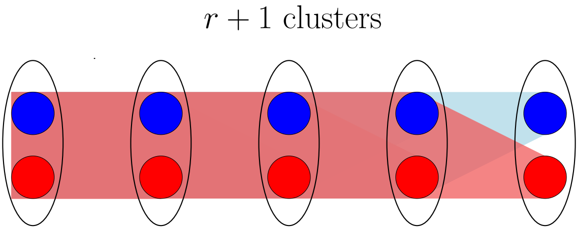

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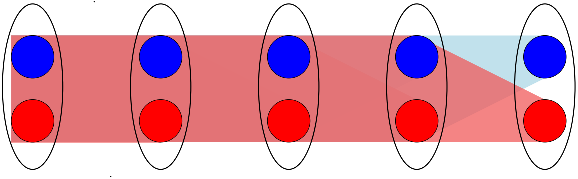

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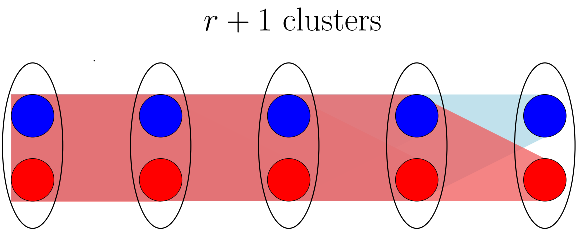

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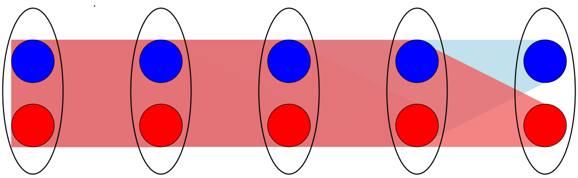

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Using Szemerédi's regularity lemma

Using a multicolored version of Szemerédi's regularity lemma, we can partition vertices into clusters $V_0, V_1, \ldots, V_\ell$. Additionally, on the vertex set $\{V_1, \ldots, V_\ell\}$ we can define the *reduced graph* R and an edge coloring $f_R \colon E(R) \to \{-1, 1\}$ such that:

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Blow-up Lemma (Komlós, Sárközy, Szemerédi, 1994)

"Regular pairs behave like complete bipartite graphs in terms of containing bounded degree subgraphs."

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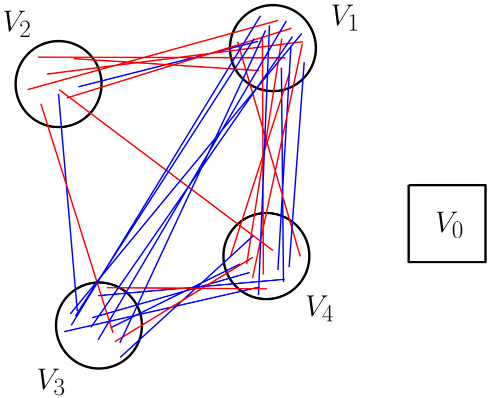

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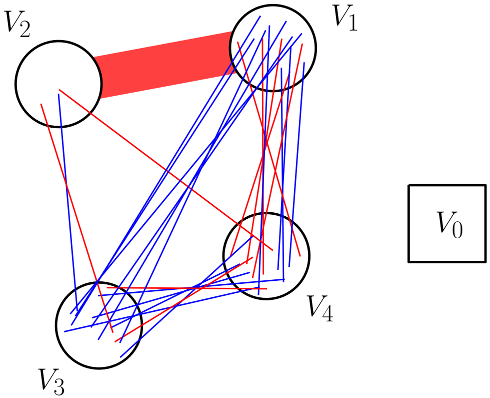

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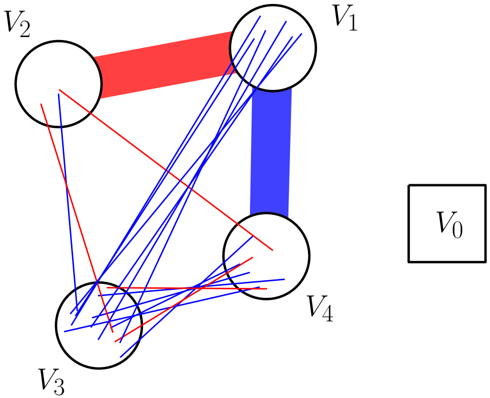

