11

Intersection Numbers and the Statement of the Disc Embedding Theorem

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We carefully state the disc embedding theorem, defining each term that appears therein. In particular, we carefully describe intersection numbers.

11.1 Immersions

Let M be a smooth 4-manifold. Recall that the disc embedding theorem begins with smooth immersions and yields topological, flat embeddings. As usual, an immersion is a local embedding and every map from a surface to a smooth 4-manifold can be approximated by an immersion. That is, every such function is homotopic to an immersion, which can be chosen to be arbitrarily close to the original function. If the boundary of the surface is already immersed, we may assume that it is fixed by the homotopy. We may assume, moreover, that the immersion is in general position; that is, all intersections are transversal, lie in the interior, and are at most double points. Henceforth, we replace maps of surfaces in 4-manifolds by immersions and assume without comment that immersed surfaces have transverse double point intersections, all of which lie in the interior, and no triple points.

A *framed immersion* of an orientable surface F in M is an immersion of F in M such that the normal bundle of the image of F is trivial. Framed immersions can also be defined topologically, as follows [FQ90, Section 1.2]. Given an abstract surface F, form the product $F \times \mathbb{R}^2$. Consider disjoint copies D and E of $\mathbb{R}^2$ in F. Perform a *plumbing* operation on $D \times \mathbb{R}^2$ and $E \times \mathbb{R}^2$. That is, identify $(x, y) \in D \times \mathbb{R}^2$ with $(y, x) \in E \times \mathbb{R}^2$. The orientations of D and E inherited from the standard orientation of $\mathbb{R}^2$ need not be restrictions of the same orientation on F, but we do require that the resulting 4-manifold be orientable. Repeated applications of this procedure for mutually disjoint D and E yields a *plumbed model* for F. A (topological) *framed immersion* of the abstract surface F in M is a map from a plumbed model for F to some open set in M that is a homeomorphism onto

Mark Powell and Arunima Ray, *Intersection Numbers and the Statement of the Disc Embedding Theorem* In: *The Disc Embedding Theorem: with an afterword by Michael H. Freedman.* Edited by: Stefan Behrens, Boldizsár Kalmár, Min Hoon Kim, Mark Powell, and Arunima Ray, Oxford University Press.
DOI: 10.1093/oso/9780198841319.003.0011

its image. Such a homeomorphism determines a map $g\colon F \to M$ when we restrict to the image of F in the framed model, and we say that the map g *extends to a framed immersion*.

The normal bundle of a smoothly immersed, connected, orientable surface is determined up to isomorphism by its Euler number. If the Euler number is even, then the surface is homotopic via local cusp homotopies [FQ90, Section 1.6] to an immersion that extends to a framed immersion. In particular, a cusp homotopy changes the Euler number of the normal bundle of F by ± 2. Performing a local cusp homotopy was called "adding a local kink" in Chapter 1 (see Figure 1.3).

Since D^2 is contractible, the normal bundle of an immersed disc in M is trivial. Fixing an orientation of the fibres determines a framing of the normal bundle, uniquely up to homotopy, again since D^2 is contractible. Recall that a *framing* of a rank n vector bundle over a space B is by definition a trivialization, namely an identification of the total space with $B \times \mathbb{R}^n$. So for $D^2 \looparrowright M$, this means an identification of the total space of the normal bundle with $D^2 \times \mathbb{R}^2$. If M and D^2 are both oriented, then the fibres of the normal bundle of D^2 inherit an orientation. But in general, orientations of the fibres are an auxiliary choice that we make in order to proceed, as in the next paragraph.

The framing of the normal bundle of an immersed disc induces a framing of the normal bundle restricted to the boundary ∂D^2. When a framing of this restricted normal bundle inducing the same orientation on fibres is independently specified, we may consider the *twisting number*, or *relative Euler number*, of the induced framing with respect to the specified framing. This twisting number is an integer (coming from $\pi_1(SO(2)) \cong \mathbb{Z}$). In such a situation, we say that the immersed disc is *framed* when the twisting number is zero, so that the two framings match up to homotopy. For us, this will mostly occur in three scenarios:

(1) when the disc is *properly immersed*; that is, when the preimage of ∂M is the boundary of the disc;

(2) when the disc is attached to a simple closed curve in an immersed surface in M, in which case we will consider the framing induced by the tangent bundle of the surface;

(3) when the disc is a Whitney disc, in which case we consider the Whitney framing. This latter case will be described in more detail soon.

In general, given an immersed surface F in M, if the boundary of F is nonempty, we will usually have a fixed framing already prescribed on the boundary. In this case, we say that a map $g\colon F \to M$ *extends to a framed immersion* if it extends to a framed immersion restricting to the given framing on ∂F. For any immersed, connected surface F with nonempty boundary $\partial F \subseteq M \smallsetminus \partial M$ in the interior of M, there is a homotopy via boundary twists, as described in Section 15.2.2 (see also [FQ90, Section 1.3]), to an immersion that extends to a framed immersion. This is because, as we will see, a single boundary twist changes the relative Euler number by ± 1.

A *regular homotopy* in the smooth category is a homotopy through immersions. It is well known that a smooth regular homotopy of immersed surfaces in a 4-manifold is generically a concatenation of smooth isotopies, finger moves, and Whitney moves with respect to smoothly embedded and framed Whitney discs whose interiors are in the complement of the surfaces. Indeed, by [GG73, Section III.3], generic immersions are dense in the space of smooth mappings, and the generic singularities for a regular homotopy between surfaces are precisely arcs of isolated double points, which may appear (finger move) or disappear (Whitney move) at finitely many distinct time values. Whitney and finger moves, which are inverse to each other, were introduced in Chapter 1, and we give further details in a moment (see also [FQ90, Chapter 1]).

A *topological regular homotopy* of immersed surfaces in a 4-manifold is by definition a concatenation of (topological) isotopies, finger moves, and Whitney moves with respect to topologically flat, embedded, and framed Whitney discs whose interiors are in the complement of the surfaces. Regular homotopies of immersed surfaces with boundary take place in the interior of the surfaces, unless stated otherwise. For example, a finger move pushing an intersection point off the boundary of a disc, such as in Figure 11.4, is not permitted as part of a regular homotopy. Having said that, we will often change surfaces by this move (see, for example, Section 15.2.4). In that case, we perform an isotopy of the surface that moved and a homotopy of the collection, but the motion does not count as a regular homotopy of the collection.

