LECTURE 7

Step 1.

– Handle Trading Lemma –

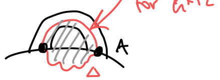
Assume $(W, \partial_0 W, \partial_1 W)$ is an h-cobordism with $\dim W = n \geq 6$ and a handle decomposition with no handles of index $\leq k-1$ for some $1 \leq k \leq n-2$.
Then the decomposition can be modified so that precisely one k-handle is removed and precisely one $(k+2)$-handle is added.

proof of Handle Trading Lemma.

Let h^k be the k-handle we wish to remove.
The idea is to use the reverse of the Cancellation Lemma to add a cancelling pair $h^{k+1} \cup h^{k+2}$ so that h^{k+1} cancels our h^k and leaves h^{k+2} behind.
In other words we will have:

$\partial_0 W \times [0,1] \cup$ handles of index $\leq k$

$$\begin{aligned}
W^{\leq k} &\cong W^{\leq k} \natural D^n && \text{by } \natural D^n \text{ does nothing (exercise)}\\
&\cong (W^{\leq k-1} \cup \text{other } k\text{-handles} \cup h^k) \cup_{\varphi_{k+1}} (h^{k+1} \cup_{\varphi_{k+2}} h^{k+2}) && \text{by reverse of the Cancellation Lemma}\\
&\cong (W^{\leq k-1} \cup \text{other } k\text{-handles}) \cup_{\varphi_k} (h^k \cup_{\varphi_{k+1}} h^{k+1}) \cup_{\varphi_{k+2}} h^{k+2} && \text{by the Reordering Lemma}\\
&\cong (W^{\leq k-1} \cup \text{other } k\text{-handles}) \cup_{\varphi_{k+2}} h^{k+2} && \text{by the Cancellation Lemma.}
\end{aligned}$$



once we find
$A := \varphi_{k+1}|_{S^k \times 0} \subseteq \partial_1 W^{\leq k}$ such that:

1° A goes over h^k geom. once (for CancL to apply) $\Leftrightarrow$ $A \pitchfork$ belt sphere of h^k = {pt}

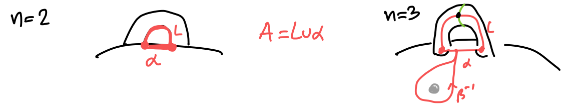
2° A is unknotted (for rev. of CancL to apply) $\Leftrightarrow$ A isotopic to the unknot.

Let us construct such an A. We need to distinguish the case $k=1$ from $k \geq 2$.

Case $k=1$.

works also for $\dim W \geq 5$

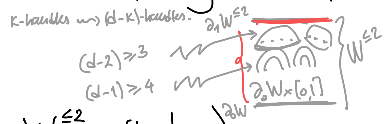
Firstly, let $L \subseteq \partial h^1$ be a push-off of the core of h^1. The endpoints $\partial L \subseteq \partial_0 W$ can be connected by an arc $\alpha \subseteq \partial_0 W$ (by connectedness assumption on $\partial_0 W$) which can be chosen to miss attaching regions of all other 1-handles. Then $A := L \cup \alpha$ is a circle in $\partial_1 W^{\leq 1}$ which can be assumed to be smooth and disjoint from all att. circles of 2-handles, so lives in $\partial_1 W^{\leq 2}$.
By construction, A goes over h^1 geometrically once. 1° ✓



$\partial: D^2 \hookrightarrow \partial_1 W^{\leq 2}$

Lemma. The arc α can be chosen so that $A := L \cup \alpha : S^1 \hookrightarrow \partial_1 W^{\leq 2}$ is nullhomotopic.
Assuming this, we will have that A is unknotted since $\dim(\partial_1 W^{\leq 2}) \geq 4$. 2° ✓ ☐

proof of Lemma. Since attaching a k-handle is homotopy equivalent to attaching a k-cell, only 1- and 2-handles can change π_1.

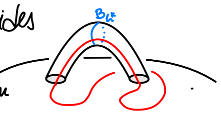


Thus: $\pi_1 W^{\leq 2} \xrightarrow{\cong} \pi_1 W$
and $\pi_1 \partial_1 (W^{\leq 2}) \xrightarrow{\cong} \pi_1 W^{\leq 2}$ (by turning $W^{\leq 2}$ upside down).
By the h-cobordism assumption $\pi_1 \partial_0 W \xrightarrow{\cong} \pi_1 W$. Therefore,
$$\pi_1 \partial_1 W^{\leq 2} \cong \pi_1 \partial_0 W.$$

- If $\pi_1 \partial_0 W \cong \{1\}$ we immediately have $A \sim *$ in $\partial_1 W^{\leq 2}$.
- More generally: A might be nontrivial $[A] \neq 0 \in \pi_1 \partial_1 W^{\leq 2} \cong \pi_1 W^{\leq 2} \cong \pi_1 \partial_0 W$.
  Let β be a loop in $\partial_0 W$ realizing this class, chosen so that it misses all att. spheres of 1- and 2-handles. Thus, β lives in $\partial_1 W^{\leq 2}$, and replacing α with $\alpha\beta^{-1}$ gives $A := L \cdot \alpha\beta^{-1} \simeq *$ in $\partial_1 W^{\leq 2}$ ☐

Case $k \geq 2$.