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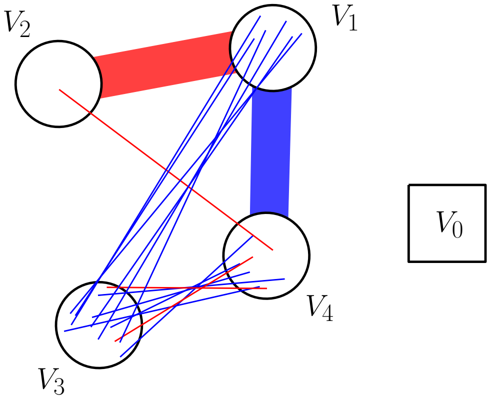

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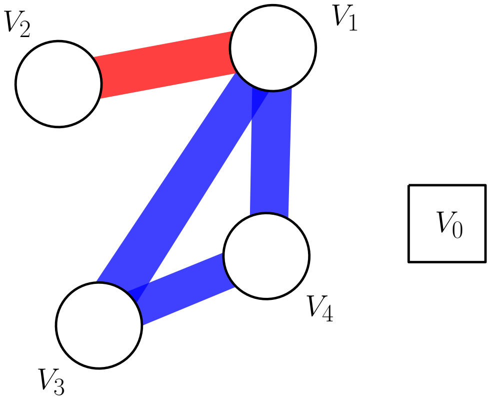

r^{th} powers of cycles in R

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C^r-tiling

A C^r-tiling $\mathcal{T}$ of R is a partition of its vertices into r^{th} powers of simple cycles. Its discrepancy is defined as $f_R(\mathcal{T}) = \sum_{C^r \in \mathcal{T}} f_R(C^r)$.

From a C^r-tiling to H^r

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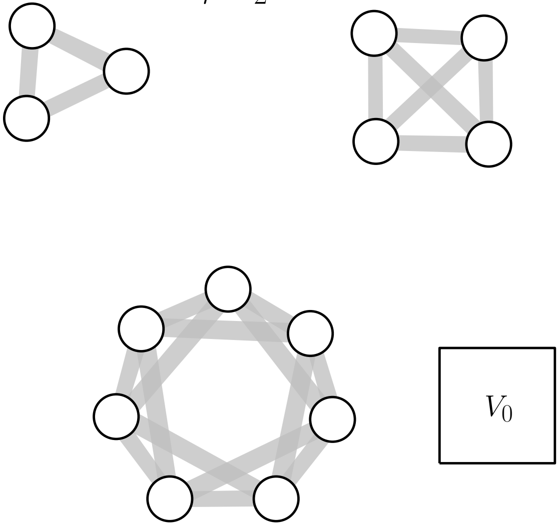

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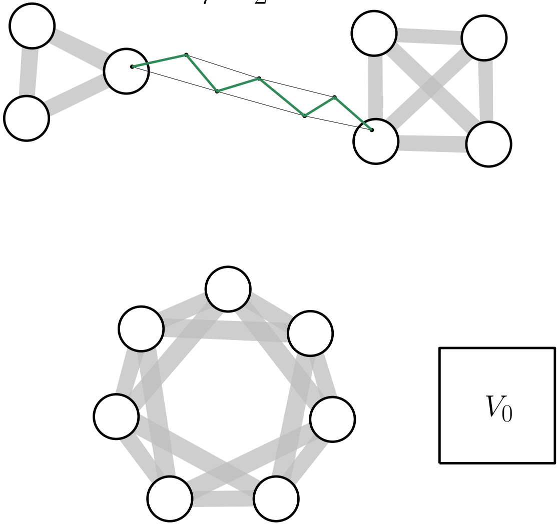

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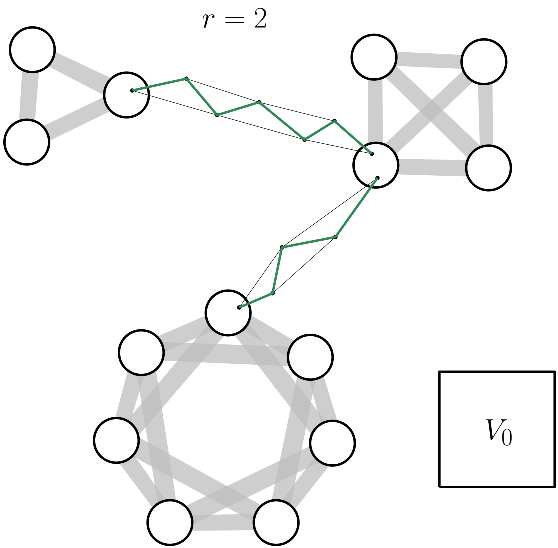