11.2 Whitney Moves and Finger Moves

11.2.1 Whitney Moves

Recall from Chapter 1 that a Whitney move is designed to remove two points of intersection between two immersed surfaces, or of an immersed surface with itself, within a smooth ambient 4-manifold M. If the associated Whitney disc is framed and embedded, with interior in the complement of the surfaces, then the resulting surfaces indeed have two fewer intersections. If the Whitney disc is framed but not embedded, or the interior intersects the surfaces, then the two original intersection points are still removed, but other double points might be introduced in the process. We now give further details.

Let f and g be two immersed oriented surfaces in a smooth 4-manifold M, with possibly $f = g$. Let p and q be two points in $f \pitchfork g$ such that there is an embedded arc γ in the interior of f from p to q and an embedded arc δ in the interior of g from p to q where the union $\gamma\delta^{-1}$ bounds an embedded disc D whose interior lies in the complement of f and g. See Figure 11.1. Fix a local orientation of M at p, and transport it along γ to q. Now, comparing with the orientations of T_pM and T_qM determined by the orientations of f and g yields a function $\operatorname{sgn}\colon \{p, q\} \to \{+, -\}$.

As before, the normal bundle of D in M is a trivial 2-plane bundle. Fix an orientation on the fibres. Consider the following 1-plane sub-bundle V of the normal bundle of D restricted to $\partial D = \gamma\delta^{-1}$. The sub-bundle along γ is given by the tangent bundle to f. This can be extended to a choice of sub-bundle along δ that is normal to g and agrees with Tf at p and q, since the intersections are transverse. This is a trivial 1-plane bundle if and only if the function $\operatorname{sgn}\colon \{p, q\} \to \{+, -\}$ is surjective; that is, if and only if the signs of p and q are opposite.

Assuming that this is the case, choose a section s of the sub-bundle V. Since V is 1-dimensional, the section s is determined up to multiplication by a continuous function $S^1 \to \mathbb{R} \setminus \{0\}$. We say that the Whitney disc D is *framed* if the section s extends to a

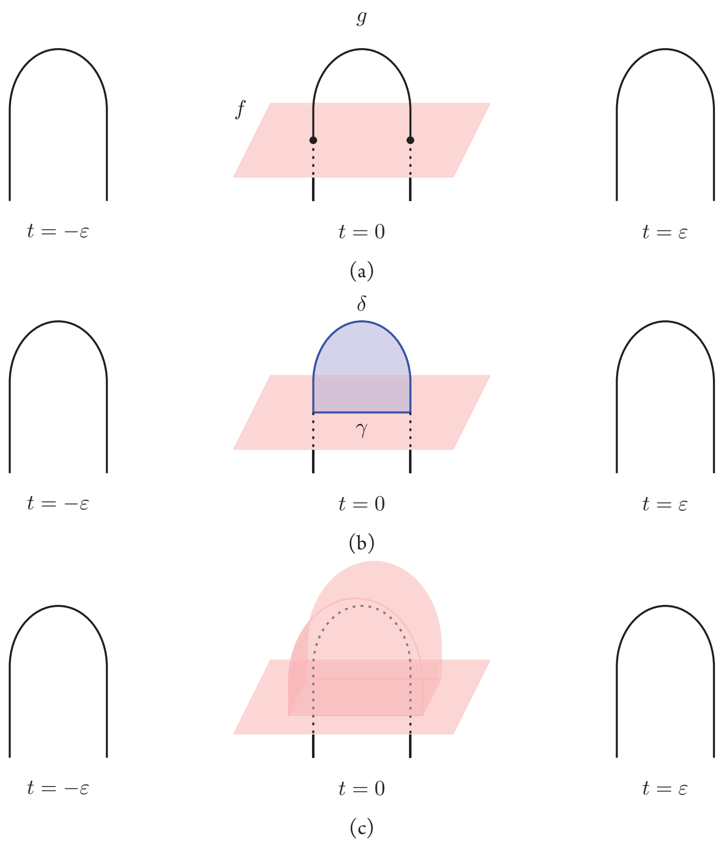

Figure 11.1 A model Whitney move. (a) A region in a smooth 4-manifold M is shown modelled on $\mathbb{R}^3 \times \mathbb{R}$, where the last co-ordinate t is interpreted as time. The central image shows a small region in an immersed surface f (red). A small region on the immersed surface g (black) is traced out by the black curves as we move backwards and forwards in time.
(b) The two points of intersection between f and g from (a) are shown to be paired by an embedded Whitney disc (blue) with interior in the complement of f and g. The boundary of the disc is given by the union of arcs $\gamma \cup \delta$.
(c) The Whitney move on f along the Whitney disc has removed the two points of intersection.

nonvanishing section on the normal bundle of all of D. The framing of the normal bundle of D restricted to ∂D, induced by s and the chosen orientation on the fibres of the normal bundle, is called the *Whitney framing*.

Now extend the Whitney disc very slightly beyond its borders; more precisely, extend γ slightly beyond p and q in A and push δ out along the radial direction of $TD|_\delta$; that is, the direction orthogonal to $T\delta$. Now consider the disc bundle $DE \cong D^2 \times D^1$, which is the sub-bundle of the normal bundle of (the extended version of) D determined by the section s, where D coincides with the zero section. The boundary of DE is a 2-sphere, with $\partial(DE) \cap f$ a neighbourhood of γ, that we denote by S. The Whitney move pushes the strip S across DE. The outcome has S replaced by two parallel copies of the Whitney disc D together with a strip whose core is parallel to δ. This is an isotopy of the surface f (if $f \neq g$) and a regular homotopy of $f \cup g$. The latter fact holds, since we have described a homotopy through local embeddings. Note that we used a framed and embedded Whitney disc with interior in the complement of $f \cup g$, and the two intersection points p and q were removed, as desired.

In the case that D is framed but not embedded, or the interior intersects $f \cup g$, the Whitney move, now called a *(framed) immersed Whitney move*, still uses the same strip S in a neighbourhood of δ and two copies of D obtained using s and $-s$, where s is a section of the normal bundle. The resulting move is a regular homotopy of f and not an isotopy, even if $f \neq g$. We state this fact here, but prove it in Section 15.3.

