Turán numbers of sunflowers

Domagoj Bradač

ETH Zürich

Joint work with Matija Bucić[1] and Benny Sudakov

November 2, 2021

[1]Princeton and Institute for Advanced Study

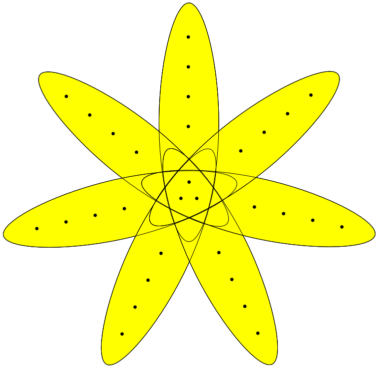
Definition

A collection of sets $A_1, \ldots, A_k$ is called a *sunflower* if the intersection of any two sets equals the intersection of all the sets.

Sunflowers

Definition

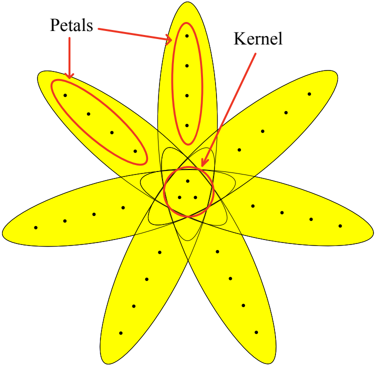
A collection of sets $A_1, \ldots, A_k$ is called a *sunflower* if the intersection of any two sets equals the intersection of all the sets.



Sunflowers

Definition

A collection of sets $A_1, \ldots, A_k$ is called a *sunflower* if the intersection of any two sets equals the intersection of all the sets.



Let $f_r(k)$ denote the smallest natural number such that any r-uniform hypergraph with $f_r(k)$ edges contains a sunflower with k petals.

Erdős-Rado sunflower conjecture

Let $f_r(k)$ denote the smallest natural number such that any r-uniform hypergraph with $f_r(k)$ edges contains a sunflower with k petals.

Theorem (Erdős, Rado '60)

$(k-1)^r \leq f_r(k) \leq (k-1)^r r! + 1.$

Erdős-Rado sunflower conjecture

Let $f_r(k)$ denote the smallest natural number such that any r-uniform hypergraph with $f_r(k)$ edges contains a sunflower with k petals.

Theorem (Erdős, Rado '60)

$(k-1)^r \leq f_r(k) \leq (k-1)^r r! + 1.$

Conjecture (Erdős, Rado '60)

For any k, there is a constant $C = C(k)$ such that $f_r(k) \leq C^r$.

Erdős-Rado sunflower conjecture

Let $f_r(k)$ denote the smallest natural number such that any r-uniform hypergraph with $f_r(k)$ edges contains a sunflower with k petals.

Theorem (Erdős, Rado '60)

$(k-1)^r \leq f_r(k) \leq (k-1)^r r! + 1.$

Conjecture (Erdős, Rado '60)

For any k, there is a constant $C = C(k)$ such that $f_r(k) \leq C^r$.

Theorem (Alweiss, Lovett, Wu, Zhang '19 + Rao '19 + Bell, Chueluecha, Warnke '20)

There is a constant C such that for all $r, k \geq 2$,
$f_r(k) \leq (Ck \log r)^r.$

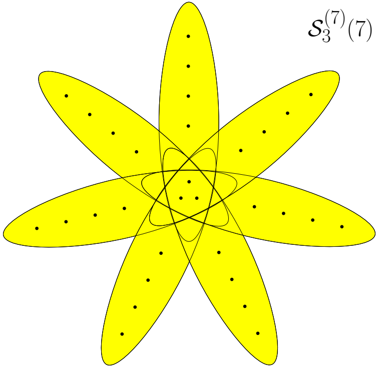
Definition

The r-uniform sunflower with kernel of size t and k petals is denoted by $\mathcal{S}_t^{(r)}(k)$.

Fixing the kernel size

Definition

The r-uniform sunflower with kernel of size t and k petals is denoted by $\mathcal{S}_t^{(r)}(k)$.



Fixing the kernel size

Definition

The r-uniform sunflower with kernel of size t and k petals is denoted by $\mathcal{S}_t^{(r)}(k)$.

Question (Duke, Erdős '77)

Determine $\mathrm{ex}(n, \mathcal{S}_t^{(r)}(k))$, *that is, the maximum number of edges in an* r*-uniform hypergraph on* n *vertices without a copy of* $\mathcal{S}_t^{(r)}(k)$.

Definition

The r-uniform sunflower with kernel of size t and k petals is denoted by $\mathcal{S}_t^{(r)}(k)$.

Question (Duke, Erdős '77)

Determine $\mathrm{ex}(n, \mathcal{S}_t^{(r)}(k))$, *that is, the maximum number of edges in an* r*-uniform hypergraph on* n *vertices without a copy of* $\mathcal{S}_t^{(r)}(k)$.

Reiterated by Füredi '91, Chung '97 and in Polymath 10.

Fixing the kernel size

Definition

The r-uniform sunflower with kernel of size t and k petals is denoted by $\mathcal{S}_t^{(r)}(k)$.