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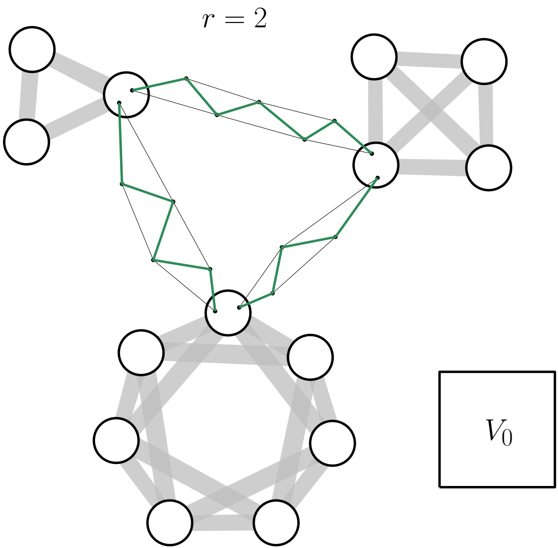

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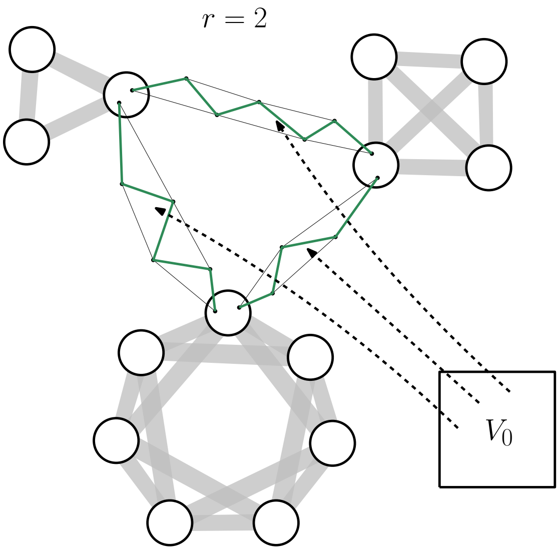

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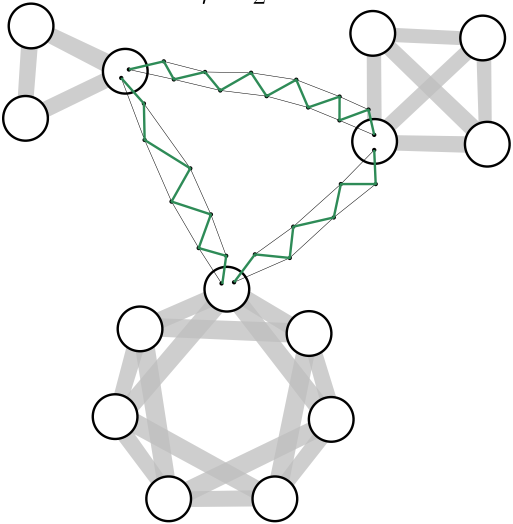

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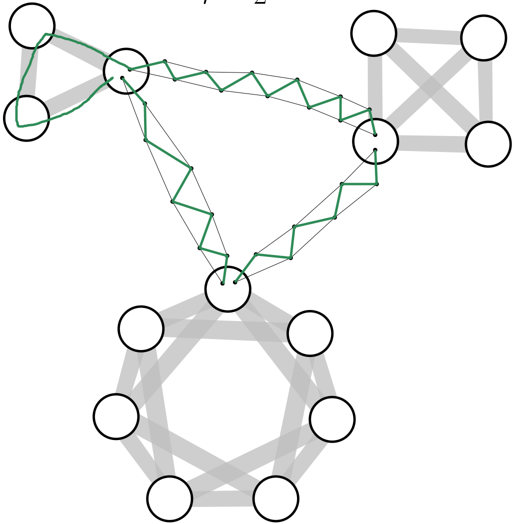