Proposition 11.1 *A (framed) immersed Whitney move is a regular homotopy.*

In particular, the intersection points p and q are removed by an immersed Whitney move, but four new self-intersection points of f are created for each self-intersection point of D, and two new intersections of $f \cup g$ are created for each intersection of the interior of D with $f \cup g$.

In more generality, if D intersects a surface Σ other than itself, where Σ may equal f or g but need not, then two intersection points of f with Σ are created for each intersection point of D with Σ.

11.2.2 Finger Moves

As we saw in Chapter 1, a *finger move* is a regular homotopy that adds two intersection points between two immersed surfaces f and g in a smooth 4-manifold M, where possibly $f = g$. It is supported in a neighbourhood of an arc. and is depicted in Figure 11.2.

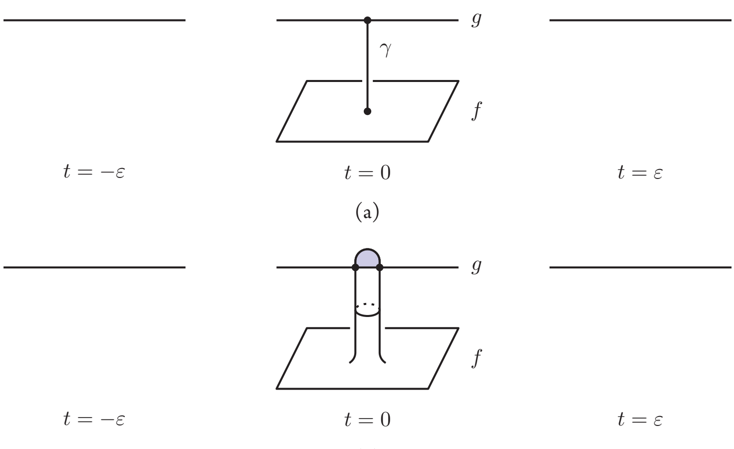

Figure 11.2 A model finger move along the arc γ, shown (a) before and (b) after the move. As usual, the ambient space is shown as $\mathbb{R}^3$ slices moving through time (see the caption to Figure 11.1).

Let $\gamma \cong [0,1]$ be an embedded path in M with an endpoint on f and an endpoint on g, away from any double points, and otherwise disjoint from $f \cup g$. We describe the finger move of f on g along γ. Thicken γ to $\gamma \times D^2$ such that $\{0\} \times D^2 \subseteq f$ is a neighbourhood of one end of γ. Extend γ slightly beyond its other endpoint on g, to an embedding of $[0, 5/4]$. Suppose that $(\{1\} \times D^2) \cap \Sigma$ is a single arc. The finger move pushes $\{0\} \times D^2 \subseteq f$ across $[0, 5/4] \times D^2$, replacing $\{0\} \times D^2$ with the rest of the boundary $([0, 5/4] \times S^1) \cup (\{5/4\} \times D^2)$.

The finger move adds two intersection points between f and g, which are paired by a framed, embedded Whitney disc with interior in the complement of $f \cup g$, should it be required. Note that if $f \neq g$, the finger move is an isotopy of f and a regular homotopy of $f \cup g$, since we have described a homotopy through local embeddings.

11.3 Intersection and Self-intersection Numbers

Let M be a smooth 4-manifold. Assume for a moment that M is compact and based and let π denote $\pi_1(M)$ based at the basepoint. The *equivariant intersection form* λ on M is the pairing

$$\begin{array}{rcl} \lambda\colon H_2(M;\mathbb{Z}\pi) \times H_2(M,\partial M;\mathbb{Z}\pi^w) & \to & \mathbb{Z}\pi \\ (x,y) & \mapsto & \langle PD^{-1}(y), x\rangle. \end{array}$$

Here $H_2(M;\mathbb{Z}\pi)$ denotes the second homology of M with twisted coefficients, which is isomorphic to the second homology of the universal cover of M; that is, to $\pi_2(M)$. The homomorphism $w\colon \pi \to \{\pm 1\}$ is the *orientation character*, which by definition satisfies that $w(\alpha) = -1$ if and only if α is orientation reversing. The orientation character makes the group ring $\mathbb{Z}\pi$ into a left $\mathbb{Z}\pi$-module denoted by $\mathbb{Z}\pi^w$ with action $g \cdot r := w(g)gr$, for $g \in \pi$ and $r \in \mathbb{Z}\pi^w$, extended linearly. The Poincaré duality map $PD\colon H^2(M;\mathbb{Z}\pi) \to H_2(M,\partial M;\mathbb{Z}\pi^w)$ is an isomorphism, and the pairing denoted by $\langle \cdot, \cdot \rangle$ is the Kronecker pairing. The cohomology group with $\mathbb{Z}\pi$ coefficients is isomorphic to cohomology with compact support of the universal cover of M.

We prefer not to restrict to compact manifolds. To avoid getting into the details of cohomology with compact support, we will use λ from now on to denote a different notion of intersection number, with a more geometric definition. The notion will be applicable to every smooth, connected 4-manifold, including noncompact and nonorientable 4-manifolds with arbitrarily many boundary components, and coincides with the above definition whenever both apply [Ran02, Proposition 7.22]. The new definition will be for intersections between smooth, based immersions of discs or spheres that intersect transversely in double points in their interiors, with no triple points. In particular, discs need not have boundaries mapping to the boundary of M and might not represent homology classes: this is another key reason that we use the geometric definition of λ in this book rather than the homological definition.

In the following definition, and in the rest of the book, we regularly abuse notation by conflating a map f and its image.