IDEA: Start from $A :=$ small unknot and isotope it using handle slides until it goes over h^k geometrically once.



Since $H_*(\widetilde{W}, \partial_0 \widetilde{W}; \mathbb{Z}) \overset{\text{h-cob. assumption}}{=} 0$ we have that $\dots \to C^{\widetilde{W}}_{k+1} \xrightarrow{\delta^{\widetilde{W}}_{k+1}} C^{\widetilde{W}}_k \xrightarrow{\delta^{\widetilde{W}}_k} C^{\widetilde{W}}_{k-1} \to \dots$ is exact.
Then $C^{\widetilde{W}}_{k-1} = 0$ implies that $\delta^{\widetilde{W}}_{k+1}$ is surjective.
So: $\exists\, z_j \in \mathbb{Z}, 1 \leq j \leq r_{k+1}, g_j \in \pi$ with $\widetilde{H}^k = \delta^{\widetilde{W}}_{k+1}\left(\sum_{j=1}^{r_{k+1}} z_j g_j \widetilde{H}_j^{k+1}\right)$.

Use HANDLE SLIDES Lemma: we can start from a small unknot $S^k \hookrightarrow \partial_1 W^{\leq k}$ and slide it over handles h_j^{k+1} with coefficients $z_j g_j$ until we have $A : S^k \hookrightarrow \partial_1 W^{\leq k}$ with $[\widetilde{A}] = \sum_{j=1}^{r_{k+1}} z_j g_j \widetilde{H}_j^{k+1}$. Since handle slides are isotopies, A is unknotted.
On the other hand, $\delta^{\widetilde{W}}_{k+1}[\widetilde{A}] = \widetilde{H}^k$ says that A goes over h^k algebraically once.
Then the Whitney Trick Lemma finishes the proof:
A can be improved to go over h^k geom. once. ☐

need: $\dim \partial_1 W^{\leq k-1} \geq 5$ so $\dim W \geq 6$

key Thm [Smale 1961] – s-cobordism Theorem –

If $(W, \partial_0 W, \partial_1 W)$ is an s-cobordism with $\dim W = n \geq 6$,

then it is smoothly trivial,

i.e. there is a diffeomorphism $(W, \partial_0 W, \partial_1 W) \cong (\partial_0 W \times [0,1], \partial_0 W \times \{0\}, \partial_0 W \times \{1\})$.

Proof.

Pick a handle decomposition of $(W, \partial_0 W, \partial_1 W)$.

Thanks to Remove 0- and n-handles Lemma, we can assume NO 0- and n-handles

Step 1. – Normal Form Lemma –

For every h-cobordism of dimension $n \geq 6$ and any $2 \leq \ell \leq n-3$

there is a handle decomposition of the form $\partial_0 W \times [0,1] \cup \bigcup_{i=1}^{r_\ell} h_i^\ell \cup \bigcup_{j=1}^{r_{\ell+1}} h_j^{\ell+1}$.

proof of Normal Form Lemma. We first prove we can remove all handles of index $\leq \ell-1$.

Indeed, using HANDLE TRADING LEMMA we trade

1- for 3-handles, then 2- for 4-handles, etc., $(\ell-1)$- for $(\ell+1)$- handles.

Thus, we have $W \underset{\mathrm{rel}\,\partial_0}{\cong} \partial_0 W \times [0,1] \cup \ell\text{-handles} \cup (\ell+1)\text{-handles} \cup \dots \cup (n-1)\text{-handles}$

Now, we can turn this handle decomposition upside down and repeat the procedure:

in effect, we will be trading $(n-1)$- for $(n-3)$-handles, …, $(\ell+2)$- for ℓ-handles.

Thus, we are left with only ℓ- and $(\ell+1)$-handles, as desired. □

Step 2.

We are left with $0 \to C_{k+1}^{\tilde{u}} \xrightarrow{\delta_{k+1}^{\tilde{u}}} C_k^{\tilde{u}} \to 0$ and we wish to remove these as well.

Since $H_*(C_*^{\tilde{u}}, \delta_*^{\tilde{u}}) = 0$, $\delta_{k+1}^{\tilde{u}}$ is an isomorphism $(\mathbb{Z}\pi)^{r_k} \longrightarrow (\mathbb{Z}\pi)^{r_k}$.

represented by the equivariant intersection matrix $J^{\tilde{u}} := \left(\tilde{I}(\tilde{A}_i \pitchfork \tilde{B}_j)\right)_{1 \leq i,j \leq r_k}$