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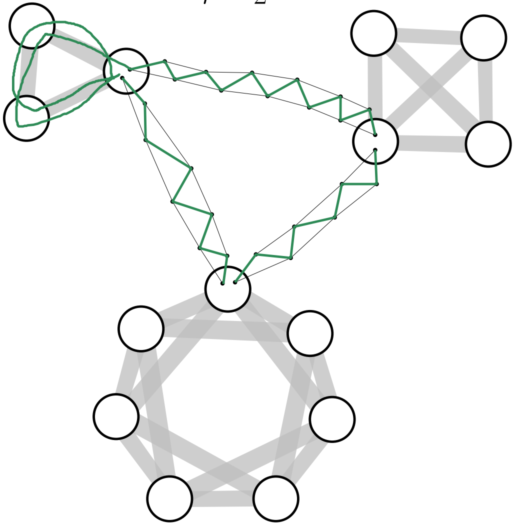

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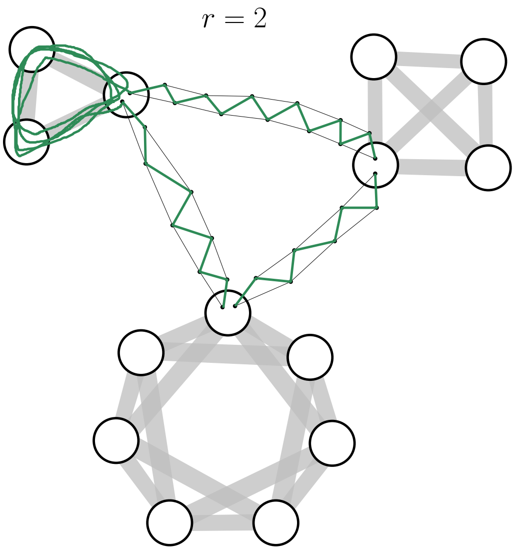

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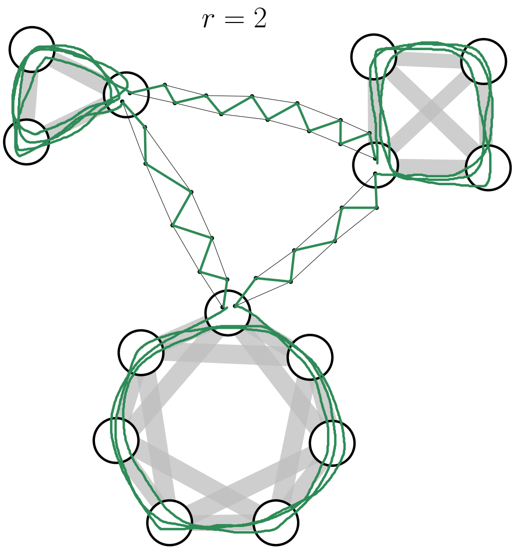

Using a tiling of high discrepancy

Lemma (Tiling Lemma)

Suppose there is a C^r*-tiling* $\mathcal{T}$ *of* R *with* $|f_R(\mathcal{T})| = \Omega(|R|)$. *Then in* G *there exists the* r^{th} *power of a Hamilton cycle* H^r *satisfying* $|f(H^r)| \geq \gamma n$.

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C^r-template

Let F be a graph. A collection of r^{th} powers of cycles $\mathcal{F} = \{C_1^r, \ldots, C_s^r\}$ is a C^r-template of F if every vertex in F appears the same number of times.
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Lemma (Template Lemma)

Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be two "small" C^r-templates on some subgraph F of R. If both $\mathcal{F}_1$ and $\mathcal{F}_2$ contain each vertex of F exactly k times, but have different discrepancies, then we are done.

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Proof outline

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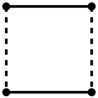

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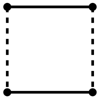

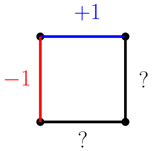

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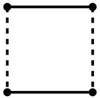

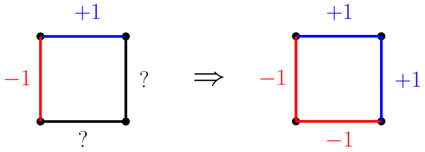

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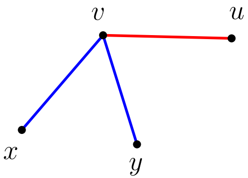

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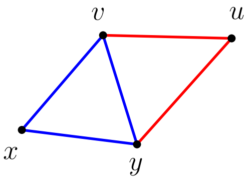

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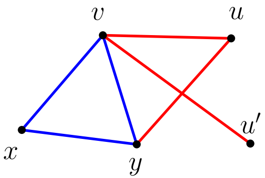