Definition 11.2 Let M be a connected, based, smooth 4-manifold with a fixed local orientation at the basepoint. Let f and g be smoothly immersed, transversely intersecting, based, oriented discs or spheres in M. If applicable, assume that f and g have disjointly embedded boundaries. Let v_f and v_g be paths in M joining the basepoint of M to the basepoint of f and g respectively. The paths v_f and v_g are called *whiskers* for f and g, respectively. Define the following sum

$$\lambda(f,g) := \sum_{p \in f \pitchfork g} \varepsilon(p)\alpha(p),$$

where

- γ_f^p is a simple path in f from the basepoint of f to p and γ_g^p is a simple path in g from the basepoint of g to p, as in Figure 11.3, such that γ_f^p and γ_g^p are disjoint from all the other points in $f \pitchfork g$;
- $\varepsilon(p) \in \{\pm 1\}$ is $+1$ when the local orientation at p induced by the orientations of f and g matches the one obtained by transporting the local orientation at the basepoint of M to p along $v_g\gamma_g^p$, and is -1 otherwise;
- $\alpha(p)$ is the element of $\pi_1(M)$ given by the concatenation $v_f\gamma_f^p(\gamma_g^p)^{-1}v_g^{-1}$.

Since f and g are (immersed) discs or spheres, distinct choices of γ_f^p and γ_g^p are homotopic, and thus $\lambda(f,g)$ is a well defined element of $\mathbb{Z}[\pi_1(M)]$.

We also define $\lambda(f,f) := \lambda(f,f^+)$, where f^+ is a push-off of f along a section of the normal bundle transverse to the zero section. If f is an immersed disc with embedded boundary equipped with a specified framing for the normal bundle restricted to the boundary, then f^+ is defined to be the push-off of f along a section restricting to one of the vectors of that framing on ∂f.

Remark 11.3 Note that in Definition 11.2, we need the simple connectivity of spheres and discs for the intersection number λ to be well defined. In surfaces with nontrivial

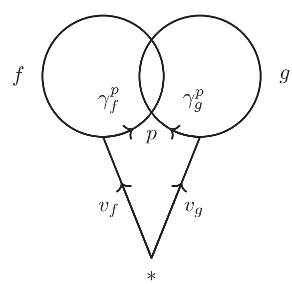

Figure 11.3 Computation of the intersection number $\lambda(f,g)$ for spheres f and g, denoted schematically as circles, in a 4-manifold M, with basepoint $*$.

fundamental group, the choice of path from the basepoint to the double point could change the value of λ, albeit in a controlled manner.

We also note that $\varepsilon(p)$ does not depend on the choice of path γ_g^p when g is an (immersed) disc or sphere, since any two choices of path are homotopic relative the endpoints and thus induce the same local orientation at p. However, $\varepsilon(p)$ might depend on the choice of whisker v_g if, for example, we preconcatenate v_g with an orientation reversing loop at the basepoint of M.

Define an involution on $\mathbb{Z}[\pi_1(M)]$ by setting

$$h = \sum_{\alpha\in\pi_1(M)} n_\alpha \alpha \mapsto \overline{h} = \sum_{\alpha\in\pi_1(M)} w(\alpha) n_\alpha \alpha^{-1},$$

where $n_\alpha \in \mathbb{Z}$ and $w\colon \pi_1(M) \to \{\pm 1\}$ is the orientation character. The next proposition summarizes the properties of the intersection number.

Proposition 11.4 *Let M be a connected, based, smooth 4-manifold with a fixed local orientation at the basepoint. Let f and g be smoothly immersed, transversely intersecting, based, oriented discs or spheres in M with whiskers v_f and v_g, respectively, and, if applicable, disjointly embedded boundaries. In particular, f and g only intersect in their interiors. The intersection number $\lambda(f, g)$ has the following properties.*

(i) The intersection number $\lambda(f, g)$ is unchanged by regular homotopies in the interiors of f and g. The intersection number is not preserved by a regular homotopy pushing an intersection point off the boundary of an immersed disc, as in Figure 11.4.

(ii) λ is hermitian; that is, $\lambda(f, g) = \overline{\lambda(g, f)}$.

(iii) A different choice of whisker v_f' for f results in multiplication of $\lambda(f, g)$ on the left by the element $v_f' v_f^{-1}$ of $\pi_1(M)$. A different choice of whisker v_g' for g results in multiplication

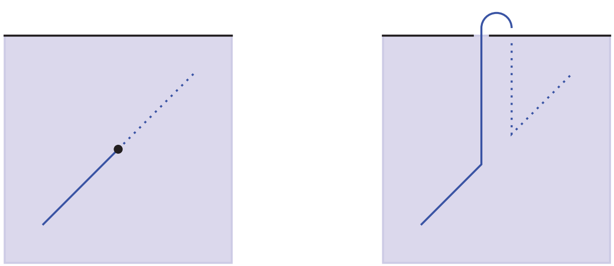

Figure 11.4 Left: A 3-dimensional snapshot of an intersection of an immersed surface f (shown as a blue interval) with an immersed disc g (shaded blue) in an ambient 4-manifold. The interval locally traces out f as we move backwards and forwards in time, as in Figures 11.1 and 11.2. Right: A finger move on f across the boundary of g removes an intersection point, and so changes the intersection number $\lambda(f, g)$.

of $\lambda(f,g)$ on the right by the element $v_g(v'_g)^{-1}$ of $\pi_1(M)$, and changes the sign by $w(v_g(v'_g)^{-1})$.

(iv) *Changing the orientation of f changes the sign of $\lambda(f,g)$, as does changing the orientation of g. Changing the orientation of both f and g leaves $\lambda(f,g)$ unchanged.*

In the homological version of λ discussed above, (iii) implies that λ is sesquilinear; that is, $\lambda(rf, sg) = r\lambda(f,g)\overline{s}$, for $r,s \in \mathbb{Z}[\pi_1(M)]$.

Proof For (i), we only need to check that the intersection number is preserved under finger moves and Whitney moves along framed, embedded Whitney discs with interiors in the complement of $f \cup g$. By the requirement that the regular homotopies occur in the interiors of f and g, double points are introduced or eliminated in pairs.