Question (Duke, Erdős '77)

Determine $\mathrm{ex}(n, \mathcal{S}_t^{(r)}(k))$, *that is, the maximum number of edges in an* r*-uniform hypergraph on* n *vertices without a copy of* $\mathcal{S}_t^{(r)}(k)$.

Reiterated by Füredi '91, Chung '97 and in Polymath 10.
Note: unlike the sunflower conjecture, the answer depends on n.

Special cases

- For $k = 2, t = 0$, we have an intersecting family and the answer is given by the Erdős-Ko-Rado theorem.

Special cases

- For $k = 2, t = 0$, we have an intersecting family and the answer is given by the Erdős-Ko-Rado theorem.
- For $k = 2$, this is the restricted intersection problem.

Special cases

- For $k = 2, t = 0$, we have an intersecting family and the answer is given by the Erdős-Ko-Rado theorem.
- For $k = 2$, this is the restricted intersection problem.
- For $t = 0$, we are looking for a matching of size k. The answer is predicted by the Erdős matching conjecture.

Special cases

- For $k = 2, t = 0$, we have an intersecting family and the answer is given by the Erdős-Ko-Rado theorem.
- For $k = 2$, this is the restricted intersection problem.
- For $t = 0$, we are looking for a matching of size k. The answer is predicted by the Erdős matching conjecture.

Theorem (Frankl, Füredi '83)

For fixed r, t, k, $\mathrm{ex}(n, \mathcal{S}_t^{(r)}(k)) = \Theta(n^{\max\{t, r-t-1\}})$.

Special cases

- For $k = 2, t = 0$, we have an intersecting family and the answer is given by the Erdős-Ko-Rado theorem.
- For $k = 2$, this is the restricted intersection problem.
- For $t = 0$, we are looking for a matching of size k. The answer is predicted by the Erdős matching conjecture.

Theorem (Frankl, Füredi '83)

For fixed r, t, k, $\mathrm{ex}(n, \mathcal{S}_t^{(r)}(k)) = \Theta(n^{\max\{t, r-t-1\}})$.

For fixed r, t, k, Frankl and Füredi '86 conjecture constructions which are optimal up to lower order terms and prove the optimality for $r \geq 2t + 3$.

What happens when k grows?

Example: $\mathrm{ex}(n, \mathcal{S}_1^{(2)}(k)) \sim \frac{nk}{2}$.

What happens when k grows?

Example: $\mathrm{ex}(n, \mathcal{S}_1^{(2)}(k)) \sim \frac{nk}{2}$.

Chung, Erdős '87: determining $\mathrm{ex}(n, \mathcal{S}_t^{(r)}(k))$ for growing k is crucial in the study of so-called unavoidable hypergraphs.

What happens when k grows?

Example: $\mathrm{ex}(n, \mathcal{S}_1^{(2)}(k)) \sim \frac{nk}{2}$.

Chung, Erdős '87: determining $\mathrm{ex}(n, \mathcal{S}_t^{(r)}(k))$ for growing k is crucial in the study of so-called unavoidable hypergraphs.

- For $r = 2$ the exact answer is given by Erdős and Kalai '61.

What happens when k grows?

Example: $\mathrm{ex}(n, \mathcal{S}_1^{(2)}(k)) \sim \frac{nk}{2}$.

Chung, Erdős '87: determining $\mathrm{ex}(n, \mathcal{S}_t^{(r)}(k))$ for growing k is crucial in the study of so-called unavoidable hypergraphs.

- For $r = 2$ the exact answer is given by Erdős and Kalai '61.
- For $r = 3$ Duke, Erdős '77 and Frankl '78 determine the answer up to constant factors; Chung and Frankl '87 determine $\mathcal{S}_1^{(3)}(k)$ precisely when $n \geq O(k^3)$.

What happens when k grows?

Example: $\mathrm{ex}(n, \mathcal{S}_1^{(2)}(k)) \sim \frac{nk}{2}$.

Chung, Erdős '87: determining $\mathrm{ex}(n, \mathcal{S}_t^{(r)}(k))$ for growing k is crucial in the study of so-called unavoidable hypergraphs.

- For $r = 2$ the exact answer is given by Erdős and Kalai '61.
- For $r = 3$ Duke, Erdős '77 and Frankl '78 determine the answer up to constant factors; Chung and Frankl '87 determine $\mathcal{S}_1^{(3)}(k)$ precisely when $n \geq O(k^3)$.
- The case $r = 4$ was resolved up to constant factors by Bucić, Draganić, Sudakov and Tran '21.

What happens when k grows?

Example: $\mathrm{ex}(n, \mathcal{S}_1^{(2)}(k)) \sim \frac{nk}{2}$.

Chung, Erdős '87: determining $\mathrm{ex}(n, \mathcal{S}_t^{(r)}(k))$ for growing k is crucial in the study of so-called unavoidable hypergraphs.

- For $r = 2$ the exact answer is given by Erdős and Kalai '61.
- For $r = 3$ Duke, Erdős '77 and Frankl '78 determine the answer up to constant factors; Chung and Frankl '87 determine $\mathcal{S}_1^{(3)}(k)$ precisely when $n \geq O(k^3)$.
- The case $r = 4$ was resolved up to constant factors by Bucić, Draganić, Sudakov and Tran '21.

Conjecture (Bucić, Draganić, Sudakov and Tran '21.)