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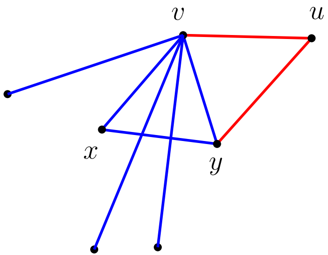

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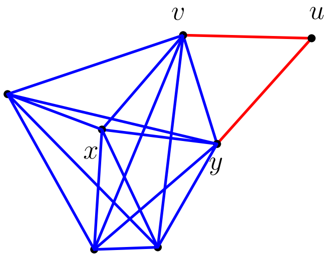

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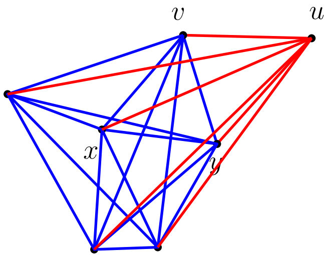

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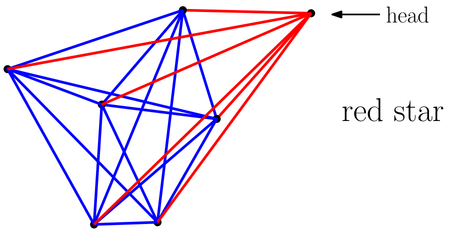

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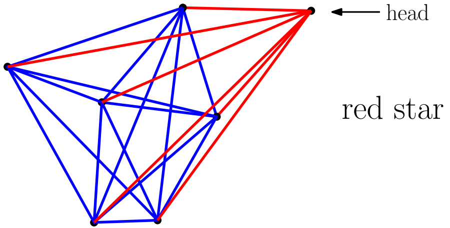

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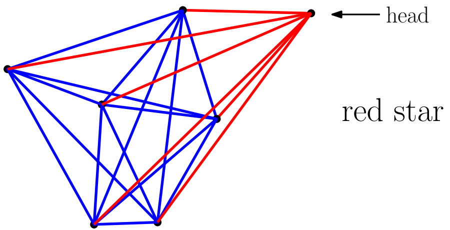

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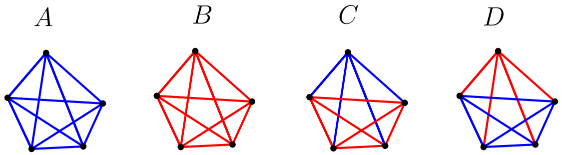
Finishing the proof



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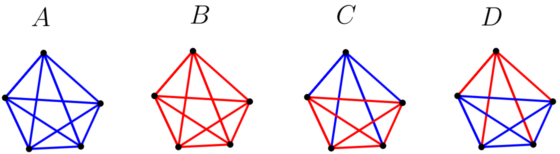

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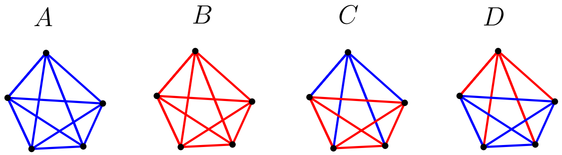

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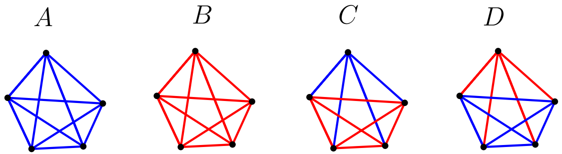

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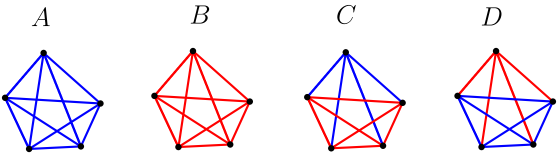

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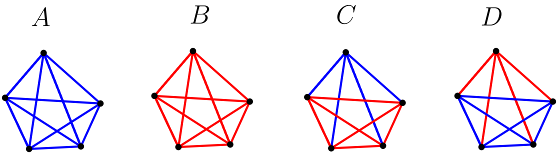

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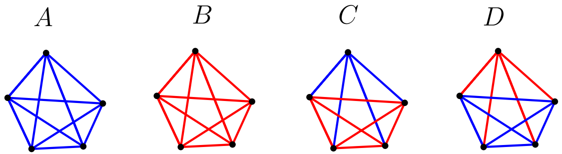