Suppose the intersection points p and q between f and g are paired by a framed, immersed Whitney disc, as described in Section 11.2.1. Then there is a path γ_f^{pq} in f from p to q and a path γ_g^{pq} in g from p to q such that $\gamma_f^{pq}(\gamma_g^{pq})^{-1}$ is null-homotopic in M. Moreover, since the Whitney framing exists, $\varepsilon(p) = -\varepsilon(q)$. Let γ_f^p be a path in f from the basepoint of f to p and let γ_g^p be a path in g from the basepoint of g to p as in Definition 11.2. Then the contribution of p and q to $\lambda(f,g)$ is the sum

$$\varepsilon(p)v_f\gamma_f^p(\gamma_g^p)^{-1}{v_g}^{-1} + \varepsilon(q)v_f\gamma_f^p\gamma_f^{pq}(\gamma_g^{pq})^{-1}(\gamma_g^p)^{-1}{v_g}^{-1},$$

which is zero in $\mathbb{Z}[\pi_1(M)]$, since $\varepsilon(p) = -\varepsilon(q)$ and $\gamma_f^{pq}(\gamma_g^{pq})^{-1}$ is null-homotopic in M. Since the Whitney disc is embedded and framed with interior in the complement of $f \cup g$, the Whitney move removes the two intersection points p and q, creates no new intersection points, and preserves the contribution of all other intersection points to $\lambda(f,g)$. It follows that the intersection number $\lambda(f,g)$ is preserved under a Whitney move along framed, embedded Whitney discs with interior in the complement of $f \cup g$. Additionally, this finishes the proof of (i), since a finger move in the interior creates a pair of intersection points paired by an embedded, framed Whitney disc, with interior in the complement of $f \cup g$.

Property (ii) is a direct consequence of the definition

$$\lambda(f,g) := \sum_{p \in f \pitchfork g} \varepsilon(p)\alpha(p),$$

noting that switching the order of f and g replaces every $\alpha(p)$ by $\alpha(p)^{-1}$, while the sign $\varepsilon(p)$ changes precisely when $\alpha(p)$ is orientation reversing.

Property (iii) is also a direct consequence of the definition, since, given another whisker v'_f for f, each $\alpha(p)$ changes by multiplication on the left by $v'_f v_f^{-1}$, and similarly, given another whisker v'_g for g, each $\alpha(p)$ changes by multiplication on the right by $v_g(v'_g)^{-1}$. The local orientation is transported along $(v'_g v_g^{-1})v_g\gamma_g = v'_g\gamma_g$ instead of $v_g\gamma_g$, so $\varepsilon(p)$ changes by $w(v'_g v_g^{-1}) = w((v'_g v_g^{-1})^{-1}) = w(v_g(v'_g)^{-1})$.

Property (iv) follows directly from the definition of $\varepsilon(p)$ for a point $p \in f \pitchfork g$. □

Next we define the *self-intersection number* of a smoothly immersed, based, oriented sphere or disc. The definition is analogous to the intersection number λ. However, for the self-intersection number, there is an ambiguity coming from the choice of sheets at each intersection point, as indicated in Proposition 11.4(ii). As a result, the self-intersection number is only well defined as an element of a quotient of $\mathbb{Z}[\pi_1(M)]$, as follows.

Definition 11.5 Let M be a connected, based, smooth 4-manifold with a fixed local orientation at the basepoint. Let f be a smoothly immersed, based, oriented disc or sphere in M with a whisker v, and embedded boundary if applicable. Let $w\colon \pi_1(M) \to \{\pm 1\}$ be the orientation character of M. Define the following sum:

$$\mu(f) := \sum_{p \in f \pitchfork f} \varepsilon(p)\alpha(p),$$

where

- γ_1^p and γ_2^p are simple paths in f from the basepoint to p along two different sheets, as in Figure 11.5, such that γ_f^p and γ_g^p are disjoint from all the other points in $f \pitchfork g$;
- $\varepsilon(p) \in \{\pm 1\}$ is $+1$ when the local orientation at p induced by the orientation of f matches the one obtained by transporting the local orientation at the basepoint of M to p along $v\gamma_2^p$, and is -1 otherwise;
- $\alpha(p)$ is the element of $\pi_1(M)$ given by the concatenation $v\gamma_1^p(\gamma_2^p)^{-1}v^{-1}$.

It follows from the proof of Proposition 11.4(ii) that $\mu(f)$ is a well defined element of the quotient group $\mathbb{Z}[\pi_1(M)]/(a \sim \overline{a})$.

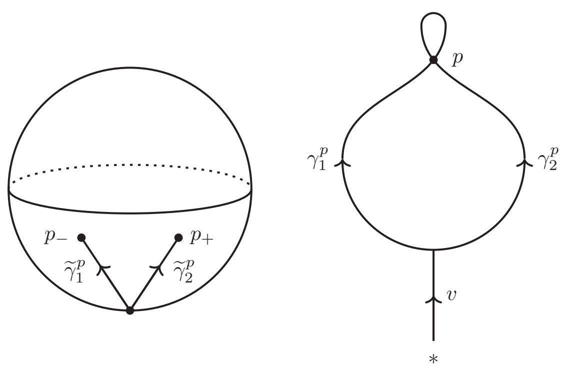

Figure 11.5 Computation of the self-intersection number $\mu(f)$ for an immersed sphere f in a 4-manifold M with basepoint $*$
Left: The points p_+ and p_- in S^2 map to the self-intersection point p of f. Lifts of γ_1^p and γ_2^p to S^2 are shown.
Right: The paths γ_1^p and γ_2^p approach p on two different sheets in the image of f.

Remark 11.6 As before for λ, we need the simple connectivity of spheres and discs in order for μ to be well defined.

Unlike before, $\varepsilon(p)$ does depend on the choice of the path γ_1^p, even when f is an (immersed) disc or sphere. In particular, the local orientation at p, induced by transporting the local orientation at the basepoint of M along $v\gamma_1^p$ and along $v\gamma_2^p$, differ exactly when the loop $\gamma_1^p(\gamma_2^p)^{-1}$ is orientation reversing. Thus in this case the value of $\varepsilon(p)$ depends on the choice of sheet at p. The difference in $\varepsilon(p)$ is encoded within the definition of the involution $\overline{h} = w(h)h^{-1}$, for $h \in \pi_1(M)$. As before, once a sheet is chosen, that is, once one of the two preimage points of p is chosen, the value of $\varepsilon(p)$ is well defined, modulo the choice of whisker v.

By virtually the same proof as Proposition 11.4, we have the following properties of the self-intersection number.