For fixed r, t,

$$\mathrm{ex}(n, \mathcal{S}_t^{(r)}(k)) = \begin{cases} \Theta(n^{r-t-1}k^{t+1}) & \text{if } 2t+1 \leq r, \\ \Theta(n^t k^{r-t}) & \text{if } 2t+1 > r. \end{cases}$$

Our result

Theorem (B., Bucić and Sudakov '21+.)

For fixed r, t,

$$\mathrm{ex}(n, \mathcal{S}_t^{(r)}(k)) = \begin{cases} \Theta(n^{r-t-1}k^{t+1}) & \text{if } 2t+1 \le r, \\ \Theta(n^t k^{r-t}) & \text{if } 2t+1 > r. \end{cases}$$

Lower bounds

- Let $V = A \mathbin{\dot\cup} B$, where $|A| = n - k + 1, |B| = k - 1$. Take $H = \{e \in \binom{V}{r} \mid |e \cap B| = t + 1\}$.

Lower bounds

- Let $V = A \cup B$, where $|A| = n - k + 1, |B| = k - 1$. Take $H = \{e \in \binom{V}{r} \mid |e \cap B| = t + 1\}$. H has no $\mathcal{S}_t^{(r)}(k)$, so

$$\mathrm{ex}(n, \mathcal{S}_t^{(r)}(k)) \geq |H| = \binom{|A|}{r - t - 1}\binom{|B|}{t + 1} = \Omega(n^{r-t-1}k^{t+1}).$$

Lower bounds

- Let $V = A \cup B$, where $|A| = n - k + 1, |B| = k - 1$. Take $H = \{e \in \binom{V}{r} \mid |e \cap B| = t + 1\}$. H has no $\mathcal{S}_t^{(r)}(k)$, so

$$\mathrm{ex}(n, \mathcal{S}_t^{(r)}(k)) \geq |H| = \binom{|A|}{r-t-1}\binom{|B|}{t+1} = \Omega(n^{r-t-1}k^{t+1}).$$

- Probabilistic construction inspired by Steiner systems giving $\mathrm{ex}(n, \mathcal{S}_t^{(r)}(k)) = \Omega(n^t k^{r-t})$.

Lower bounds

- Let $V = A \cup B$, where $|A| = n - k + 1, |B| = k - 1$. Take $H = \{e \in \binom{V}{r} \mid |e \cap B| = t + 1\}$. H has no $\mathcal{S}_t^{(r)}(k)$, so
$$\mathrm{ex}(n, \mathcal{S}_t^{(r)}(k)) \geq |H| = \binom{|A|}{r-t-1}\binom{|B|}{t+1} = \Omega(n^{r-t-1}k^{t+1}).$$
- Probabilistic construction inspired by Steiner systems giving $\mathrm{ex}(n, \mathcal{S}_t^{(r)}(k)) = \Omega(n^t k^{r-t})$.
- Note that both constructions give the same bound when $r = 2t + 1$.

Upper bound proof outline

Theorem

For fixed r, t,

$$\mathrm{ex}(n, \mathcal{S}_t^{(r)}(k)) = \begin{cases} O(n^{r-t-1}k^{t+1}) & \text{if } 2t+1 \le r, \\ O(n^t k^{r-t}) & \text{if } 2t+1 > r. \end{cases}$$

Upper bound proof outline

Theorem

For fixed r, t,

$$\mathrm{ex}(n, \mathcal{S}_t^{(r)}(k)) = \begin{cases} O(n^{r-t-1}k^{t+1}) & \text{if } 2t+1 \le r, \\ O(n^t k^{r-t}) & \text{if } 2t+1 > r. \end{cases}$$

- Reduce the general problem to the *balanced* case $r = 2t + 1$.

Upper bound proof outline

Theorem

For fixed r, t,

$$\mathrm{ex}(n, \mathcal{S}_t^{(r)}(k)) = \begin{cases} O(n^{r-t-1}k^{t+1}) & \text{if } 2t+1 \le r, \\ O(n^t k^{r-t}) & \text{if } 2t+1 > r. \end{cases}$$

- Reduce the general problem to the *balanced* case $r = 2t + 1$.
- Reduce $\mathrm{ex}(n, \mathcal{S}_t^{(2t+1)}(k)) = O(n^t k^{t+1})$ to non-existence of a certain set system on $[2t+1]$.

Upper bound proof outline

Theorem

For fixed r, t,

$$\mathrm{ex}(n, \mathcal{S}_t^{(r)}(k)) = \begin{cases} O(n^{r-t-1}k^{t+1}) & \text{if } 2t+1 \leq r, \\ O(n^t k^{r-t}) & \text{if } 2t+1 > r. \end{cases}$$

- Reduce the general problem to the *balanced* case $r = 2t+1$.
- Reduce $\mathrm{ex}(n, \mathcal{S}_t^{(2t+1)}(k)) = O(n^t k^{t+1})$ to non-existence of a certain set system on $[2t+1]$.
- Prove such a set system does not exist.