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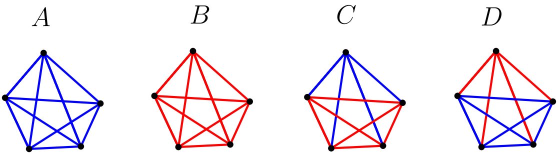
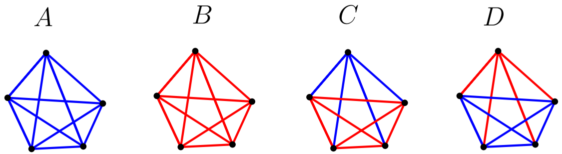
- Let $X \in A$ and $v \in X$. Then $d(v) \leq (r-1)(|B| + |C|) + (r+1)(|A| + |D|) \leq \frac{r}{r+1}|R|$.

Finishing the proof



- We can assume $|B| + |C| \geq |A| + |D|$.
- Consider two cliques X and Y in $\mathcal{T}$ and a vertex $v \in X$. We show $d(v, Y) \leq r - 1$ if:
  - $X \in A$ and $Y \in B$ or
  - $X \in A$ and $Y \in C$ or
  - $X \in C$, $Y \in D$ and v is the head of X.
- Let $X \in A$ and $v \in X$. Then $d(v) \leq (r-1)(|B| + |C|) + (r+1)(|A| + |D|) \leq \frac{r}{r+1}|R|$. So, $A = \emptyset$.

Finishing the proof



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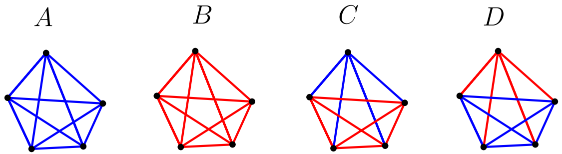
- $|f_R(\mathcal{T})| \leq \beta|R|$

Finishing the proof



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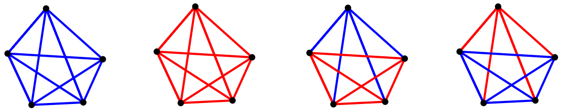
- Let $X \in A$ and $v \in X$. Then $d(v) \leq (r-1)(|B| + |C|) + (r+1)(|A| + |D|) \leq \frac{r}{r+1}|R|$. So, $A = \emptyset$.
- $|f_R(\mathcal{T})| \leq \beta|R| \implies |B| \leq \beta|R|$ and $|B| + |C| - |D| \leq \beta|R|$.



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  - $X \in A$ and $Y \in B$ or
  - $X \in A$ and $Y \in C$ or
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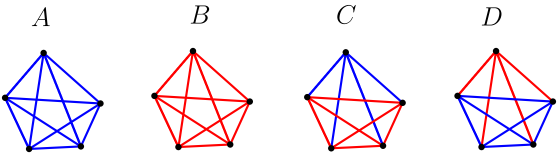
- Let $X \in A$ and $v \in X$. Then $d(v) \leq (r-1)(|B| + |C|) + (r+1)(|A| + |D|) \leq \frac{r}{r+1}|R|$. So, $A = \emptyset$.
- $|f_R(\mathcal{T})| \leq \beta|R| \implies |B| \leq \beta|R|$ and $|B| + |C| - |D| \leq \beta|R|$.
- Let $X \in C$ and v be the head of X. Then $d(v) \leq (r-1)|D| + (r+1)(|B| + |C|) \leq \left(\frac{r}{r+1} + \beta\right)|R|$.

Finishing the proof



- We can assume $|B| + |C| \geq |A| + |D|$.
- Consider two cliques X and Y in $\mathcal{T}$ and a vertex $v \in X$. We show $d(v, Y) \leq r - 1$ if:
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  - $X \in A$ and $Y \in C$ or
  - $X \in C$, $Y \in D$ and v is the head of X.
- Let $X \in A$ and $v \in X$. Then $d(v) \leq (r-1)(|B| + |C|) + (r+1)(|A| + |D|) \leq \frac{r}{r+1}|R|$. So, $A = \emptyset$.
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- Let $X \in C$ and v be the head of X. Then $d(v) \leq (r-1)|D| + (r+1)(|B| + |C|) \leq \left(\frac{r}{r+1} + \beta\right)|R|$.
- $C = \emptyset$, contradiction.

Thank you!