Proposition 11.7 *Let M be a connected, based, smooth 4-manifold with a fixed local orientation at the basepoint. Let f be a smoothly immersed, based, oriented disc or sphere in M with a whisker v, and embedded boundary if applicable. The self-intersection number $\mu(f)$ has the following properties.*

(i) The self-intersection number $\mu(f)$ is unchanged by regular homotopies in the interior of f. The intersection number is not preserved by a regular homotopy pushing an intersection point off the boundary of an immersed disc, as in Figure 11.4.

(ii) A different choice of whisker v' for f results in conjugation of $\mu(f)$ by the element $v'v^{-1}$ of $\pi_1(M)$ and multiplication by $w(v'v^{-1})$.

Moreover, the intersection and self-intersection numbers of an immersed sphere or disc are related by the following helpful formula.

Proposition 11.8 *Let M be a connected, based, smooth 4-manifold with a fixed local orientation at the basepoint. Let f be a smoothly immersed, based, oriented disc or sphere in M. In the case that f is an immersed disc, assume that the boundary is embedded and the normal bundle of the disc restricted to the boundary has a specified framing. Let v be a whisker for f. Then*

$$\lambda(f,f) = \mu(f) + \overline{\mu(f)} + \chi,$$

where $\chi \in \mathbb{Z}$ is the Euler number of the normal bundle of f if f is a sphere, or is the twisting number of the framing induced on the boundary by the restriction of the canonical framing of the normal bundle of the immersed disc with respect to the specified framing when f is an immersed disc.

Note that $\mu(f)$ and $\lambda(f,f)$ do not lie in the same group. However, the formula above still holds, since $\mu(f) + \overline{\mu(f)} \in \mathbb{Z}[\pi_1(M)]$ is well defined; that is, it is independent of the choice of lift of $\mu(f)$ to $\mathbb{Z}[\pi_1(M)]$. Such a lift corresponds to a choice of ordering of the sheets of f at each double point of f.

Proof When $\chi = 0$, this is a straightforward consequence of the definitions, observing that each self-intersection point p of f gives rise to a pair of intersection points between f and

f^+. Here, recall that $\lambda(f,f) := \lambda(f,f^+)$ where f^+ is a push-off of f along a section of the normal bundle transverse to the zero section if f is a sphere and is defined to be the push-off along a section restricting to the specified framing on the normal bundle of the disc restricted to the boundary if f is a disc. The key point is that if one of the new intersection points contributes $\varepsilon(p)\alpha(p)$ to $\lambda(f,f)$, then the other contributes $\overline{\varepsilon(p)\alpha(p)}$.

By the definition of f^+, there are χ intersection points of f and f^+ which do not arise from self-intersection points of f. Such an intersection point r corresponds to the trivial element of $\pi_1(M)$: if γ^r is a path in f joining the basepoint to r, then $\alpha(r)$ may be chosen to be the concatenation $v\gamma^r((\gamma^r)^+)^{-1}(v^+)^{-1}$, where $(\gamma^r)^+$ and v^+ are push-offs of γ^r and v respectively. This concatenation is null-homotopic in M. □

By Proposition 11.8, if f is a immersed sphere with $\lambda(f,f) = 0$, then the Euler number of the normal bundle is even, and so f is homotopic to a map that extends to a framed immersion, via local cusp homotopies. If both $\lambda(f,f)$ and $\mu(f)$ vanish, then the Euler number must already be zero and f extends to a framed immersion. Moreover, we have the following corollary, showing that in many cases $\lambda(f,f) = 0$ implies that $\mu(f) = 0$ for an immersed sphere or disc f.

Corollary 11.9 *Let M be a connected, based, smooth 4-manifold with a fixed local orientation at the basepoint. Suppose that the orientation character vanishes on all the order two elements of $\pi_1(M)$. Let f be a smoothly immersed, based, oriented disc or sphere in M. In case f is an immersed disc, assume that the boundary is embedded and the normal bundle of the disc restricted to the boundary has a specified framing. Assume that f is framed.*

Then $\lambda(f,f) = 0$ implies that $\mu(f) = 0$.

Proof For each equivalence class $\{\alpha,\alpha^{-1}\} \in \pi_1(M)/\sim$, where $\alpha \sim \beta$ if $\alpha = \beta$ or $\alpha^{-1} = \beta$ for any $\alpha,\beta \in \pi_1(M)$, choose a representative $r(\alpha)$. That is, $r(\alpha)$ is either α or α^{-1} (or both) for every $\alpha \in \pi_1(M)$. Write

$$\mu(f) = \sum_{\{\alpha,\alpha^{-1}\}\in\pi_1(M)/\sim} n_\alpha r(\alpha).$$

Then, using Proposition 11.8, since f is framed, we have

$$\begin{aligned} 0 = \lambda(f,f) &= \mu(f) + \overline{\mu(f)} \\ &= \sum_{\{\alpha,\alpha^{-1}\}\in\pi_1(M)/\sim} n_\alpha r(\alpha) + w(\alpha) n_\alpha r(\alpha)^{-1}. \end{aligned}$$

Equating coefficients, and using that if $\alpha = \alpha^{-1}$, then $w(\alpha) = 1$ by hypothesis, we see that $n_\alpha = 0$ for all $\alpha \in \pi_1(M)$. □

On the other hand, if there exists $\alpha \in \pi_1(M)$ with $w(\alpha) = -1$ and $\alpha^2 = 1$ and f is such that $\mu(f) = \alpha$, then $\lambda(f,f) = 0$. So the hypothesis of the corollary cannot be removed.

So far, we have discussed intersection and self-intersection numbers in general. In the forthcoming proof of the disc embedding theorem, we will primarily use only the following proposition.