$(t+1,t)$-systems

Definition (Nägele, Sudakov, Zenklusen '19)

A set family $\mathcal{A} \subseteq \mathcal{P}([N])$ is said to be a $(t+1,t)$-system if:

- $\forall A, B \in \mathcal{A}$ we also have $A \cap B \in \mathcal{A}$,
- any subset of $[N]$ of size t is contained in some set in $\mathcal{A}$ and
- for any $A \in \mathcal{A}$ we have $|A| \not\equiv N \pmod{t+1}$.

Definition (Nägele, Sudakov, Zenklusen '19)

A set family $\mathcal{A} \subseteq \mathcal{P}([N])$ is said to be a $(t+1,t)$-system if:

- $\forall A, B \in \mathcal{A}$ we also have $A \cap B \in \mathcal{A}$,
- any subset of $[N]$ of size t is contained in some set in $\mathcal{A}$ and
- for any $A \in \mathcal{A}$ we have $|A| \not\equiv N \pmod{t+1}$.

Lemma (Nägele, Sudakov, Zenklusen '19)

If $t+1$ is a prime power, there is no $(t+1,t)$-system.

Definition (Nägele, Sudakov, Zenklusen '19)

A set family $\mathcal{A} \subseteq \mathcal{P}([N])$ is said to be a $(t+1, t)$-system if:

- $\forall A, B \in \mathcal{A}$ we also have $A \cap B \in \mathcal{A}$,
- any subset of $[N]$ of size t is contained in some set in $\mathcal{A}$ and
- for any $A \in \mathcal{A}$ we have $|A| \not\equiv N \pmod{t+1}$.

Lemma (Nägele, Sudakov, Zenklusen '19)

If $t+1$ is a prime power, there is no $(t+1, t)$-system.

Lemma (Brakensiek, Gopi, Guruswam '19)

If $t+1$ has at least two prime divisors, there exist $(t+1, t)$-systems.

$(t+1, t)$-systems

Definition (Nägele, Sudakov, Zenklusen '19)

A set family $\mathcal{A} \subseteq \mathcal{P}([N])$ is said to be a $(t+1, t)$-system if:

- $\forall A, B \in \mathcal{A}$ we also have $A \cap B \in \mathcal{A}$,
- any subset of $[N]$ of size t is contained in some set in $\mathcal{A}$ and
- for any $A \in \mathcal{A}$ we have $|A| \not\equiv N \pmod{t+1}$.

Lemma (Nägele, Sudakov, Zenklusen '19)

If $t+1$ is a prime power, there is no $(t+1, t)$-system.

Lemma (Brakensiek, Gopi, Guruswam '19)

If $t+1$ has at least two prime divisors, there exist $(t+1, t)$-systems.

Lemma

For any t, there is no $(t+1, t)$-system with $N = 2t+1$.

Non-existence of $(t+1,t)$-systems with $N = 2t+1$.

Lemma

For any t, there is no $(t+1,t)$-system with $N = 2t+1$.

In other words, such a set system $\mathcal{A} \subseteq \mathcal{P}([2t+1])$ would satisfy:

- $\forall A, B \in \mathcal{A}$, we have $A \cap B \in \mathcal{A}$,
- any t-subset of $[2t+1]$ is contained in some set in $\mathcal{A}$ and
- for any $A \in \mathcal{A}$, we have $|A| \notin \{t, 2t+1\}$.

Non-existence of $(t+1, t)$-systems with $N = 2t+1$.

Lemma

For any t, there is no $(t+1, t)$-system with $N = 2t+1$.

In other words, such a set system $\mathcal{A} \subseteq \mathcal{P}([2t+1])$ would satisfy:

- $\forall A, B \in \mathcal{A}$, we have $A \cap B \in \mathcal{A}$,
- any t-subset of $[2t+1]$ is contained in some set in $\mathcal{A}$ and
- for any $A \in \mathcal{A}$, we have $|A| \notin \{t, 2t+1\}$.

The key proof ingredient is the following theorem.

Theorem (Frankl, Katona '79)

Given $m+1$ not necessarily distinct subsets of $[m]$, there are s of them whose intersection has size $s-1$, for some s, $1 \leq s \leq m+1$.

The key lemma

Lemma

Suppose there exists no $(t+1,t)$*-system with* $N = 2t+1$*. Then,* $\mathrm{ex}(n, \mathcal{S}_t^{(2t+1)}(k)) = O(n^t k^{t+1})$.

The key lemma

Lemma

Suppose there exists no $(t+1,t)$*-system with* $N = 2t+1$*. Then,* $\mathrm{ex}(n, \mathcal{S}_t^{(2t+1)}(k)) = O(n^t k^{t+1})$.

Observation

The link graph of any set of t *vertices can be covered by* $(k-1)(t+1) = O(k)$ *vertices.*

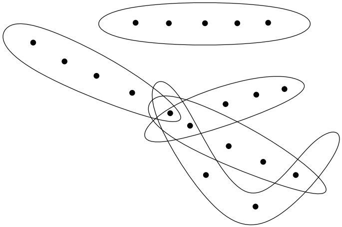
The key lemma

Lemma

Suppose there exists no $(t+1,t)$-system with $N = 2t+1$. Then, $\mathrm{ex}(n, \mathcal{S}_t^{(2t+1)}(k)) = O(n^t k^{t+1})$.

Observation

The link graph of any set of t vertices can be covered by $(k-1)(t+1) = O(k)$ *vertices.*



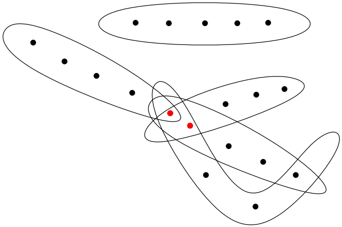
The key lemma

Lemma

Suppose there exists no $(t+1,t)$-system with $N = 2t+1$. Then, $\mathrm{ex}(n, \mathcal{S}_t^{(2t+1)}(k)) = O(n^t k^{t+1})$.

Observation

The link graph of any set of t vertices can be covered by $(k-1)(t+1) = O(k)$ *vertices.*



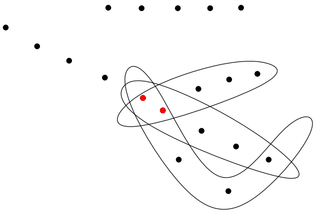
The key lemma

Lemma

Suppose there exists no $(t+1,t)$*-system with* $N = 2t+1$*. Then,* $\mathrm{ex}(n, \mathcal{S}_t^{(2t+1)}(k)) = O(n^t k^{t+1})$.

Observation

The link graph of any set of t *vertices can be covered by* $(k-1)(t+1) = O(k)$ *vertices.*



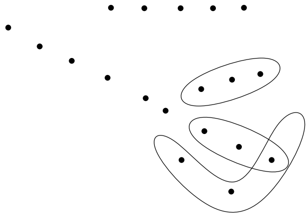
The key lemma

Lemma

Suppose there exists no $(t+1,t)$-system with $N = 2t+1$. Then, $\mathrm{ex}(n, \mathcal{S}_t^{(2t+1)}(k)) = O(n^t k^{t+1})$.

Observation

The link graph of any set of t vertices can be covered by $(k-1)(t+1) = O(k)$ vertices.

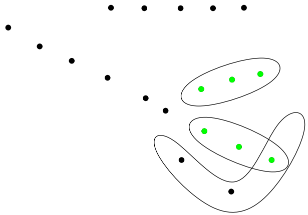


Lemma

Suppose there exists no $(t+1,t)$*-system with* $N=2t+1$*. Then,* $\mathrm{ex}(n,\mathcal{S}_t^{(2t+1)}(k)) = O(n^t k^{t+1})$.

Observation

The link graph of any set of t *vertices can be covered by* $(k-1)(t+1) = O(k)$ *vertices.*



The key lemma

- For each t-set T fix a cover $S(T)$ of its link graph with $|S(T)| = O(k)$.

The key lemma

- For each t-set T fix a cover $S(T)$ of its link graph with $|S(T)| = O(k)$.
- Enumerate all the edges.

The key lemma

- For each t-set T fix a cover $S(T)$ of its link graph with $|S(T)| = O(k)$.
- Enumerate all the edges.
- Simple enumeration:

The key lemma

- For each t-set T fix a cover $S(T)$ of its link graph with $|S(T)| = O(k)$.
- Enumerate all the edges.
- Simple enumeration:
  - $v_1, \ldots, v_t$ arbitrary $\rightarrow O(n^t)$ ways;

The key lemma

- For each t-set T fix a cover $S(T)$ of its link graph with $|S(T)| = O(k)$.
- Enumerate all the edges.
- Simple enumeration:
  - $v_1, \ldots, v_t$ arbitrary $\rightarrow O(n^t)$ ways;
  - $v_{t+1} \in S(\{v_1, \ldots, v_t\}) \rightarrow O(k)$ ways;

The key lemma

- For each t-set T fix a cover $S(T)$ of its link graph with $|S(T)| = O(k)$.
- Enumerate all the edges.
- Simple enumeration:
  - $v_1, \dots, v_t$ arbitrary $\to O(n^t)$ ways;
  - $v_{t+1} \in S(\{v_1, \dots, v_t\}) \to O(k)$ ways;
  - $v_{t+2}, \dots, v_{2t+1}$ arbitrary $\to O(n^t)$ ways.

The key lemma

- For each t-set T fix a cover $S(T)$ of its link graph with $|S(T)| = O(k)$.
- Enumerate all the edges.
- Simple enumeration:
  - $v_1, \ldots, v_t$ arbitrary $\to O(n^t)$ ways;
  - $v_{t+1} \in S(\{v_1, \ldots, v_t\}) \to O(k)$ ways;
  - $v_{t+2}, \ldots, v_{2t+1}$ arbitrary $\to O(n^t)$ ways.

We obtain $\mathrm{ex}(n, \mathcal{S}_t^{(r)}(k)) \leq O(n^{2t}k)$.

The key lemma

- For each t-set T fix a cover $S(T)$ of its link graph with $|S(T)| = O(k)$.
- Enumerate all the edges.
- Simple enumeration:
  - $v_1, \ldots, v_t$ arbitrary $\to O(n^t)$ ways;
  - $v_{t+1} \in S(\{v_1, \ldots, v_t\}) \to O(k)$ ways;
  - $v_{t+2}, \ldots, v_{2t+1}$ arbitrary $\to O(n^t)$ ways.