Proposition 11.10 *Let M be a connected, smooth 4-manifold. Let f and g be immersed discs or spheres in M intersecting transversely, and with embedded boundaries if applicable.*

(1) *The quantity $\lambda(f,g)=0$ for some choice of basepoints for M, f, and g and some choice of whiskers for f and g, if and only if all the intersection points of f and g can be paired up by framed, immersed Whitney discs in M with disjointly embedded boundaries. These discs may intersect one another, themselves, and f and g. Similarly, $\lambda(f,g)=1$ for some choice of basepoints for M, f, and g, and some choice of orientations and whiskers for f and g, if and only if all but one intersection point can be paired up in such a manner.*

(2) *The quantity $\mu(f)=0$ for some choice of basepoint for M and f, and some choice of whisker for f, if and only if all the self-intersection points of f can be paired up by framed, immersed Whitney discs in M with disjointly embedded boundaries. These discs may intersect one another, themselves, and f.*

Remark 11.11 In the statement of Proposition 11.10, the vanishing of $\lambda(f,g)$ or $\mu(f)$ for some choice of basepoints for M, f, and g and some choice of whisker for f and g implies the vanishing for all choices of basepoints and whiskers by Propositions 11.4(iii) and 11.7(ii). Thus in these cases it is meaningful to refer to the vanishing of intersection and self-intersection numbers for immersed discs or spheres with no specified choice of basepoints or whiskers. A different choice of whisker or basepoint changes nonzero values of $\lambda(f,g)$ or $\mu(f)$, as dictated by Propositions 11.4(iii) and 11.7(ii). By Proposition 11.10, from now on, for spheres or discs f and g with no specified choice of whisker or basepoint, when we say $\lambda(f,g)=1$ we mean that all but one of the intersection points may be paired up by Whitney discs in the ambient 4-manifold.

Proof Suppose that $\lambda(f,g)=0$ with respect to whiskers v_f and v_g for f and g respectively, corresponding to some chosen basepoints. Then the contributions of the intersection points between f and g to $\lambda(f,g)$ must cancel in pairs. Let p and q be intersection points of f and g such that the contributions of p and q to $\lambda(f,g)$ cancel each other. That is, $\varepsilon(p)=-\varepsilon(q)$ and $\alpha(p)=\alpha(q)$ in $\pi_1(M)$. In other words, $\alpha(p)\alpha(q)^{-1}$ is the trivial element of $\pi_1(M)$. Let γ_f^p be a path in f from the basepoint of f to p as in Definition 11.2 and let γ_f^{pq} be a path in f from p to q. Similarly, let γ_g^p be a path in g from the basepoint of g to p and let γ_g^{pq} be a path in g from p to q. We may assume that all these paths are disjointly embedded. Then

$$\begin{aligned}\alpha(p)\alpha(q)^{-1} &= (v_f\gamma_f^p(\gamma_g^p)^{-1}v_g^{-1})(v_g\gamma_g^p\gamma_g^{pq}(\gamma_f^{pq})^{-1}(\gamma_f^p)^{-1}v_f^{-1})\\ &= v_f\gamma_f^p\gamma_f^{pq}(\gamma_g^{pq})^{-1}(\gamma_f^p)^{-1}v_f^{-1}\end{aligned}$$

is trivial in $\pi_1(M)$. Thus $\alpha(p)\alpha(q)^{-1}$ is a basepoint-changing conjugation away from the loop $\gamma_f^{pq}(\gamma_g^{pq})^{-1}$, so this loop is also null-homotopic in M. The trace of this null homotopy gives a Whitney disc pairing p and q in M. This gives one direction of the argument. The reverse direction holds, since double points paired by a Whitney disc have opposite signs and they have the same element of $\pi_1(M)$ associated with them. A similar argument applies when $\lambda(f,g)=1$.

As we will show in Section 15.2.3, after an isotopy of the Whitney discs and their boundaries, we may arrange for any collection of Whitney circles to be mutually disjoint and embedded. Then by boundary twisting, explained in detail in Section 15.2.2, there is a homotopy of each Whitney disc to a framed Whitney disc. The homotopy is supported in a neighbourhood of one of the boundary arcs. See Remark 15.1.

We have to be a bit more careful when considering the self-intersection number, since it takes values in the quotient group $\mathbb{Z}[\pi_1(M)]/(a \sim \overline{a})$. Assume that $\mu(f) = 0$ with respect to a whisker v. We now introduce some new notation for the rest of the proof. Given a self-intersection point p of f and an arc γ from the basepoint of f to p, let $\varepsilon_\gamma(p) \in \{\pm 1\}$ equal $+1$ when the local orientation at p induced by the orientation of f matches the one obtained by transporting the local orientation at the basepoint of M to p along $v\gamma$, and equal -1 otherwise.

As before, since $\mu(f) = 0$, the contributions of the self-intersection points of f to $\mu(f)$ must cancel in pairs. Let p and q be self-intersection points of f such that the contributions of p and q to $\mu(f)$ cancel each other. Then there exist lifts to $\mathbb{Z}[\pi_1(M)]$ of these contributions which cancel each other as elements of $\mathbb{Z}[\pi_1(M)]$. Recall that such a lift to $\mathbb{Z}[\pi_1(M)]$ corresponds precisely to a choice of ordering of the sheets of f at both p and q.

Let γ_i, for $i \in \{1, 2, 3, 4\}$, be disjointly embedded arcs in f such that γ_1 goes from the basepoint of f to p along the second sheet of f at p, γ_2 goes from p to itself, leaving on the second sheet of f at p and returning on the first, γ_3 goes from p to q, leaving on the first sheet of f at p and ending on the second sheet of f at q, and γ_4 goes from q to itself, leaving on the second sheet of f at q and returning on the first (see Figure 11.6). Then the contribution of p to $\mu(f)$, lifted to $\mathbb{Z}[\pi_1(M)]$, is

$$\varepsilon(p)\alpha(p) = \varepsilon_{\gamma_1}(p) v\gamma_1\gamma_2\gamma_1^{-1}v^{-1}.$$

The sheet change occurs between γ_2 and γ_1^{-1}. Similarly, the contribution of q to $\mu(f)$, lifted to $\mathbb{Z}[\pi_1(M)]$, is

$$\varepsilon(q)\alpha(q) = \varepsilon_{\gamma_1\gamma_2\gamma_3}(q) v\gamma_1\gamma_2\gamma_3\gamma_4\gamma_3^{-1}\gamma_2^{-1}\gamma_1^{-1}v^{-1},$$

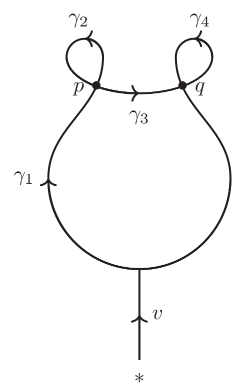

Figure 11.6 Finding Whitney discs pairing self-intersection points of an immersed sphere f when $\mu(f) = 0$.

where the sheet change occurs between γ_4 and γ_3^{-1}. Since these lifts cancel in $\mathbb{Z}[\pi_1(M)]$ by the choice of ordering of the sheets, we have $\varepsilon(p)\alpha(p) = -\varepsilon(q)\alpha(q)$. That is,

$$\varepsilon_{\gamma_1}(p) = -\varepsilon_{\gamma_1\gamma_2\gamma_3}(q)$$

and

$$v\gamma_1\gamma_2\gamma_1^{-1}v^{-1} = v\gamma_1\gamma_2\gamma_3\gamma_4\gamma_3^{-1}\gamma_2^{-1}\gamma_1^{-1}v^{-1}$$

in $\pi_1(M)$. In other words,

$$\begin{aligned}(v\gamma_1\gamma_2^{-1}\gamma_1^{-1}v^{-1})(v\gamma_1\gamma_2\gamma_3\gamma_4\gamma_3^{-1}\gamma_2^{-1}\gamma_1^{-1}v^{-1})\\ = v\gamma_1\gamma_3\gamma_4\gamma_3^{-1}\gamma_2^{-1}\gamma_1^{-1}v^{-1}\end{aligned}$$

is trivial in $\pi_1(M)$. This is a basepoint-changing conjugation away from the loop $\gamma_3\gamma_4\gamma_3^{-1}\gamma_2^{-1}$, so that loop is null-homotopic in M. The trace of this null homotopy produces a map of a disc in M bounded by the curve $\gamma_3\gamma_4\gamma_3^{-1}\gamma_2^{-1}$. Note that the change of sheets occurs between γ_4 and γ_3^{-1}. We know that $\varepsilon_{\gamma_1}(p) = -\varepsilon_{\gamma_1\gamma_2\gamma_3}(q)$, which implies that $\varepsilon(p) = -\varepsilon(q)$ in the definition of $\mu(f)$. Again, as we shall show in Section 15.2.3, we may arrange for the Whitney circles to be disjointly embedded and, by boundary twisting (Section 15.2.2), for each Whitney disc to be framed (see Remark 15.1). As before, the reverse direction holds, since double points paired by a Whitney disc have algebraically cancelling contributions to $\mu(f)$. This completes the proof of Proposition 11.10. □

Given an immersed disc or sphere f in a smooth, connected 4-manifold M, an immersion $g\colon S^2 \looparrowright M$ is said to be a *transverse sphere* or a *dual sphere* for f if f and g intersect transversely and $\lambda(f,g) = 1$. This was called *algebraically transverse* in Chapters 1 and 2. By Proposition 11.10, this means that all but one intersection point between f and g can be paired by Whitney discs in M. If $\lambda(f,g) = 0$, we say that f and g have *algebraically cancelling intersections*. More generally, for a set $\{f_i\}$ of immersed discs or spheres in M, a set of immersed spheres $\{g_i\}$ is said to be a collection of (algebraically) *transverse* or *dual* spheres if, for every i, j and some choices of orientations and whiskers, we have that f_i and g_j intersect transversely and $\lambda(f_i, g_j) = \delta_{ij}$. The collection $\{g_i\}$ is said to be a *geometrically transverse* or *geometrically dual* collection of spheres if, in addition, $f_i \pitchfork g_j$ is a single point when $i = j$ and is empty otherwise.

11.4 Statement of the Disc Embedding Theorem

Now we state the disc embedding theorem (see [FQ90, Theorem 5.1A; PRT20]). Recall that, since D^2 is contractible, the normal bundle of every immersed disc in a 4-manifold is trivial. For an immersed disc, a choice of orientation of the fibres of the normal bundle determines a framing of the normal bundle, inducing a framing of the normal bundle restricted to the boundary. We say that two immersed discs $f, \overline{f}\colon (D^2, S^1) \looparrowright (M, \partial M)$ have *the same framed boundary* if $f(S^1) = \overline{f}(S^1) \subseteq \partial M$, and there are choices of orientations of the fibres of the normal bundles of f and $\overline{f}$ such that the induced framings on the boundaries are homotopic.

Disc embedding theorem *Let M be a smooth, connected 4-manifold with nonempty boundary and such that $\pi_1(M)$ is a good group. Let*

$$F = (f_1, \ldots, f_n)\colon (D^2 \sqcup \cdots \sqcup D^2, S^1 \sqcup \cdots \sqcup S^1) \looparrowright (M, \partial M)$$

be an immersed collection of discs in M with pairwise disjoint, embedded boundaries. Suppose that F has an immersed collection of framed, algebraically dual 2-spheres

$$G = (g_1, \ldots, g_n)\colon S^2 \sqcup \cdots \sqcup S^2 \looparrowright M;$$

that is, $\lambda(f_i, g_j) = \delta_{ij}$ with $\lambda(g_i, g_j) = 0 = \mu(g_i)$ for all $i, j = 1, \ldots, n$.

Then there exists a collection of pairwise disjoint, flat, topologically embedded discs

$$\overline{F} = (\overline{f}_1, \ldots, \overline{f}_n)\colon (D^2 \sqcup \cdots \sqcup D^2, S^1 \sqcup \cdots \sqcup S^1) \hookrightarrow (M, \partial M),$$

with geometrically dual, framed, immersed spheres

$$\overline{G} = (\overline{g}_1, \ldots, \overline{g}_n)\colon S^2 \sqcup \cdots \sqcup S^2 \looparrowright M,$$

such that, for every i, the discs $\overline{f}_i$ and f_i have the same framed boundary and $\overline{g}_i$ is homotopic to g_i.

In other words, within a smooth 4-manifold M with good fundamental group, we can replace a collection of immersed discs $\{f_i\}$, equipped with a collection of algebraically transverse spheres $\{g_i\}$ with vanishing intersection and self-intersection numbers, with pairwise disjoint, flat, embedded discs $\{\overline{f}_i\}$ equipped with a collection of geometrically transverse spheres $\{\overline{g}_i\}$, such that, for every i, f_i and $\overline{f}_i$ have the same framed boundary.

We do not require M to be compact, since we use the geometric definition of the intersection number λ. In the upcoming proof, we will only use intersection number information to find Whitney discs, as described by Proposition 11.10.

We will define good groups in the next chapter (Definition 12.12), once we have the requisite terminology.