We obtain $\mathrm{ex}(n, \mathcal{S}_t^{(r)}(k)) \leq O(n^{2t}k)$.
How to improve? Suppose we chose $v_1, \ldots, v_{t+1}$ as before and for some i, $v_i \notin S\left(\{v_1 \ldots, v_{t+1}\} \setminus \{v_i\}\right)$. Then we can choose v_{t+2} from $S(\{v_1 \ldots, v_{t+1}\} \setminus \{v_i\})$. Similarly choose all remaining vertices.

- Fix an arbitrary ordering of the vertices in each edge.

The key lemma – grouping edges

- Fix an arbitrary ordering of the vertices in each edge.
- For an edge $e = (v_1, \dots, v_{2t+1})$ and a t-set $I \in \binom{[2t+1]}{t}$, find an index j such that $v_j \in S(\{v_i \mid i \in I\})$.

The key lemma – grouping edges

- Fix an arbitrary ordering of the vertices in each edge.
- For an edge $e = (v_1, \ldots, v_{2t+1})$ and a t-set $I \in \binom{[2t+1]}{t}$, find an index j such that $v_j \in S(\{v_i \mid i \in I\})$.
- Assign to each edge a function $f_e \colon \binom{[2t+1]}{t} \to [2t+1]$ which satisfies $f_e(I) \notin I, \forall I \in \binom{[2t+1]}{t}$.

The key lemma – grouping edges

- Fix an arbitrary ordering of the vertices in each edge.
- For an edge $e = (v_1, \ldots, v_{2t+1})$ and a t-set $I \in \binom{[2t+1]}{t}$, find an index j such that $v_j \in S(\{v_i \mid i \in I\})$.
- Assign to each edge a function $f_e \colon \binom{[2t+1]}{t} \to [2t+1]$ which satisfies $f_e(I) \notin I, \forall I \in \binom{[2t+1]}{t}$.
- There are $O(1)$ such functions, so it is enough to fix one of them and prove there are $O(n^t k^{t+1})$ edges with this function assigned to them.

The key lemma

Suppose $t = 2$ and we wish to enumerate all edges e with $f_e = f$, where:

$$f(\{1,2\}) = 3,\ f(\{1,3\}) = 2,\ f(\{1,4\}) = 2,\ f(\{1,5\}) = 2,$$
$$f(\{2,3\}) = 4,\ f(\{2,4\}) = 3,\ f(\{2,5\}) = 1,\ f(\{3,4\}) = 1,$$
$$f(\{3,5\}) = 2,\ f(\{4,5\}) = 3.$$

The key lemma

Suppose $t = 2$ and we wish to enumerate all edges e with $f_e = f$, where:

$$f(\{1,2\}) = 3,\ f(\{1,3\}) = 2,\ f(\{1,4\}) = 2,\ f(\{1,5\}) = 2,$$
$$f(\{2,3\}) = 4,\ f(\{2,4\}) = 3,\ f(\{2,5\}) = 1,\ f(\{3,4\}) = 1,$$
$$f(\{3,5\}) = 2,\ f(\{4,5\}) = 3.$$

- Choose v_1, v_2 arbitrarily;

The key lemma

Suppose $t = 2$ and we wish to enumerate all edges e with $f_e = f$, where:

$$f(\{1,2\}) = 3,\ f(\{1,3\}) = 2,\ f(\{1,4\}) = 2,\ f(\{1,5\}) = 2,$$
$$f(\{2,3\}) = 4,\ f(\{2,4\}) = 3,\ f(\{2,5\}) = 1,\ f(\{3,4\}) = 1,$$
$$f(\{3,5\}) = 2,\ f(\{4,5\}) = 3.$$

- Choose v_1, v_2 arbitrarily; choose v_3 from $S(\{v_1, v_2\})$;

Suppose $t = 2$ and we wish to enumerate all edges e with $f_e = f$, where:

$$f(\{1,2\}) = 3,\ f(\{1,3\}) = 2,\ f(\{1,4\}) = 2,\ f(\{1,5\}) = 2,$$
$$f(\{2,3\}) = 4,\ f(\{2,4\}) = 3,\ f(\{2,5\}) = 1,\ f(\{3,4\}) = 1,$$
$$f(\{3,5\}) = 2,\ f(\{4,5\}) = 3.$$

- Choose v_1, v_2 arbitrarily; choose v_3 from $S(\{v_1, v_2\})$;

The key lemma

Suppose $t = 2$ and we wish to enumerate all edges e with $f_e = f$, where:

$$f(\{1,2\}) = 3,\ f(\{1,3\}) = 2,\ f(\{1,4\}) = 2,\ f(\{1,5\}) = 2,$$
$$f(\{2,3\}) = 4,\ f(\{2,4\}) = 3,\ f(\{2,5\}) = 1,\ f(\{3,4\}) = 1,$$
$$f(\{3,5\}) = 2,\ f(\{4,5\}) = 3.$$

- Choose v_1, v_2 arbitrarily; choose v_3 from $S(\{v_1, v_2\})$; choose v_4 from $S(\{v_2, v_3\})$;

The key lemma

Suppose $t = 2$ and we wish to enumerate all edges e with $f_e = f$, where:

$$f(\{1,2\}) = 3,\ f(\{1,3\}) = 2,\ f(\{1,4\}) = 2,\ f(\{1,5\}) = 2,$$
$$f(\{2,3\}) = 4,\ f(\{2,4\}) = 3,\ f(\{2,5\}) = 1,\ f(\{3,4\}) = 1,$$
$$f(\{3,5\}) = 2,\ f(\{4,5\}) = 3.$$

- Choose v_1, v_2 arbitrarily; choose v_3 from $S(\{v_1, v_2\})$; choose v_4 from $S(\{v_2, v_3\})$;

The key lemma

Suppose $t = 2$ and we wish to enumerate all edges e with $f_e = f$, where:

$$f(\{1,2\}) = 3,\ f(\{1,3\}) = 2,\ f(\{1,4\}) = 2,\ f(\{1,5\}) = 2,$$
$$f(\{2,3\}) = 4,\ f(\{2,4\}) = 3,\ f(\{2,5\}) = 1,\ f(\{3,4\}) = 1,$$
$$f(\{3,5\}) = 2,\ f(\{4,5\}) = 3.$$

- Choose v_1, v_2 arbitrarily; choose v_3 from $S(\{v_1, v_2\})$; choose v_4 from $S(\{v_2, v_3\})$; stuck at $\{1,2,3,4\}$.

The key lemma

Suppose $t = 2$ and we wish to enumerate all edges e with $f_e = f$, where:

$$f(\{1,2\}) = 3,\ f(\{1,3\}) = 2,\ f(\{1,4\}) = 2,\ f(\{1,5\}) = 2,$$
$$f(\{2,3\}) = 4,\ f(\{2,4\}) = 3,\ f(\{2,5\}) = 1,\ f(\{3,4\}) = 1,$$
$$f(\{3,5\}) = 2,\ f(\{4,5\}) = 3.$$

- Choose v_1, v_2 arbitrarily; choose v_3 from $S(\{v_1, v_2\})$; choose v_4 from $S(\{v_2, v_3\})$; stuck at $\{1, 2, 3, 4\}$.
- Choose v_1, v_5 arbitrarily;

The key lemma

Suppose $t = 2$ and we wish to enumerate all edges e with $f_e = f$, where:

$$f(\{1,2\}) = 3,\ f(\{1,3\}) = 2,\ f(\{1,4\}) = 2,\ f(\{1,5\}) = 2,$$
$$f(\{2,3\}) = 4,\ f(\{2,4\}) = 3,\ f(\{2,5\}) = 1,\ f(\{3,4\}) = 1,$$
$$f(\{3,5\}) = 2,\ f(\{4,5\}) = 3.$$

- Choose v_1, v_2 arbitrarily; choose v_3 from $S(\{v_1, v_2\})$; choose v_4 from $S(\{v_2, v_3\})$; stuck at $\{1,2,3,4\}$.
- Choose v_1, v_5 arbitrarily; choose v_2 from $S(\{v_1, v_5\})$;

Suppose $t = 2$ and we wish to enumerate all edges e with $f_e = f$, where:

$$f(\{1,2\}) = 3,\ f(\{1,3\}) = 2,\ f(\{1,4\}) = 2,\ f(\{1,5\}) = 2,$$
$$f(\{2,3\}) = 4,\ f(\{2,4\}) = 3,\ f(\{2,5\}) = 1,\ f(\{3,4\}) = 1,$$
$$f(\{3,5\}) = 2,\ f(\{4,5\}) = 3.$$

- Choose v_1, v_2 arbitrarily; choose v_3 from $S(\{v_1, v_2\})$; choose v_4 from $S(\{v_2, v_3\})$; stuck at $\{1, 2, 3, 4\}$.
- Choose v_1, v_5 arbitrarily; choose v_2 from $S(\{v_1, v_5\})$; choose v_3 from $S(\{v_1, v_2\})$;

The key lemma

Suppose $t = 2$ and we wish to enumerate all edges e with $f_e = f$, where:

$$f(\{1,2\}) = 3,\ f(\{1,3\}) = 2,\ f(\{1,4\}) = 2,\ f(\{1,5\}) = 2,$$
$$f(\{2,3\}) = 4,\ f(\{2,4\}) = 3,\ f(\{2,5\}) = 1,\ f(\{3,4\}) = 1,$$
$$f(\{3,5\}) = 2,\ f(\{4,5\}) = 3.$$

- Choose v_1, v_2 arbitrarily; choose v_3 from $S(\{v_1, v_2\})$; choose v_4 from $S(\{v_2, v_3\})$; stuck at $\{1, 2, 3, 4\}$.
- Choose v_1, v_5 arbitrarily; choose v_2 from $S(\{v_1, v_5\})$; choose v_3 from $S(\{v_1, v_2\})$; choose v_4 from $S(\{v_2, v_3\})$.

Suppose $t = 2$ and we wish to enumerate all edges e with $f_e = f$, where:

$$f(\{1,2\}) = 3,\ f(\{1,3\}) = 2,\ f(\{1,4\}) = 2,\ f(\{1,5\}) = 2,$$
$$f(\{2,3\}) = 4,\ f(\{2,4\}) = 3,\ f(\{2,5\}) = 1,\ f(\{3,4\}) = 1,$$
$$f(\{3,5\}) = 2,\ f(\{4,5\}) = 3.$$

- Choose v_1, v_2 arbitrarily; choose v_3 from $S(\{v_1, v_2\})$; choose v_4 from $S(\{v_2, v_3\})$; stuck at $\{1, 2, 3, 4\}$.
- Choose v_1, v_5 arbitrarily; choose v_2 from $S(\{v_1, v_5\})$; choose v_3 from $S(\{v_1, v_2\})$; choose v_4 from $S(\{v_2, v_3\})$.
  We enumerated $O(n^t k^{t+1})$ $(2t + 1)$-tuples containing all edges with $f_e = f$.

The key lemma

Suppose we get stuck for any t-set $J \in \binom{[2t+1]}{t}$.

The key lemma

Suppose we get stuck for any t-set $J \in \binom{[2t+1]}{t}$. Define a set system $\mathcal{A} \in \mathcal{P}([2t+1])$ as follows:

$$\mathcal{A} = \left\{ S \subsetneq [2t+1] \mid f(I) \in S, \forall I \in \binom{S}{t} \right\}.$$

The key lemma

Suppose we get stuck for any t-set $J \in \binom{[2t+1]}{t}$. Define a set system $\mathcal{A} \in \mathcal{P}([2t+1])$ as follows:

$$\mathcal{A} = \left\{ S \subsetneq [2t+1] \mid f(I) \in S, \forall I \in \binom{S}{t} \right\}.$$

Then, $\mathcal{A}$ is a $(t+1, t)$-system:

Suppose we get stuck for any t-set $J \in \binom{[2t+1]}{t}$. Define a set system $\mathcal{A} \in \mathcal{P}([2t+1])$ as follows:

$$\mathcal{A} = \left\{ S \subsetneq [2t+1] \mid f(I) \in S, \forall I \in \binom{S}{t} \right\}.$$

Then, $\mathcal{A}$ is a $(t+1, t)$-system:

- If $A, B \in \mathcal{A}$, then for any $I \in \binom{A \cap B}{t}$, $f(I) \in A \cap B$, so $A \cap B \in \mathcal{A}$.

The key lemma

Suppose we get stuck for any t-set $J \in \binom{[2t+1]}{t}$. Define a set system $\mathcal{A} \in \mathcal{P}([2t+1])$ as follows:

$$\mathcal{A} = \left\{ S \subsetneq [2t+1] \mid f(I) \in S, \forall I \in \binom{S}{t} \right\}.$$

Then, $\mathcal{A}$ is a $(t+1, t)$-system:

- If $A, B \in \mathcal{A}$, then for any $I \in \binom{A \cap B}{t}$, $f(I) \in A \cap B$, so $A \cap B \in \mathcal{A}$.
- For any $J \in \binom{[2t+1]}{t}$, we get stuck in our enumeration, so J is contained in some set in $\mathcal{A}$.

The key lemma

Suppose we get stuck for any t-set $J \in \binom{[2t+1]}{t}$. Define a set system $\mathcal{A} \in \mathcal{P}([2t+1])$ as follows:

$$\mathcal{A} = \left\{ S \subsetneq [2t+1] \mid f(I) \in S, \forall I \in \binom{S}{t} \right\}.$$

Then, $\mathcal{A}$ is a $(t+1, t)$-system:

- If $A, B \in \mathcal{A}$, then for any $I \in \binom{A \cap B}{t}$, $f(I) \in A \cap B$, so $A \cap B \in \mathcal{A}$.
- For any $J \in \binom{[2t+1]}{t}$, we get stuck in our enumeration, so J is contained in some set in $\mathcal{A}$.
- $S \in \mathcal{A} \implies |S| \notin \{t, 2t+1\}$.

We determined, up to constant factors, the Turán number of sunflowers when the uniformity and kernel size are fixed and the size of the sunflower is allowed to grow with n.

Conclusion

We determined, up to constant factors, the Turán number of sunflowers when the uniformity and kernel size are fixed and the size of the sunflower is allowed to grow with n.

Open problems:

Conclusion

We determined, up to constant factors, the Turán number of sunflowers when the uniformity and kernel size are fixed and the size of the sunflower is allowed to grow with n.

Open problems:

- Getting the correct dependence on r.

Conclusion

We determined, up to constant factors, the Turán number of sunflowers when the uniformity and kernel size are fixed and the size of the sunflower is allowed to grow with n.

Open problems:

- Getting the correct dependence on r.
- Determining the extremal number of so-called generalized stars appearing in the unavoidability problem.

Conclusion

We determined, up to constant factors, the Turán number of sunflowers when the uniformity and kernel size are fixed and the size of the sunflower is allowed to grow with n.

Open problems:

- Getting the correct dependence on r.
- Determining the extremal number of so-called generalized stars appearing in the unavoidability problem.
- Excluding a subset of kernel sizes $T \subseteq \{0, \ldots, r-1\}$. This is hard even for $k = 2$.

We determined, up to constant factors, the Turán number of sunflowers when the uniformity and kernel size are fixed and the size of the sunflower is allowed to grow with n.

Open problems:

- Getting the correct dependence on r.
- Determining the extremal number of so-called generalized stars appearing in the unavoidability problem.
- Excluding a subset of kernel sizes $T \subseteq \{0, \ldots, r-1\}$. This is hard even for $k = 2$.
- Resolving the cases $T = \{0, \ldots, t-1\}$ and $T = \{\ell, \ldots, r-1\}$ with correct dependence on r.

Thank you!